Universität Bonn
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Generalized Verlinde Formula and Unifying W-Algebras

by

Ralf Hübel

Abstract

In the first part of this thesis we prove a generalization of the Verlinde formula to fermionic rational conformal field theories. Using 'simple current' arguments one finds that the fusion coefficients of the fermionic theory are equal to sums of fusion coefficients of the bosonic projection. We argue that one is forced to weaken the axioms of fusion algebras for fermionic theories. Our results are applied to several examples e.g. fermionic $\mathcal{W}(2,\delta)$-algebras and minimal models of the $N = 1$ super Virasoro algebra.
In the second part we consider quantum $\mathcal{W}$-algebras belonging to the new class of finitely non-freely generated $\mathcal{W}$-algebras. These algebras occur in cosets and orbifolds of $\mathcal{W}$-algebras and have the property that the classical analogs are infinitely non-freely generated. We demonstrate that orbifolding of $\mathcal{W}$-algebras has quite different properties in the quantum and the classical case respectively. We proceed with the study of truncations of quantum Casimir $\mathcal{W}$-algebras at degenerate values of the central charge c. Choosing a suitable parametrization of the central charge c in terms of the rank of the Casimir algebra, the rank of the truncated algebra is fixed if the rank of the Casimir algebra is high enough. This gives rise to the so-called 'unifying' $\mathcal{W}$-algebras which belong to the class of finitely non-freely generated quantum $\mathcal{W}$-algebras. They can also be inferred from level-rank duality of dual coset pairs leading to coset realizations. We present coset realizations of other unifying $\mathcal{W}$-algebras generalizing level-rank duality.

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Wisdom and Spirit of the universe!
Thou Soul that art the eternity of thought!
That giv'st to forms and images a breath
And everlasting motion! not in vain,
By day or star-light thus from my first dawn
Of Childhood didst Thou intertwine for me
The passions that build up our human Soul,
Not with the mean and vulgar works of Man,
But with high objects, with enduring things,
With life and nature, purifying thus
The elements of feeling and thought,
And sanctifying, by such discipline,
Both pain and fear, until we recognize
A grandeur in the beatings of the heart.

from *The Prelude, W. Wordsworth*

meinen Eltern

Contents

Philosophy is written in this great book (by which I mean the universe) which stands always open to our view, but it cannot be understood unless one first learns how to comprehend the language and interpret the symbols in which it is written, and its symbols are triangles, circles, and other geometric figures, without which it is not humanly possible to comprehend even one word of it; without these one wanders in a dark labyrinth.

Galileo Galilei (1623)

1 Introduction

If one reviews the enormous achievements in our understanding of the fundamental processes in nature made in the 20^{th} century, one has to admit that symmetry principles play a very prominent rôle. The Noether theorem provides a connection of symmetry and conservation laws observed in experiments. It states that to each generator of the symmetry group of transformations leaving invariant a given physical system one obtains a conserved charge. According to quantum mechanics the states of a physical system are represented as elements of a (in general infinite-dimensional) projective Hilbertspace. Hence one naturally has to study the projective representations of symmetry groups. In various cases of interest this question is simplified by the fact that a universal central extension of the symmetry group exists for which all projective representations are induced by linear representations. A beautiful demonstration of the power of this theoretical method is the prediction of the possible values of the spin of massive particles and helicities of massless particles gained from the study of the irreducible linear representations of the universal central extension of the restricted Lorentz group [Si]. Even much more impressive is the great success of gauge theories in elementary particle physics providing us with a unified description of all interactions besides the gravitational force. Despite of these fascinating insights in the structure of matter, quantum field theory (QFT) has not been formulated on a satisfactory rigorous mathematical level so far. Apart from mathematical problems theoretical physicists are faced with the problem of finding a unified quantum field theory of all four interactions including gravity.

The question of quantum gravity led to a renaissance of string theory which still seems to be the most promising way for constructing such a theory. String theory is the natural playground for two dimensional conformally invariant quantum field theory which developed as an independent branch in mathematical physics in the last years. Among various connections to pure mathematical areas of research like knot theory and classification of three manifolds, two-dimensional conformal quantum field theory (CFT) is intimately related to the theory of critical phenomena. Two dimensional statistical systems with second order phase transitions possess conformal invariance at the critical point and can thus be described by 2d CFTs (see e.g. [CH]). The critical exponents are related to the representation theory of the underlying symmetry algebra of the conformal field theory.

One of the most interesting questions in two dimensional conformal field theory is the classification of rational conformal field theories (RCFT) which have the property that their Hilbert space decomposes into a finite number of irreducible representations of the maximal symmetry algebra. It is not only a question of mathematical interest but has important consequences for the classification of the universality classes of 2d second order phase transitions and the classification of all possible string vacua in super string theory. RCFTs with effective central charge $\tilde{c} < 1$ can be classified exploiting the representation theory of the Virasoro algebra. In order to find RCFTs with $\tilde{c} \geq 1$ it is necessary to study extended conformal algebras and their representations. The investigation of these $\mathcal{W}$-symmetries in conformal field theory is an important step towards the desired classification (see e.g. [BS] and references therein). In this approach one tries to construct an (associative) algebra of local quantum fields and searches for rational models investigating the representation theory thereof.

A promising way to characterize a RCFT is given by their fusion algebras which are finite-dimensional commutative associative algebras with integer valued structure constants in a special basis given by the finite number of primary fields of the RCFT (see e.g. [Fu]). The fusion coefficients are introduced as dimensions of intertwiner spaces of representations of the chiral algebra underlying the RCFT. The fusion rules of a RCFT are directly linked to the modular properties of the theory via the famous Verlinde formula [Ver,MS]. The characters of a RCFT provide a representation of the modular group $SL(2,\mathbb{Z})$ and Verlinde's formula relates the fusion coefficients to the representation matrix of the modular transformation $S : \tau \to -1/\tau$.

In the first part of this thesis we inquire into the fusion algebras of fermionic rational conformal field theories, i.e. RCFTs where the underlying chiral algebra includes fermionic fields. We prove a generalization of the Verlinde formula under certain assumptions and apply it to several examples, e.g. minimal models of the $N = 1$ super Virasoro algebra and fermionic $\mathcal{W}$-algebras. Furthermore, we find that the requirement of a unitary S-matrix is not tenable for fermionic RCFTs any more. An immediate consequence is the observation that for fermionic RCFTs fusion coefficients connecting two conjugate Ramond fields with the identity can take the values one or two. In particular, the 'fusion charge conjugation matrix' does in general not equal the 'modular charge conjugation matrix'. We come to the conclusion that one has to weaken the axioms of fusion algebras of fermionic RCFTs.

As indicated above a very important tool in the classification of RCFTs is provided by extended conformal algebras ($\mathcal{W}$-algebras) (see e.g. [BS,FL2,Za]). Although $\mathcal{W}$-algebras have been the object of intense studies in the last years, a satisfactory classification of $\mathcal{W}$-algebras and their representations has not been achieved yet. One has to distinguish between $\mathcal{W}$-algebras which exist only for fixed values of the central charge c, called nondeformable, and deformable (or generic) $\mathcal{W}$-algebras existing for generic central charge c. For the latter class the structure constants are continuous functions of c apart from a finite set of singularities. Intense studies revealed that it is possible to explain most of the nondeformable $\mathcal{W}$-algebras as extensions and truncations of other $\mathcal{W}$-algebras (see e.g. [Bl1,KW,Eh1,Hf2,Hf3]).

It has been noticed very recently that the class of deformable $\mathcal{W}$-algebras contains at least two subclasses whose classical counterparts have different features [BFH]. The first class consists of deformable $\mathcal{W}$-algebras originating from quantum Drinfeld-Sokolov (DS) type hamiltonian reduction of affine Kac-Moody algebras (see e.g. [Fe3,BT,Fe1]). These algebras are by far the best understood ones and their main property is that the classical counterparts are Poisson bracket (PB) algebras based on finitely and freely generated rings of differential polynomials. It is believed that these algebras can be classified by $sl(2,\mathbb{R})$ embeddings into simple Lie algebras [BW,Fe3,Fe2]. The so-called Casimir algebras [BBSS] correspond to the principal embedding. In contrast hereto the second class of generically existing quantum $\mathcal{W}$-algebras have as classical counterparts infinitely generated rings of differential polynomials with infinitely many relations, i.e. the ring is non-freely generated [BFH].

Apart from the coset construction there is another construction which leads to $\mathcal{W}$-algebras with infinitely non-freely generated classical counterparts: orbifolding of $\mathcal{W}$-algebras (see e.g. [Di,Ho3,FRS]). If a given $\mathcal{W}$-algebra possesses an outer automorphism one considers the projection onto the invariant subspace. A special case of this construction is the bosonic projection of a $\mathcal{W}$-algebra containing fermionic fields. We show that in contrast to the classical case [FRS] the orbifold of a quantum Casimir algebra does in general not contain a Casimir algebra as a subalgebra for generic central charge.

In the second part of this thesis we discuss features of quantum $\mathcal{W}$-algebras belonging to this new class. The first observation is that although the classical PB algebra is infinitely generated the quantum $\mathcal{W}$-algebra is generated by a finite number of simple fields in all cases studied up to now. This is due to the fact that on the quantum level normal ordered versions of the classical relations between the generators eliminate all but a finite number of generators. A very intriguing point is that the unexplained solutions of $\mathcal{W}(2,4,6)$ [KW,Eh3] and $\mathcal{W}(2,3,4,5)$ [Hf2] with generic null fields obtained by direct construction are of this type and can be explained in terms of coset constructions (compare [BFH]). In [BFH] it has been argued that all classical coset algebras are infinitely generated. Here we argue for certain cases that the corresponding quantum $\mathcal{W}$-algebras are *finitely* generated. In the case of $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ we give arguments that the commutant of the $\widehat{U(1)}$-current yields the unexplained solution of $\mathcal{W}(2,3,4,5)$. We also present supporting arguments for the realization [BFH] of the previously unexplained solution of $\mathcal{W}(2,4,6)$ as the diagonal coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$.

Furthermore, we show that the $\mathcal{W}$-algebras of the second class have the interesting property of being unifying $\mathcal{W}$-algebras for special series of minimal models of Casimir $\mathcal{W}$-algebras. To this end identifications between a priori different algebras $\mathcal{W}$-algebras for special values of c are studied which in turn imply truncations of the algebras under consideration (some of the generators become null fields). From the study of the Kac determinant of Casimir $\mathcal{W}$-algebras we conclude that such truncations are a very common phenomenon predicting whole series of unifying $\mathcal{W}$-algebras. We also give explicit coset realizations for many of these unifying algebras which generalize level-rank-duality of coset pairs [BG,Al]. In special cases like the unitary models of the $\mathcal{WA}_{k-1}$ algebras the unifying algebras can be inferred directly from level-rank-duality. The coset $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ for example describes the first unitary model of the $\mathcal{WA}_{k-1}$ Casimir algebra, the so-called $\mathbb{Z}_k$ parafermions [FZ]. For the corresponding values of the central charge $c = \frac{2(k-1)}{k+2}$ the algebras $\mathcal{WA}_{k-1}$ truncate to $\mathcal{W}(2,3,4,5)$. Even in the cases where level-rank-duality can be exploited our considerations go beyond the 'T-equivalence' of coset pairs [BS,BG,Al]. We compute the spins of the finite generating set of the unifying algebras and check in some examples the isomorphism of the extended symmetry algebras.

The unifying algebras for $\mathcal{WC}_k$ minimal models are the new algebras $\mathcal{WD}_{-n}$ [Hf3,Bl4]. We argue that these algebras $\mathcal{WD}_{-n}$ can be realized in terms of a diagonal $\widehat{sp(2n)}$ coset. The first member $\mathcal{WD}_{-1}$ is the formerly unexplained $\mathcal{W}(2,4,6)$. The existence of $\mathcal{WD}_{-n}$ can be expected from the connection of the negative-dimensional orthogonal groups $SO(-2n)$ (see e.g. [Du]) and the symplectic groups $Sp(2n)$.

This thesis is organized as follows. In section 2 we give a brief introduction into the necessary material for reading the main body of the thesis. In particular the notion of $\mathcal{W}$-algebras (section 2.1) and fusion algebras of rational conformal field theories (section 2.2) are reviewed.

In section 3 some fundamental properties of fermionic RCFTs are discussed and the explicit form of the generalized Verlinde formula is presented. The results are applied to various examples, e.g. fusion algebras of the (unitary as well as non-unitary) minimal models of the $N=1$ super Virasoro algebra.

In section 4.1 we work out explicitly the algebraic structure of some cosets ($\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$, $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$) and argue that they lead to finitely generated algebras. As a further important example we investigate the diagonal coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$ which explains for $\mu = -\frac{1}{2}$ the special solution of $\mathcal{W}(2,4,6)$. We continue with a short study of orbifolds of quantum $\mathcal{W}$-algebras and determine the spin content of several orbifolds. Furthermore, we discuss the vacuum preserving algebra (VPA) and classical limits of some prominent examples of deformable $\mathcal{W}$-algebras outside the Drinfeld-Sokolov class. It turns out that the VPA is actually infinite dimensional in contrast to algebras in the DS class. Section 4.2 deals with the truncation and identification of Casimir $\mathcal{W}$-algebras gained from a study of structure constants. Next the discussion is turned to the realization of unifying $\mathcal{W}$-algebras. Using character techniques we compute the spin content of these coset algebras and discuss the relation to level-rank-duality.

The calculations in the thesis have been performed in parts with a small special purpose computer algebra system [Ho1] which we used for calculations with $\mathcal{W}$-algebra modules. For the more complicated OPE calculations we used the Mathematica-package of ref. [Th].

The essential parts of this thesis are based on the following publications:

[EH] W. Eholzer, R. Hübel, *Fusion Algebras of Fermionic Rational Conformal Field Theories via a Generalized Verlinde Formula*, Nucl. Phys. **B414** (1994) p. 348

[Bl4] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hübel, *Unifying $\mathcal{W}$-Algebras*, Phys. Lett. **B332** (1994) p. 51

[Bl5] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck, R. Hübel, *Coset Realization of Unifying $\mathcal{W}$-Algebras*, to appear in Int. J. Mod. Phys. **A**

2 Preliminaries: $\mathcal{W}$-algebras, Cosets, Fusion algebras

In this section we will give a very short account on $\mathcal{W}$-algebras, cosets and fusion algebras in order to fix the notation for the following sections. A satisfactory discussion of the concepts repeated here is beyond the scope of this thesis and we refer the reader therefore to the literature (see e.g. [BS,Gi1] and references therein).

2.1 $\mathcal{W}$-algebras, Cosets

In order to present the notion of a $\mathcal{W}$-algebra, we have to recall some basic facts of two-dimensional conformal field theory (CFT). If one introduces complex coordinates $z = x+iy, \bar{z} = x-iy$ in the underlying 2-dimensional euclidian manifold, the conformal transformations are given by the holomorphic and antiholomorphic mappings $z \to f(z), \bar{z} \to \bar{f}(\bar{z})$. Thus the conformal symmetry group has the intriguing property of being infinitely generated in two dimensions in contrast to the higher-dimensional case. Because the left-moving and right-moving components of the theory can be treated analogously we restrict to the left-moving z-component. The unique (modulo isomorphisms) one-dimensional central extension of the conformal algebra Lie(Diff(S^1)) is the well known Virasoro-algebra:

$$[L_m, L_n] = (n-m)L_{n+m} + \tfrac{C}{12}(n^3 - n)\delta_{n+m,0}. \tag{2.1}$$

According to Schur's lemma the central element C is represented in each irreducible representation by $c\mathbb{I}$, where c is called the 'central charge'. The Virasoro algebra possesses an $sl(2,\mathbb{C}) = \langle L_{-1}, L_0, L_1 \rangle$ subalgebra which corresponds to the Möbiustransformations or rational conformal transformations. Using the Laurent expansion of the energy-momentum tensor $T(z)$ of the conformal field theory into the 'modes' L_n ($\mathcal{C}_0$ is a contour around the origin)

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{n-2}, \qquad L_n = \oint_{\mathcal{C}_0} \frac{dz}{2\pi i}\, T(z)\, z^{-n+1}, \tag{2.2}$$

the Virasoro algebra is equivalent to the singular part of the OPE of $T(z)$ with itself [1]) :

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.. \tag{2.3}$$

Notice that the product $T(z)T(w)$ has to be understood radially ordered. Objects of particular importance in CFT are primary fields of scale dimension $\Delta = d(\phi)$ which transform under an element f of the conformal group according to the rule

$$\phi(z) \mapsto \phi'(z) = \left(\frac{\partial f(z)}{\partial z}\right)^{\Delta} \phi(f(z)) \tag{2.4}$$

If (2.4) is valid only for the Möbius subgroup $\phi(z)$ is called quasi-primary. The commutation relations of the modes of a primary field ϕ_n ($\phi(z) = \sum_{n-d(\phi)\in\mathbb{Z}} \phi_n\, z^{n-d(\phi)}$) with the generators L_n of the conformal algebra look as follows:

$$[L_n, \phi_m] = (m - n(d(\phi)-1))\phi_{n+m}. \tag{2.5}$$

[1]) We will use the following shorthand notation for the OPE of two fields ϕ_i, ϕ_j: $\phi_i \star \phi_j = \sum C_{ij}^k \phi_k$. C_{ij}^k are the coupling constants between the respective conformal families.

For a quasi-primary field this equation is valid for $n \in \{0, \pm 1\}$ only; the energy-momentum tensor is thus a quasi-primary field of conformal dimension 2.
A further requirement to the theory is the existence of a complex vector space $\mathcal{H}_0$ (vacuum representation) with a cyclic highest weight state $|0\rangle$ ('vacuum') which is annihilated by all negative modes of the fields in the field algebra $\mathcal{F}$ of the CFT. The modes of the fields are endomorphisms of $\mathcal{H}_0$ and $\mathcal{H}_0$ provides an irreducible representation of the mode algebra corresponding to $\mathcal{F}$. The vacuum state obeys the additional property that for quasi-primary fields $\phi(z)$ $\lim_{z\to 0}\phi(z)|0\rangle$ is regular. This implies that all modes ϕ_n with $n < d(\phi)$ of a quasi-primary field of dimension $d(\phi)$ annihilate $|0\rangle$ (for $\phi = L$ this implies the $sl(2)$-invariance of $|0\rangle$). Additionally one requires that $\mathcal{F}$ is spanned by all quasi-primary fields and their derivatives and that the continuation to $\mathcal{F}$ of the map $\phi(z) \to \lim_{z\to 0}\phi(z)|0\rangle = \phi_{d(\phi)}|0\rangle$ for quasi-primary $\phi \in \mathcal{F}$ defines an isomorphism between $\mathcal{F}$ and $\mathcal{H}_0$. Otherwise formulated one requires that $\mathcal{H}_0$ decomposes into $sl(2) = \langle L_{-1}, L_0, L_1\rangle$ highest weight representations with the highest weight states corresponding to the quasi-primary fields in $\mathcal{F}$. From (2.5) one obtains an L_0 grading of $\mathcal{H}_0$. It is also assumed that the spectrum of L_0 in $\mathcal{H}_0$ is bounded from below by 0 and that the graded components of $\mathcal{H}_0$ are finite dimensional. A very important property for all concrete calculations is the existence of nondegenerate invariant bilinear form $(.,.)$ on $\mathcal{H}_0$ satisfying $(\phi_n u, v) = (u, \phi_n^\dagger v)$ with $\phi_n^\dagger = \phi_{-n}$ for ϕ quasi-primary.
We continue with the analysis of the algebraic structure of the algebra of local chiral fields $\mathcal{F}$ underlying the CFT in order to reveal the basic features of $\mathcal{W}$-algebras. As already indicated above the multiplicative structure of $\mathcal{F}$ is given by the operator product expansion (OPE). The form of the OPE of two quasi-primary fields ϕ_i, ϕ_j is constrained by conformal invariance so that only some structure constants C_{ij}^k - depending only on the dimensions h_i, h_j, h_k of the fields involved - remain free [Na1]:

$$\phi_i(z)\phi_j(w) = \frac{d_{ij}}{(z-w)^{h_i+h_j}} + \sum_k C_{ij}^k \sum_{n=0}^{\infty} \frac{a_{ijk}^n}{n!} \frac{\partial^n \phi_k(w)}{(z-w)^{h_i+h_j-h_k-n}}, \tag{2.6}$$

where the first sum runs over the quasi-primary fields and

$$a_{ijk}^n = \binom{2h_k+n-1}{n}^{-1} \binom{h_i+h_j+h_k+n-1}{n}. \tag{2.7}$$

d_{ij} is the 'metric' on the space of quasi-primary fields induced by the invariant bilinear form on $\mathcal{H}_0$. Apart from the derivative the field algebra $\mathcal{F}$ is equipped with the operation of normal ordering. The modes of the normal ordered product $N(\psi, \phi)$ of two local chiral fields ψ, ϕ are defined as:

$$N(\psi,\phi)_n := \epsilon_{\phi\psi} \sum_{k<d(\psi)} \phi_{n-k}\psi_k + \sum_{k\geq d(\psi)} \psi_k \phi_{n-k}. \tag{2.8}$$

$\epsilon_{\phi\psi}$ is equal to -1 for ϕ and ψ fermionic (half-integer conformal dimension) and otherwise equal to 1. Normal ordered products occur naturally in the regular part of the OPE of the corresponding fields. For several reasons it is necessary to write down the quasi-primary projection $\mathcal{N}(\phi, \psi)$ of the field $N(\phi, \psi)$. The explicit formula is omitted but can be found e.g. in ref. [Bl1]. The normal ordered product can be used to define a reduced generating set of the algebra $\mathcal{F}$. A quasi-primary field is simple if it is nonderivative and orthogonal to all quasi-primary normal ordered products with respect to the invariant bilinear form on $\mathcal{H}_0$. A set of simple fields is said to generate $\mathcal{F}$ if their normal ordered products and derivatives span $\mathcal{F}$. If $\mathcal{F}$ is generated by a finite number of simple fields it is called finitely generated.

Expressed in terms of the Laurent modes of the local chiral quasi-primary fields ϕ_i with dimensions $h_i \in \mathbb{N}/2$ (due to locality) the singular part of the OPE (2.6) is equivalent to the following commutator formula

$$[\phi_{i,m}, \phi_{j,n}]_{\pm} = \sum_{k \in I} C_{ij}^{k}\, p_{ijk}(m,n) \phi_{k,m+n} + d_{ij}\, \delta_{n,-m} \binom{n + h(i) - 1}{2h(i) - 1}, \tag{2.9}$$

with $C_{ij}^{l} d_{lk} = C_{ijk}$ and $d_{ij} = \langle 0 | \phi_{i,-h(i)} \phi_{j,h(j)} | 0 \rangle$, $C_{ijk} = \langle 0 | \phi_{k,-h(k)} \phi_{i,h(k)-h(j)} \phi_{j,h(j)} | 0 \rangle$. The universal polynomials $p_{ijk}(m,n)$ depend only on the dimensions of the quasi-primary fields involved and can be found in [Bl1,Eh5]. As stated above the structure constants C_{ij}^{k} describing the interactions of the CFT are not restricted by conformal invariance. The main additional assumption allowing a determination of these three-point functions is the associativity of the OPE which is equivalent to the Jacobi identity of the commutator (2.9). An equivalent way to formulate this 'dynamical symmetry' is to require crossing symmetry of the four-point functions of the CFT. Now we have collected the basic facts to present the notion of a $\mathcal{W}$-algebra:

> The associative algebra of local chiral fields which is generated by normal ordered products and derivatives of the finitely many simple fields $T(z) = \phi_1(z)$, $\phi_2(z), \ldots, \phi_n(z)$ with dimensions $d(\phi_1) = 2, d(\phi_2), \ldots, d(\phi_n)$ is called a finitely generated $\mathcal{W}$-algebra of type $\mathcal{W}(2, d(\phi_2), \ldots, d(\phi_n))$.

This definition does not exclude the existence of different $\mathcal{W}$-algebras with the same spectrum of generators. The concept introduced so far allows the direct construction of $\mathcal{W}$-algebras with a small number of generators with dimensions not too high in an algorithmic manner [Bl1,KW,K1,Hf2,Bl3]. The first example of a 'true' $\mathcal{W}$-algebra not being just an infinite-dimensional Lie-algebra like the Virasoro algebra and Kac-Moody algebras is the $\mathcal{W}(2,3)$ constructed first by Zamolodchikov [Za]. The existence of different solutions of the $\mathcal{W}$-algebra $\mathcal{W}(2,4,6)$ [KW] has been the germ for the discovery of 'unifying $\mathcal{W}$-algebras' which are discussed in this thesis (see also [Eh3,BFH,Bl4,Bl5]).

The framework for a rigorous mathematical treatment of the statements presented above is given by the theory of Vertex Operator Algebras (VOA) developed in the last few years. Here one starts with a $\mathbb{Z}$-graded complex vector space $\mathcal{H}_0$ ('vacuum representation') and associates to each element of $\mathcal{H}_0$ a vertex operator which is a formal power series in z, z^{-1} with coefficients in the endomorphisms of $\mathcal{H}_0$ ('fields'). These objects satisfy several axioms which embody the requirements of a CFT. A nice fact is that the commutator formula and the notion of the normal ordered product can be unified into a single axiom. $\mathcal{W}$-algebras are a special kind of VOA's. A short introduction and the relevant literature can be found in [Eh5].

It is straight-forward to define highest weight representations (HWRs) of $\mathcal{W}$-algebras. A HWR of a finitely generated $\mathcal{W}$-algebra $\mathcal{A}$ is a complex $\mathbb{N}$-graded vector space $\mathcal{H}$ (the graded components being finite dimensional) containing a cyclic vector $|v\rangle$ (highest weight state) together with a linear map ρ from the mode algebra of $\mathcal{A}$ to the endomorphism-algebra of $\mathcal{H}$ such that the images of the lexicographic ordered monomials of the non-negative modes of the generating fields of $\mathcal{A}$ applied to $|v\rangle$ generate the whole space $\mathcal{H}$, whereas $\rho(\phi_n)|v\rangle = |0\rangle$ $(\phi \in \mathcal{A})$ for negative n. Of course the map ρ has to satisfy several covariance properties in order to be compatible with the algebraic structure of $\mathcal{A}$. In the language of VOA's one requires the same properties as in the definition of the VOA but replaces the endomorphisms by their images under ρ (for a precise definition see e.g. [Eh5]). We also assume that $\rho(L_0)$ (defining the grading in $\mathcal{H}$) is diagonalizable in the graded components of $\mathcal{H}$. The algebraic compatibility conditions have some important consequences. Whereas for infinite-dimensional Lie-algebras like the Virasoro-algebra there exists an irreducible HWR for each highest weight this is in general not the case for $\mathcal{W}$-algebras. For particular values of the central charge of a $\mathcal{W}$-algebra $\mathcal{A}$ there exist differential polynomials in $\mathcal{A}$ (linear combinations of normal ordered products of the fields and their derivatives) which are identically zero in the vacuum representation $\mathcal{H}_0$. They are called 'null fields'. Due to the compatibility conditions of the representation map ρ these null fields have to act trivially in each representation. In particular, this constraint can restrict the number of 'allowed' representations of a $\mathcal{W}$-algebra to a finite set for special values of the central charge ('minimal series of the $\mathcal{W}$-algebra'). The most prominent example is the minimal series of the Virasoro-algebra. If one considers the Virasoro-algebra as a $\mathcal{W}$-algebra it has been shown [BPZ] that for c lying in the minimal series $c_{\text{Vir}}(p,q) = 1 - 6(p-q)^2/(pq)$ (p, q coprime integers) there exist only finitely many HWRs with the highest weights $h_{p,q}(r,s) = \left((pr-qs)^2 - (p-q)^2\right)/(4pq)$, $0 < r < q, 0 < s < p$.

If a $\mathcal{W}$-algebra has only finitely many inequivalent irreducible highest weight representations it is called 'rational'. In this case one also speaks of a rational model of a $\mathcal{W}$-algebra. More generally, a rational CFT (RCFT) is a CFT where the Hilbert space of the theory decomposes into finitely many irreducible representations of the maximally extended symmetry algebra ($\mathcal{W}$-algebra). One of the most important examples are the RCFTs provided by the rational models of the Virasoro-algebra for c lying in the minimal series mentioned above. The character $\chi_{\mathcal{H}}(\tau)$ of an irreducible HWR $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ of a $\mathcal{W}$-algebra is defined as $\chi_{\mathcal{H}}(\tau) := \text{tr}_{\mathcal{H}}\left(q^{\rho(L_0)-c/24}\right) = q^{h-c/24}\sum_{n\in\mathbb{N}} \dim(\mathcal{H}_n) q^n$ with $q = e^{2\pi i\tau}$ and h being the lowest eigenvalue of $\rho(L_0)$. For rational VOA's it has been shown recently that under certain finiteness conditions the characters of the (finitely many) irreducible modules are holomorphic functions on the upper complex half plane and that they span a representation space of the modular group $SL(2,\mathbb{Z})$ with respect to the natural action $(\phi_A(\chi))(\tau) = \chi(A\tau), \forall A \in SL(2,\mathbb{Z})$ [Zhu]. Here, the modular group acts on the upper complex half plane via Möbius transformations. The partition function of a RCFT is defined as $Z(q,\bar{q}) = \text{tr}_{\mathcal{V}\otimes\bar{\mathcal{V}}}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right)$, where $\mathcal{V}(\bar{\mathcal{V}})$ is the full Hilbert space of the (anti-)holomorphic sector of the theory, and can be written as a sesquilinear form in the finitely many characters of the underlying (maximally extended) $\mathcal{W}$-algebra

$$Z(q,\bar{q}) = \sum_{h,\bar{h}} \mathcal{N}_{h\bar{h}} \chi_h(q) \bar{\chi}_{\bar{h}}(\bar{q}). \tag{2.10}$$

A rather restrictive constraint to the possible partition functions is the requirement of modular invariance of Z (the matrix $\mathcal{N}$ has to commute with the representation matrices of the generating elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $SL(2,\mathbb{Z})$ [2])). The modular invariant partition functions of Virasoro minimal models have been classified in the celebrated ADE-classification of [CIZ]. In the meanwhile the mathematically very interesting classification of the modular invariants of the Kac-Moody algebra $\widehat{\mathcal{A}}_2$ (involving e.g. Galois theory) has been successfully completed [Ga].

The next topic of our short review through CFT is the coset or GKO construction [GKO]. In order to study cosets we have to consider Kac-Moody algebras first. Affine Kac-Moody algebras (AKM) play a profound rôle in CFT, in particular as a tool for the construction of $\mathcal{W}$-algebras. Currently it is believed that all $\mathcal{W}$-algebras arise from AKMs by applying the coset construction and the so-called Drinfeld-Sokolov (DS) reduction. They can also be viewed as the most simple examples of $\mathcal{W}$-algebras generated only by primary fields of dimension one (usually called 'currents'). Let $\mathbf{g}$ be a finite-dimensional simple Lie-algebra with a (anti-hermitian) basis $\{T_i, i = 1,\ldots,\dim(\mathbf{g})\}$ (in the fundamental representation) and commutation relations $[T_a, T_b] = f_{ab}^c T_c$. The Cartan-Killing form will be denoted by $d_{ab} = \mathrm{tr}(T_a T_b)$ and is used to raise and lower indices. We take now a $\mathbf{g}$-valued current $J(z) = \sum J^a(z) T_a$ satisfying the OPEs:

$$J^a(z) J^b(w) = \frac{k d^{ab}}{(z-w)^2} + \frac{f_c^{ab} J^c(w)}{z-w} + reg.. \tag{2.11}$$

The number k is called 'level'. The corresponding commutators of the Laurent modes are given by:

$$[J_n^a, J_m^b] = f_c^{ab}\, J_{n+m}^c + knd^{ab} \delta_{n+m,0}. \tag{2.12}$$

The infinite-dimensional Lie-algebra (2.12) is called the untwisted affine Kac-Moody algebra $\widehat{\mathbf{g}}$. We will also consider $\widehat{\mathbf{g}}$ as a $\mathcal{W}$-algebra, i.e. the chiral algebra $\mathcal{W}(1,\ldots,1)$ generated by the $\dim(\mathbf{g})$ currents $J^a(z)$ subject to the OPEs (2.11). Note that the zero-modes J_0^a span a subalgebra satisfying the commutation relations of the underlying Lie-algebra $\mathbf{g}$, the so-called horizontal subalgebra of $\widehat{\mathbf{g}}$. Physically AKMs arise as maximal symmetry algebras of Wess-Zumino-Witten (WZW) models which are nonlinear sigma models on compact group manifolds G (with Lie-algebra $\mathbf{g}$) (see e.g. [Fe1,GW]).

The energy-momentum tensor of the CFT with an AKM as the underlying chiral symmetry algebra is given by the Sugawara construction:

$$T^{\widehat{\mathbf{g}}}(z) = \frac{1}{2(k+h^\vee)} d_{ab} N(J^a, J^b)(z). \tag{2.13}$$

[2]) T and S satisfy the relations $S^2 = (TS)^3 = -\mathbb{1}$.

$h^\vee$ is the dual Coxeter number of $\mathbf{g}$. $T(z)$ satifies the Virasoro algebra with central charge $c(\mathbf{g},k) = \frac{k\dim(\mathbf{g})}{k+h^\vee}$. The currents themselves are primary fields of conformal dimension 1 with respect to $T(z)$. Irreducible highest weight representations $L(\Lambda)$ (with highest weight Λ) of a AKM are defined in the standard way. Of particular interest for unitary CFTs are the integrable HWRs which occur for integer level k and for which Λ is an integral dominant weight. For a given integer level k there exists only a finite number of unitary ($L(\Lambda)$ admits a positive definite invariant bilinear form) irreducible HWRs $L(\Lambda)$ satisfying the condition $\frac{2(\theta,\Lambda)}{(\theta,\theta)} \leq k$, where $(.,.)$ is the invariant bilinear form on the weight lattice and θ the highest root. The associated conformal dimension h_Λ (lowest L_0 eigenvalue) of the HWR $L(\Lambda)$ is given by $h_\Lambda = \frac{(\Lambda,\Lambda+2\rho)}{2(k+h^\vee)}$ (2ρ is the sum of all positive roots). In other words for integer level k the AKMs provide us with unitary RCFTs with central charge $c(\mathbf{g},k)$. For fractional levels the situation is much more delicate. Here one can introduce a finite number of 'modular invariant HWRs' $L(\lambda)$ with λ an 'admissible' weight, which define a non-unitary RCFT. In [KacW] it has been shown that the characters of $L(\lambda)$ for admissible λ are holomorphic modular functions on the upper half plane.
An important tool to construct more complicated RCFTs is the coset construction [GKO]. One starts with a AKM $\widehat{\mathbf{g}}$ at level k and AKM subalgebra $\widehat{\mathbf{h}}$ at level m. The ratio k/m is equal to the Dynkin index of the embedding $\mathbf{h} \subset \mathbf{g}$. The coset $\mathcal{W}$-algebra $\widehat{\mathbf{g}}/\widehat{\mathbf{h}}$ is defined as the commutant of $\widehat{\mathbf{h}}$ in $\widehat{\mathbf{g}}$, i.e. the polynomials in the currents of $\widehat{\mathbf{g}}$ which have a regular OPE with the currents of $\widehat{\mathbf{h}}$ [3]). It will be a main topic of this thesis to demonstrate for several examples that these coset $\mathcal{W}$-algebras are finitely generated. It can be shown that the difference of the Sugawara energy-momentum tensors $T^{\widehat{\mathbf{g}}/\widehat{\mathbf{h}}}(z) := T^{\widehat{\mathbf{g}}}(z) - T^{\widehat{\mathbf{h}}}(z)$ lies in the commutant $\widehat{\mathbf{g}}/\widehat{\mathbf{h}}$. It is the energy-momentum tensor of the coset CFT and satisfies a Virasoro algebra with central charge $c(\widehat{\mathbf{g}},k) - c(\widehat{\mathbf{h}},m)$. Take an integrable HWR $L(\Lambda)$ of $\widehat{\mathbf{g}}$ (level k) and define the vector space $U(\Lambda,\lambda)$ to be the span of highest weight vectors for $\widehat{\mathbf{h}}$ with integrable weight λ (level m) inside $L(\Lambda)$. Then one has the decomposition

$$L(\Lambda) = \bigoplus_\lambda L(\lambda) \otimes U(\Lambda,\lambda), \tag{2.14}$$

where $U(\Lambda,\lambda)$ carries a highest weight representation of the coset algebra (in general not irreducible!) and in particular of the Virasoro algebra given by $T^{\widehat{\mathbf{g}}/\widehat{\mathbf{h}}}(z)$. Here, λ runs through the integral dominant weights of $\widehat{\mathbf{h}}$ at level m. The characters $b^\Lambda_\lambda(\tau) = \mathrm{tr}_{U(\Lambda,\lambda)} q^{L_0^{\widehat{\mathbf{g}}/\widehat{\mathbf{h}}} - c(\widehat{\mathbf{g}}/\widehat{\mathbf{h}})/24}$ of the modules $U(\Lambda,\lambda)$ are called 'branching functions' of the coset $\widehat{\mathbf{g}}/\widehat{\mathbf{h}}$ which becomes clear in view of the implication of (2.14) for the characters $\chi^{\widehat{\mathbf{g}}}_{L(\Lambda)}(\tau,z) = \sum_\lambda b^\Lambda_\lambda(\tau) \chi^{\widehat{\mathbf{h}}}_{L(\lambda)}(\tau,z')$ ($q = e^{2\pi i\tau}$, τ in the upper half-plane). The character $\chi_{L(\Lambda)}(\tau,z)$ of an integrable HWR $L(\Lambda)$ of an AKM $\widehat{\mathbf{g}}$ is defined as

$$\chi_{L(\Lambda)}(\tau,z) = q^{-c(\mathbf{g},\Lambda)/24}\ \mathrm{tr}_{L(\Lambda)}\ q^{L_0^{\widehat{\mathbf{g}}}}\, e^{2\pi i z}, \tag{2.15}$$

[3]) This concept can be extended to cosets $\widehat{\mathbf{g}}/\mathbf{g}$ and $\mathcal{A}/\widehat{\mathbf{g}}$, if the $\mathcal{W}$-algebra $\mathcal{A}$ possesses an AKM subalgebra $\widehat{\mathbf{g}}$.

where z is an element of the Cartan subalgebra of $\mathbf{g}$ and $e^{2\pi i z}$ is a linear form on the weight lattice of $\widehat{\mathbf{g}}$. A case of special interest is the choice $\mathbf{h}$ to be the Cartan subalgebra of $\mathbf{g}$. The branching functions take the form $b_\lambda^\Lambda(q) = \eta(q)^l c_\lambda^\Lambda(q)$, where $\eta(q) = q^{1/24}\prod_{n=1}^\infty(1-q^n)$ is Dedekind's η-function, l the rank of $\mathbf{g}$ and $c_\lambda^\Lambda(q)$ are the Kac-Peterson string functions [KP]. Making use of the string functions one can give an explicit formula for the branching functions of the coset $\widehat{\mathbf{g}}/\mathbf{g}$:

$$b_\lambda^\Lambda(q) = \sum_{w\in W} \epsilon(w) c^\Lambda_{w\star\lambda+k\Lambda_0} q^{\frac{1}{2k}|w\star\lambda|^2}. \tag{2.16}$$

W is the Weyl group of $\mathbf{g}$, Λ_0 the first fundamental weight and $\star$ denotes the shifted Weyl group action on the weights $w\star\lambda = w(\lambda+\rho)-\rho$. This formula is very useful since it can be shown to be the $l\to\infty$ limit of the branching functions $b^{\Lambda_3^{k+l}}_{\Lambda_1^k,\Lambda_2^l}$ of the diagonal coset $\widehat{\mathbf{g}}\oplus\widehat{\mathbf{g}}/\widehat{\mathbf{g}}$ where Λ^k denotes an integrable weight at level k (see e.g. [BS] for details).

We conclude with some notational remarks. The Casimir algebra corresponding to a simple Lie algebra $\mathcal{L}_n$ is denoted by $\mathcal{W}\mathcal{L}_n$. They arise from quantum Drinfeld-Sokolov reduction with the principal $sl(2)$ embedding into $\mathcal{L}_n$. Note that we reserve the notation $\mathcal{W}\mathcal{B}_n$ for the purely bosonic algebras $\mathcal{W}\mathcal{B}_n \cong \mathcal{W}(2,4,\ldots,2n)$. We denote the algebras $\mathcal{W}(2,4,\ldots,2n,n+\frac{1}{2})$ containing one fermionic field by $\mathcal{W}\mathcal{B}(0,n)$ because they can be obtained from quantum DS reduction for the Super Lie algebras $\mathcal{B}(0,n)$. For a $\mathcal{W}$-algebra coming from DS reduction of a nonprincipal $sl(2)$ embedding into $\mathcal{L}_n$ we use the notation $\mathcal{W}_\mathcal{S}^{\mathcal{L}_n}$ where $\mathcal{S}$ denotes the embedding. For example, for $\mathcal{L}_n = \mathcal{A}_n$, $\mathcal{S}$ is the r-tuple of the dimensions of the irreducible $sl(2)$ representations which appear in the defining representation of $\mathcal{L}_n$.

2.2 Fusion algebras

In this subsection we follow partially ref. [Eh5] where the definitions of some notions presented here may be found. For a detailled review we refer e.g. to ref. [Fu]. We start with a rational model of a $\mathcal{W}$-algebra $\mathcal{A}$, i.e. a RCFT where the Hilbert space $\mathcal{H}$ decomposes into the finite number n of irreducible inequivalent HWRs $(\mathcal{H}_i,\rho_i), i=0,\ldots,n-1$ of $\mathcal{A}$. Usually one introduces the dual (or conjugate) module $\mathcal{H}_i^*$ of $\mathcal{H}_i$ which also carries an irreducible highest weight representation of $\mathcal{A}$. Since we supposed that $\mathcal{A}$ is rational and possesses only n inequivalent irreducible HWRs, the HWR $\mathcal{H}_i^*$ must be equivalent to one of the given n HWRs, say $\mathcal{H}_{i'}$. Therefore the conjugation defines a permutation $\pi\in S_n$ of order two on the set of irreducible HWRs. Naively one would like to apply the standard concept of tensor products of HWRs of (infinite-dimensional) Lie-algebras to $\mathcal{W}$-algebras. However, the tensor product of two HWRs of a $\mathcal{W}$-algebra does in general not carry the structure of a representation. Nevertheless, it is possible to generalize the notion of the 'multiplicity' $N_{i,j}^k$ of a representation $(\mathcal{H}_k,\rho_k)$ in the tensor product of two other representations $(\mathcal{H}_i,\rho_i)$, $(\mathcal{H}_j,\rho_j)$ as the dimension of the vector space of intertwiner operators of the modules $\mathcal{H}_i,\mathcal{H}_j,\mathcal{H}_k$. In the language of VOAs an intertwiner operator of three HWRs $\mathcal{H}_i,\mathcal{H}_j,\mathcal{H}_k$ is defined as a linear map from $\mathcal{H}_i$ into the space of formal power series in z,z^{-1} with coefficients in the vector space of linear maps from $\mathcal{H}_j$ to $\mathcal{H}_k$

satisfying some algebraic compatibility conditions (see [Eh5] for details). The integers $N_{i,j}^k$ are called the 'fusion rule coefficients' of the rational model of the $\mathcal{W}$-algebra and satisfy several interesting properties. Under certain assumptions it can proven that the fusion rule coefficients are finite and satisfy

$$N_{0,i}^j = \delta_{i,j},\ N_{i,j}^0 = \delta_{i,\pi(j)},\ N_{i,j}^k = N_{j,i}^k,\ N_{i,j}^k = N_{\pi(i),\pi(j)}^{\pi(k)},\ \sum_{k=0}^{n-1} N_{i,j}^k N_{k,l}^m = \sum_{k=0}^{n-1} N_{i,k}^m N_{j,l}^k, \tag{2.17}$$

where $i,j,l,m = 0,\ldots,n-1$. This enables us to interpret the fusion rule coefficients as the structure constants of an associative and commutative algebra with n basis elements ϕ_i corresponding to the irreducible HWRs $\mathcal{H}_i$ of the RCFT. The unit element ϕ_0 corresponds to the vacuum representation $\mathcal{H}_0$. This algebra is called the 'fusion algebra' of the rational model. In an abstract way one defines a fusion algebra $\mathcal{F}$ as a finite-dimensional algebra over $\mathbb{Q}$ ($\dim(\mathcal{F}) = n$) with a special basis $\phi_0,\ldots,\phi_{n-1}$ and unit ϕ_0. $\mathcal{F}$ is required to be an associative, commutative algebra with integer structure constants $N_{i,j}^k$ with respect to the special basis and to admit a permutation $\pi \in S_n$ of order two such that the conditions $N_{i,j}^0 = \delta_{i,\pi(j)}$ and $N_{i,j}^k = N_{\pi(i),\pi(j)}^{\pi(k)}$ are satisfied. The permutation will be called 'charge conjugation' in the sequel.

For (bosonic) RCFTs there exists a remarkable relation of the fusion algebra and the modular properties of the characters of the theory. As mentioned earlier the conformal characters $\chi_i(q)$ of a rational model of a $\mathcal{W}$-algebra are holomorphic functions on the upper complex half plane ($q = e^{2\pi i\tau}$) and span a representation space of a representation $\rho : SL(2,\mathbb{Z}) \to GL(n,\mathbb{C})$ of the modular group. Assume that the representation ρ acts as $\chi_i(A\tau) = \sum_{j=0}^{n-1} \rho(A)_{i,j}\chi_j(\tau),\ A \in SL(2,\mathbb{Z})$. The famous 'Verlinde formula' [Ver] connects the representation matrix $\rho(S)$ to the fusion rule coefficients of the rational model:

$$N_{i,j}^k = \sum_{m=0}^{n-1} \frac{\rho(S)_{m,i}\rho(S)_{m,j}\rho(S^{-1})_{k,m}}{\rho(S)_{m,0}} \tag{2.18}$$

For the charge conjugation the equality $N_{i,j}^0 = \rho(S^2)_{i,j}$ holds. In section 3 we will present a generalization of this formula to fermionic RCFTs.

In order to classify fusion algebras of rational models one has to exploit additional properties. Apart from the rationality of the central charge and the conformal dimensions of the model the so-called Fuchs conditions relating c, h_i with $N_{i,j}^k$ impose severe constraints on the possible fusion algebras [CPR]. In [Eh5] a different ansatz has been made in order to advance to a classification of fusion algebras related to RCFTs. Here strongly-modular fusion algebras admitting a representation of the modular group which yields the fusion coefficients via Verlinde's formula and factors through a congruence subgroup of $SL(2,\mathbb{Z})$ have been introduced.

Finally, we mention the notion of 'simple currents' of fusion algebras [SY,In]. Consider a fusion algebra $[\phi_i][\phi_j] = \sum_{k=0}^{n-1} N_{i,j}^k[\phi_k]$ of a RCFT. A basis element $[\phi_i] =: [J]$ will be called a 'simple current' of the fusion algebra if $\sum_k N_{i,j}^k = 1$ for all labels $j = 0,\ldots,n-1$. Consequently, for each label j only one fusion coefficient $N_{i,j}^k$ can be non-vanishing, i.e. $[J][\phi_j] = [\phi_k]$. The smallest positive integer N with $[J]^N = \mathbb{1} = [\phi_0]$ is called the order of the simple current.

3 Fusion algebras of Fermionic RCFTs via a Generalized Verlinde Formula

In a rational conformal field theory (RCFT) a representation of the modular group is given by the natural action of $SL(2,\mathbb{Z})$ on the characters of the highest weight representations (HWRs) of the (maximally extended) chiral symmetry algebra $\mathcal{W}$ underlying the RCFT [Car,Na1]. As described above the Verlinde formula [Ver,MS] establishes the connection between the representation matrix S of the modular transformation $\tau \to -\frac{1}{\tau}$ and the fusion coefficients of the model. The main goal of this section is to put forward a generalized Verlinde formula describing the fusion in all sectors of *fermionic* theories. This generalization reproduces the correct sector structure of the fusion algebra [Mu]. It is shown that the fusion algebras given by the generalized Verlinde formula can be obtained from the fusion algebras of the corresponding bosonic projections applying 'simple current' arguments. Such 'simple current' arguments have been first proposed by Intrilligator [In] and Schellekens et al. [SY]. Furthermore, the representations of the horizontal subalgebras on the highest weights in the Ramond sector are considered. For fermionic $\mathcal{W}(2,\delta)$-algebras their dimensions are directly encoded in the fusion algebra.

In section 3.1 we present the generalized Verlinde formula and formulate the main statements of our analysis. In the next section we prove the generalized Verlinde formula under certain assertions. We apply the formula to $\mathcal{W}(2,\delta)$-algebras ($\delta \geq \frac{5}{2}$) in section 3.3 and proceed with a discussion of the fusion algebras of the (unitary as well as non-unitary) minimal models of the $N=1$ super Virasoro algebra in section 3.4. Finally, we treat $N=1$ super $\mathcal{W}$-algebras with two generators ($\mathcal{SW}(\frac{3}{2},\delta)$).

3.1 Generalized Verlinde Formula for Fermionic RCFTs

Let $\mathcal{R}$ be a fermionic RCFT with (maximally extended) chiral symmetry algebra $\mathcal{W}$ containing the Virasoro algebra. One has to distinguish between the Neveu-Schwarz sector (NS) and the Ramond sector (R) because the symmetry algebra contains bosonic and fermionic fields. We denote the $\mathcal{W}$-primary fields and the corresponding highest weight representations of the symmetry algebra enlarged by the (involutive) 'chirality'-operator $\Gamma = (-1)^{\mathcal{F}}$ ($\mathcal{F}$ is the fermion number operator) by

$$\phi_i^{NS} \;\leftrightarrow\; \mathcal{H}_i^{NS} \quad (\forall\, i \in \mathcal{I}_{NS}), \qquad \phi_j^{R} \;\leftrightarrow\; \mathcal{H}_j^{R} \qquad (\forall\, j \in \mathcal{I}_{R}), \tag{3.1}$$

where $\mathcal{H}_1^{NS}$ is the vacuum representation. The corresponding characters are defined by:

$$\chi_k^{NS} := \mathrm{tr}_{\mathcal{H}_k^{NS}}\big(q^{L_0-c/24}\big), \qquad \chi_k^{R} := \mathrm{tr}_{\mathcal{H}_k^{R}}\big(q^{L_0-c/24}\big). \tag{3.2a}$$

The sector structure of such a fermionic theory is reflected by the modular properties of the characters. Let T (S) be the representation matrix of the modular transformation $\tau \to \tau+1$ ($\tau \to -\frac{1}{\tau}$) for the representation of the modular group given by its natural action on the characters. Because the span of the characters in the NS sector is invariant only under the subgroup of the modular group generated by T^2 and S, it is useful to define a third (physically irrelevant) sector $\widetilde{NS}$ (see e.g. [CIZ,Cap]) in order to obtain a representation space of the full modular group:

$$\chi_k^{\widetilde{NS}} := e^{-2\pi i (h(\phi_k^{NS})-c/24)}\;\; T\chi_k^{NS} = \mathrm{tr}_{\mathcal{H}_k^{NS}}\big((-1)^{\mathcal{F}} q^{L_0-c/24}\big). \tag{3.2b}$$

The modular transformation TST intertwines between the NS and the R sectors. Together with the R sector, which is invariant under T and ST^2S, these three sectors have the structure of a $SL(2,\mathbb{Z})$ module.
The 'horizontal' subalgebra is defined as the subalgebra consisting of the zero modes of all fields in $\mathcal{W}$ and the 'chirality'-operator Γ. We stress that in our convention the Ramond characters χ^R begin with $q^{h-\frac{c}{24}}(l+\ldots)$, where l is the dimension of the L_0 eigenspace $\mathcal{V}_0$ to the lowest eigenvalue h in the representation. A highest weight vector is an eigenvector of a maximal number of zero-modes of fields in the bosonic part of the horizontal subalgebra which commute in $\mathcal{V}_0$. Therefore, l is in general greater than one because zero-modes of fermionic fields act nontrivially on the highest weight vector generating $\mathcal{V}_0$. In the following we call the characters (3.2) 'energy' characters. In particular, we do not include a factor $\sqrt{2}$ in the R characters like in ref. [Cap]. It may happen that some of the (inequivalent) irreducible representations of the algebra $\mathcal{W}$ are degenerate in the sense that they have equal energy characters (ignoring additional $\mathcal{W}$ quantum numbers). In this case the sector structure of the representation of the modular group is respected only if one considers the energy characters and identifies the degenerate ones.
In analogy to the $\widetilde{NS}$ sector one can define characters in the $\widetilde{R}$ sector by

$$\chi_k^{\widetilde{R}} := \mathrm{tr}_{\mathcal{H}_k^R}\big((-1)^{\mathcal{F}} q^{L_0-c/24}\big). \tag{3.2c}$$

This sector is invariant under the action of the modular group, and the $\widetilde{R}$ characters (3.2c) are constant. These constants can be identified with Witten indices of the corresponding highest weight representations because they indicate whether the boson-fermion symmetry of the ground state is broken or not [Wi, Ka].

The transformation properties of the characters lead to the following form for the representation matrices of S and T (we omit the $\widetilde{R}$ sector because of its modular invariance) [CIZ,Cap]:

$$S = \begin{pmatrix} S^{NS\to NS} & 0 & 0 \\ 0 & 0 & S^{R\to\widetilde{NS}} \\ 0 & S^{\widetilde{NS}\to R} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & T^{\widetilde{NS}\to NS} & 0 \\ T^{NS\to\widetilde{NS}} & 0 & 0 \\ 0 & 0 & T^{R\to R} \end{pmatrix} \tag{3.3}$$

with $T_{k,l}^{\widetilde{NS}\to NS} = T_{k,l}^{NS\to\widetilde{NS}} = \delta_{k,l}\ e^{2\pi i\left(h(\phi_k^{NS})-c/24\right)}$.
It is well known that the fusion rules respect the sector structure of the theory [Mu]:

$$\begin{aligned} [\Phi_i^{NS}][\Phi_j^{NS}] &= \sum_{k\in\mathcal{I}_{NS}} \big(\mathcal{N}_{NS,NS}^{NS}\big)_{i,j}^{\ k}[\Phi_k^{NS}] \\ [\Phi_l^{R}][\Phi_m^{R}] &= \sum_{i\in\mathcal{I}_{NS}} \big(\mathcal{N}_{R,R}^{NS}\big)_{l,m}^{\ i}[\Phi_i^{NS}] \\ [\Phi_l^{R}][\Phi_i^{NS}] &= \sum_{m\in\mathcal{I}_{R}} \big(\mathcal{N}_{R,NS}^{R}\big)_{l,i}^{\ m}[\Phi_m^{R}]. \end{aligned} \tag{3.4}$$

This can be interpreted as the conservation of an additional additive $\mathbb{Z}_2$ charge (the NS sector is neutral and the R sector carries charge 1). Taking into account the sector structure of the fusion rules for fermionic theories the generalized Verlinde formula can be written as (here the vacuum carries index 1):

$$
\begin{aligned}
(\mathcal{N}^{NS}_{NS,NS})_{i,j}^{\ k} &= \sum_{n\in\mathcal{I}_{NS}} \frac{S^{NS\to NS}_{n,i} S^{NS\to NS}_{n,j} (S^{NS\to NS})^{-1}{}_{k,n}}{S^{NS\to NS}_{n,1}} \\
(\mathcal{N}^{NS}_{R,R})_{l,m}^{\ i} &= d_l d_m \sum_{n\in\mathcal{I}_{NS}} \frac{S^{R\to\widetilde{NS}}_{n,l} S^{R\to\widetilde{NS}}_{n,m} (S^{NS\to NS})^{-1}{}_{i,n}}{S^{NS\to NS}_{n,1}} \\
(\mathcal{N}^{R}_{R,NS})_{l,i}^{\ m} = (\mathcal{N}^{R}_{NS,R})_{i,l}^{\ m} &= \frac{d_l}{d_m} \sum_{n\in\mathcal{I}_{NS}} \frac{S^{R\to\widetilde{NS}}_{n,l} S^{NS\to NS}_{n,i} (S^{R\to\widetilde{NS}})^{-1}{}_{m,n}}{S^{NS\to NS}_{n,1}}
\end{aligned}
\tag{3.5}
$$

with $i,j,k\in\mathcal{I}_{NS}$ and $l,m\in\mathcal{I}_R$. The first equation is the usual Verlinde formula [Ver,MS]. The $\{d_l \mid l\in\mathcal{I}_R\}$ are defined as follows. Consider the bosonic projection $\mathcal{PW}$ of the symmetry algebra $\mathcal{W}$ and the corresponding rational model of $\mathcal{PW}$. The fusion algebra of this rational model contains a 'simple current' of order two describing its extended symmetry. The orbits under this 'simple current' correspond to the fields in the fusion algebra of $\mathcal{W}$. Now d_l is defined as the order of the 'simple current' divided by the length of the orbit corresponding to the l^{th} field in the fusion algebra of $\mathcal{W}$. For the fields in the NS sector d is equal to one (cf. section 3.2). We combine these integers to a diagonal matrix $D^R = diag(\{d_l \mid l\in\mathcal{I}_R\})$ and define the D-matrix as

$$
D = \begin{pmatrix} 1\!\!1 & 0 & 0 \\ 0 & 1\!\!1 & 0 \\ 0 & 0 & D^R \end{pmatrix}
\tag{3.6}
$$

The definition (3.5) directly implies that the fusion constants lead to a well-defined, commutative and associative algebra. Furthermore, fusion with the identity field acts trivially in the fusion algebra. However, it is not apparent that the fusion coefficients defined in (3.5) are positive integers. In section 3.2 we will show that this is an immediate consequence of the fusion coefficients of the bosonic fusion algebra being positive integers.
In general, the representations are degenerate and give rise to a diagonal 'multiplicity' matrix M defined as

$$
M = \begin{pmatrix} M^{NS} & 0 & 0 \\ 0 & M^{\widetilde{NS}} & 0 \\ 0 & 0 & M^R \end{pmatrix}, \quad M^{NS} = M^{\widetilde{NS}}.
\tag{3.7}
$$

Here the entries of the three diagonal submatrices are the multiplicities of the respective representations. In all known cases degeneracies can be removed by considering the eigenvalues of some additional zero modes of bosonic fields $\mathcal{B}_0$ which commute with L_0. In general, there are several inequivalent HWRs with identical energy characters but with different eigenvalues of the operators in $\mathcal{B}_0$. We must use the energy characters in order to preserve the sector structure because in general there exist different numbers of inequivalent HWRs in the NS and R sector. Thus we have to deal with the multiplicity matrix M.

Instead of being unitary, S obeys the equation

$$S^\dagger H S = H \qquad H = MD^{-1}, \tag{3.8}$$

as will be explained in the next section. One should note that if S and M are known D is fixed by this equation. Due to (3.8) the 'fusion charge conjugation matrix' $\mathcal{N}_{ij}^{1}$ is in general not equal to the usual charge conjugation matrix $C = S^2$ but satisfies $\mathcal{N}_{ij}^{1} = (MD)_{i,j}$. As we will discuss below, it is not possible to avoid non-trivial D-matrices by means of an extension of the fusion algebra in contrast to the degeneracies encoded in M which can be resolved (see e.g. [SY]). Thus, one is forced to weaken the axioms of fusion algebras of fermionic RCFTs and has to allow more general fusion charge conjugation matrices.

Finally, we add some remarks concerning the chirality operator Γ which we always include into the chiral symmetry algebra. In general it is very unphysical to consider representations with diagonal fermionic operators. However, if the chirality operator (anticommuting with all fermionic operators) is included into the symmetry algebra the fermionic operators act non-diagonal and do not preserve the Γ-eigenspaces. Irreducible representations of the fermionic algebra (including Γ) which correspond to orbits of length two in the fusion algebra of the bosonic projection are also irreducible with respect to the chiral algebra without Γ. In contrast, irreducible representations corresponding to fixed points in the bosonic fusion algebra are not irreducible with respect to the chiral algebra without Γ but decompose into a direct sum of two irreducible representations. This may explain the fact that the fusion coefficients connecting two such conjugate fields with the identity are equal to two. As we will see in section 3.3 it is not possible to resolve these nontrivial fusion coefficients by extending the fusion algebra.

In section 3.3 we verify for fermionic $\mathcal{W}(2,\delta)$-algebras that the diagonal entries of D^R give exactly the dimensions of the L_0-eigenspaces $\mathcal{V}_0$ in the respective representations. However, this observation is in general not valid for symmetry algebras with more than one fermionic generator like $\mathcal{SW}(\frac{3}{2},\delta)$-algebras (cf. section 3.5).

3.2 Proof of the Generalized Verlinde Formula

Assume that we have a bosonic RCFT with characters χ_i^{bos} ($i \in \mathcal{I}_{bos}$) and unitary symmetric S-matrix S^{bos}. Furthermore, we suppose for simplicity that $\left(S^{bos}\right)^2 = \mathbb{1}$, i.e. trivial charge conjugation and that there are no degenerate HWRs so that $N_{i,i}^{1} = 1$ ($\forall i \in \mathcal{I}_{bos}$) is valid. The bosonic RCFT shall also admit a fermionic extension of its underlying chiral symmetry algebra. We require therefore that the fusion algebra $\mathcal{A}$ obtained from S^{bos} via the (ordinary) Verlinde formula possesses a 'simple current' $[J]$ of order two ($[J][J] = \mathbb{1}$) with conformal dimension $h([J]) = \delta \in \mathbb{N} + \frac{1}{2}$. Assume furthermore that the fermionic symmetry algebra is obtained from its bosonic projection by extension with this 'simple current' (this is at least valid for $N = 1$ supersymmetric theories and fermionic $\mathcal{W}(2,\delta)$-algebras).

The basis elements $[i]$ ($i \in \mathcal{I}_{bos}$) of $\mathcal{A}$ are organized into orbits of the 'simple current' $[J]$ of length one (fixed points) or two. These orbits thus define the multiplets of the extended fermionic RCFT. Accordingly, the characters of the fermionic theory are defined as $\chi_i^{fer} := \chi_i^{bos} + \chi_{Ji}^{bos}$. As shown in ref. [SY] it is possible to define a conserved charge Q

on the fusion algebra $\mathcal{A}$ in the following way: $Q([i]) := \big(h([i]) + h([J]) - h([J][i])\big)\ mod\ 1$. It obeys the addition rule $Q([i][j]) = Q([i]) + Q([j])\ mod\ 1$. The fields have either charge 0 or charge $\frac{1}{2}$ with respect to the 'simple current' $[J]$. Due to charge conservation the fusion algebra $\mathcal{A}$ is $\mathbb{Z}_2$ graded, i.e. we have $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_{\frac{1}{2}}$. Here $\mathcal{A}_0$ ($\mathcal{A}_{\frac{1}{2}}$) denotes the subspace spanned by fields of charge $0\,(\frac{1}{2})$. The fusion rules respect this structure in the following way:

$$\mathcal{A}_0 \cdot \mathcal{A}_0 \subset \mathcal{A}_0 \qquad\qquad \mathcal{A}_0 \cdot \mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_{\frac{1}{2}} \qquad\qquad \mathcal{A}_{\frac{1}{2}} \cdot \mathcal{A}_{\frac{1}{2}} \subset \mathcal{A}_0. \tag{3.9}$$

We observe that the fields in $\mathcal{A}_0$ form a subalgebra of $\mathcal{A}$. It is thus natural to identify the orbits in $\mathcal{A}_0(\mathcal{A}_{\frac{1}{2}})$ with the fields of the Neveu-Schwarz (Ramond) sector of the fermionic RCFT. Note that in $\mathcal{A}_0$ all orbits have length two, i.e. there are no fixed points in the NS sector. However, in $\mathcal{A}_{\frac{1}{2}}$ fixed points as well as orbits of length two are possible.

It is now straightforward to calculate the fusion algebra of the fermionic RCFT. We just have to perform the change of basis prescribed by the orbits of $[J]$. The new basis is defined as

$$[\hat{\imath}] := \tfrac{1}{2}\big([i] + [J][i]\big) \qquad \alpha(i) = \hat{\imath} \in \mathcal{I}_{NS} \cup \mathcal{I}_R, \quad i \in \mathcal{I}'_{NS} \cup \mathcal{I}'_R = \mathcal{I}_{bos}$$

where $\mathcal{I}_{NS}$ and $\mathcal{I}_R$ label the orbits in $\mathcal{A}_0$ and $\mathcal{A}_{\frac{1}{2}}$ respectively. $\mathcal{I}'_{NS}$ and $\mathcal{I}'_R$ label *all* basis elements of $\mathcal{A}_0$ and $\mathcal{A}_{\frac{1}{2}}$. The action of $[J]$ induces a natural surjective map of $\mathcal{I}'_{NS}(\mathcal{I}'_R)$ onto $\mathcal{I}_{NS}(\mathcal{I}_R)$ which we denote by α. The fusion coefficients are then given by:

$$\mathcal{N}_{\alpha(i),\alpha(j)}^{\alpha(k)} = N_{i,j}^{k} + N_{i,j}^{Jk} \quad k \in \mathcal{I}'_{NS},\ i,j \in \mathcal{I}'_{NS} \text{ or } i,j \in \mathcal{I}'_R. \tag{3.10}$$

$N_{i,j}^k$ are the fusion coefficients of the bosonic fusion algebra $\mathcal{A}$ obtained from S^{bos} and $\mathcal{N}_{\alpha(i),\alpha(j)}^{\alpha(k)}$ denote the fusion coefficients of the corresponding fusion algebra of the fermionic RCFT. Note that for $i \in \mathcal{I}'_{NS}$ either $N_{i,\bar{i}}^{1}$ or $N_{i,\bar{i}}^{J}$ has to be zero ($\bar{i}$ denotes the field conjugate to i) because there are no fixed points in $\mathcal{A}_0$. Hence, the fusion charge conjugation matrix is equal to the usual charge conjugation matrix in the NS sector.

Is order to prove the generalized Verlinde formula we have to show that these fusion coefficients are equal to the coefficients calculated with formulae $(3.5) - (3.8)$ using the S-matrix S^{fer}. We obtain S^{fer} from S^{bos} by performing the appropriate change of basis on the characters $\chi_{\alpha(i)}^{fer} := \chi_i^{bos} + \chi_{Ji}^{bos}$. This implies that S^{fer} in not unitary but obeys eq. (3.8). To formulate it differently we have to show that the following diagram commutes:

$$\begin{array}{ccc} S^{bos} & \xrightarrow{\chi_{\alpha(i)}^{fer} := \chi_i^{bos} + \chi_{Ji}^{bos}} & S^{fer} \\ \Big\downarrow {\scriptstyle \text{Verlinde}} & & \Big\downarrow {\scriptstyle \text{generalized Verlinde}} \\ \text{bosonic fusion algebra} & \xrightarrow{[\hat{\imath}] := \frac{1}{2}\left([i]+[J][i]\right)} & \text{fermionic fusion algebra} \end{array} \tag{3.11}$$

Equation (3.10) corresponds to the bottom line of the diagram (3.11).

Let us first check the result for $\mathcal{N}_{i,j}^{k}$ with $i,j,k \in \mathcal{I}_{NS}$. The S-matrix $S^{fer}_{NS\to NS}$ is given by $(S^{fer}_{NS\to NS})_{\alpha(i),\alpha(j)} = S^{bos}_{i,j} + S^{bos}_{Ji,j}$. Inserting this in the first relation of (3.5) we obtain (using the fact that S^{bos} and $S^{fer}_{NS\to NS}$ are symmetric and orthogonal):

$$
\begin{aligned}
\mathcal{N}_{\alpha(i),\alpha(j)}^{\alpha(k)} &= \frac{1}{2}\sum_{n\in\mathcal{I}'_{NS}} \frac{(S^{bos}_{n,i}+S^{bos}_{n,Ji})(S^{bos}_{n,j}+S^{bos}_{n,Jj})(S^{bos}_{k,n}+S^{bos}_{k,Jn})}{S^{bos}_{n,0}+S^{bos}_{Jn,0}} \\
&= \frac{1}{4}\sum_{n\in\mathcal{I}'_{NS}} \frac{(S^{bos}_{n,i}+S^{bos}_{n,Ji})(S^{bos}_{n,j}+S^{bos}_{n,Jj})(S^{bos}_{k,n}+S^{bos}_{k,Jn})}{S^{bos}_{n,0}} \\
&= \frac{1}{4}\big(N_{i,j}^{k}+N_{i,j}^{Jk}+N_{i,Jj}^{k}+N_{i,Jj}^{Jk}+N_{Ji,j}^{k}+N_{Ji,j}^{Jk}+N_{Ji,Jj}^{k}+N_{Ji,Jj}^{Jk}\big) \\
&= N_{i,j}^{k}+N_{Ji,j}^{k}.
\end{aligned}
$$

Note that we used the identities $S^{bos}_{Ji,j} = S^{bos}_{i,Jj}$ and $S^{bos}_{n,0} = S^{bos}_{Jn,0}$ (formula (4.2) in ref. [SY]). This follows directly from the Verlinde formula and the defining property of the 'simple current' $[J]$ (fusion of $[J]$ with a basis element of $\mathcal{A}$ yields only one (different) basis element).

Consider now the by far more interesting case of $\mathcal{N}_{\alpha(i),\alpha(j)}^{\alpha(k)}$ with $i,j \in \mathcal{I}'_{R}$ (in the case $i\in\mathcal{I}'_R, j\in\mathcal{I}'_{NS}$ one proceeds in the same way). For simplicity we treat only the case where i,j correspond to fixed points in $\mathcal{A}_{\frac{1}{2}}$. For fixed points the characters of the fermionic theory are twice the corresponding bosonic characters. Due to the chirality operator Γ in the underlying chiral symmetry algebra these characters are indeed the characters of irreducible HWRs of the fermionic theory. In particular, for fixed points the dimension d of the irreducible representation of the horizontal subalgebra of the symmetry algebra in $\mathcal{V}_0$ is two. For HWRs corresponding to orbit length two in $\mathcal{A}_{\frac{1}{2}}$ the dimension d equals one or two depending on whether the conformal dimensions of the fields in the orbit are different or not. With $(S^{fer}_{R\to\widetilde{NS}})_{\alpha(i),\alpha(j)} = S^{bos}_{i,j}$ ($i\in\mathcal{I}'_{NS}$, $j\in\mathcal{I}'_R$, j corresponding to a fixed point) we obtain with the second relation of (3.5) the following result (inserting $d_i = d_j = 2$):

$$
\begin{aligned}
\mathcal{N}_{\alpha(i),\alpha(j)}^{\alpha(k)} &= \frac{1}{2}d_i d_j \sum_{n\in\mathcal{I}'_{NS}} \frac{S^{bos}_{n,i}S^{bos}_{n,j}(S^{bos}_{k,n}+S^{bos}_{k,Jn})}{S^{bos}_{n,0}+S^{bos}_{Jn,0}} = \sum_{n\in\mathcal{I}'_{NS}} \frac{S^{bos}_{n,i}S^{bos}_{n,j}(S^{bos}_{k,n}+S^{bos}_{k,Jn})}{S^{bos}_{n,0}} \\
&= N_{i,j}^{k}+N_{i,j}^{Jk} = 2N_{i,j}^{k}.
\end{aligned}
$$

The case with orbits of length two in $\mathcal{A}_{\frac{1}{2}}$ can be treated similarly. We see that the diagram (3.11) is indeed commutative thus proving our assertion.

3.3 Fermionic $\mathcal{W}(2,\delta)$-algebras

In this section we discuss $\mathcal{W}(2,\delta)$-algebras with one additional generator W with half-integer conformal dimension $\delta \geq \frac{5}{2}$ [Bl1,Va,Eh1]. Besides the parabolic cases [Fl1] these algebras exist for Virasoro minimal values of the central charge which can be understood by the ADE-classification of modular invariant partition functions [CIZ]. These values of the central charge can be organized in two series according to the type of the partition function which is diagonalized by the $\mathcal{W}$-characters. From table 1 below one can read off these $\mathcal{W}$-characters in terms of Virasoro characters. The $\mathcal{W}$-characters are the quantities which appear in $Z = Z^{NS} + \widetilde{Z^{NS}} + Z^R$ with their absolute value squared. The irreducible representations of the horizontal subalgebra of $\mathcal{W}$ in the L_0-eigenspace $\mathcal{V}_0$ are either one or two-dimensional in the R sector. In the one-dimensional (trivial) representation W_0 is represented by zero whereas in the two-dimensional representation W_0 acts non-trivially. The representation of the horizontal subalgebra in $\mathcal{V}_0$ is equivalent to the irreducible two-dimensional representation of the Clifford algebra $Cl(2,0)$ of a two-dimensional euclidian vectorspace:

$$\{W_0, W_0\} = 2w^2, \qquad \{W_0, \Gamma\} = 0, \qquad \{\Gamma, \Gamma\} = 2.$$

The trivial representation occurs only in those highest weight modules for which the $\mathcal{W}$-character is a sum of Virasoro characters with an equal number of terms as in the NS sector. From the representation theory of these algebras we see that this is exactly the case for the HWRs with $w = 0$ [Eh1]. In the case of $\mathcal{W}(2,\delta)$-algebras M is equal to the identity matrix $\mathbb{1}$ because there are no additional 'quantum numbers' present (and necessary). So one can read off the D-matrix from table 1 and verify that the diagonal entries coincide with the dimensions of the corresponding vector spaces $\mathcal{V}_0$.

algebra	$c_{\mathrm{Vir}}(p,q)$	series	Z^{NS}, Z^R
$\mathcal{W}\left(2, \frac{p-4}{2}\right)$	$c(p,12)$	(E_6, A_{p-1})	$Z^{NS} = \sum_{s=1}^{\frac{p-1}{2}} \lvert\chi_{1,s}+\chi_{7,s}+\chi_{5,s}+\chi_{11,s}\rvert^2$ $Z^R = \sum_{s=1}^{\frac{p-1}{2}} \frac{1}{2}\lvert 2(\chi_{4,s}+\chi_{8,s})\rvert^2$
$\mathcal{W}\left(2, \frac{(p-2)(2k-1)}{2}\right)$	$c(p,4k)$	(D_{2k+1}, A_{p-1})	$Z^{NS} = \sum_{s=1}^{\frac{p-1}{2}} \sum_{\substack{r=1 \\ odd}}^{2k-1} \lvert\chi_{r,s}+\chi_{r,p-s}\rvert^2$ $Z^R = \sum_{s=1}^{\frac{p-1}{2}} \frac{1}{2}\lvert 2\chi_{2k,s}\rvert^2 + \sum_{s=1}^{\frac{p-1}{2}} \sum_{\substack{r=2 \\ even}}^{2k-2} \lvert\chi_{r,s}+\chi_{r,p-s}\rvert^2$

Table 1: partition functions and series of fermionic $\mathcal{W}(2,\delta)$-algebras [CIZ,Bl1,Eh1,KM]

Let us also discuss the parabolic fermionic $\mathcal{W}(2,\delta)$-algebras. The series is given by the algebras $\mathcal{W}(2,3k)$ existing for $c = 1 - 24k$ with $k \in \mathbb{N} + \frac{1}{2}$. The following HWRs are permitted [Eh1] ($h_{r,r} = k(r^2-1)$, $h_{r,-r} = h_{r,r} + r^2$)

$$\begin{array}{lll}
NS: & h_{\frac{m}{2k},\frac{m}{2k}} & m = 0,\ldots,\lfloor k\rfloor, 2k \\
 & h_{\frac{m}{2k+2},-\frac{m}{2k+2}} & m = 1,\ldots,\lfloor k+\frac{1}{2}\rfloor \\
R: & h_{\frac{2m+1}{4k},\frac{2m+1}{4k}} & m = 0,\ldots,\lfloor k\rfloor - 1, \lfloor k\rfloor \\
 & h_{\frac{2m+1}{4k+4},-\frac{2m+1}{4k+4}} & m = 0,\ldots,\lfloor k+\frac{1}{2}\rfloor - 1, \lfloor k+\frac{1}{2}\rfloor
\end{array}$$

The modular invariant partition function $Z = Z^{NS} + Z^{\widetilde{NS}} + Z^R$ is given by the expressions [Fl1]:

$$
\begin{aligned}
Z^{NS} &= \frac{1}{\eta^2}\Big(\mid \tfrac{1}{2}(\theta_{0,k} - \theta_{0,k+1}) \mid^2 + \mid \tfrac{1}{2}(\theta_{0,k} + \theta_{0,k+1}) \mid^2 + \sum_{m=1}^{\lfloor k \rfloor} \mid \theta_{m,k} \mid^2 + \sum_{m=1}^{\lfloor k+\frac{1}{2} \rfloor} \mid \theta_{m,k+1} \mid^2 \Big) \\
Z^R &= \frac{1}{\eta^2}\Big(\tfrac{1}{2} \mid \theta_{k,k} \mid^2 + \tfrac{1}{2} \mid \theta_{k+1,k+1} \mid^2 + \sum_{m=0}^{\lfloor k \rfloor -1} \mid \theta_{m+\frac{1}{2},k} \mid^2 + \sum_{m=0}^{\lfloor k+\frac{1}{2} \rfloor -1} \mid \theta_{m+\frac{1}{2},k+1} \mid^2 \Big),
\end{aligned}
$$

where we have introduced Dedekind's eta function and the Riemann-Jacobi theta functions

$$
\eta(\tau) = q^{\frac{1}{24}} \prod_{n\in\mathbb{N}} (1-q^n), \qquad \theta_{\lambda,k}(\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2kn+\lambda)^2}{4k}} \qquad \text{with} \quad q = e^{2\pi i \tau}. \tag{3.12}
$$

The part $Z^{\widetilde{NS}}$ of the partition function is obtained from Z^{NS} by applying the modular transformation $\tau \to \tau + 1$. The $\mathcal{W}$-characters are the quantities which appear in Z with their absolute value squared divided by the η-function, e.g. $\chi^{\mathcal{W}}_{vac} = \eta^{-1}\frac{1}{2}(\theta_{0,k} - \theta_{0,k+1})$. From the partition function one can immediately read off the matrix $D^R = diag(\{2,2,1,\ldots,1\})$ taking into account $M = 1\!\!1$. Also in this case the diagonal elements of D^R are equal to the dimensions of the corresponding spaces $\mathcal{V}_0$. The fusion algebra can easily be obtained by the well known transformation rules of the theta functions [Fl1].

In the rest of this section we discuss the fusion rules emerging from the generalized Verlinde formula for two series of $\mathcal{W}(2,\delta)$-algebras, namely those corresponding to (D_3, A_{p-1}) and (E_6, A_{p-1}). As an example for the general (D,A)-series we discuss the case (D_5, A_2) and verify that it is not possible to extend the fusion algebra such that the fusion charge conjugation becomes equal to the usual charge conjugation.

Take as the first concrete example $\mathcal{W}(2, \frac{p-2}{2})$ at the Virasoro minimal value of the central charge $c = c(p,4) = 1 - \frac{3}{2}\frac{(p-4)^2}{p}$ for odd $p \geq 3$. These algebras are related to the partition functions of the type (D_3, A_{p-1}) (specializing the second case in the table to $k=1$). In order to give the characters of these algebras explicitly, recall the general form of the Virasoro minimal characters [Ro1]

$$
\chi^{Vir}_{r,s}(\tau) = \eta(\tau)^{-1}\big(\theta_{pr-qs,pq}(\tau) - \theta_{pr+qs,pq}(\tau)\big) \quad \text{for} \;\; 1 \leq r \leq q-1, \;\; 1 \leq s \leq p-1. \tag{3.13}
$$

The $\mathcal{W}$-algebra characters can be expressed in terms of the Virasoro minimal characters as [Va]

$$
\begin{aligned}
NS: \quad & \chi^{\mathcal{W},NS}_i = \chi_{1,i} + \chi_{1,p-i} \quad && \text{for} \;\; i \in \mathcal{I}_{NS} = \{1,\ldots,\tfrac{p-1}{2}\} \\
R: \quad & \chi^{\mathcal{W},R}_i = 2\chi_{2,i} && \text{for} \;\; i \in \mathcal{I}_R = \{1,\ldots,\tfrac{p-1}{2}\}.
\end{aligned} \tag{3.14}
$$

Using the arguments given above, we infer from (3.14) that the matrix D^R is equal to $2 1\!\!1$. Rewriting the modular transformation matrix S of the Virasoro minimal models [Ro1] in the $\mathcal{W}$-algebra character basis, one obtains (for $S^{NS,NS}$ see [Va])

$$\begin{aligned} S^{NS,NS}_{i,j} &= \frac{2}{\sqrt{p}}(-1)^{i+j+1+\frac{p-1}{2}+\lfloor\frac{p+1}{4}\rfloor}\sin\left(\frac{4\pi ij}{p}\right) && \text{for } \ i,j\in\mathcal{I}_{NS} \\ S^{\widetilde{NS},R}_{i,l} &= \frac{1}{2}S^{R,\widetilde{NS}}_{l,i} = \frac{2}{\sqrt{2p}}(-1)^{l+1+\frac{p-1}{2}}\sin\left(\frac{4\pi il}{p}\right) && \text{for } \ i\in\mathcal{I}_{NS},\ l\in\mathcal{I}_R. \end{aligned} \tag{3.15}$$

Inserting this result into (3.5) gives the fusion algebra. This fusion algebra is isomorphic to the fusion algebra of the Virasoro minimal model with $c=c(2,p)$ tensor an element ω with $\omega^2=2$, i.e. an additional $\mathbb{Z}_2$ grading. In particular, the vacuum occurs with multiplicity 2 in the fusion of a R field with itself indicating that all R fields correspond to fixed points in the fusion algebra of the bosonic projection. The structure of the fusion algebra is evident because the two matrices $S^{NS,NS}$ and $S^{\widetilde{NS},R}$ are, up to a constant, equal to the S-matrix of the Virasoro minimal $c(2,p)$ model. The element ω of order 2 leads to the correct sector structure due to $\mathbb{Z}_2$ charge conservation. Note that a rescaling of the fields in the R sector with a factor $\frac{1}{\sqrt{2}}$ leads to a fusion algebra where all coefficients are equal to zero or one and the fusion charge conjugation is equal to S^2. However, this rescaling is unnatural as our considerations in section 3.2 have shown.

Note that the fusion algebras of these $\mathcal{W}(2,\delta)$-algebras were also determined in ref. [KRV] using path space realizations of the corresponding characters.

Let us now discuss the algebras $\mathcal{W}(2,\frac{p-4}{2})$ existing for the Virasoro minimal values $c(p,12)$ with $p\geq 5$ odd and $p,3$ coprime. The partition function is of the type (E_6,A_{p-1}). From table 1 we find that the $\mathcal{W}$-characters are given by [Bl1,Eh1,Va]:

$$\begin{aligned} NS:&\quad \chi^{\mathcal{W},NS}_i = \chi_{1,i}+\chi_{7,i}+\chi_{5,i}+\chi_{11,i} && \text{for } \ i\in\mathcal{I}_{NS}=\{1,\ldots,\tfrac{p-1}{2}\} \\ \widetilde{NS}:&\quad \chi^{\mathcal{W},\widetilde{NS}}_i = \mathrm{sgn}(p-4i)\big(\chi_{1,i}+\chi_{7,i}-\chi_{5,i}-\chi_{11,i}\big) && \text{for } \ i\in\mathcal{I}_{NS}=\{1,\ldots,\tfrac{p-1}{2}\} \\ R:&\quad \chi^{\mathcal{W},R}_i = 2(\chi_{4,i}+\chi_{8,i}) && \text{for } \ i\in\mathcal{I}_R=\{1,\ldots,\tfrac{p-1}{2}\}. \end{aligned} \tag{3.16}$$

Using the S-matrix of the minimal models of the Virasoro algebra we obtain for the S-matrix in the $\mathcal{W}$-character basis:

$$\begin{aligned} S^{NS,NS}_{i,j} &= \frac{2}{\sqrt{p}}(-1)^{i+j+1+\lfloor\frac{p}{12}\rfloor}\sin\left(\frac{12\pi ij}{p}\right) && \text{for } \ i,j\in\mathcal{I}_{NS} \\ S^{\widetilde{NS},R}_{i,l} &= \frac{1}{2}S^{R,\widetilde{NS}}_{l,i} = \frac{2}{\sqrt{2p}}\mathrm{sgn}(p-4i)(-1)^{l+(p \bmod 3)}\sin\left(\frac{12\pi il}{p}\right) && \text{for } \ i\in\mathcal{I}_{NS},\ l\in\mathcal{I}_R \end{aligned} \tag{3.17}$$

The S-matrix obeys (3.8) with $H^R=(D^R)^{-1}=\frac{1}{2}\mathbb{1}$. As in the first example the resulting fusion algebra is isomorphic to the fusion algebra of the Virasoro $(2,p)$ model tensor an element of order two generating the right sector structure. This can be inferred from the form (3.17) of the S-matrix being equal to that of the $(2,p)$ model modulo factors.

We remark that for $c(p,12)$ also the bosonic $\mathcal{W}$-algebras $\mathcal{W}(2,p-3)$ exist and diagonalize the modular invariant partition functions given by

$$Z_{bos} = \sum_{s=1}^{(p-1)/2}\left(|\chi_{1,s}+\chi_{7,s}|^2+|\chi_{5,s}+\chi_{11,s}|^2+|\chi_{4,s}+\chi_{8,s}|^2\right) \tag{3.18}$$

The first realization for low spins is $\mathcal{W}(2,8) \subset \mathcal{W}(2,\frac{7}{2})$ at $c = \frac{21}{22}$ [Bl1]. The subalgebra $\mathcal{W}(2,p-3)$ is the bosonic projection of the fermionic algebra $\mathcal{W}(2,\frac{p-4}{2})$ for $c = c(p,12)$ with $p \geq 5$ odd and $p,3$ coprime. One can show that the field with conformal dimension $\frac{p-4}{2}$ is a 'simple current' in the fusion algebra of the bosonic algebra. It is not very hard to find the fusion algebra of the fermionic algebra from that of the bosonic subalgebra (which can be calculated with the ordinary Verlinde formula) using a 'simple current' argument. In appendix A we discuss this in detail for the $N = 1$ super Virasoro algebra. For a detailed discussion of 'simple currents' in the context of modular invariants of RCFTs we refer the reader to ref. [SY].

In the two series discussed above it was possible to obtain a fusion algebra with fusion coefficients $\mathcal{N}_{ii}^{1} = 1$ by rescaling the R fields with a factor $\frac{1}{\sqrt{2}}$. In the general case of the (A,D) series this rescaling would result in irrational fusion coefficients. Already the simplest nontrivial example demonstrates this: Now we show that the coefficients $\mathcal{N}_{ii}^{1} = 2$ (for some $i \in \mathcal{I}_R$) are essential and cannot be removed by an extension of the fusion algebra.

Our example is $\mathcal{W}(2,\frac{3}{2})$ at $c = c(3,8) = -\frac{21}{4}$ with the modular invariant (D_5, A_2). The model is the non-unitary super Virasoro minimal model $c(2,8)$ and consists of two HWRs in each sector:

$$NS:\ \{0,-\tfrac{1}{4}\} \equiv \{\mathbb{1},\sigma\} \qquad\qquad R:\ \{-\tfrac{3}{32},-\tfrac{7}{32}\} \equiv \{\phi,\psi\}.$$

The $\mathcal{W}$-characters in terms of Virasoro characters are

$$\begin{aligned}
\chi_0^{\mathcal{W},NS} &= \chi_{1,1} + \chi_{1,2} & \qquad \chi_{-\frac{1}{4}}^{\mathcal{W},NS} &= \chi_{3,1} + \chi_{3,2}\\
\chi_{-\frac{3}{32}}^{\mathcal{W},R} &= 2\chi_{4,1} & \qquad \chi_{-\frac{7}{32}}^{\mathcal{W},R} &= \chi_{2,1} + \chi_{2,2}\,.
\end{aligned}$$

The matrix D^R is equal to $diag(\{2,1\})$. The fusion rules are given by (using the formulae of section 3.1):

$$\begin{aligned}
&[\sigma][\sigma] = [\mathbb{1}] + 2[\sigma] & \quad &[\sigma][\phi] = [\phi] + 2[\psi] & \quad &[\sigma][\psi] = [\phi] + [\psi]\\
&[\phi][\phi] = 2[\mathbb{1}] + 2[\sigma] & &[\phi][\psi] = 2[\sigma] & &[\psi][\psi] = [\mathbb{1}] + [\sigma].
\end{aligned}$$

We have indeed $\mathcal{N}_{ii}^{1} = D_{i,i}^{R}$ $(i \in \mathcal{I}_R)$ but a simple rescaling of the R characters does not remove the coefficient $\mathcal{N}_{\phi\phi}^{1} = 2$. Let us now show that an extension of this fusion algebra by splitting the field ϕ is not possible. This is in contrast to the case of degeneracies. To this end one makes the most general ansatz $(a, b, a^{\pm}, b^{\pm}, d, e, f^{\pm}, g^{\pm}, h^{\pm} \in \mathbb{N}_0)$:

$$\begin{aligned}
&[\phi] = [\phi^+] + [\phi^-] & \quad &[\phi^+][\psi] = a[\sigma] & \quad &[\phi^-][\psi] = b[\sigma]\\
&[\phi^+][\phi^+] = a^+[\mathbb{1}] + b^+[\sigma] & &[\phi^-][\phi^-] = a^-[\mathbb{1}] + b^-[\sigma] & &[\phi^+][\phi^-] = d[\mathbb{1}] + e[\sigma]\\
&[\phi^+][\sigma] = f^+[\phi^+] + f^-[\phi^-] + h^+[\psi] & &[\phi^-][\sigma] = g^+[\phi^+] + g^-[\phi^-] + h^-[\psi]. & &
\end{aligned}$$

Using the fusion rules for $[\phi][\psi]$, $[\phi][\phi]$ and $[\sigma][\phi]$ we get from this ansatz the following equations:

$$\begin{aligned}
&a + b = 2 & \quad &a^+ + a^- + 2d = 2 & \quad &b^+ + b^- + 2e = 2\\
&f^+ + g^+ = 1 & &f^- + g^- = 1 & &h^+ + h^- = 2.
\end{aligned}$$

From the associativity of $[\psi][\phi^+][\phi^+]$ and $[\psi][\phi^-][\phi^-]$ we have

$$af^+ = b^+ \qquad af^- = b^+ \qquad ah^+ = a^+ + b^+ \qquad bg^+ = b^- \qquad bg^- = b^- \qquad bh^- = a^- + b^-$$

and from $[\psi][\phi^+][\phi^-]$ one obtains

$$ag^+ = e \qquad ag^- = e \qquad ah^- = d + e \qquad bf^+ = e \qquad bf^- = e \qquad bh^+ = d + e.$$

Because of the triviality of the solutions $a = 0$ or $b = 0$ (in the sense that one recovers the original fusion algebra), we conclude from $a + b = 2$ that $a = b = 1$. Inserting this into $ag^+ + bf^+ = 2e$, we get the contradiction $1 = f^+ + g^+ = 2e$ (q.e.d.).

3.4 $N = 1$ super Virasoro minimal models

For the $N = 1$ super Virasoro minimal models the central charge, the conformal dimensions of the HWRs and the characters are given by [Ro2]

$$\begin{aligned}
c = c(p,q) &= \tfrac{3}{2}\left(1 - 2\frac{(p-q)^2}{pq}\right) \qquad p,q \in \mathbb{N},\ (p \mid q) = 1,\ p+q \in 2\mathbb{N} \quad \text{or} \\
&\qquad\qquad p,q \in 2\mathbb{N},\ (\tfrac{p}{2} \mid \tfrac{q}{2}) = 1,\ \tfrac{p}{2} + \tfrac{q}{2} \notin 2\mathbb{N} \\
h(r,s) &= \frac{(pr-qs)^2 - (p-q)^2}{8pq} + \frac{1-(-1)^{r+s}}{32} \qquad 1 \le r \le q-1,\ 1 \le s \le p-1 \\
\chi^{SVir,NS}_{r,s}(\tau) &= \frac{e^{-2\pi i \frac{1}{48}}\eta(\frac{\tau+1}{2})}{\eta(\tau)^2}\big(\theta_{pr-qs,pq}(\tfrac{\tau}{2}) - \theta_{pr+qs,pq}(\tfrac{\tau}{2})\big) \\
\chi^{SVir,R}_{r,s}(\tau) &= \frac{\eta(2\tau)}{\eta(\tau)^2}\big(2 - \delta_{r,\frac{q}{2}}\delta_{s,\frac{p}{2}}\big)\big(\theta_{pr-qs,pq}(\tfrac{\tau}{2}) - \theta_{pr+qs,pq}(\tfrac{\tau}{2})\big)
\end{aligned} \tag{3.19}$$

where $r + s$ even (odd) corresponds to the NS (R) sector. We do not include a 'global' factor $\sqrt{2}$ in the R characters as ref. [Cap] but use the definition (3.2). Using the reflection symmetry in the superconformal grid the set of linear independent characters is labelled by:

$$\begin{aligned}
NS:\ \mathcal{I}_{NS} &= \{\,(r,s) \mid r+s \text{ even},\ 1 \le r \le q-1,\ 1 \le s \le \lfloor\tfrac{p-1}{2}\rfloor \text{ or } 1 \le r \le \tfrac{q}{2},\ s = \tfrac{p}{2}\} \\
R:\ \mathcal{I}_{R} &= \{\,(r,s) \mid r+s \text{ odd},\ 1 \le r \le q-1,\ 1 \le s \le \lfloor\tfrac{p-1}{2}\rfloor \text{ or } 1 \le r \le \tfrac{q}{2},\ s = \tfrac{p}{2}\}.
\end{aligned} \tag{3.20}$$

From the well known transformation properties of the theta functions under modular transformations we obtain the following S-matrix:

$$\begin{aligned}
S^{NS,NS}_{r_1,s_1;r_2,s_2} &= \frac{2}{\sqrt{pq}}\left(\cos\left(\frac{2\pi\lambda_1\lambda_2}{4pq}\right) - \cos\left(\frac{2\pi\bar{\lambda}_1\lambda_2}{4pq}\right)\right) \\
S^{R,\widetilde{NS}}_{r_1,s_1;r_2,s_2} &= \frac{2}{\sqrt{2pq}}\left(\cos\left(\frac{2\pi\lambda_1\lambda_2}{4pq}\right) - (-1)^{r_2 s_2}\cos\left(\frac{2\pi\lambda_1\bar{\lambda}_2}{4pq}\right)\right) \\
S^{\widetilde{NS},R}_{r_1,s_1;r_2,s_2} &= \frac{2}{\sqrt{2pq}}\left(1 + \delta_{r_2,\frac{q}{2}}\delta_{s_2,\frac{p}{2}}\right)\left(\cos\left(\frac{2\pi\lambda_1\lambda_2}{4pq}\right) - \cos\left(\frac{2\pi\lambda_1\bar{\lambda}_2}{4pq}\right)\right)
\end{aligned} \tag{3.21}$$

with $\lambda_i = pr_i - qs_i, \bar{\lambda}_i = pr_i + qs_i$. One can check that with the standard definition of T these formulae define a proper representation of the modular group. It is well known that the *unitary* minimal models are given by $(p,q) = (m, m+2)$ for $m \geq 2$. For this special choice of (p,q) formula (3.21) reduces to the corresponding one already given in [MY]. We emphasize that the multiplicity matrix M is equal to the identity matrix in the case of the $N=1$ super Virasoro minimal models because there are no additional independent 'quantum numbers'.

Let us first consider the case p and q odd. Here the reflection symmetry in the superconformal grid has no fixed point and the D-matrix is given by $D^R = 2\mathbb{1}$ since the S-matrix obeys (3.8) with $H^R = \frac{1}{2}\mathbb{1}$. In order to calculate the fusion algebra corresponding to the super Virasoro minimal models one has to insert (3.21) into (3.5). It is not necessary to calculate the fusion coefficients directly if one remembers that the matrices $S^{NS,NS}$ and $S^{\widetilde{NS},R}$ are (modulo constants) equal to the S-matrix of the Virasoro minimal models with central charge $c = c(p,q)$, so that the well known selection rules for the Virasoro minimal models can be applied to the fusion coefficients of the super Virasoro minimal models. Since the selection rules of the Virasoro minimal models respect the sector stucture given by odd or even $r+s$ in the same way as the super minimal models, the corresponding fusion algebras are isomorphic. Obviously, the only difference between the fusion algebras is the fact that the fusion coefficients connecting two R fields with a NS field are elements of $2\mathbb{N}$ for the $N=1$ supersymmetric model. Note that the fusion algebra has a $\mathbb{Z}_2$-structure like the first two examples in section 3.3. As in the case of the $\mathcal{W}(2, \frac{k-2}{2})$ algebras, $\mathcal{N}_{ii}^1 = 2$ ($\forall i \in \mathcal{I}_R$) indicates that the fields in the R sector correspond to fixed points in the fusion algebra of the bosonic projection. Furthermore, the dimensions of the representations of the horizontal subalgebra in the L_0 eigenspace $\mathcal{V}_0$ are equal to the corresponding diagonal entries of D^R. A concrete example is given at the end of this section.

In the case p and q even the structure is different. Because M is trivial the matrix D can be immediately read off from (3.21) and (3.8): $D^R_{i,j} = (2 - \delta_{i,i_0})\delta_{i,j}$, where i_0 is the label of the HWR $h = \frac{c}{24}$ in the R sector. Note that this is exactly the only fixed point under the reflection symmetry in the superconformal grid: $i_0 \equiv (r,s) = (\frac{q}{2}, \frac{p}{2})$. This implies that in this representation G_0^2 is represented by zero so that the irreducible representation of the horizontal subalgebra in $\mathcal{V}_0$ is one-dimensional (in contrast to the generic case $G_0^2 \neq 0$ where the irreducible representations are two-dimensional). Consequently, the corresponding $\widetilde{R}$ character is nontrivial and encodes the broken boson-fermion symmetry of the ground state. Hence the Witten index [Wi,Ka] of this HWR is nontrivial. This shows that the diagonal entries of D^R are equal to the dimensions of the spaces $\mathcal{V}_0$. Calculating the fusion algebra with (3.5) yields an associative commutative algebra with nontrivial fusion charge conjugation $\mathcal{N}_{ij}^1 = D^R_{i,j}, i,j \in \mathcal{I}_R$. The example $c(2,8)$ has been treated in section 3.3.

In the remaining part of this section the formulae of section 3.1 are verified explicitly by considering the bosonic projection of the $N=1$ super Virasoro algebra. As we will see in section 4.1.4 this orbifold leads to an algebra of type $\mathcal{W}(2,4,6)$ for generic values of the central charge. We calculate the fusion algebra – using the ordinary Verlinde formula – of the orbifold for two special minimal values of c where this orbifold truncates to $\mathcal{W}(2,4)$.

We will recognize that these fusion algebras contain a 'simple current' of dimension $\frac{3}{2}$. The definition of a new basis, given by the sum of the fields lying in one orbit, divided by the length of the orbit, yields the fusion algebra of the $N=1$ super Virasoro minimal model. The fusion algebras obtained this way coincide exactly with the fusion algebras calculated directly using the generalized Verlinde formula.

As one can infer from eq. (4.46) in section 4.1.4, the primary field of scale dimension 6 in the bosonic projection of the $N=1$ super Virasoro algebra turns out to be a null field for some special values of c, among them -11 and $-\frac{11}{14}$, so that the $\mathcal{W}(2,4,6)$ reduces to a $\mathcal{W}(2,4)$ (the primary field of dimension four has nonzero norm for these values of c). It has been found earlier that at these two values of c rational models of $\mathcal{W}(2,4)$ exist and the possible h-values are known [Eh1].

Firstly let us determine the fusion algebra of the $N=1$ super Virasoro minimal model at $c(2,12)=-11$ using the explicit form of the S-matrix (3.21) and the generalized Verlinde formula (3.5). There are three HWRs in each sector with the following highest weights: $NS: h\in\{0,-\frac{1}{3},-\frac{1}{2}\}$, $\quad R: h\in\{-\frac{1}{8},-\frac{3}{8},-\frac{11}{24}\}$. Note that the last h-value is the fixed point of the superconformal grid and is equal to $\frac{c}{24}$. The fusion algebra reads:

$$\begin{array}{ll}
[\Phi^{NS}_{-\frac{1}{3}}][\Phi^{NS}_{-\frac{1}{3}}]=[\Phi^{NS}_{0}]+[\Phi^{NS}_{-\frac{1}{3}}]+[\Phi^{NS}_{-\frac{1}{2}}] & [\Phi^{NS}_{-\frac{1}{3}}][\Phi^{NS}_{-\frac{1}{2}}]=[\Phi^{NS}_{-\frac{1}{3}}]+2[\Phi^{NS}_{-\frac{1}{2}}] \\
[\Phi^{NS}_{-\frac{1}{2}}][\Phi^{NS}_{-\frac{1}{2}}]=[\Phi^{NS}_{0}]+2[\Phi^{NS}_{-\frac{1}{3}}]+2[\Phi^{NS}_{-\frac{1}{2}}] & \\
[\Phi^{R}_{-\frac{1}{8}}][\Phi^{R}_{-\frac{1}{8}}]=2[\Phi^{NS}_{0}]+2[\Phi^{NS}_{-\frac{1}{3}}] & [\Phi^{R}_{-\frac{1}{8}}][\Phi^{R}_{-\frac{3}{8}}]=2[\Phi^{NS}_{-\frac{1}{3}}]+2[\Phi^{NS}_{-\frac{1}{2}}] \\
[\Phi^{R}_{-\frac{1}{8}}][\Phi^{R}_{-\frac{11}{24}}]=2[\Phi^{NS}_{-\frac{1}{2}}] & [\Phi^{R}_{-\frac{3}{8}}][\Phi^{R}_{-\frac{3}{8}}]=2[\Phi^{NS}_{0}]+2[\Phi^{NS}_{-\frac{1}{3}}]+4[\Phi^{NS}_{-\frac{1}{2}}] \\
[\Phi^{R}_{-\frac{3}{8}}][\Phi^{R}_{-\frac{11}{24}}]=2[\Phi^{NS}_{-\frac{1}{3}}]+2[\Phi^{NS}_{-\frac{1}{2}}] & [\Phi^{R}_{-\frac{11}{24}}][\Phi^{R}_{-\frac{11}{24}}]=[\Phi^{NS}_{0}]+[\Phi^{NS}_{-\frac{1}{3}}]+[\Phi^{NS}_{-\frac{1}{2}}].
\end{array} \tag{3.22}$$

For $\mathcal{W}(2,4)$ at $c=-11$ the effective central charge $\tilde{c}$ is equal to 1. Consequently this model belongs to the parabolic $\mathcal{W}(2,\delta)$-algebras which have been studied by Flohr [Fl1]. Using the explicit form of the S-matrix one is able to calculate the fusion algebra of this model [Fl2,Fl1]. The model has the 10 HWRs $h\in\{0,\frac{3}{2},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},\tilde{0},-\frac{1}{8},-\frac{3}{8},-\frac{11}{24},\frac{13}{24}\}$. The fusion algebra can be found in appendix A (eq. (A.1)). One recognizes immediately that the field $[\Phi_{\frac{3}{2}}]$ is a 'simple current' reflecting the supersymmetric structure of this model. We conclude that the symmetry algebra can be extended by the 'simple current'. This extension can be viewed as the inverse procedure of the projection onto the bosonic part of the fermionic algebra. Doing so we sum up the fields lying in one orbit under $[\Phi_{\frac{3}{2}}]$ and arrive at the following natural definitions:

$$\begin{array}{ll}
[\Phi^{NS}_{0}]:=\frac{1}{2}\big([\Phi_{0}]+[\Phi_{\frac{3}{2}}]\big) & [\Phi^{R}_{-\frac{1}{8}}]:=[\Phi_{-\frac{1}{8}}] \\
[\Phi^{NS}_{-\frac{1}{3}}]:=\frac{1}{2}\big([\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]\big) & [\Phi^{R}_{-\frac{3}{8}}]:=[\Phi_{-\frac{3}{8}}] \\
[\Phi^{NS}_{-\frac{1}{2}}]:=\frac{1}{2}\big([\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}]\big) & [\Phi^{R}_{-\frac{11}{24}}]:=\frac{1}{2}\big([\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}]\big).
\end{array} \tag{3.23}$$

With this definition we recover exactly the fusion algebra (3.22) of the $N=1$ super Virasoro minimal model $c(2,12)$ from the fusion algebra (A.1) of the $\mathcal{W}(2,4)$ rational model. This verifies the consistency of the formulae presented in section 3.1.

As an example for the (odd,odd) case we consider the fusion algebra of the $N=1$ super Virasoro minimal model $c(3,7)=-\frac{11}{14}$. This model has three HWRs in each sector:

$NS: h \in \{0, \frac{2}{7}, -\frac{1}{14}\}$, $\quad R: h \in \{\frac{11}{16}, -\frac{3}{112}, \frac{13}{122}\}$. There is no fixed point in the superconformal grid. With the generalized Verlinde formula we obtain with the S-matrix eq. (3.21) the following fusion algebra:

$$
\begin{array}{ll}
[\Phi^{NS}_{\frac{2}{7}}][\Phi^{NS}_{\frac{2}{7}}] = [\Phi^{NS}_{0}] + [\Phi^{NS}_{-\frac{1}{14}}] & [\Phi^{NS}_{\frac{2}{7}}][\Phi^{NS}_{-\frac{1}{14}}] = [\Phi^{NS}_{\frac{2}{7}}] + [\Phi^{NS}_{-\frac{1}{14}}] \\
[\Phi^{NS}_{-\frac{1}{14}}][\Phi^{NS}_{-\frac{1}{14}}] = [\Phi^{NS}_{0}] + [\Phi^{NS}_{\frac{2}{7}}] + [\Phi^{NS}_{-\frac{1}{14}}] & \\
[\Phi^{R}_{\frac{11}{16}}][\Phi^{R}_{\frac{11}{16}}] = 2[\Phi^{NS}_{0}] & [\Phi^{R}_{\frac{11}{16}}][\Phi^{R}_{-\frac{3}{112}}] = 2[\Phi^{NS}_{\frac{2}{7}}] \\
[\Phi^{R}_{\frac{11}{16}}][\Phi^{R}_{\frac{13}{122}}] = 2[\Phi^{NS}_{-\frac{1}{14}}] & [\Phi^{R}_{-\frac{3}{112}}][\Phi^{R}_{-\frac{3}{112}}] = 2[\Phi^{NS}_{0}] + 2[\Phi^{NS}_{-\frac{1}{14}}] \\
[\Phi^{R}_{-\frac{3}{112}}][\Phi^{R}_{\frac{13}{122}}] = 2[\Phi^{NS}_{\frac{2}{7}}] + 2[\Phi^{NS}_{-\frac{1}{14}}] & [\Phi^{R}_{\frac{13}{122}}][\Phi^{R}_{\frac{13}{122}}] = 2[\Phi^{NS}_{0}] + 2[\Phi^{NS}_{\frac{2}{7}}] + 2[\Phi^{NS}_{-\frac{1}{14}}].
\end{array}
\tag{3.24}
$$

We already pointed out that this fusion algebra has a $\mathbb{Z}_2$-structure. As mentioned above the bosonic projection of the $N=1$ super Virasoro algebra yields a $\mathcal{W}(2,4)$ for $c=-\frac{11}{14}$. To obtain the fusion algebra of this $\mathcal{W}(2,4)$ model we note that this value of the central charge is contained in the minimal series of the Virasoro algebra and can be parametrized by $c = c_{\text{Vir}}(7,12) = -\frac{11}{14}$. The change of basis of the Virasoro characters to be performed and the resulting fusion algebra are given in appendix A. Again we recognize that the field $[\Phi_{\frac{3}{2}}]$ is a 'simple current' reflecting the supersymmetry of the model. Repeating the argument from above we make the following definition:

$$
\begin{array}{ll}
[\Phi^{NS}_{0}] := \frac{1}{2}\big([\Phi_0] + [\Phi_{\frac{3}{2}}]\big) & [\Phi^{R}_{\frac{11}{16}}] := [\Phi_{\frac{11}{16}}] \\
[\Phi^{NS}_{\frac{2}{7}}] := \frac{1}{2}\big([\Phi_{\frac{2}{7}}] + [\Phi_{\frac{11}{14}}]\big) & [\Phi^{R}_{-\frac{3}{112}}] := [\Phi_{-\frac{3}{112}}] \\
[\Phi^{NS}_{-\frac{1}{14}}] := \frac{1}{2}\big([\Phi_{-\frac{1}{14}}] + [\Phi_{\frac{3}{7}}]\big) & [\Phi^{R}_{\frac{13}{122}}] := [\Phi_{\frac{13}{112}}].
\end{array}
\tag{3.25}
$$

With this change of basis we recover the fusion algebra (3.24) from the fusion algebra (A.2) of the $\mathcal{W}(2,4)$-minimal model.

3.5 $N=1$ $\mathcal{SW}(\frac{3}{2},\delta)$-algebras

In this section we investigate $N=1$ super $\mathcal{W}$-algebras $\mathcal{SW}(\frac{3}{2},\delta)$ with two generators for $\delta \geq 2$. These are the most complicated examples of fermionic RCFTs as far as the fusion algebra is concerned because both the multiplicity matrix M as well as the D-matrix are in general different from the identity matrix. After some general comments on the representation theory in the Ramond sector we discuss the $\mathcal{SW}(\frac{3}{2},\delta)$-algebras fitting into the ADE-classification [Cap]. We proceed with parabolic $N=1$ $\mathcal{SW}$-algebras with vanishing and non-vanishing self-coupling constant and give the matrices M, D for all these series. As an example for ADE-cases we present the fusion algebra for the rational model of $\mathcal{SW}(\frac{3}{2},2)$ at $c=-\frac{6}{5}$.

We found by explicit computer calculations that for the $\mathcal{SW}(\frac{3}{2},\delta)$-algebras with $2 \leq \delta \leq \frac{9}{2}$ the identity $[\psi_0, G_0\phi_0] = 0$ holds on the corresponding highest weight vectors where ψ (ϕ) is the bosonic (fermionic) component of the additional super field. Therefore, one considers only HWRs in which $G_0\phi_0$ is represented by a scalar on the highest weight leading to an additional quantum number which can take at most two values (since $G_0\phi_0$ satisfies a quadratic equation for fixed L_0 and ψ_0 eigenvalues of the highest weight). However, this quantum number is redundant in the supersymmetric theory because $\mathcal{V}_0$ contains for all possible eigenvalues of $G_0\phi_0$ an eigenvector. Nevertheless, in the bosonic projection it distinguishes between different highest weight representations. Because $G_0\phi_0$ is represented by a scalar on the highest weight the dimension of $\mathcal{V}_0$ is at most two, it is one-dimensional exactly if $G_0^2 = \phi_0^2 = 0$ holds on the highest weight. Indeed, for $\delta \in \mathbb{N}$ the identity $G_0^2 = 0$ implies $\phi_0^2 = 0$ due to $[G_0, \psi_0] = \phi_0$. However, computer results show that even for $\frac{9}{2} \geq \delta \in \mathbb{N} + \frac{1}{2}$ this implication is true. In the case $\delta \in \mathbb{N} + \frac{1}{2}$, $G_0^2 \neq 0$ the representation of the horizontal subalgebra in $\mathcal{V}_0$ is equal to the two-dimensional representation of the Clifford algebra $Cl(2,0)$. For $\delta \in \mathbb{N}$ the structure of the representation of the horizontal subalgebra is more complicated.

The algebras existing for super Virasoro minimal values of c can be organized into four series according to the partition function which is diagonalized by the $\mathcal{SW}$-characters. From table 2 below we can directly read off these characters in terms of super Virasoro characters (the quantities appearing with their absolute value in Z are the $\mathcal{SW}$-characters). Obviously, one can obtain D from the form of the partition functions given in table 2 if the multiplicity matix M is known. This matrix follows from earlier studies of the degeneracies of the representations [Eh2,H]. The representations whose characters are a sum of Virasoro characters with maximal number of summands are non-degenerate. The representations whose characters are a sum of Virasoro characters with half the number of summands are doubly degenerate. Note that one has to consider the NS and R sectors separately.

We continue with the discussion of parabolic $\mathcal{SW}$-algebras. There are two series, one with non-vanishing and one with vanishing self-coupling constant [Fl1]. The series with $C_{\phi\phi}^{\phi} \neq 0$ consists of the algebras $\mathcal{SW}(\frac{3}{2}, 8k)$ at $c = \frac{3}{2}(1-16k)$ with $4k \in \mathbb{N}$. They possess the following HWRs [Eh2,H] ($h_{r,r} = k(r^2-1)$, $h_{r,-r} = h_{r,r} + \frac{1}{2}r^2$):

$$
\begin{array}{lll}
NS: & h_{\frac{m}{4k},\frac{m}{4k}} & m = 0,\dots,4k,8k \\
 & h_{\frac{m}{4k+2},-\frac{m}{4k+2}} & m = 1,\dots,4k+1 \\
R: & h_{\frac{m}{4k},\frac{m}{4k}} + \frac{1}{16} & m = 0,\dots,4k \\
 & h_{\frac{m}{4k+2},-\frac{m}{4k+2}} + \frac{1}{16} & m = 0,\dots,4k+2.
\end{array}
$$

The modular invariant partition function Z is given by the expressions [Fl1]:

$$
\begin{aligned}
Z^{NS}(\tau) = \left|\frac{\eta(\frac{\tau+1}{2})}{\eta(\tau)^2}\right|^2 \Big(& \mid \tfrac{1}{2}(\theta_{4k,4k}(\tau) - \theta_{4k,4k+2}(\tau)) \mid^2 + \mid \tfrac{1}{2}(\theta_{4k,4k}(\tau) + \theta_{4k,4k+2}(\tau)) \mid^2 + \\
& \mid \tfrac{1}{2}(\theta_{0,4k}(\tau) - \theta_{0,4k+2}(\tau)) \mid^2 + \mid \tfrac{1}{2}(\theta_{0,4k}(\tau) + \theta_{0,4k+2}(\tau)) \mid^2 + \\
& \sum_{m=1}^{4k+1} \mid \theta_{4k+2+m,4k+2}(\tau) \mid^2 + \sum_{m=1}^{4k-1} \mid \theta_{4k+m,4k}(\tau) \mid^2 \Big) \\
Z^{R}(\tau) = \left|\frac{\eta(2\tau)}{\eta(\tau)^2}\right|^2 \Big(& \mid \theta_{0,4k+2}(\tau) \mid^2 + \mid \theta_{0,4k}(\tau) \mid^2 + \mid \theta_{4k,4k}(\tau) \mid^2 + \mid \theta_{4k+2,4k+2}(\tau) \mid^2 + \\
& \sum_{m=1}^{4k-1} \tfrac{1}{2} \mid 2\theta_{4k+m,4k}(\tau) \mid^2 + \sum_{m=1}^{4k+1} \tfrac{1}{2} \mid 2\theta_{4k+2+m,4k+2}(\tau) \mid^2 \Big).
\end{aligned}
\tag{3.26}
$$

$\mathcal{SW}(\frac{3}{2},\delta)$	$c(p,q)$	series	Z^{NS}, Z^R
$\delta=\frac{k-2}{2}$	$c(12,2k)$ k odd $(6,k)=1$	(D_{k+1},E_6)	$Z^{NS}=\sum_{\substack{r=1\\ odd}}^{k-1}\lvert\chi_{r,1}+\chi_{r,7}+\chi_{r,5}+\chi_{r,11}\rvert^2+ 2\lvert\chi_{k,1}+\chi_{k,5}\rvert^2$ $Z^R=\sum_{\substack{r=1\\ odd}}^{k-1}\lvert\chi_{r,4}+\chi_{r,8}\rvert^2+2\lvert\chi_{k,4}\rvert^2$
$\delta=\frac{2k-3}{2}$	$c(12,2k)$ k odd $(6,k)=1$	(A_{2k-1},E_6)	$Z^{NS}=\sum_{\substack{r=1\\ odd}}^{k-1}\left(\lvert\chi_{r,1}+\chi_{r,7}\rvert^2+\lvert\chi_{r,5}+\chi_{r,11}\rvert^2\right)+ \lvert\chi_{k,1}+\chi_{k,7}\rvert^2+\sum_{\substack{r=2\\ even}}^{k-1}\lvert\chi_{r,4}+\chi_{r,8}\rvert^2$ $Z^R=\sum_{\substack{r=2\\ even}}^{k-1}\left(\frac{1}{2}\lvert\chi_{r,1}+\chi_{r,7}\rvert^2+\frac{1}{2}\lvert\chi_{r,5}+\chi_{r,11}\rvert^2\right)+ \sum_{\substack{r=1\\ odd}}^{k-1}\frac{1}{2}\lvert\chi_{r,4}+\chi_{r,8}\rvert^2+\lvert\chi_{k,4}\rvert^2$
$\delta=\frac{2k-5}{2}$	$c(30,2k)$ k even $(15,k)=1$	(A_{2k-1},E_8)	$Z^{NS}=\sum_{\substack{r=1\\ odd}}^{k-1}\left(\lvert\chi_{r,1}+\chi_{r,11}+\chi_{r,19}+\chi_{r,29}\rvert^2+ \lvert\chi_{r,7}+\chi_{r,13}+\chi_{r,17}+\chi_{r,23}\rvert^2\right)$ $Z^R=\sum_{\substack{r=2\\ even}}^{k-1}\left(\frac{1}{2}\lvert\chi_{r,1}+\chi_{r,11}+\chi_{r,19}+\chi_{r,29}\rvert^2+ \frac{1}{2}\lvert\chi_{r,7}+\chi_{r,13}+\chi_{r,17}+\chi_{r,23}\rvert^2\right)+ \lvert\chi_{k,1}+\chi_{k,11}\rvert^2+\lvert\chi_{k,7}+\chi_{k,13}\rvert^2$
$\delta=\frac{(q-2)(k-1)}{4}$	$c(2k,q)$ k odd $(k,\frac{q}{2})=1$	(A_{q-1},D_{k+1})	$Z^{NS}=\sum_{\substack{r=1\\ odd}}^{\frac{q}{2}-1}\left(\sum_{\substack{s=1\\ odd}}^{k-1}\lvert\chi_{r,s}+\chi_{q-r,s}\rvert^2+2\lvert\chi_{r,k}\rvert^2\right)$ $Z^R=\sum_{\substack{r=2\\ even}}^{\frac{q}{2}-1}\sum_{\substack{s=1\\ odd}}^{k-1}\frac{1}{2}\lvert\chi_{r,s}+\chi_{q-r,s}\rvert^2+\sum_{\substack{s=1\\ odd}}^{k-1}\lvert\chi_{\frac{q}{2},s}\rvert^2+ \sum_{\substack{r=2\\ even}}^{\frac{q}{2}-1}\lvert\chi_{r,k}\rvert^2+2\lvert\chi_{\frac{q}{2},k}\rvert^2$

Table 2: partition functions and series of $\mathcal{SW}(\frac{3}{2},\delta)$-algebras [CIZ,Bl3,Eh2]

Firstly, we state that the *two* representations to the h-value $h_{0,0}+\frac{1}{16}=\frac{c}{24}$ – corresponding to the first two summands in Z^R – are different and that the dimension of the representation of the horizontal subalgebra in $\mathcal{V}_0$ is equal to one in both cases. In all other representations this dimension is equal to two. Furthermore, the representations to $h_{1,1}+\frac{1}{16}$ – third summand – and $h_{1,-1}+\frac{1}{16}$ – fourth summand – are doubly degenerate, whereas all other representations occur only once. We conclude that $M^R=diag(\{1,1,2,2,1,\ldots,1\})$ so that $D^R=diag(\{1,1,2,2,2,\ldots,2\})$ (again the entries of D^R equal the dimensions of the corresponding spaces $\mathcal{V}_0$).

The series with $C_{\phi\phi}^{\phi}=0$ is given by the algebras $\mathcal{SW}(\frac{3}{2},3k)$ existing for $c=\frac{3}{2}(1-16k)$ with $2k\in\mathbb{N}$. They possess the following HWRs [Eh2,H]:

$$\begin{array}{lll} NS: & h_{\frac{m}{2k},\frac{m}{2k}} & m=0,\ldots,\lfloor k\rfloor,2k \\ & h_{\frac{m}{2k+1},-\frac{m}{2k+1}} & m=0,\ldots,\lfloor k+\frac{1}{2}\rfloor \\ R: & h_{\frac{m}{2k},\frac{m}{2k}}+\frac{1}{16} & m=k-\lfloor k\rfloor,\ldots,k \\ & h_{\frac{m}{2k+1},-\frac{m}{2k+1}}+\frac{1}{16} & m=\frac{1}{2}-(k-\lfloor k\rfloor),\ldots,k+\frac{1}{2}. \end{array}$$

It turned out in the course of our calculations that in the Ramond sector one has to distinguish the two cases $k\in\mathbb{N}$ and $k\in\mathbb{N}+\frac{1}{2}$. In the case $k\in\mathbb{N}$ there are two doubly degenerate HWRs with two-dimensional – $d=2$ – representation of the horizontal subalgebra ($h_{\frac{1}{2},\frac{1}{2}}+\frac{1}{16},h_{\frac{1}{2},-\frac{1}{2}}+\frac{1}{16}$). In the single representation with $h_{0,0}+\frac{1}{16}=\frac{c}{24}$ this dimension d is equal to one. All other representations are non-degenerate and have $d=2$. For $k\in\mathbb{N}+\frac{1}{2}$ there is only one doubly degenerate representation ($h_{\frac{1}{2},\frac{1}{2}}+\frac{1}{16}$) which has $d=2$. Furthermore, the representation $h_{0,0}+\frac{1}{16}=\frac{c}{24}$ has $d=1$ and all other representations have $d=2$ and are non-degenerate, clarifying some unexplained subtleties in ref. [Fl1]. From the corresponding modular invariant partition function Z one obtains D^R. For $k\in\mathbb{N}$ the diagonal entries of D^R are given by the dimensions d of $\mathcal{V}_0$. For $k\in\mathbb{N}+\frac{1}{2}$ this is only different for the representation with conformal dimension ($h_{\frac{1}{2},-\frac{1}{2}}+\frac{1}{16}$) where $d=2$ but the D^R entry is equal to one. Here we see that in general the entries of the diagonal matrix D^R are different from the corresponding dimensions of $\mathcal{V}_0$.

We present now an example for a fusion algebra of a rational model of a $\mathcal{SW}(\frac{3}{2},\delta)$-algebra which shows the most general features of fusion algebras of fermionic RCFTs. Our example is $\mathcal{SW}(\frac{3}{2},2)$ at $c=-\frac{6}{5}$ corresponding to the partition function (A_3,D_6). The model consists of three HWRs in each sector: $NS: h\in\{0,-\frac{1}{10},\frac{1}{5}\}$, $R: h\in\{\frac{3}{4},\frac{3}{20},-\frac{1}{20}\}$. The last h-value is the fixed point of the superconformal grid and hence equal to $\frac{c}{24}$. The representations with $h=\frac{1}{5},\frac{3}{4},\frac{3}{20},-\frac{1}{20}$ are doubly degenerate. The dimensions of the representations of the horizontal subalgebra in $\mathcal{V}_0$ are equal to *two* with the exception of the representation $h=-\frac{1}{20}=\frac{c}{24}$, where it is equal to *one*. These dimensions coincide again with the corresponding diagonal entries of D^R obtained from M and S. Using the explicit form of the $\mathcal{SW}$-characters in terms of super Virasoro characters, we obtain the fusion algebra via formulae (3.21) and (3.5):

$$\begin{array}{ll} [\Phi^{NS}_{-\frac{1}{10}}][\Phi^{NS}_{-\frac{1}{10}}]=[\Phi^{NS}_{0}]+[\Phi^{NS}_{-\frac{1}{10}}]+[\Phi^{NS}_{\frac{1}{5}}] & [\Phi^{NS}_{-\frac{1}{10}}][\Phi^{NS}_{\frac{1}{5}}]=2[\Phi^{NS}_{-\frac{1}{10}}]+[\Phi^{NS}_{\frac{1}{5}}] \\ [\Phi^{NS}_{\frac{1}{5}}][\Phi^{NS}_{\frac{1}{5}}]=2[\Phi^{NS}_{0}]+2[\Phi^{NS}_{-\frac{1}{10}}]+[\Phi^{NS}_{\frac{1}{5}}] & \\ & \\ [\Phi^{R}_{\frac{3}{4}}][\Phi^{R}_{\frac{3}{4}}]=4[\Phi^{NS}_{0}] & [\Phi^{R}_{\frac{3}{4}}][\Phi^{R}_{\frac{3}{20}}]=4[\Phi^{NS}_{-\frac{1}{10}}] \\ [\Phi^{R}_{\frac{3}{4}}][\Phi^{R}_{-\frac{1}{20}}]=2[\Phi^{NS}_{\frac{1}{5}}] & [\Phi^{R}_{\frac{3}{20}}][\Phi^{R}_{\frac{3}{20}}]=4[\Phi^{NS}_{0}]+4[\Phi^{NS}_{-\frac{1}{10}}]+4[\Phi^{NS}_{\frac{1}{5}}] \\ [\Phi^{R}_{\frac{3}{20}}][\Phi^{R}_{-\frac{1}{20}}]=4[\Phi^{NS}_{-\frac{1}{10}}]+2[\Phi^{NS}_{\frac{1}{5}}] & [\Phi^{R}_{-\frac{1}{20}}][\Phi^{R}_{-\frac{1}{20}}]=2[\Phi^{NS}_{0}]+2[\Phi^{NS}_{-\frac{1}{10}}]+[\Phi^{NS}_{\frac{1}{5}}]. \end{array} \tag{3.27}$$

We recognize that the equality $\mathcal{N}_{ii}^{1}=(MD)_{i,i}$ $(i\in\mathcal{I}_R)$ is indeed satisfied in this example. Using the fact that the fusion algebra of the super Virasoro minimal model $c(10,4)=-\frac{6}{5}$

possesses a $\mathbb{Z}_2$ 'simple current' of conformal dimension 2 one can recover (3.27) from the fusion algebra of the super Virasoro minimal model.

Finally, note that it is possible to obtain the fusion algebras of the $\mathcal{SW}(\frac{3}{2},\delta)$-algebra rational models related to the $(A_{q-1}, D_{\frac{p+2}{2}})$-series in the ADE-classification from the super Virasoro fusion algebras using the simple current of order two and conformal dimension δ. However, this is not possible for $\mathcal{SW}(\frac{3}{2},\delta)$-algebras fitting into one of the other three series because the field with conformal dimension δ is no 'simple current' in the fusion algebra of the corresponding super Virasoro minimal model any more. Nevertheless, it is possible to obtain the fusion algebra with the generalized Verlinde formula from the S-matrix of the $\mathcal{SW}(\frac{3}{2},\delta)$-minimal model. Furthermore, one can show that $-$ in perfect analogy to the second example in section 3.3 $-$ the fusion algebras of the models related to (D_{k+1}, E_6) factorize into a $\mathbb{Z}_2 \otimes \mathcal{A}$ fusion algebra ($\mathcal{A}$ denotes the fusion algebra of the NS sector). In the first example of this series $-$ $\mathcal{SW}(\frac{3}{2},\frac{5}{2})$ at $c = c(12,14) = \frac{10}{7}$ $-$ we verified by explicit calculation that it is possible to resolve the degeneracies in the NS sector by a suitable extension of the fusion algebra. Unfortunately, we are not able to calculate the fusion algebras of the exceptional rational models of $\mathcal{SW}(\frac{3}{2},\delta)$ algebras [Eh2] since no explicit formulae for the S-matrices are known.

4 Unifying $\mathcal{W}$-algebras and their Coset Realization

4.1 Non-freely generated $\mathcal{W}$-algebras: cosets and orbifolds

A very important step towards a classification of rational conformal field theories (RCFTs) is a good understanding of (quantum) $\mathcal{W}$-algebras. The investigations initiated by Zamolodchikov [Za] have shown that firstly one has to distinguish between algebras existing for isolated values of the Virasoro central charge c (nondeformable $\mathcal{W}$-algebras) and those existing for generic values of c called deformable or generic $\mathcal{W}$-algebras (for a recent review see e.g. [BS]). Nondeformable $\mathcal{W}$-algebras with few generators (and their representations) have been studied intensely in [Bl1,KW,Va,Eh1,Eh3]. On the basis of these results it is now generally believed that nondeformable $\mathcal{W}$-algebras can be regarded as extensions or truncations of deformable ones with known exceptions [K1,Fl1].
The finitely generated deformable quantum $\mathcal{W}$-algebras fall at least in two classes which have completely different features of the classical counterparts [BFH]. The first class consists of *freely* generated quantum $\mathcal{W}$-algebras which are believed to be obtainable by quantum Drinfeld-Sokolov (DS) type hamiltonian reduction of affine Kac-Moody algebras (see e.g. [BW,Fe2,Fe3]). The classical counterparts are Poisson bracket algebras based on finitely and freely generated differential polynomial rings. The other class of deformable quantum $\mathcal{W}$-algebras, however, have as classical counterparts *infinitely* generated rings of differential polynomials, the generators satisfying *infinitely* many relations, i.e. the ring is non-freely generated [BFH]. Due to cancellations between generators and relations upon normal ordering, the quantum $\mathcal{W}$-algebras are nevertheless finitely but non-freely generated. This results in the appearance of generic null fields which is common to quantum $\mathcal{W}$-algebras of this type.
In this section we discuss the important features of $\mathcal{W}$-algebras of the second class occuring in cosets and orbifolds. To this end we investigate some typical examples both on the classical and the quantum level to demonstrate the main ideas of the topic. The very unexpected and exciting rôle which a part of these algebras play as *unifying* $\mathcal{W}$-algebras [Bl4,Bl5] will be discussed in detail in section 4.2.

4.1.1 The paradigm: $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$

Below we study the classical and the quantum version of the coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$. The coset is infinitely generated with infinitely many relations in the classical case [BFH]. However, we will argue that the quantum coset has at least a finitely generated subalgebra $\mathcal{W}(2,3,4,5)$ (most probably the coset is equal to this algebra) what is already indicated by counting the invariants in the classical coset. By explicit calculation of the first (quantum) generators in the commutant of $\widehat{U(1)}$ in $\widehat{sl(2,\mathbb{R})}$ we find that no new generators with conformal dimension 6,7 or 8 appear. This is contrary to the claim of [BK1] [4]) that one needs a new generator at each integer conformal dimension greater than one. Furthermore, by computing structure constants explicitly we find agreement with the solution for

[4]) It has been known to the authors of [BK1] shortly after its publication that there is no independent spin 6 field [BK2].

$\mathcal{W}(2,3,4,5)$ possessing generic null fields [Hf2]. In fact, a character argument indicates that the coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ is isomorphic to $\mathcal{W}(2,3,4,5)$.

We start with the currents $J^{\pm}(z), J^0(z)$ defined on the unit circle satisfying the Poisson brackets (PBs) of the classical $\widehat{sl(2)}_k$:

$$\begin{aligned} &\{J^0(z), J^{\pm}(w)\} = \pm 2J^{\pm}(w)\delta(z-w), \quad \{J^+(z), J^-(w)\} = J^0(w)\delta(z-w) - k\delta'(z-w) \\ &\{J^0(z), J^0(w)\} = -2k\delta'(z-w). \end{aligned} \tag{4.1}$$

Our aim is to describe the commutant of the current J^0 in the chiral algebra carried by the differential ring spanned by the polynomials in $J^{\pm,0}_{(i)} := \partial^i J^{\pm,0}$. To this end one introduces a covariant derivative $\mathcal{D}$:

$$\mathcal{D}J^{\pm} = \partial J^{\pm} \mp \frac{1}{k} J^0 J^{\pm}. \tag{4.2}$$

Then a redundant set of generators for the coset $\widehat{sl(2)}_k/\widehat{U(1)}$ is given by [BFH]

$$U^{i,j} := \mathcal{D}^i J^+ \mathcal{D}^j J^- \quad i,j = 0,1,2,\ldots\ . \tag{4.3}$$

All relations between these generators are generated by the following basic relations (forgetting about the action of $\mathcal{D}$ as a derivation):

$$U^{i,j}U^{m,n} - U^{i,n}U^{m,j} = 0. \tag{4.4}$$

In the present case the proof of the above statements is fairly easy (see e.g. [BFH]). Using the action of the covariant derivative $\mathcal{D}$ on the generators $U^{i,j}$

$$\mathcal{D}U^{i,j} = U^{i+1,j} + U^{i,j+1}, \tag{4.5}$$

one can obtain a *nonredundant* set of generators, e.g. $U^{0,j}$. The complete set of relations obeyed by the nonredundant set $U^{0,j}$ can be inferred from the set of relations satisfied by the redundant set and the derivation property of $\mathcal{D}$.

We stress the important fact that the classical coset $\widehat{sl(2)}_k/\widehat{U(1)}$ is infinitely generated with infinitely many relations satisfied by the generators. Otherwise stated the invariant ring is non-freely generated. This is a quite general feature of classical coset theories as discussed in [BFH]. As stated above one has one generator for each integer dimension greater than one. The first nontrivial relation appears for dimension 6 and is given by

$$U^{0,0}U^{0,2} - U^{0,0}\mathcal{D}U^{0,1} - U^{0,1}U^{0,1} = 0. \tag{4.6}$$

For dimension 7 there exists also a single relation whereas for dimension 8 one has already 3 independent quadratic relations. This will become important in the study of the quantum version below for the explanation of the appearance of two generic null fields with conformal dimension 8.

Let us now study the quantum version of the above coset. We present here the explicit calculations even if they agree in part with those performed by other authors [BK1,NQ], because we would like to put some emphasis on the fact that the quantum coset is finitely generated. We start now from the $\widehat{sl(2,\mathbb{R})}$ Kac-Moody algebra, where we use a standard shorthand notation $\phi_i \star \phi_j = \sum_k C_{i\,j}^{k} \phi_k$ for singular part of the OPE of two fields ϕ_i, ϕ_j:

$$J^o \star J^o = 2\,k\,I \qquad J^o \star J^{\pm} = \pm\,2\,J^{\pm} \qquad J^+ \star J^- = k\,I\,+\,J^o. \tag{4.7}$$

It is easily verified that the generator L with conformal dimension 2 in the commutant of J^o in $\widehat{sl(2,\mathbb{R})}$ ($L \star J^o = 0$) is given by

$$L \;=\; \frac{1}{2k\,(k+2)}\left(2\,k\,J^+\,J^- - J^o\,J^o - k\,\partial J^o\right). \tag{4.8}$$

L satisfies the Virasoro algebra with central charge $c = \frac{2(k-1)}{k+2}$. We proceed with the construction of the spin-3 generator W_3 of the commutant of J^o in $\widehat{sl(2,\mathbb{R})}$ which shall also be primary with respect to the Virasoro operator L. W_3 is a linear combination of operators with zero $U(1)$-charge and we find [5])

$$W_3 = -6k^2 J^+\partial J^- - 12kJ^oJ^+J^- + 4J^oJ^oJ^o + 6k^2\partial J^+J^- + 6k\partial J^oJ^o + k^2\partial^2J^o. \tag{4.9}$$

Starting from W_3 we find the complete algebraic structure of the commutant by successively calculating the fundamental OPEs:

$$\begin{aligned}
&W_3 \star W_3 = d_{3,3} I + W_4, \qquad W_3 \star W_4 = \frac{d_{4,4}}{d_{3,3}} W_3 + W_5, \qquad W_3 \star W_5 = \frac{d_{5,5}}{d_{4,4}} W_4 + \widetilde{C}_{3\,5}^{P_6^{33}}\,P_6^{33}\\
&W_4 \star W_4 = d_{4,4}\,I\,+\,\widetilde{C}_{4\,4}^{4}\,W_4\,+\,\widetilde{C}_{4\,4}^{P_6^{33}}\,P_6^{33}\\
&W_4 \star W_5 = \frac{d_{5,5}}{d_{4,4}}\,W_3\,+\,\widetilde{C}_{4\,5}^{5}\,W_5\,+\,\widetilde{C}_{4\,5}^{P_7^{34}}\,P_7^{34}\,+\,\widetilde{C}_{4\,5}^{P_8^{34}}\,P_8^{34}\\
&W_5 \star W_5 = d_{5,5}\,I + \frac{d_{5,5}}{d_{4,4}}\widetilde{C}_{4\,5}^{5} W_4 + \widetilde{C}_{5\,5}^{P_6^{33}}\,P_6^{33} + \widetilde{C}_{5\,5}^{P_8^{33}}\,P_8^{33} + \widetilde{C}_{5\,5}^{P_8^{44}}\,P_8^{44} + \widetilde{C}_{5\,5}^{P_8^{35}}\,P_8^{35}.
\end{aligned} \tag{4.10}$$

Note that primary fields composed of two simple fields and their derivatives are sometimes abbreviated using their dimensions if these dimensions determine the simple fields uniquely. For example, we write $P_{d(\phi_1)+d(\phi_2)+2}^{d(\phi_1)\,d(\phi_2)}$ for the primary field appearing in the OPE $\phi_1(z)\phi_2(w)$ at the zero of order 2 with normalization $C_{\phi_1\,\phi_2}^{P_\bullet^\bullet} = 1$.
The primary spin-4 and spin-5 generators as well as some two- and three-point functions are given in appendix B. The tilde for the structure constants $\widetilde{C}_{i\,j}^{k}$ distinguishes them from the structure constants $C_{i\,j}^{k}$ used for fields in standard normalization.

We make the important observation that in eq. (4.10) new generators of dimensions > 5 do not appear in the quantum coset so that the algebra closes within the fields L, W_3, W_4

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[5]) In repeated normal ordered products we use the following convention: $\phi_k \ldots \phi_2\phi_1 = (\phi_k(\ldots(\phi_2\phi_1)))$.

and W_5. This is a very strong indication that the coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ is finitely generated on the quantum level. Furthermore, the algebra can be identified with the solution of $\mathcal{W}(2,3,4,5)$ involving null fields by comparison of the structure constants with those given in [Hf2] for this algebra. For example, we obtain for the structure constant $C_{3\,3}^{4}$ in standard normalization

$$(C_{3\,3}^{4})^2 = \frac{128(k-3)(2k+1)(2k+3)^2}{(k-2)(k+2)(3k+4)(16k+17)} = \frac{16}{3}\frac{(c+2)(c+10)^2(5c-4)}{(c+7)(2c-1)(5c+22)} \tag{4.11}$$

in accordance with the $\mathcal{W}(2,3,4,5)$ of [Hf2] different from $\mathcal{WA}_4$. The corresponding structure constant in [NQ,BK1] and in [DW] (derived from a construction using $SU(2)$ parafermions) agrees with eq. (4.11).

Let us now understand why we fail to get a new generator at conformal dimension 6 in the quantum coset. In the classical case we have at each dimension greater or equal to two one generator. However, the first nontrivial relation appears exactly at dimension 6. The mechanism responsible for eliminating the classical dimension 6 generator on the quantum level is the following: upon normal ordering the classical relation we receive quantum correction terms due to the nonassociativity of the normal ordered product. These terms allow to express the invariant field with dimension 6 through normal ordered products of generators with smaller dimension so that it is not included to the generating set on the quantum level. In the classical limit, however, these correction terms vanish and consequently the invariant field of dimension 6 belongs to the generating set. In the limit $k \to \infty$ we can explicitly calculate the leading order term in k of the normal ordered version of the classical relation at dimension 6 because the ordinary derivative equals the covariant derivative. One obtains that this term cannot be expressed as a derivative of the (leading order terms of the) invariants L, W_3, W_4, W_5 of the coset. This result indicates that the quantum correction term of the classical relation contains indeed a term which corresponds to the new classical invariant of dimension 6.

At dimension 7 we have one generator and one independent relation which cancel upon quantization. At scale dimension 8 we have already 3 independent relations but only one generator. Therefore, on the quantum level the two additional relations not needed for cancellation give rise to the generic null fields observed in the solution of $\mathcal{W}(2,3,4,5)$ which we identified with the coset under consideration.

The identification of $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ with the special $\mathcal{W}(2,3,4,5)$ is also supported by the following calculation of the vacuum character of $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ (see also [Bl6]). To this end one has to count the uncharged states in the module generated freely by $J^{\pm}$:

$$\frac{1}{\prod_{n\geq 1}(1-z^2q^n)(1-z^{-2}q^n)} = \frac{(1-z^2)\sum_{m\in\mathbb{Z}}\phi_m(q)z^{2m}}{\prod_{n\geq 1}(1-q^n)^2} = \frac{\sum_{m\in\mathbb{Z}}(\phi_m(q)-\phi_{m-1}(q))z^{2m}}{\prod_{n\geq 1}(1-q^n)^2} \tag{4.12}$$

where the first manipulation follows from the well known identity

$$\frac{1}{\prod_{n\geq 0}(1-z^2q^n)(1-z^{-2}q^{n+1})} = \frac{\sum_{m\in\mathbb{Z}}\phi_m(q)z^{2m}}{\prod_{n\geq 1}(1-q^n)^2} \tag{4.13a}$$

(see e.g. [BCP]) with

$$\phi_m(q) = \sum_{r\geq 0}(-1)^r q^{\frac{r(r+1)}{2}+mr}, \qquad \phi_{-m}(q) = q^m \phi_m(q). \tag{4.13b}$$

The vacuum character $B_0^0(q)$ of $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ is given by the $m=0$ term of the r.h.s. of eq. (4.12) which is the generating function of the $U(1)$ singlets:

$$B_0^0(q) = \frac{\phi_0(q) - q\phi_1(q)}{\prod_{n\geq 1}(1-q^n)^2}. \tag{4.14}$$

This agrees with the corresponding formula of [BK1] after correcting the following misprint: the exponent of $f(q)$ in formula (4.14) of [BK1] should read 2 not 3. For the difference of $B_0^0(q)$ and the character $\chi_{2,3,4,5}$ of the vacuum module freely generated by fields of dimension 2, 3, 4 and 5 we obtain

$$B_0^0(q) - \chi_{2,3,4,5} = -2q^8 - 4q^9 - 9q^{10} + \mathcal{O}(q^{11}). \tag{4.15}$$

This supports the identification of the coset algebra $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ with the special solution of $\mathcal{W}(2,3,4,5)$ because both algebras have two generic null fields at conformal dimension 8 [Hf2]. We remark that the origin of this solution has been a puzzle for some time as well as the explanation of the forth solution of $\mathcal{W}(2,4,6)$ involving generic null fields (compare e.g. [Eh3]) which will be discussed below.

We conclude this subsection with some remarks concerning the coset $SVIR(N=2)/\widehat{U(1)}$, i.e. the commutant of the $\widehat{U(1)}$-current J in the $N=2$ super Virasoro algebra. The classical version can be worked out in complete analogy to above treatment of the coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ (see [BFH]). As expected one obtains an infinite set of generators (one per dimension ≥ 2) and infinitely many relations, the first nontrivial one occuring at conformal dimension 6. In [Bl5] we calculated the first four generating fields with dimension 2,3,4,5 of the quantum coset explicitly and calculated the structure constants connecting the primary fields with spin 3,4,5 (see also [Bl6] for some explicit expressions omitted in [Bl5]). We found out that the structure constants are identical with those of the special solution $\mathcal{W}(2,3,4,5)$ involving null fields. The identification is supported by the calculation of the vacuum character of $SVIR(N=2)/\widehat{U(1)}$ [Bl5,Bl6]. Furthermore, we demonstrated explicitly how the normal ordered version of the classical relation at conformal dimension 6 yields correction terms that cancel contributions which might give rise to a new generator with conformal dimension 6. Thus we strongly suspect that the coset considered here coincides with $\mathcal{W}(2,3,4,5)$. However, this is not a great surprise since it is a well known fact that the quantum cosets $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ and $SVIR(N=2)/\widehat{U(1)}$ are in fact isomorphic. The equivalence of these cosets has been used in the investigation of the representation theory of the unitary minimal series of the $N=2$ super Virasoro algebra (see e.g. [Q]).

4.1.2 The coset $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$

Like the $SVIR(N=2)$-algebra, the $\mathcal{W}_{2,1}^{sl(3)}$-algebra of Polyakov and Bershadsky [Po,Be] has a primary field of dimension 1 (J^o) and two of dimension $\frac{3}{2}$ ($G^{\pm}$) in addition to the energy momentum tensor L. $\mathcal{W}_{2,1}^{sl(3)}$ is obtained by quantum hamiltonian reduction for the nonprincipal embedding of $sl(2)$ in $sl(3)$. In contrast to the $SVIR(N=2)$-algebra the spin $\frac{3}{2}$ fields $G^{\pm}$ obey bosonic statistics. One important consequence is that the OPE of the generators does not close linearly, but quadratically in J^o.
In the normalization of Bershadsky [Be] the algebra reads

$$\begin{aligned} &J^o \star J^o = \tfrac{1}{3}(2k+3)\, I \qquad J^o \star G^{\pm} = \pm\, G^{\pm} \qquad G^{\pm} \star G^{\pm} = 0 \\ &G^+ \star G^- = (k+1)\,(2k+3)\, I \,+\, 3\,(k+1)\, J^o \,+\, 3\, P_2^{11} \end{aligned} \tag{4.16}$$

where the central charge c of this algebra is connected to k by

$$c = -(2\,k+3)\,(3\,k+1)(k+3)^{-1} \tag{4.17}$$

and the field P_2^{11} is the composite primary field:

$$P_2^{11} = J^o J^o + \tfrac{2}{3}(k+3)(3k+1)^{-1}\; L. \tag{4.18}$$

The classical version of the coset $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$ has been treated in [De] and an infinite generating set of the commutant has been given. A nonredundant (infinite) set of generators and the full (infinite) generating set of relations between these generators has been presented in [BFH]. We discuss now briefly the quantum version of this coset model and give supporting arguments that the quantum version is a finitely generated algebra. We will make use of the explicit data presented here in section 4.2 when we discuss the unifying $\mathcal{W}$-algebras of certain minimal models of the $\mathcal{WA}_{n-1}$ Casimir algebras.
As for $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$, we construct $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$ by computing a Virasoro-operator $\hat{L}$ and a primary W_3-field both commuting with J^o [6]), and recursive evaluation of the OPEs starting with $W_3 \star W_3$. The bosonic statistics of the fields $G^{\pm}$ implies that the classical limit of P_6^{33} whose leading term is $((G^+\,G^-)\,(G^+\,G^-))$ does not vanish. Therefore, in the coset-algebra a W_6-field appears, contrary to the coset $SVIR(N=2)/\widehat{U(1)}$. Analogously the only composite field with scale dimension 7, P_7^{34}, does not vanish in the classical limit. From counting the invariant fields (commuting with J^o) up to scale dimension 8 we expect that the quantum coset does not contain a spin 8 generator which is confirmed by our explicit calculations below.
Firstly, we have to construct the modified energy momentum tensor and the W_3-field both commuting with the $\widehat{U(1)}$-current J^o. These fields are given by:

$$\hat{L} = L - \tfrac{3}{2}(2k+3)^{-1} J^o J^o \tag{4.19}$$

$$\begin{aligned} W_3 = \frac{1}{2(2k+3)^2}\Big(&2(3+2k)^2 G^+G^- - 18(2+k)J^oJ^oJ^o + 6(3+k)(3+2k)LJ^o - \\ &6(3+2k)^2\partial J^oJ^o + (3+k)(3+2k)^2\partial L - 2(3+2k)(6+4k+k^2)\partial^2 J^o\Big) \end{aligned} \tag{4.20}$$

[6]) Since W_3 commutes with J^o it is simultaneously primary under L and $\hat{L}$.

where the central charge is shifted by one,

$$\hat{c} = c - 1 = -6(k+1)^2(k+3)^{-1}. \tag{4.21}$$

We verified the following OPEs:

$$\begin{aligned}
&W_3 \star W_3 = d_{3,3} I + W_4, \qquad W_3 \star W_4 = \frac{d_{4,4}}{d_{3,3}} W_3 + W_5, \qquad W_3 \star W_5 = \frac{d_{5,5}}{d_{4,4}} W_4 + W_6 \\
&W_3 \star W_6 = \frac{d_6}{d_{3,3}} W_3 + \frac{d_{6,6}}{d_{5,5}} W_5 + W_7 + \widetilde{C}_{3\,6}^{P_8^{34}} P_8^{34} \\
&W_4 \star W_4 = d_{4,4} I + \widetilde{C}_{4\,4}^{4} W_4 + \widetilde{C}_{4\,4}^{6} W_6 + \widetilde{C}_{4\,4}^{P_6^{33}} P_6^{33} \\
&W_4 \star W_5 = \frac{d_{5,5}}{d_{3,3}} W_3 + \widetilde{C}_{4\,5}^{5} W_5 + \widetilde{C}_{4\,5}^{7} W_7 + \widetilde{C}_{4\,5}^{P_7^{34}} P_7^{34} + \widetilde{C}_{4\,5}^{P_8^{34}} P_8^{34} \\
&W_4 \star W_6 = \widetilde{C}_{4\,6}^{4} W_4 + \widetilde{C}_{4\,6}^{6} W_6 + \widetilde{C}_{4\,6}^{P_6^{33}} P_6^{33} + \widetilde{C}_{4\,6}^{P_8^{33}} P_8^{33} + \widetilde{C}_{4\,6}^{P_8^{35}} P_8^{35} + \widetilde{C}_{4\,6}^{P_8^{44}} P_8^{44} + \widetilde{C}_{4\,6}^{P_9^{35}} P_9^{35}.
\end{aligned} \tag{4.22}$$

The fields W_4 to W_7 are again defined recursively by these OPEs. In order to save space we omit the lengthy expressions of these fields. For our purpose (i.e. to show that there is no new spin-8 field) it is not necessary to define the new fields W_6 and W_7 to be orthogonal to the composite fields P_6^{33} and P_7^{34}, i.e. they are just defined by eq. (4.22). The appearance of W_3 in the OPE of W_3 with W_6 reflects this fact, where d_6 is the off-diagonal element of the spin-6 fields: $W_6 \star P_6^{33} = d_6\, I + \ldots$. The OPE of W_4 with W_6 does not show a new spin-8 field. We expect therefore that the $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$-coset has a subalgebra of type $\mathcal{W}(2,3,4,5,6,7)$ (being different from the $\mathcal{WA}_6$ Casimir algebra). The OPEs eq. (4.22) do not exclude completely the existence of fields with dimension ≥ 9 in the coset, but supporting counting arguments on the classical level [BFH] can be given. Moreover, these counting arguments indicate two generic null fields at dimension 10.

The structure constant $C_{3\,3}^4$ in standard normalization for this algebra is given by

$$\left(C_{3\,3}^4\right)^2 = \frac{128k^2\,(3+k)\,(5+3k)\,(12+5k)}{(1+2k)\,(9+4k)\,(15k^2+19k-18)}. \tag{4.23}$$

Rewriting it in terms of the new central charge $\hat{c}$ would lead to square roots in the expression. Additional structure constants are given in appendix C. As already mentioned the explicit data will be needed in the discussion of the rôle of $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$ as a unifying $\mathcal{W}$-algebra for a non-unitary model of the $\mathcal{WA}_{n-1}$ Casimir algebras.

4.1.3 The coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$

In this section we study the diagonal $\widehat{sl(2,\mathbb{R})}$-coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$. It can be shown that

$$\lim_{\mu\to\infty} \frac{\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu}{\widehat{sl(2,\mathbb{R})}_{\kappa+\mu}} = \frac{\widehat{sl(2,\mathbb{R})}_\kappa}{sl(2,\mathbb{R})} \tag{4.24}$$

where the equality is valid on the level of algebras [GS,BG,BFH]. Furthermore, the diagonal coset on the l.h.s. at generic μ is a deformation of the coset on the r.h.s. and both cosets have the same spectrum of generators. We conclude that much information about the coset we are interested in can already be obtained from the simpler coset $\widehat{sl(2,\mathbb{R})}_\kappa / sl(2,\mathbb{R})$. Therefore it is reasonable to treat this coset first.

Upon a mode expansion of the OPEs (4.7) we obtain the following commutation relations for the Kac-Moody algebra $\widehat{sl(2,\mathbb{R})}_\kappa$:

$$\begin{aligned} [J_m^{(\pm)}, J_n^{(\pm)}] &= 0, & [J_m^{(0)}, J_n^{(0)}] &= 2\kappa\delta_{m,-n}n, \\ [J_m^{(0)}, J_n^{(\pm)}] &= \pm\, 2J_{m+n}^{(\pm)}, & [J_m^{(+)}, J_n^{(-)}] &= J_{m+n}^{(0)} + \kappa\delta_{m,-n}n. \end{aligned} \tag{4.25}$$

We want to construct the invariant subspace under the natural action of the horizontal subalgebra $sl(2,\mathbb{R})$ on $\widehat{sl(2,\mathbb{R})}$. This means that we have to construct all fields commuting with $\{J_0^{(\pm)}, J_0^{(0)}\}$. For explicit calculations it is sometimes convenient to use a Vertex-operator approach. First we note that

$$[J_m^{(\pm,0)}, X_n] = 0 \quad \forall m,n \qquad \Longleftrightarrow \qquad J_{-m}^{(\pm,0)} X_{d(X)}|0\rangle = 0\ , \quad m = 0,1,\ldots,d(X) \tag{4.26}$$

for any field X with fixed scale-dimension $d(X)$. We also note a similar result for primary fields. The equivalence is given by

$$\begin{aligned} & [\hat{L}_m, X_n] = (n - (d(X)-1)\ m)\ X_{m+n} \quad \forall m,n \\ \Longleftrightarrow \qquad & \hat{L}_{-m} X_{d(X)}|0\rangle = 0\ , \quad m = 1,\ldots,d(X) \end{aligned} \tag{4.27}$$

for the primarity of a field X with scale-dimension $d(X)$ with respect to some energy momentum tensor $\hat{L}$.

To obtain the commutant one may proceed as follows: First one makes the most general ansatz at a given scale dimension for an invariant field. In the ansatz we can restrict to $J_0^{(0)}$-uncharged composite fields (fields with $J_0^{(0)}$-grade 0). This automatically ensures that the condition $J_0^{(0)} X_{d(X)}|0\rangle = 0$ is satisfied. Then one uses the conditions (4.26) to determine the unknowns in the ansatz for the invariant field.

It is straightforward to make the most general ansatz in uncharged fields of scale dimension 2 and determine the field(s) that also commute(s) with $J_0^{(\pm)}$. It is no surprise to find a unique invariant field (up to normalization) at dimension 2, which is just given by the Sugawara construction:

$$\begin{aligned} 2(\kappa+2)\ L :&= \mathcal{N}(J^{(+)}, J^{(-)}) + \tfrac{1}{2}\mathcal{N}(J^{(0)}, J^{(0)}) + \mathcal{N}(J^{(-)}, J^{(+)}) \\ &= J^{(+)}\, J^{(-)} + \tfrac{1}{2} J^{(0)}\, J^{(0)} + J^{(-)}\, J^{(+)} \end{aligned} \tag{4.28}$$

with $\mathcal{N}(J^{(0)}, J^{(0)}) = J^{(0)}\, J^{(0)}$, $\mathcal{N}(J^{(\pm)}, J^{(\mp)}) = J^{(\mp)}\, J^{(\pm)} \pm \frac{1}{2}\partial J^{(0)}$. It is easy to check that L satisfies the Virasoro algebra with central charge $c = 3\kappa(\kappa+2)^{-1}$. Note that the original currents are primary with respect to L. Therefore, one can conveniently use quasi-primary normal ordered products in the calculations which we shall do below.

One may proceed along the same lines to find the next independent invariant field at scale dimension 4. However, this approach becomes unfeasible for higher dimensions. Therefore, we use group theoretic knowledge about the generators of this coset [BFH]. In the present case invariant theory supplies us with the explicit form of the generators and relations of the ring of invariant polynomials [BFH]. Using this knowledge one arrives at the following counting function for the classical minimal generating set (a_i denotes the number of (classical) generators of scale dimension i) [BFH]:

$$\phi(z) := \sum_i a_i z^i = \frac{z^6 - z^5 + z^2}{(1-z^2)(1-z^3)} = z^2 + z^4 + 2z^6 + 2z^8 + z^9 + 2z^{10} + z^{11} + 2z^{12} + z^{13} + O(z^{14}).$$

Counting the classical relations we expect the quantum coset $\widehat{sl(2,\mathbb{R})}_\kappa / sl(2,\mathbb{R})$ to be finitely generated with the generating function $\widetilde{\phi}(z) = z^2 + z^4 + 2z^6 + 2z^8 + z^9 + 2z^{10} + z^{12}$. We proceed with the explicit calculation of further invariant generating fields in the quantum coset using the classical results of [BFH]. It is an important fact that in the present case the coset can be quantized by simply normal ordering the classical invariants because one is not forced to introduce a covariant derivative [BFH].
Let g^{ij} be the metric on the simple Lie algebra and f^k_{ij} be the structure constants. With the inverse of the metric g_{ij} we define $\epsilon_{ijk} := \sum_l g_{kl} f^l_{ij}$. Then, the $SL(2,\mathbb{R})$ invariant generators are given by [BFH]

$$S_{m,n} := \sum_{i,j} g_{ij}\ \partial^n J^{(j)} \partial^m J^{(i)}, \tag{4.29a}$$

$$\begin{aligned} S_{m,n,k} :=& \sum_{i,j,l} \epsilon_{ijl}\ \partial^k J^{(l)} \partial^n J^{(j)} \partial^m J^{(i)} \\ =& \partial^k J^{(-)}(\partial^n J^{(0)} \partial^m J^{(+)}) - \partial^k J^{(-)}(\partial^n J^{(+)} \partial^m J^{(0)}) - \partial^k J^{(0)}(\partial^n J^{(-)} \partial^m J^{(+)}) \\ &+ \partial^k J^{(0)}(\partial^n J^{(+)} \partial^m J^{(-)}) + \partial^k J^{(+)}(\partial^n J^{(-)} \partial^m J^{(0)}) - \partial^k J^{(+)}(\partial^n J^{(0)} \partial^m J^{(-)}). \end{aligned} \tag{4.29b}$$

Instead of the second order invariants $S_{m,n}$ we use their quasi-primary projections:

$$W^{(n+2)} := \mathcal{N}(J^{(+)}, \partial^n J^{(-)}) + \tfrac{1}{2}\mathcal{N}(J^{(0)}, \partial^n J^{(0)}) + \mathcal{N}(J^{(-)}, \partial^n J^{(+)}) \tag{4.30}$$

for all *even* n. For the third order invariants it is more complicated to obtain quasi-primary projections; we will come back to this problem below. Note that it can also be verified case by case that $W^{(n)}$ and $S_{m,n,k}$ indeed commute with $J_0^{(\pm)}$ (they commute with $J_0^{(0)}$ by construction) using the fact that $[J_0^{(\pm)}, X] = 0$ is equivalent to $J_0^{(\pm)} X_{d(X)}|0\rangle = 0$ according to (4.26).

In order to obtain a generating set for the quasi-primary projections of the third order invariants we first note that the third order invariants are classically completely antisymmetric and therefore the quantization $S_{m,n,k}$ can be expressed in terms of lower order invariants if any of the arguments coincide. Next, the derivative ∂ maps third order invariants to third order invariants. Keeping all this in mind we may choose as independent generators

$$S_{0,1,2},\ S_{0,1,4},\ S_{0,1,5},\ S_{0,1,6},\ S_{0,1,7},\ S_{0,1,8} \tag{4.31}$$

up to dimension 12. The quasi-primary projection of these fields can now be calculated by orthogonalization with respect to all derivatives – using again the vacuum representation. One obtains the following quasi-primary third order invariants:

$$\begin{aligned}
\hat{S}_{0,1,2} &:= S_{0,1,2} - \tfrac{1}{35}\partial^4 \hat{L} - \tfrac{1}{18}\partial^2 W^{(4)}, \quad \hat{S}_{0,1,4} := S_{0,1,4} - \tfrac{1}{126}\partial^6 \hat{L} + \tfrac{19}{39}\partial^4 W^{(4)} - \tfrac{10}{13}\partial^2 \hat{S}_{0,1,2}, \\
\hat{S}_{0,1,5} &:= S_{0,1,5} - \tfrac{1}{210}\partial^7 \hat{L} - \tfrac{1}{198}\partial^5 W^{(4)} + \tfrac{45}{91}\partial^3 W^{(6)} - \tfrac{50}{91}\partial^3 \hat{S}_{0,1,2} + \tfrac{1}{4}\partial W^{(8)} - \tfrac{15}{8}\partial \hat{S}_{0,1,4}, \\
\hat{S}_{0,1,6} &:= S_{0,1,6} - \tfrac{1}{330}\partial^8 \hat{L} - \tfrac{7}{858}\partial^6 W^{(4)} + \tfrac{17}{39}\partial^4 W^{(6)} - \tfrac{5}{13}\partial^4 \hat{S}_{0,1,2} + \tfrac{41}{68}\partial^2 W^{(8)} \\
&\quad - \tfrac{315}{136}\partial^2 \hat{S}_{0,1,4} - \tfrac{7}{3}\partial \hat{S}_{0,1,5}, \\
\hat{S}_{0,1,7} &:= S_{0,1,7} - \tfrac{1}{495}\partial^9 \hat{L} - \tfrac{4}{429}\partial^7 W^{(4)} + \tfrac{14}{39}\partial^5 W^{(6)} - \tfrac{7}{26}\partial^5 \hat{S}_{0,1,2} + \tfrac{140}{153}\partial^3 W^{(8)} \\
&\quad - \tfrac{245}{102}\partial^3 \hat{S}_{0,1,4} - \tfrac{196}{57}\partial^2 \hat{S}_{0,1,5} + \tfrac{1}{5}\partial W^{(10)} - \tfrac{14}{5}\partial \hat{S}_{0,1,6}.
\end{aligned} \tag{4.32}$$

Using (4.32) we have computed determinants of all quasi-primary fields up to scale dimension 9. From the zeroes of the determinants one can read off those values of κ where truncations of the coset algebra occur. All determinants have a singularity at $\kappa = -2$ which reflects the fact that the $c(\kappa) = 3\kappa(\kappa+2)^{-1} \to \infty$ limit of this coset algebra is not well defined (see section 4.1.5). Furthermore, all fields are null fields at $\kappa = c = 0$. The remaining exceptional values of κ are listed in table 3. For these values of κ we have further determined the kernels of the d-matrices giving us precise information which generators drop out. The resulting generating sets are also presented in table 3.

κ	c	*algebra*
generic	*generic*	$\mathcal{W}(2,4,6,6,8,8,9,10,10,12)$
1	1	$\mathcal{W}(2)$
2	$\frac{3}{2}$	$\mathcal{W}(2,4,6)$
3	$\frac{9}{5}$	$\mathcal{W}(2,4,6,6,8,9)$
4	2	$\mathcal{W}(2,4,6,6,8,8,9,10)$
5	$\frac{15}{7}$	$\mathcal{W}(2,4,6,6,8,8,9,10,10)$
$-\frac{1}{2}$	-1	$\mathcal{W}(2,4,6)$
$-\frac{4}{3}$	-6	$\mathcal{W}(2,6,8,10,12)$
$-\frac{8}{5}$	-12	$\mathcal{W}(2,4,6,8,9,10,12)$
$-\frac{12}{7}$	-18	$\mathcal{W}(2,4,6,6,8,9,10)$

Table 3: Truncations of the $\widehat{sl(2,\mathbb{R})}_\kappa / sl(2,\mathbb{R})$-coset algebra

The information beyond scale dimension 9 for positive integer κ in table 3 is taken from character arguments presented in [Bl2] (see also [BS]). For the remaining cases information about dimensions higher than 9 is obtained from character arguments explained in detail in section 2.1.3 of [Bl5]. The algebras of type $\mathcal{W}(2,4,6)$ appearing in table 3 are not identical; the case $\kappa = -\frac{1}{2}$ was discussed in [BFH] and corresponds to the solution which was unexplained for some time [KW,Eh3]. The case $\kappa = 2$ is the bosonic projection of the $N = 1$ super Virasoro algebra which will be discussed below in section 4.1.4.

It is now straightforward to obtain a *primary* set of generators by orthogonalization of (4.30) and (4.32) with respect to normal ordered products $\mathcal{N}$. Up to scale dimension 6 one can choose for example the following primary generators in addition to L:

$$\begin{aligned}
\Phi^{(4)} &= 24(\kappa+2)^2\mathcal{N}(L,L) - (37\kappa+44)W^{(4)}, \\
\Phi^{(6\mathrm{a})} &= 1920(\kappa+2)^3\mathcal{N}(\mathcal{N}(L,L),L) - 40(209\kappa+328)(\kappa+2)\mathcal{N}(W^{(4)},L) \\
&\quad + 18(31\kappa+24)(9\kappa+16)W^{(6)} + 45(9\kappa+16)(\kappa+8)\hat{S}_{0,1,2}, \\
\Phi^{(6\mathrm{b})} &= 3840(675\kappa^3+75\kappa^2-1403\kappa+4328)(\kappa+2)^3\mathcal{N}(\mathcal{N}(L,L),L) + \mathcal{M}(\kappa+2)^2\mathcal{N}(L,\partial^2 L) \\
&\quad - 360(89\kappa+136)(15\kappa-8)(5\kappa+13)(5\kappa-2)(\kappa+2)\mathcal{N}(W^{(4)},L) \\
&\quad + 5(3645\kappa^3+19947\kappa^2-200\kappa-40192)(89\kappa+136)(5\kappa-2)W^{(6)} \\
&\quad - 30(645\kappa^2-928\kappa-3392)(89\kappa+136)(5\kappa-2)\hat{S}_{0,1,2},
\end{aligned} \tag{4.33}$$

where we have used the abbreviation $\mathcal{M} := 36(18225\kappa^4+485475\kappa^3+637424\kappa^2-1738048\kappa-2584576)$. The two point functions (or central terms) read

$$\begin{aligned}
d_{4,4} &= 120(37\kappa+44)(3\kappa+4)(\kappa-1)\kappa, \qquad d_{6\mathrm{a},6\mathrm{a}} = 8\,\mathcal{M}\,(9\kappa+16)(\kappa-1)\kappa, \\
d_{6\mathrm{b},6\mathrm{b}} &= 252000\,\mathcal{M}\,(89\kappa+136)(5\kappa+8)(5\kappa-2)(3\kappa+4)(2\kappa+1)(\kappa-1)(\kappa-2)\kappa.
\end{aligned} \tag{4.34}$$

Note that we have chosen $d_{6\mathrm{a},6\mathrm{b}} = d_{6\mathrm{b},6\mathrm{a}} = 0$. The normalization constants (4.34) indeed vanish for the corresponding values of κ in table 3. The additional zeroes belong to Virasoro-minimal values of c (where a correction term becomes a null field and thus the primary projection fails).

Finally, we present the first nontrivial structure constant of this coset algebra in standard normalization ($\hat{d}_{4,4} = \frac{c}{4}$):

$$\left(C_{4\,4}^{4}\right)^2 = \frac{2(99\kappa^2-25\kappa-236)^2}{5(37\kappa+44)(3\kappa+4)(\kappa+2)(\kappa-1)} = \frac{(35c^2+211c-354)^2}{15(5c+22)(c+6)(c-1)}. \tag{4.35}$$

Further structure constants would strongly depend on the choice of basis and are therefore omitted.

We continue with the general diagonal coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$. To this end we consider two commuting copies of the Kac-Moody algebra $J^{(i,\pm,0)}$ ($i=1,2$) based on $sl(2,\mathbb{R})$. The $J^{(i,\pm,0)}$ each satisfy eq. (4.25) independently, the mixed commutators being zero. We denote the first level by κ and the second level by μ. All the currents are primary spin 1 fields with respect to some energy momentum tensor. We are however not going to use any information about the nature of this energy momentum tensor or its central charge.

We want to construct the invariant subspace under the action of the diagonally embedded $\widehat{sl(2,\mathbb{R})}$. This means that we have to construct all fields commuting with

$$J^{(\pm)} := J^{(1,\pm)} + J^{(2,\pm)}, \qquad J^{(0)} := J^{(1,0)} + J^{(2,0)}. \tag{4.36}$$

This task is simplified by the observation that this $\widehat{sl(2,\mathbb{R})}$ is generated by the horizontal $sl(2,\mathbb{R})$ subalgebra of (4.36) and all modes of the current $J^{(0)}$. So the commutant consists of those $sl(2,\mathbb{R})$-invariant polynomials which also commute with $J^{(0)}$. The $sl(2,\mathbb{R})$-invariant fields are generated by (4.29) where we have now the freedom to insert the currents $J^{(1)}$ and $J^{(2)}$ for any of the arguments. The second order invariants coming from (4.29a) are

$$S_{m,n}(r,s) := \partial^n J^{(s,-)}\partial^m J^{(r,+)} + \tfrac{1}{2}\partial^n J^{(s,0)}\partial^m J^{(r,0)} + \partial^n J^{(s,+)}\partial^m J^{(r,-)}. \tag{4.37}$$

The third order invariants $S_{m,n,k}(r,s,t)$ are obtained from (4.29b) in the same manner.

For explicit calculations we use again the Vertex-operator approach mentioned before. First, condition (4.26) is used to find the invariant fields. Then, the primary generators can be computed from (4.27).

The first invariant we find with the ansatz (4.37) and the condition (4.26) is the coset energy momentum tensor

$$L = \frac{1}{\kappa+\mu+2}\left(\frac{\mu}{2(\kappa+2)}S_{0,0}(1,1) - S_{0,0}(1,2) + \frac{\kappa}{2(\mu+2)}S_{0,0}(2,2)\right). \tag{4.38}$$

The central charge c of the coset theory is found immediately to be

$$c = \frac{3\mu}{\mu+2}\left(1 - 2\frac{\mu+2}{(\kappa+\mu+2)(\kappa+2)}\right). \tag{4.39}$$

Note that for the diagonal coset the original currents are *not* primary with respect to the coset energy momentum tensor (4.38). This is the reason why it is not advantageous to use quasi-primary normal ordered products $\mathcal{N}$.

The next primary generator can be constructed at scale dimension 4 using eqs. (4.37), (4.26), (4.27). The explicit form is given by:

$$\begin{aligned}
\Phi^{(4)} :=\; & (3\mu+11)\mu S_{0,0}(1,1)\, S_{0,0}(1,1) - 4(3\mu+11)(\kappa+2)\, S_{0,0}(1,2)S_{0,0}(1,1)\\
&+ (2\mu\kappa - 11\mu - 11\kappa - 22)(3\kappa+4)(\mu-1)^{-1}\, S_{0,0}(2,2)S_{0,0}(1,1)\\
&+ (4\mu\kappa + 23\mu + 23\kappa + 76)(3\kappa+4)(\mu-1)^{-1}\, S_{0,0}(1,2)S_{0,0}(1,2)\\
&- 4(\mu+2)(3\kappa+11)(3\kappa+4)(\kappa-1)\big((3\mu+4)(\mu-1)\big)^{-1} S_{0,0}(2,2)S_{0,0}(1,2)\\
&+ (3\kappa+11)(3\kappa+4)(\kappa-1)\kappa\big((3\mu+4)(\mu-1)\big)^{-1} S_{0,0}(2,2)S_{0,0}(2,2) + (37\mu^2\kappa + 44\mu^2\\
&+ 37\mu\kappa^2 + 192\mu\kappa + 176\mu + 44\kappa^2 + 176\kappa + 176)(3\kappa+4)\big((3\mu+4)(\mu-1)\big)^{-1} S_{0,0,1}(1,2,2)\\
&+ (37\mu^2\kappa + 44\mu^2 + 37\mu\kappa^2 + 192\mu\kappa + 176\mu + 44\kappa^2 + 176\kappa + 176)(\mu-1)^{-1} S_{0,0,1}(2,1,1)\\
&+ \tfrac{3}{2}(5\mu\kappa + 4\mu + 5\kappa^2 + 20\kappa + 8)\mu\, S_{1,1}(1,1) - (22\mu^2\kappa + 32\mu^2 + 22\mu\kappa^2 + 147\mu\kappa + 164\mu\\
&+ 59\kappa^2 + 236\kappa + 200)\mu(2(\mu-1))^{-1} S_{0,2}(1,1) + (5\mu\kappa^2 - 23\mu\kappa - 36\mu + 5\kappa^3 + 5\kappa^2 - 96\kappa\\
&- 112)S_{2,0}(1,2) - (5\mu^2\kappa - 5\mu^2 + 5\mu\kappa^2 + 6\mu\kappa - 38\mu - 5\kappa^2 - 38\kappa - 56)(3\kappa+4)(\mu-1)^{-1} S_{1,1}(1,2)\\
&+ \big(5(3\mu+4)(\mu-1)\big)^{-1}\big((4\mu\kappa + 23\mu + 23\kappa + 76)(3\mu+4)(3\mu\kappa^2 + \mu\kappa + 16\mu - 9\kappa^2 - 23\kappa + 12)\\
&+ (\mu+2)(3\kappa+11)(\kappa-1)(-12\mu^2\kappa + 44\mu^2 + 60\mu\kappa + 280\mu + 168\kappa + 384)\big) S_{0,2}(1,2) - (22\mu^2\kappa\\
&+ 59\mu^2 + 22\mu\kappa^2 + 147\mu\kappa + 236\mu + 32\kappa^2 + 164\kappa + 200)(3\kappa+4)\kappa\big(2(3\mu+4)(\mu-1)\big)^{-1} S_{0,2}(2,2)\\
&+ 3(5\mu^2 + 5\mu\kappa + 20\mu + 4\kappa + 8)(3\kappa+4)(\kappa-1)\kappa\big(2(3\mu+4)(\mu-1)\big)^{-1} S_{1,1}(2,2).
\end{aligned}$$

From this realization one obtains the central term (or normalization constant) which is omitted here. A tedious calculation yields the self-coupling constant of the dimension 4 generator. In standard normalization it reads

$$\begin{aligned}
\left(C_{44}^{4}\right)^2 = 18\Big(&99(\kappa^2\mu^4+\kappa^4\mu^2)-25(\kappa\mu^4+\kappa^4\mu)-236(\mu^4+\kappa^4)+198\kappa^3\mu^3+742(\kappa^2\mu^3+\kappa^3\mu^2) \\
&-672(\kappa\mu^3+\kappa^3\mu)-1888(\mu^3+\kappa^3)+576\kappa^2\mu^2-4088(\kappa\mu^2+\kappa^2\mu)-5168(\mu^2+\kappa^2)-8592\kappa\mu \\
&-5568(\kappa+\mu)-1792\Big)^2\Big(5(\kappa-1)(\mu-1)(2+\kappa)(2+\mu)(4+3\kappa)(4+3\mu)(2+\kappa+\mu) \\
&(5+\kappa+\mu)(8+3(\kappa+\mu))(176(1+\kappa+\mu)+44(\kappa^2+\mu^2)+192\kappa\mu+37(\kappa^2\mu+\kappa\mu^2))\Big)^{-1}.
\end{aligned} \tag{4.40}$$

A few remarks about the structure constant eq. (4.40) are in place. It is symmetric in the levels κ and μ reflecting the symmetry of the construction. Furthermore, eq. (4.35) is recovered from eq. (4.40) in the limit $\mu \to \infty$, as it should be. Also the singularities for one of the levels κ or μ equal to 1 or $-\frac{4}{3}$ are expected because in these cases the field $\Phi^{(4)}$ should be a null field. For $\mu = 2$ one recovers the structure constants of the bosonic projection of the $N = 1$ super Virasoro algebra presented in [KW,Ho1]. For $\mu = -\frac{1}{2}$ the structure constant of the $\mathcal{W}(2,4,6)$ in [KW,Eh3] is reproduced. This last identity confirms the identification in [BFH] of the last unexplained solution $\mathcal{W}(2,4,6)$ with the above $sl(2,\mathrm{I\!R})$ coset at $\mu = -\frac{1}{2}$.

One should mention that one of the levels can be replaced by the central charge c. The other level still occurs as a parameter in the structure constants. Thus, the resulting $\mathcal{W}$-algebra can be regarded as a 1-parameter family of algebras of type $\mathcal{W}(2,4,6,6,8,8,9,10,10,12)$ for generic c.

4.1.4 Orbifolds of $\mathcal{W}$-algebras

In the previous subsections we discussed cosets. Coset constructions can be viewed as projections onto the subspace invariant under an *inner* symmetry (realized by a subalgebra) of a $\mathcal{W}$-algebra. Orbifolds are projections onto subspaces invariant under *outer* automorphisms (leaving the algebraic structure invariant) and behave therefore similarly to coset constructions. On the classical level both constructions lead to infinitely nonfreely generated $\mathcal{W}$-algebras [BFH]. On the quantum level all known examples of orbifolds are finitely generated. The responsible mechanism is the same as for cosets namely the cancellation of generators and relations upon normal ordering.

There are two further questions, both turning up in the general context of classification of $\mathcal{W}$-symmetries, which motivate the study of orbifold constructions of $\mathcal{W}$-algebras. The first motivation is that recently projections of the generators onto invariant subspaces were calculated for the classical Casimir algebras $\mathcal{WA}_{n-1}$ in [FRS] and were shown to give rise to other Casimir algebras. In all cases of quantum Casimir $\mathcal{W}$-algebras where one knows the structure constants one realizes that such an identity is not true on the quantum level. We will demonstrate that indeed such an identity is never true for the quantum Casimir algebras $\mathcal{WA}_{n-1}$.

The second motivation is the observation that instead of considering symmetry algebras containing fermions – in particular super $\mathcal{W}$-algebras – it might be simpler to use their bosonic projections [EH] (see also section 3). In particular, a classification program might turn out to be simpler for the bosonic case. From this point of view it is certainly important to study orbifolds of $\mathcal{W}$-algebras.

Orbifolds turn up in various applications of CFT. For example they occur as the chiral algebras of the GSO projected models used in superstring theory [GSO]. Orbifolds are also useful for applications to statistical mechanics (see e.g. [Ho4]).

In the following we will take an algebraic approach to orbifolds which is different from the usual orbifolds that deal with the complete CFT (see e.g. [Di]). If a $\mathcal{W}$-algebra has a nontrivial outer automorphism group one can consider the projection onto the invariant subspace. We will denote this projection by 'orbifold'. Available results on outer automorphisms of $\mathcal{W}$-algebras have been collected in [Ho3]. In all known cases the group of outer automorphisms is discrete, in many cases the automorphism group is just $\mathbb{Z}_2$. For simplicity, we will restrict to the case of $\mathbb{Z}_2$ automorphisms. Note that this covers in particular the bosonic projections of fermionic $\mathcal{W}$-algebras. Here the automorphism maps any fermion ψ to $-\psi$ and the invariant subspace is precisely given by the space of the bosonic fields.

Classical orbifolds

We start with a short discussion of the orbifold of a classical $\mathcal{W}$-algebra [BFH]. The purpose is to demonstrate the similarity of the algebraic structure of cosets and orbifolds, i.e. that the differential polynomial rings are non-freely generated. The $\mathcal{W}$-algebra shall have a generating set $\{W_a | a \in \mathcal{I} \cup \mathcal{J}\}$ and shall admit a $\mathbb{Z}_2$ automorphism ρ acting as follows:

$$\rho(W_a) = W_a \quad \forall a \in \mathcal{J} \quad \text{and} \quad \rho(W_b) = -W_b \quad \forall b \in \mathcal{I}. \tag{4.41}$$

The ρ-invariant subspace is then generated by the quadratic expressions in the generators belonging to the index set $\mathcal{I}$ and the invariant generators belonging to the index set $\mathcal{J}$ [BFH]. A redundant set of the quadratic generators of the orbifold is given by

$$U^{i,j}_{b,c} := \partial^i W_b \partial^j W_c \quad b, c \in \mathcal{I}, \quad i, j = 0, 1, 2, ... \tag{4.42}$$

Using the action of the derivative $\partial U^{i,j}_{b,c} = U^{i+1,j}_{b,c} + U^{i,j+1}_{b,c}$ and taking care of the Pauli principle for fermionic fields, a minimal generating set of the orbifold consists of the invariant original generators and the following quadratic generators

$$\begin{aligned} &U^{0,j}_{b,c}, \quad b < c, \quad b, c \in \mathcal{I}, \quad j = 0, 1, 2, ... \\ &U^{0,k}_{d,d}, \quad d \in \mathcal{I}_1, \quad k = 0, 2, 4, ... \\ &U^{0,l}_{e,e}, \quad e \in \mathcal{I}_2, \quad l = 1, 3, 5, ... \,. \end{aligned} \tag{4.43}$$

Here $b < c$ refers to a certain ordering of the generators $\{W_b | \, b \in \mathcal{I}\}$ and the index sets $\mathcal{I}_1, \mathcal{I}_2$ refer to the bosonic and fermionic generators in $\{W_b | \, b \in \mathcal{I}\}$. The complete set of relations satisfied by the redundant generating set is generated by [BFH]:

$$U^{i,j}_{b,c} - \epsilon_{b,c} U^{j,i}_{c,b} = 0, \quad U^{i,j}_{b,c} U^{k,l}_{d,e} - \epsilon_{c,d} U^{i,k}_{b,d} U^{j,l}_{c,e} = 0, \tag{4.44}$$

where $\epsilon_{b,c} = -1$ if both W_b and W_c are fermionic fields and 1 otherwise. We conclude that the classical orbifolds are infinitely generated with infinitely many relations.

The paradigm: orbifold of the $N = 1$ super Virasoro algebra

Let us now illustrate the main ideas of the subject in the well known example of the orbifold (bosonic projection) of the $N = 1$ super Virasoro algebra [Bo,BFH,K2,Ho1], the extension of the Virasoro algebra L by a primary fermionic field G of conformal dimension $\frac{3}{2}$ (remember that we made use of the quantum orbifold in section 3). The minimal generating set of the classical orbifold is then given by $\{L, G\partial^k G,\ k \text{ odd}\}$ (one generator at each *even* conformal dimension), whereas the infinitely many relations are generated by (4.44). The first nontrivial relation $(G\partial G)(G\partial G) = 0$ occurs at scale dimension 8. At scale dimension 10 one has 2 independent relations: $(G\partial G)(G\partial^3 G) = 0$ and $(G\partial G)\partial^2(G\partial G) = 0$. As shown in [BFH] one obtains additional terms proportional to $(G\partial^5 G)$ and $(G\partial^7 G)$ upon normal ordering the relation on scale dimension 8 and the first one on scale dimension 10 respectively. These quantum corrections allow in complete analogy to the cosets discussed earlier to remove the invariants $(G\partial^5 G)$ and $(G\partial^7 G)$ from the generating set of the quantum orbifold (see below). Thus the orbifold contains a $\mathcal{W}(2,4,6)$ as a subalgebra (from character arguments one expects the orbifold to equal this algebra). Furthermore, the second relation on scale dimension 10 yields in the quantum case a generic null field which allows to identify the orbifold among the different solutions of $\mathcal{W}(2,4,6)$ [BFH] (we will come back to this point below).

We review some results of an explicit calculation of the quantum orbifold of the $N = 1$ super Virasoro algebra. The orbifold has been proposed originally by P. Bouwknegt who also determined the field content [Bo]. Explicit constructions were carried out before in [K2,Ho1]. For our calculations we adopt the (noncovariant) conventions for the extension of the Virasoro algebra by a spin $\frac{3}{2}$ fermion G used e.g. in [Bl3]. Orthogonalizing the $\mathbb{Z}_2$ invariant normal ordered products $\mathcal{N}(G, \partial G)$ and $\mathcal{N}(G, \partial^3 G)$ with respect to other normal ordered products one obtains the composite *primary* fields in the projection as:

$$\begin{aligned}
\Phi^{(4)} &= (5c + 22)\mathcal{N}(G, \partial G) - 17\mathcal{N}(L, L) \\
\Phi^{(6)} &= 5(7c + 68)(2c - 1)(c + 24)\mathcal{N}(G, \partial^3 G) - 130(7c + 68)(2c - 1)\mathcal{N}(\mathcal{N}(G, \partial G), L) \\
&\quad + 20(218c - 293)\mathcal{N}(\mathcal{N}(L, L), L) - 6(11c - 86)(c + 24)\mathcal{N}(L, \partial^2 L).
\end{aligned} \tag{4.45}$$

The two point functions of these fields can easily computed to be

$$\begin{aligned}
d_{4,4} &= \tfrac{1}{6}c(10c - 7)(5c + 22)(4c + 21) \\
d_{6,6} &= 50c(14c + 11)(10c - 7)(7c + 68)(4c + 21)(2c - 1)(c + 24)(c + 11).
\end{aligned} \tag{4.46}$$

For the structure constants one obtains exactly those of the first solution (Set 1) in [KW]. Furthermore, we can compare the structure constant $(C_{4\,4}^{4})^2$ with the corresponding expression of the diagonal coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$. Taking $\mu = 2$ we get as expected an agreement of the structure constant since it is well known that the coset $\widehat{sl(k,\mathbb{R})}_\kappa \oplus \widehat{sl(k,\mathbb{R})}_2 / \widehat{sl(k,\mathbb{R})}_{\kappa+2}$ realizes the bosonic sector of the $N = 1$ super Virasoro algebra (cf. table 3).

In order to demonstrate the notational advantages of quasi-primary fields we also present a quasi-primary analogue of the relations presented in appendix A of [BFH]. The quasi-primary version of (A.2) in [BFH] is:

$$\begin{aligned}(192-31c)\,\mathcal{N}(G,\partial^5 G)+90\,\mathcal{N}(\mathcal{N}(G,\partial G),\mathcal{N}(G,\partial G))+\tfrac{81}{10}\,\mathcal{N}(L,\partial^4 L)\\ +154\,\mathcal{N}(\mathcal{N}(G,\partial G),\partial^2 L)-420\,\mathcal{N}(\mathcal{N}(G,\partial^3 G),L)=0.\end{aligned} \tag{4.47}$$

Hence, $\mathcal{N}(G,\partial^5 G)$ can be expressed in terms of normal ordered products of invariant fields with lower dimension.

The commutators of the simple fields of an algebra of type $\mathcal{W}(2,4,6)$ involve fields up to dimension 10. Therefore, closure of the algebra is ensured if in addition to eq. (4.47) also $\mathcal{N}(G,\partial^7 G)$ can be expressed in terms of L, $\mathcal{N}(G,\partial G)$ and $\mathcal{N}(G,\partial^3 G)$. A suitable relation in the spirit of (A.4) in [BFH] can easily be established but is omitted. It is also no difficulty to calculate an explicit expression for the generic null field with conformal dimension 10.

Note that it is straightforward to derive the representations of this orbifold $\mathcal{W}$-algebra from the well known representations of the $N=1$ super Virasoro algebra. In particular, the representation theory of $\mathcal{W}(2,4)$ and the $N=1$ super Virasoro algebra at $c=-11$ is much the same which is clear keeping the above truncations of the orbifold in mind (see also [Eh1,EH]). We continue with some general remarks on quantum orbifolds and present some explicit results of our investigations.

General remarks and results

By definition, the energy momentum tensor L of any $\mathcal{W}$-algebra is invariant under all outer automorphisms. Therefore, orbifold constructions never change the energy momentum tensor L (in contrast to coset constructions). This fact can be exploited using a quasi-primary basis from the very beginning which simplifies the transition to primary generators. The normal ordered version of the minimal generating set of the classical orbifold is also a (redundant) generating set for the quantum orbifold. From this we obtain in the case of $\mathbb{Z}_2$ orbifolds that the nonzero quasi-primary normal ordered products

$$\mathcal{N}(\phi^{(1)},\partial^n\phi^{(2)}) \tag{4.48}$$

of any two generators $\phi^{(j)}$ transforming under the automorphism as $\phi^{(j)}\mapsto -\phi^{(j)}$ constitute a generating set for the orbifold together with the invariant generators. One can further restrict to those normal ordered products eq. (4.48) where the fields $\phi^{(j)}$ appear in a certain order (cf. (4.41)-(4.44)).

The spin content of an orbifold can be easily predicted using a character argument which also enables one (at least in principle) to determine the representations of the orbifold. Let V be an irreducible highest weight module of the underlying $\mathcal{W}$-algebra. Then we define a character $\chi(q,z)$ which also encodes the $\mathbb{Z}_2$ automorphism by

$$\chi(q,z):=\mathrm{tr}_V\left(q^{(L_0-\frac{c}{24})}z^P\right) \tag{4.49}$$

where P denotes the parity of a state in V with respect to the automorphism. The parity of invariant states is zero ($P = 0$). The states transforming with a sign under the automorphism group are defined to have parity $P = 1$. In the case of fermions, P is equal to the fermion number operator $\mathcal{F}$ modulo 2. The subspace of V invariant under the automorphism as well as the subspace of states transforming with a sign both provide irreducible representations of the orbifold $\mathcal{W}$-algebra. From this we conclude that the decomposition

$$\chi(q, z) = \chi^{(0)}(q) + z\,\chi^{(1)}(q) \tag{4.50}$$

yields the two characters $\chi^{(i)}(q)$ for the two representations of the orbifold obtained from V. In particular, the constant part $\chi_0^{(0)}(q)$ in z of the vacuum character $\chi_0(q, z)$ of the underlying $\mathcal{W}$-algebra is the vacuum character of the orbifold. It is straightforward to compute this character for a $\mathcal{W}$-algebra without null fields (which applies to Drinfeld-Sokolov $\mathcal{W}$-algebras at generic c) up to a finite order. The field content of the orbifold can now be read off by determining the minimal set of fields whose free vacuum module is at least as large.

If $\chi_i(q, z)$ are the characters of a rational model of the underlying $\mathcal{W}$-algebra, the characters obtained from (4.50) are the characters of the associated rational model of the orbifold. Note, however, that for this statement to be valid one has to take the representations of all sectors of the original algebra into account, in particular for bosonic algebras also the twisted sector [Ho3]. Then the statement about rational models follows from the identity (3.6) in [Ho3] for the partition functions.

Note that instead of determining the field content of the orbifold using this character argument it can also be computed by applying a basis algorithm to those simple fields that will drop out. The invariant normal ordered products must be contained in the orbifold. Either they can be considered as composite or must be added to the generating set. This approach is slightly more involved but has the small advantage that one can trace invariant fields and up to the cancellation of correction terms in relations and generators (see [BFH]) ensure closure of the orbifold $\mathcal{W}$-algebra.

The *primary* generators in the orbifold can be efficiently determined using the definition of *simple* fields (cf. section 2). Instead of taking the quasi-primary normal ordered products (4.48) we orthogonalize them with respect to all quasi-primary normal ordered products in the orbifold.

Below we will use the following notations: $W^{(\delta)}$ denotes a simple field of dimension δ in the original algebra, $\Phi^{(n)}$ denotes a simple field of dimension n in the orbifold. $d_{n,n}$ is the central term in the commutator of $\Phi^{(n)}$ with itself which equals its norm squared. The structure constants in the original algebra are denoted by $C_{W^{(\kappa)}\,W^{(\lambda)}}^{W^{(\mu)}}$ whereas for the orbifold we just write the dimensions, i.e. $C_{k\,l}^{m}$. Note that the structure constants for the orbifold are given in the standard normalization $\hat{d}_{n,n} = \frac{c}{n}$ and *not* in the induced one given by $d_{n,n}$.

Throwing out one field

Before presenting a collection of results for $\mathbb{Z}_2$ orbifolds we will first consider the simplest case where we project out a single field $W^{(\delta)}$ using an automorphism $W^{(\delta)} \mapsto -W^{(\delta)}$. In order to determine the field content of the orbifold let us look at all invariant fields that can be built out of $W^{(\delta)}$ alone. This argumentation will rely on the absence of null fields, i.e. it will apply to generic c only.

Firstly, we discuss the bosonic case where δ is an integer $\delta = n$. According to eq. (4.48) all fields $\mathcal{N}(W^{(n)}, \partial^{2k} W^{(n)})$ $(k \geq 0)$ are invariant under the automorphism. Their primary projections $\Phi^{(2(n+k))}$ cannot be composite for $k < n$. At dimension $4n$ we have invariant fields of the form $\mathcal{N}(W^{(n)}, \partial^{2n} W^{(n)})$ and $\mathcal{N}(\mathcal{N}(\mathcal{N}(W^{(n)}, W^{(n)}), W^{(n)}), W^{(n)})$. These two fields are equivalent to $\Phi^{(4n)}$ and $\mathcal{N}(\Phi^{(2n)}, \Phi^{(2n)})$. Clearly, there is precisely one additional generator $\Phi^{(4n)}$ at scale dimension $4n$. At dimension $4n + 2$ the invariant fields are $\mathcal{N}(W^{(n)}, \partial^{2n+2} W^{(n)})$ and $\mathcal{N}(\mathcal{N}(\mathcal{N}(W^{(n)}, W^{(n)}), W^{(n)}), \partial^2 W^{(n)})$. Assuming the cancellation procedure described in [BFH] to be valid in general, this space is also spanned by $\mathcal{N}(\Phi^{(2n)}, \partial^2 \Phi^{(2n)})$ and $\mathcal{N}(\Phi^{(2n+2)}, \Phi^{(2n)})$. Finally, at dimension $4n + 4$ there are two fourth order and one second order invariant fields (three altogether). At this dimension we have in the orbifold $\mathcal{N}(\Phi^{(2n)}, \partial^4 \Phi^{(2n)})$, $\mathcal{N}(\Phi^{(2n+2)}, \partial^2 \Phi^{(2n)})$, $\mathcal{N}(\Phi^{(2n+2)}, \Phi^{(2n+2)})$ and $\mathcal{N}(\Phi^{(2n+4)}, \Phi^{(2n)})$. Thus, there must be a generic null field at dimension $4n + 4$. In summary, we have generators $\Phi^{(2k)}$, $n \leq k \leq 2n$ in the orbifold and a first generic null field at dimension $4n + 4$.

The same procedure can be applied to a fermionic field, i.e. δ half-integer. Again, we obtain generators of the orbifold $\Phi^{(2\delta+2k+1)}$ from the primary projection of $\mathcal{N}(W^{(\delta)}, \partial^{2k+1} W^{(\delta)})$ for $0 \leq k \leq \delta - \frac{1}{2}$ and the first generic null field at dimension $4\delta + 4$.

Note that the Casimir algebras $\mathcal{WD}_n$ and $\mathcal{WB}(0, n)$ have this property. $\mathcal{WD}_n$ $(n > 4)$ contains precisely one simple bosonic field which can be projected out [Ho3]. This field has dimension n. $\mathcal{WB}(0, n)$ has precisely one simple fermionic field. The bosonic subalgebra can therefore be obtained by projecting out the fermion with dimension $n + \frac{1}{2}$. It was noted in [BS] that this bosonic subspace is realized in terms of the diagonal coset $(\hat{\mathcal{B}}_n)_k \oplus (\hat{\mathcal{B}}_n)_1 / (\hat{\mathcal{B}}_n)_{k+1}$. Eq. (7.19) of [BS] is a closed formula for the vacuum character of this coset.

Results

From the result of the preceding paragraph and the character argument explained above one can predict the generating set for many orbifolds. Table 4 contains a collection of results [7]). Strictly speaking, one would have to check in each case separately that the cancellation mechanism of [BFH] indeed takes place.

To determine the orbifold of the cosets $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$ and $\widehat{sl(2,\mathbb{R})}_\kappa / \widehat{U(1)}$ one has to use the explicit knowledge of the invariants (and relations) generating the coset because of the presence of generic null fields. In the case of $\widehat{sl(2,\mathbb{R})}_\kappa / \widehat{U(1)}$ the outer automorphism of the coset comes from the inner automorphism of the $\widehat{sl(2,\mathbb{R})}$ Kac-Moody algebra (4.7) that maps $J^+ \leftrightarrow J^-$, $J^o \mapsto -J^o$. This automorphism leaves the even dimensional generators of the coset invariant and changes the sign of the odd dimensional ones.

The automorphism of the coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$ is induced by the map

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[7]) For the field content of the orbifolds of $\mathcal{WA}_2$ and $\mathcal{WA}_3$ see also [BS].

$J^{(1,\cdot)} \leftrightarrow J^{(2,\cdot)}$. The effect is that for $\kappa = \mu$ the quadratic fields (4.29a) are left invariant and the third order invariants (4.29b) change their sign. With this information the field content of the orbifold can easily be inferred from [BFH] by dropping the third order invariants from the generating set and the relations.

algebra	*projection for generic c*	*dimension of first generic null field*
$\mathcal{W}(2,1)$	$\mathcal{W}(2,2,4)$	8
$\mathcal{W}(2,\frac{3}{2})$	$\mathcal{W}(2,4,6)$	10
$\mathcal{W}(2,2)$	$\mathcal{W}(2,4,6,8)$	12
$\mathcal{WA}_2 \cong \mathcal{W}(2,3)$	$\mathcal{W}(2,6,8,10,12)$	16
$\mathcal{WA}_3 \cong \mathcal{W}(2,3,4)$	$\mathcal{W}(2,4,6,8,10,12)$	16
$\mathcal{WA}_4 \cong \mathcal{W}(2,3,4,5)$	$\mathcal{W}(2,4,6,8^2,9,10^3,11,12^3,13,14^2)$	16
$\mathcal{WA}_5 \cong \mathcal{W}(2,3,4,5,6)$	$\mathcal{W}(2,4,6^2,8^2,9,10^3,11,12^3,13,14^2)$	16
$\widehat{sl(2,\mathbb{R})}_\kappa/\widehat{U(1)} \cong \mathcal{W}(2,3,4,5)$	$\mathcal{W}(2,4,6,8,10)$	14
$\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\kappa/\widehat{sl(2,\mathbb{R})}_{2\kappa}$	$\mathcal{W}(2,4,6,8,10,12,14,16,18)$	22
$\mathcal{WD}_n \cong \mathcal{W}(2,4,\ldots,2n-2,n)$	$\mathcal{W}(2,4,\ldots,4n)$	$4n+4$
$\mathcal{WB}(0,n) \cong \mathcal{W}(2,4,\ldots,2n,n+\frac{1}{2})$	$\mathcal{W}(2,4,\ldots,4n+2)$	$4n+6$
$\mathcal{SW}(\frac{3}{2},2)$	$\mathcal{W}(2^2,4^2,5,6^3,7)$	9

Table 4: Field content of some orbifolds of $\mathcal{W}$-algebras

In the simplest cases we also determined some primary generators of the orbifold and calculated the corresponding structure constants but omit the results since they do not provide new insights (for details see [Bl5]).

Remarks on the orbifold of $\mathcal{WA}_{n-1}$

It has been shown in [FRS] that the orbifolds of the classical $\mathcal{WA}_{n-1}$ possess other classical Casimir algebras as subalgebras. In this subsection we will show that such a relation does not hold true for the quantum orbifolds.

For $\mathcal{WA}_{n-1} \cong \mathcal{W}(2,\ldots,n)$ some structure constants are known generally [Hf1,Bl5]:

$$\begin{aligned}
\left(C_{3\,3}^{4}\right)^2 &= \frac{64(n-3)(c+2)(c(n+3)+2(4n+3)(n-1))}{(n-2)(5c+22)(c(n+2)+(3n+2)(n-1))} \\
C_{3\,3}^{4}C_{4\,4}^{4} &= \frac{48(c^2(n^2-19)+3c(6n^3-25n^2+15)+2(n-1)(6n^2-41n-41))}{(n-2)(5c+22)(c(n+2)+(3n+2)(n-1))} \\
\left(C_{3\,4}^{5}\right)^2 &= \frac{25(cn+4c+15n^2-3n-12)(5c+22)(n-4)}{(cn+2c+3n^2-n-2)(7c+114)(n-2)} \\
C_{4\,5}^{5} &= \tfrac{15}{8}\left((2+c)\,(114+7c)\,(n-3)\left(8n^2+cn-2n+3c-6\right)\right)^{-1}C_{3\,3}^{4} \\
&\times(6756+4120c-483c^2-97c^3-6972n^2-5192cn^2-467c^2n^2+3c^3n^2+216n^3+856cn^3+94c^2n^3).
\end{aligned} \tag{4.51}$$

The first primary composite field in the $\mathbb{Z}_2$ orbifold (the $\mathbb{Z}_2$ automorphism changes the sign of the odd dimensional simple fields) can be calculated for all $n \geq 4$:

$$\begin{aligned}\Phi^{(6)} = {} & 27(43c-844)(5c+22)(c+24)\mathcal{N}(L,\partial^2 L) + 1440(191c+22)(c+24)\mathcal{N}(\mathcal{N}(L,L),L) \\ & + 1980(7c+68)(5c+22)(2c-1)C_{W^{(3)}\,W^{(3)}}^{W^{(4)}}\,\mathcal{N}(W^{(4)},L) \\ & - 270(7c+68)(5c+22)(2c-1)(c+24)\mathcal{N}(W^{(3)},W^{(3)}).\end{aligned} \tag{4.52}$$

Using the expression for $C_{W^{(3)}\,W^{(3)}}^{W^{(4)}}$ this induces the following normalization

$$\begin{aligned}d_{6,6} = & \Big(32400c\big(7(n+2)(n-2)c^4 + (21n^3+380n^2-1800)c^3 + (1399n^3+1585n^2 \\ & - 7700)c^2 + 4(1179n^3+4375n^2-1770)c - 32(473n^2-81n-81)(n-1)\big) \\ & (7c+68)(5c+22)^2(2c-1)(c+24)(c+2)\Big)\Big((cn+2c+3n^2-n-2)(n-2)\Big)^{-1}.\end{aligned} \tag{4.53}$$

The simplest nontrivial structure constant of the orbifold reads in the standard normalization ($\hat{d}_{6,6} = \frac{c}{6}$)

$$\begin{aligned}\left(C_{4\,4}^{6}\right)^2 = {} & 6\Big(29c^3n^2 - 284c^3 + 255c^2n^3 - 427c^2n^2 - 368c^2 + 540cn^3 + 3016cn^2 - 1636c \\ & - 1920n^3 + 2032n^2 - 112\Big)^2(5c+22)^2\Bigg(\Big(7c^4n^2 - 28c^4 + 21c^3n^3 + 380c^3n^2 - 1800c^3 \\ & + 1399c^2n^3 + 1585c^2n^2 - 7700c^2 + 4716cn^3 + 17500cn^2 - 7080c - 15136n^3 \\ & + 17728n^2 - 2592\Big)(cn+2c+3n^2-n-2)(7c+68)(2c-1)(c+24)(c+2)(n-2)\Bigg)^{-1},\end{aligned} \tag{4.54}$$

where we have used (4.51). Fortunately the structure constant $C_{W^{(3)}\,W^{(4)}}^{W^{(5)}}$ in (4.51) vanishes for $n = 4$ such that we can apply it to $\mathcal{W}(2,3,4)$ as well. Note that the structure constant (4.54) is nonzero for any n and the central charge c generic. This means that the invariant original generators do not close among themselves and one is forced to include the dimension 6 generator (4.52) into the orbifold. In particular, for $n \geq 6$ the orbifold contains two simple fields of scale dimension 6. This has to be contrasted with the classical situation where it has been shown in [FRS] that already the invariant original generators generate a closed subalgebra of the complete orbifold. The fact that (4.54) vanishes in the limit $c \to \infty$ is consistent with this statement because this limit should correspond to a classical limit (see below).

4.1.5 Vacuum preserving algebras (VPA) and classical limits

Once a construction of a $\mathcal{W}$-algebra as a reduction of a Kac-Moody algebra or a similar linear system is known, it is usually well under control. In particular, one can easily discuss its classical counterpart, i.e. the analogous reduction of the corresponding classical linear system. Note, however, that in general several constructions of the same quantum

$\mathcal{W}$-algebra are possible and can lead to different classical counterparts. For a classification one needs more general methods which do not refer to any particular construction. Two closely related ideas in this direction have been put forward in [BW] for deformable $\mathcal{W}$-algebras: The vacuum preserving algebra (VPA) as well as a particular classical limit of a $\mathcal{W}$-algebra. These methods work nicely for $\mathcal{W}$-algebras obtained by Drinfeld-Sokolov reduction [BW,Fe2]. The $\mathcal{W}$-algebras discussed in this section are not in the DS class and therefore it is interesting to see to what extent these methods work for them.

We start with a discussion of a general approach to the VPA. First, one introduces the 'vacuum preserving modes' of all quasi-primary fields. The space spanned by them carries a Lie algebra structure. Next, one considers the limit $c \to \infty$ of this algebraic structure. The VPA is the smallest subalgebra of this algebra containing the vacuum preserving modes of the simple fields. To be more precise, the vacuum preserving modes of a quasi-primary field Φ are given by

$$\{\Phi_n \mid |n| < d(\Phi)\}. \tag{4.55}$$

The vacuum preserving modes of all quasi-primary fields have the important property [BW,Fe2] that the commutator closes among them and does not have any central term. Note that the vacuum preserving modes $L_{\pm 1}, L_0$ of the energy momentum tensor L form an $sl(2)$ subalgebra of all vacuum preserving modes. The space spanned by all vacuum preserving modes (4.55) is in general infinite dimensional and the commutator still depends continuously on c. In order to cure the second property one takes the limit $c \to \infty$. In general, one will have to rescale the generators $W^{(i)}$ of the finitely generated quantum $\mathcal{W}$-algebra in order to make sense of the limit $c \to \infty$:

$$\hat{W}^{(i)} := c^{-\alpha_i} W^{(i)}. \tag{4.56}$$

The exponents α_i have to be adapted in order to make all structure constants connecting the fields $\hat{W}^{(i)}$ bounded and nontrivial for $c \to \infty$. Even in this limit, the algebra of all vacuum preserving modes is a very complicated object. Therefore, the VPA is defined as the smallest subalgebra of the limit $c \to \infty$ of the algebra of all vacuum preserving modes that contains the vacuum preserving modes of the simple fields.

It was shown in [BW] that this works nicely for the algebras in the DS class: One can associate to them a finite dimensional Lie algebra with an $sl(2)$ embedding that encodes the spin content of these algebras. For the $\mathcal{W}$-algebras in the DS class one can set all $\alpha_i := 0$ and then the VPA is defined by

$$\mathcal{V} := \text{span}\{W_n^{(i)} \mid |n| < d(W^{(i)})\} \tag{4.57}$$

where $W^{(i)}$ are the simple fields of the $\mathcal{W}$-algebra. On this space a Lie bracket is induced by taking the limit $c \to \infty$ of the commutator. The crucial point is that the commutator linearizes, i.e. that the induced Lie bracket closes in the space (4.57). The data (4.57) together with the Lie bracket and the $sl(2)$ embedding is equivalent to the original data used for the DS reduction [BW,Fe2].

The situation is much less clear for $\mathcal{W}$-algebras outside the DS class, i.e. those $\mathcal{W}$-algebras which do not have 'nice' asymptotic properties of the structure constants. We will first discuss one example in detail: The bosonic projection of the $N = 1$ super Virasoro algebra. Let us for the moment forget about the construction of this $\mathcal{W}(2,4,6)$ and look what can be said about the VPA solely from inspection of the structure constants (Set 1 in section 6.2 of [KW]). Denote the generators of scale dimension 2, 4 and 6 by L, $W^{(i)}$ $(i = 4,6)$. We observe from the structure constants in [KW] that we have to rescale

$$\hat{L} := L\,, \qquad \hat{W}^{(i)} := \frac{1}{\sqrt{c}} W^{(i)}. \tag{4.58}$$

Then, all structure constants connecting these three simple fields are bounded and nonzero for $c \to \infty$. Unlike for $\mathcal{W}$-algebras in the DS class the commutator does not linearize for this $\mathcal{W}(2,4,6)$, i.e. the commutator does not close in the space (4.57). The structure constants $C_{W^{(4)}\,W^{(6)}}^{(W^{(4)}W^{(4)})}$ and $C_{W^{(6)}\,W^{(6)}}^{(W^{(4)}W^{(4)})}$ are invariant under the rescaling (4.58). Furthermore, these structure constants tend to a nonzero constant in the limit $c \to \infty$. This means that the vacuum preserving modes $\hat{W}_n^{(8)}$, $|n| < 8$ of the primary projection $\hat{W}^{(8)}$ of $(\hat{W}^{(4)}\hat{W}^{(4)})$ have to be included into the VPA. We have checked that the same happens at scale dimension 10. At scale dimension 10 one finds the quadratic fields $(\partial^2\hat{W}^{(4)}\hat{W}^{(4)})$ and $(\hat{W}^{(4)}\hat{W}^{(6)})$. Due to the presence of a generic null field at scale dimension 10 [Bo] these two fields give rise to precisely one primary field at scale dimension 10. Since both $(\partial^2\hat{W}^{(4)}\hat{W}^{(4)})$ and $(\hat{W}^{(4)}\hat{W}^{(6)})$ do not decouple in the limit $c \to \infty$, the vacuum preserving modes of the primary field with dimension 10 have to be included into the VPA. Thus, the VPA does definitely not close on the vacuum preserving modes of the simple fields only but one has to include the vacuum preserving modes of further primary fields (at least up to scale dimension 10). We expect that one must indeed include the vacuum preserving modes of infinitely many primary fields because there is no reason to expect any of the crucial structure constants to vanish. Therefore, we actually expect the VPA of this $\mathcal{W}(2,4,6)$ to be infinite dimensional.

Similar reasoning applies to the second $\mathcal{W}(2,4,6)$ outside the DS class (Set 2 in section 6.2 of [KW] and eq. (6) in [Eh3]). The situation is slightly different at scale dimension 10. Here, no null field is present and therefore we do indeed have two primary fields which turn up in the commutator of $\hat{W}^{(6)}$ with itself. However, up to that stage only a particular linear combination of these fields appears in the commutators and therefore it might be sufficient to include the vacuum preserving modes of only one field at scale dimension 10 into the VPA. As before the realization of this $\mathcal{W}(2,4,6)$ in terms of the coset $\widehat{sl(2,\mathbb{R})}_k \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}} / \widehat{sl(2,\mathbb{R})}_{k-\frac{1}{2}}$ [BFH] (compare also the remarks after eq. (4.40)) is not needed for the discussion of the VPA.

Let us now turn to the $\mathcal{W}(2,3,4,5)$ which we discussed in section 4.1.1. The structure constants for this algebra were first calculated in [Hf2] checking associativity of the OPE without reference to the cosets $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ or $SVIR(N = 2)/\widehat{U(1)}$. Denote the generators of this $\mathcal{W}(2,3,4,5)$ by L and $W^{(i)}$ $(i = 3,4,5)$. Then we see from the structure constants in table 2 of ref. [Hf2] that we have to rescale as in eq. (4.58). Up to scale dimension 8 there are 6 further quadratic fields in terms of the $W^{(i)}$, two of them at scale dimension 8 give rise to null fields (see eq. (2.11) and (2.12) of ref. [Hf2]) leaving us with 4 primary fields: One each at scale dimensions 6 and 7, two at scale dimension 8. As in the previous cases, the structure constants connecting the simple primary fields with these composite primary fields are invariant under the rescaling (4.58) and tend to a nonzero constant for $c \to \infty$ (compare table 2 of ref. [Hf2]). This means that the vacuum preserving modes of the fields at scale dimensions 6 and 7 have to be included into the VPA. Only a linear combination of the two fields at scale dimension 8 appears in the OPEs of the simple fields. Therefore, so far we have to include the vacuum preserving modes of only one primary field at scale dimension 8 into the VPA. As before, we have not calculated structure constants involving fields of higher dimension, but there is no reason to expect those structure constants involving primary fields to vanish. Thus, we expect also the VPA of this algebra to be infinite dimensional.

The above discussion can further be supported by looking at the classical counterparts of these algebras that correspond to the above constructions [BFH]. These classical counterparts are infinitely generated. From [BW,Fe2] we know that in the classical case the VPA consists of the vacuum preserving modes of all generators. In particular, for the examples under consideration it is definitely infinite dimensional. Even more, the composite primary fields which had to be added precisely correspond to the additional generators of the classical counterparts. This means that the VPA of a $\mathcal{W}$-algebra is sensitive to underlying constructions even if they are not known.

We have seen in three examples outside the DS class that their VPA is probably an infinite dimensional Lie algebra with $sl(2)$ embedding which decomposes into finite dimensional representations under this $sl(2)$. This indicates that probably all $\mathcal{W}$-algebras outside the DS class have infinite dimensional VPAs which makes the VPA as a tool for classification unhandy. At least, for any explicitly known algebra the construction of the VPA is purely algorithmic but does unfortunately not necessarily stop after finitely many steps. It should be noted that the impact of the realization of composite fields contributing to the VPA in terms of finitely many simple ones is not completely clear to us, neither is the impact of the relations satisfied by the generators of these $\mathcal{W}$-algebras [BFH] on the VPA.

Next we consider classical limits of $\mathcal{W}$-algebras. A classical Kac-Moody algebra is a Lie algebra and can therefore be quantized according to Dirac's rule. In particular, the r.h.s. of the commutator is multiplied by $\hbar$, and the canonical classical limit of the quantum Kac-Moody algebra is the limit $\hbar \to 0$. For a $\mathcal{W}$-algebra that arises as some reduction of a Kac-Moody algebra, this classical limit of the Kac-Moody algebra induces a classical limit of the $\mathcal{W}$-algebra. If we have a particular construction in mind we refer to the induced classical limit as 'the classical limit'. In contrast hereto, the procedure of [BW] is a set of rules to re-institute $\hbar$'s in a $\mathcal{W}$-algebra without referring to any particular reduction and afterwards define a classical limit $\hbar \to 0$. We will refer to this classical limit as the 'BW classical limit'.

To be more explicit, the BW classical limit is defined by the following set of rules

$$W^{(i)} \mapsto \frac{\hat{W}_\hbar^{(i)}}{\hbar^{1+\alpha_i}}\,, \qquad c \mapsto \frac{\hat{c}}{\hbar} \tag{4.59}$$

where $W^{(i)}$ are the simple generators of the $\mathcal{W}$-algebra. Then, the quantum fields $W^{(i)}(z)$ are replaced by classical fields $w^{(i)}(z) := \lim_{\hbar\to 0} \hat{W}_\hbar^{(i)}(z)$ and Poisson brackets as well as the ring structure are defined by the following identifications

$$\{w^{(i)}(z), w^{(j)}(w)\} = \lim_{\hbar\to 0} \frac{1}{\hbar}[\hat{W}_\hbar^{(i)}(z), \hat{W}_\hbar^{(j)}(w)]\,, \qquad w^{(i)}(z)w^{(j)}(z) = \lim_{\hbar\to 0}(\hat{W}_\hbar^{(i)}\hat{W}_\hbar^{(j)})(z) \tag{4.60}$$

where the choice of normal ordering prescription on the quantum level is actually irrelevant. The crucial point is to choose the exponents α_i in (4.59) such that the limits in (4.60) exist and are nontrivial. The reader should note that in general the existence of such a set of exponents α_i is not guaranteed. For $\mathcal{W}$-algebras in the DS class one can choose all $\alpha_i = 0$ [BW] and then (4.59) and (4.60) are one-to-one maps between the one-parameter families of $\mathcal{W}$-algebras on the quantum level and on the classical level where the parameter is the central charge. It should be noted that the BW classical limit according to (4.59) and (4.60) is not necessarily the same as the one induced by a reduction. Close inspection shows that even in the DS class the central charges of these two classical limits are indeed different, and therefore the classical limits are equivalent only when looking at one-parameter families of $\mathcal{W}$-algebras. Note that also nondeformable $\mathcal{W}$-algebras can have a classical limit (compare e.g. the $\beta - \gamma$ system of [BFH]). Thus, deformability and existence of a classical limit should not be confused. However, only for deformable $\mathcal{W}$-algebras the rules (4.59) can be applied.

Let us now apply these ideas to the algebras outside the DS class. Requiring the linear term (in terms of the generators) on the r.h.s. of the commutator to be bounded and nonzero for $\hbar \to 0$ we conclude that the exponents α_i in (4.59) and (4.56) are actually identical. In particular, for the three examples already discussed above the exponent for the Virasoro field L is $\alpha_0 = 0$ and all other $\alpha_i = \frac{1}{2}$. In order to have a well-defined limit of the commutator in (4.60) all structure constants connecting the additional simple fields with quadratic fields not containing L must be at most of order $\mathcal{O}(c^{-1})$ as $c \to \infty$. However, from the discussion of the VPA we conclude that the coupling constants to the quadratic fields in terms of the generators tend to a nonzero constant for $c \to \infty$. This means that one must decouple the quadratic fields from the ring and rescale them independently, i.e. one must introduce further relations and further generators in order to make sense of the BW classical limit. This is similar to the VPA, in particular those fields have to be introduced as new generators whose modes had to be included into the VPA. This indicates that the BW classical limits of these three algebras are probably infinitely generated and satisfy infinitely many constraints.

At least for the bosonic projection of the $N = 1$ super Virasoro algebra we can verify in terms of the underlying realization that one field each at scale dimensions 8 and 10 decouples in the classical limit and gives rise to a new generator. Recall that the field G has to be rescaled with $\alpha_1 = 0$ (the exponent α_0 for L is also zero). Denote the classical limits corresponding to $L(z), G(z)$ by $l(z), g(z)$. After these substitutions, the coefficient of $g\partial^5 g$ in (4.47) vanishes in this classical limit and one has to add its primary projection $\mathcal{P}(g\partial^5 g)$ to the generating set. A similar phenomenon happens at scale dimension 10 where the primary projection of $g\partial^7 g$ has to be added to the generating set (compare also the discussion in [BFH], in particular appendix A loc. cit.).

For these three examples (the $\mathcal{W}(2,3,4,5)$ and the two algebras of type $\mathcal{W}(2,4,6)$) the realization in terms of a reduction is known both on the quantum as well as on the classical level. Therefore, we can compare the result of the limiting procedure (4.59) and (4.60) to the corresponding classical algebras [BFH]. First, we remark that the fields which we had to introduce as generators in the BW classical limit indeed turn up as generators in the classical reduction and that the classical counterparts of these algebras are indeed infinitely generated [BFH]. Rescaling of the generators with exponents $\alpha_i > 0$ implies a vanishing central term in the limit $\hbar \to 0$. On the classical level, all generators (with exception of the Virasoro field $l(z)$) are at least second order in the fields of the underlying algebra and the Poisson brackets of such fields do indeed not contain any central term. For any orbifold (including in particular the bosonic projection of the $N = 1$ super Virasoro algebra), the energy momentum tensor L in the projection is noncomposite and the central charge c is a free parameter on the classical level. Thus, the BW classical limit of this $\mathcal{W}(2,4,6)$ has a chance to be identical to the $\mathcal{W}$-algebra obtained from the classical orbifold. The situation is different for the other $\mathcal{W}(2,4,6)$ and the $\mathcal{W}(2,3,4,5)$ which we realized in terms of cosets. Note that the classical coset energy momentum tensor $l(z)$ is composite and therefore has no central term, i.e. $c = 0$ on the classical level. However, the BW limiting procedure (4.59) and (4.60) gives rise to nonzero Virasoro centre c showing that the BW classical limit is not identical to the classical coset. Even more, close inspection of the structure constants of the $\mathcal{W}(2,4,6)$ arising in the diagonal $\widehat{sl(2,\mathbb{R})}$ coset shows that some structure constants still are proportional to $\frac{1}{c^r}$ after taking the BW classical limit. Because this c-dependence cannot be completely scaled away, one cannot simply set $c = 0$ in the BW classical limit. This is not very surprising because for cosets of Kac-Moody algebras the energy momentum tensor L satisfies $[L_m, L_n] = \hbar(n-m)L_{m+n} + \hbar^2 c(n^3 - n)\delta_{m+n,0}$. This means that one should modify the BW procedure by substituting $c \mapsto \hat{c}$ instead of (4.59) in order to obtain at least the correct classical form of the Virasoro algebra. However, this substitution does not introduce any $\hbar$'s in the structure constants and would therefore leave us with c-dependent structure constants in this modified BW classical limit – something we do not want either. Furthermore, in the case of the $\mathcal{W}(2,3,4,5)$, the classical counterpart of the realization in terms of the coset $SVIR(N=2)/\widehat{U(1)}$ has a non-vanishing Virasoro centre in contrast to the classical coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$. So, the limiting procedure (4.59), (4.60) might correspond to the classical coset $SVIR(N=2)/\widehat{U(1)}$ but it is not clear how to obtain the classical counterpart of the other coset realization by a limiting procedure of this type. Even more, one can check that the classical coset $SVIR(N=2)/\widehat{U(1)}$ admits a *primary* generating set whereas the classical coset $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ does not (cf. [Ho4]). This shows that the Poisson brackets carried by these two classical cosets are not at all related to each other. These ambiguities for the classical counterpart might be related to the possibility that the BW procedure (4.59), (4.60) does in these cases not give rise to a classical $\mathcal{W}$-algebra at all.

In summary, we have seen that one can introduce the VPA and study the BW classical limit of any quantum $\mathcal{W}$-algebra in an algorithmic manner. In doing so one recovers many features of a corresponding classical counterpart without using any knowledge about the underlying construction (although the attempt to construct the BW classical limit could fail). This gives rise to the hope that all $\mathcal{W}$-algebras outside the DS class which are finitely generated on the quantum level belong to the same class of $\mathcal{W}$-algebras with infinitely generated classical counterparts [BFH]. From this point of view coset constructions and orbifolds behave in a very similar manner.

4.2 Unifying $\mathcal{W}$-algebras

In this section we will show that the second class of deformable $\mathcal{W}$-algebras, i.e. the finitely non-freely generated ones, contain 'unifying $\mathcal{W}$-algebras' interpolating the rank n of Casimir algebras $\mathcal{WL}_n$ at particular values of the central charge. For example, $\mathcal{W}(2,3,4,5) \cong \widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ unifies the first unitary minimal models of $\mathcal{WA}_{n-1}$. Similarly, the $\mathcal{W}(2,4,6) \cong \widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$ corresponds to certain minimal models of $\mathcal{WC}_n$. Using the coset realization of these algebras, the unifying $\mathcal{W}$-algebras can be regarded as a generalization of level-rank-duality [BG,Al]. These identifications between a priori different $\mathcal{W}$-algebras for particular values of the central charge c are closely related to the fact that for certain values of the central charge c some generators become null fields leading to a 'truncation' of the $\mathcal{W}$-algebra. We will show from inspection of the Kac determinant for Casimir $\mathcal{W}$-algebras that truncations of them are in fact a very general phenomenon.

We introduce the notion of 'unifying $\mathcal{W}$-algebras' before establishing their existence. The central charges of the minimal series of a Casimir algebra $\mathcal{WL}_n$ based on a simple Lie algebra $\mathcal{L}_n$ is denoted by $c_{\mathcal{L}_n}(p,q)$. A unifying $\mathcal{W}$-algebra is a finitely generated deformable quantum $\mathcal{W}$-algebra (i.e. existing for generic c) with the following properties:

- There exists an integer n_0 and simple Lie algebras $\mathcal{L}_n$ such that for all $n \geq n_0$ and $c = c_{\mathcal{L}_n}(p(n), q(n))$ it is isomorphic to the Casimir algebra $\mathcal{WL}_n$.
- $\mathcal{L}_n$ is either of type $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ independent of n.

This implies that for all $n > n_0$ the Casimir algebras $\mathcal{WL}_n$ at central charge $c_{\mathcal{L}_n}(p(n), q(n))$ truncate to the unifying $\mathcal{W}$-algebra whose spin content is *independent* of n.

Since unifying $\mathcal{W}$-algebras exist for generic c, they can be considered as continuations to real values of the rank n of Casimir algebras $\mathcal{WL}_n$ at $c_{\mathcal{L}_n}(p(n), q(n))$. Some Casimir algebras play the rôle of unifying $\mathcal{W}$-algebras. In contrast to them, the 'true' unifying $\mathcal{W}$-algebras are non-freely generated and belong to the second class of quantum $\mathcal{W}$-algebras introduced in section 4.1.

4.2.1 Truncations and identifications from structure constants

We begin with an introductory discussion of the simplest examples of unifying $\mathcal{W}$-algebras, $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ and $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$. It is well known that the first unitary models of the Casimir algebras $\mathcal{WA}_{k-1}$ describe the $\mathbb{Z}_k$ parafermionic theories [BBSS,FL1]. The central charge is given by $c_{\mathcal{A}_{k-1}}(k+1, k+2) = \frac{2(k-1)}{k+2} = c(k)$ and one observes that this coincides with the central charge of the coset $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)} \cong \mathcal{W}(2,3,4,5)$. Thus one might expect that the cosets $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)} \cong SVIR(N=2)/\widehat{U(1)}$ are *unifying* objects for the first unitary minimal models of $\mathcal{WA}_{k-1}$. There are several arguments supporting our statement. Inserting $c(k)$ into the general expression for the simplest structure constants $(C_{3\ 3}^{4})^2, (C_{4\ 4}^{4})^2, (C_{3\ 4}^{5})^2$ eq. (4.51) one reproduces the corresponding structure constants of $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ eq. (4.11) (compare also appendix B). We conclude that at this special value of the central charge $\mathcal{WA}_{k-1}$ truncates to the algebra $\mathcal{W}(2,3,4,5)$, i.e. the generating fields with spin ≥ 6 turn out to be null fields. Indeed, this can be checked explicitly in the first example $\mathcal{WA}_5$ [Hf2]. Furthermore, recalling that the central charge of the unitary minimal series of the

$N = 2$ super Virasoro algebra ($SVIR(N=2)$) is given by $c = \frac{3k}{k+2}, 0 < k \in \mathbb{N}$, we expect from the realization $SVIR(N=2)/\widehat{U(1)} \cong \mathcal{W}(2,3,4,5)$ that the first unitary models of $\mathcal{WA}_{k-1}$ agree with the rational models of $SVIR(N=2)/\widehat{U(1)}$ coming from the unitary minimal models of $SVIR(N=2)$. Indeed, one can compute the conformal dimensions of these representations of $SVIR(N=2)/\widehat{U(1)}$ and finds perfect agreement with the first unitary model of $\mathcal{WA}_{k-1}$. This unifying property can be interpreted in a more general way if one inspects the 'dual coset pairs' given in [BS,BG,Al]. Here dual coset pairs possessing equivalent energy momentum tensors ('T-equivalence') have been presented, e.g. $\widehat{sl(k,\mathbb{R})}_\kappa \oplus \widehat{sl(k,\mathbb{R})}_\mu / \widehat{sl(k,\mathbb{R})}_{\kappa+\mu} \cong \widehat{sl(\kappa+\mu,\mathbb{R})}_k / (\widehat{sl(\kappa,\mathbb{R})}_k \oplus \widehat{U(\mu)}_k)$. Specializing to $\kappa = \mu = 1$ we know that the left hand side yields the first unitary model of $\mathcal{WA}_{k-1}$ at $c(k)$ [BBSS]. The right hand side reduces to the coset $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ and we recover the equivalence considered above.

Similarly, the coset $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)} \cong \mathcal{W}(2,3,4,5,6,7)$ discussed in section 4.1.2 is a unifying algebra for the *non-unitary* minimal models $c_{\mathcal{A}_{n-1}}(n+1,n+3)$ of $\mathcal{WA}_{n-1}$. Indeed, with the relation $k = \frac{n-3}{2}$ (k is the level in the Bershadsky algebra eq. (4.16)) we find that the central charge of the $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$-coset eq. (4.21) coincides with the central charge $c_{\mathcal{A}_{n-1}}(n+1,n+3) = -3\frac{(n-1)^2}{n+3}$ of the non-unitary model $p = n+1,\ q = p+2$ of $\mathcal{WA}_{n-1}$. Furthermore, we find agreement of the structure constants eq. (4.51) of $\mathcal{WA}_{n-1}$ and those of the $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$-coset eq. (4.23) (compare appendix C) for these particular choices of k and c. Thus we expect that the generators of $\mathcal{WA}_{n-1}$ with scale dimension ≥ 8 are null fields for these values of the central charge so that the Casimir algebras $\mathcal{WA}_{n-1}$ truncate to this $\mathcal{W}(2,3,4,5,6,7)$. This is also supported by the study of Kac-determinants below.

In order to study these truncations and identifications in a systematic way we make use of the observation [Hf3] that deformable $\mathcal{W}$-algebras can be parametrized with 'universal' formulae in which only the conformal dimension of a certain representation enters as a parameter. More precisely, the structure constants of many $\mathcal{W}$-algebras fall into different classes, so that in each class the structure constants can be described by universal formulae $C_{n\,m}^{l}(c,h(c))$. The parametrizing function $h(c)$ varies for each of these algebras, being in fact any one of the two conformal dimensions which correspond to representations with a particularly low-lying null state. Usually, there is also a third parametrizing function $h(c)$ that does not correspond to degenerate representations.

One of these classes are $\mathcal{W}$-algebras of type $\mathcal{W}(2,3,4,\ldots)$, that is $\mathcal{W}$-algebras with primary fields of dimensions 3 and 4 and any number of fields with dimension ≥ 5. Therefore, all $\mathcal{WA}_{n-1}$-algebras and all truncations of them including in particular the special algebras $\mathcal{W}(2,3,4,5) \cong \widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ and $\mathcal{W}(2,3,4,5,6,7) \cong \mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$ belong to this class. Another class is formed by some $\mathcal{W}$-algebras of type $\mathcal{W}(2,4,\ldots)$ which includes the algebras $\mathcal{WB}_n$ and $\mathcal{WC}_n$ as well as the orbifolds of $\mathcal{WD}_n$ and $\mathcal{WB}(0,n)$. In this approach, the structure constants of the algebras $\mathcal{WD}_n$ can be continued to negative values of the rank such that the algebra $\mathcal{W}(2,4,6) \cong \mathcal{WD}_{-1}$ belongs to this class [Hf3]. Below we will give a realization of $\mathcal{WD}_{-n}$ in terms of the coset $\widehat{sp(2n)}_k \oplus \widehat{sp(2n)}_{-\frac{1}{2}}/\widehat{sp(2n)}_{k-\frac{1}{2}}$ and show that they are of type $\mathcal{W}(2,4,\ldots,2n(n+2))$.

The fact that the structure constants in each class can be described solely by the set of values of the three parameters for $h(c)$, $H = \{h_1(c), h_2(c), h_3(c)\}$, has the immediate consequence that the structure constants for two algebras X and V, and hence the algebras, become identical, whenever the sets H_X and H_V are equal. In general, the identification will lead to a truncation of the algebra with the higher number of simple fields, say X, to an algebra V with less fields at a *finite* set of values for the central charge. We shall denote this truncation by the symbol '$\triangleright$', i.e.

$$X \triangleright V \quad \text{if} \quad H_X = H_V. \tag{4.61}$$

However, we remark that the two sets H_X und H_V are also equal if V is a subalgebra of X (at some fixed c-values) and no truncation takes place. There are indeed special cases known where this occurs. Thus, one has to prove the existence of null fields in order to confirm the truncation expected from this argument. Indications for the presence of null fields will be presented below.

Note that (4.61) is usually automatically satisfied if one $h_i \in H_X$ is equal to one $h_j \in H_V$. The exceptions are related to singularities that occur e.g. at $c = 0$ or at $c = -2$ (see (4.62)).

Identifications of $\mathcal{W}$-algebras of type $\mathcal{W}(2, 3, 4, \ldots)$

The first few structure constants for these algebras are given in [Hf3]. It is immediately clear from the structure constants that any $\mathcal{W}$-algebra of type $\mathcal{W}(2, 3, 4, \ldots)$ (which can be described by these structure constants) truncates to the algebra $\mathcal{W}(2, 3)$ at $c = -2$:

$$\left.\begin{array}{rcl} \mathcal{WA}_{n-1} & \triangleright & \mathcal{W}(2,3) \\ \mathcal{W}(2,3,4,5) & \triangleright & \mathcal{W}(2,3) \\ \mathcal{W}(2,3,4,5,6,7) & \triangleright & \mathcal{W}(2,3) \end{array}\right\} \quad \text{at} \quad c = -2. \tag{4.62}$$

These truncations are independent of the values of h whence in these cases it is not necessary that the corresponding sets H agree. We will omit such cases in the following and restrict our discussion to cases where the sets H agree.

For the three algebras mentioned the set $H = \{h_1, h_2, h_3\}$ consists of the following values:

$$\begin{array}{rl} \mathcal{WA}_{n-1}: & 4n^2 h_{1,2}^2 + 2\big(c-(n-1)(2n+1)\big)h_{1,2} + c(n-1) = 0; \quad h_3 = \dfrac{c(n+1)}{2(c+1-n)} \\ \mathcal{W}(2,3,4,5): & h_1 = \dfrac{3c}{2(c+1)}; \quad h_2 = \dfrac{c(2-c)}{8(c+1)}; \qquad h_3 = -\dfrac{c+4}{2(c+1)} \\ \mathcal{W}(2,3,4,5,6,7): & h_1 = \dfrac{3(k+1)}{2k+3}; \quad h_2 = \dfrac{3(k+1)(k+2)}{2(2k+3)}; \quad h_3 = -\dfrac{(k+1)^2}{(k+3)(2k+3)} \\ & \text{with} \quad c = -6\dfrac{(k+1)^2}{k+3}. \end{array} \tag{4.63}$$

By identifying the sets $H_{\mathcal{WA}_{m-1}}$ and $H_{\mathcal{WA}_{n-1}}$ we find a solution for the central charge where $\mathcal{WA}_{m-1}$ truncates to $\mathcal{WA}_{n-1}$ $(m > n)$:

$$\mathcal{WA}_{m-1} \triangleright \mathcal{WA}_{n-1} \qquad \text{at} \quad c_m(n) = -\frac{(m-1)(n-1)(m+n+mn)}{m+n}. \tag{4.64}$$

The identification (4.64) has already been proposed in [KNS]. Let us now investigate the truncations of $\mathcal{WA}_{n-1}$ to $\mathcal{W}(2,3,4,5)$. From eq. (4.63) we find the solutions [8])

$$\mathcal{WA}_{n-1} \lhd\!\rhd \mathcal{W}(2,3,4,5) \cong \frac{\widehat{sl(2,\mathbb{R})}_n}{\widehat{U(1)}} \qquad \text{at} \quad \begin{array}{l} c(n) = \frac{2(n-1)}{n+2} \\ c(n) = -1-3n \\ c(n) = \frac{2(1-2n)}{n-2}\,. \end{array} \tag{4.65}$$

The first truncation is the well known identification of the coset $\widehat{sl(2,\mathbb{R})}_n/\widehat{U(1)}$ with the $\mathcal{WA}_{n-1}$-algebra for the first unitary minimal model. The other two truncations do not correspond to minimal models.

We proceed in the same way for truncations of $\mathcal{WA}_{n-1}$ to $\mathcal{W}(2,3,4,5,6,7)$:

$$\mathcal{WA}_{n-1} \lhd\!\rhd \mathcal{W}(2,3,4,5,6,7) \cong \frac{\mathcal{W}^{sl(3)}_{2,1}}{\widehat{U(1)}} \qquad \text{at} \quad \begin{array}{ll} k(n) = \frac{-3(n+1)}{n+2} & c(n) = \frac{-2(2n+1)^2}{n+2} \\ k(n) = \frac{n-3}{2} & c(n) = \frac{-3(n-1)^2}{n+3} \\ k(n) = \frac{3-2n}{n-2} & c(n) = \frac{6(n-1)^2}{(2-n)(n-3)}\,. \end{array} \tag{4.66}$$

Here, 'k' is the level entering $\mathcal{W}^{sl(3)}_{2,1}$ [Bl5,Po,Be] (see (4.16)). Note that the second line of (4.66) corresponds to minimal models of $\mathcal{W}^{sl(3)}_{2,1}$ and that this truncation is the identification of $\mathcal{WA}_{n-1}$ with $\mathcal{W}^{sl(3)}_{2,1}/\widehat{U(1)}$ at $c_{\mathcal{A}_{n-1}}(n+1,n+3)$ mentioned at the beginning of this subsection.

Identification of $\mathcal{W}$-algebras of type $\mathcal{W}(2,4,\ldots)$

We will now look for identifications between algebras of type $\mathcal{W}(2,4,\ldots)$. Since the bosonic projections of the superalgebras $\mathcal{WB}(0,m)$ are described by the formulae for the orbifold of $\mathcal{WD}_n$ with $n = m + \frac{1}{2}$ half-integer, we shall not consider these bosonic projections separately, but use the notation of the orbifold of $\mathcal{WD}_n$ for both of them, having in mind that we treat the superalgebra $\mathcal{WB}(0,n-\frac{1}{2})$ whenever n is half-integer. A further argument for this notation is that both algebras are realized in terms of diagonal $so(k)$-cosets with even and odd $k = 2n$ respectively (see e.g. [BS]). The orbifold of $\mathcal{WD}_{\frac{3}{2}}$ for example is the bosonic projection of the $N = 1$ super Virasoro algebra. Moreover, also negative values of n are allowed for $\mathcal{WD}_n$.

The first structure constants for this class of $\mathcal{W}$-algebras have been presented in [Hf3]. We give only the classifying sets H:

$$\begin{array}{rl} \mathrm{Orb}\,(\mathcal{WD}_n): & 4n(n-1)h_{1,2}^2 + \big(c+n(3-4n)\big)h_{1,2} + c(n-1) = 0; \quad h_3 = n; \\ \mathcal{WB}_n, \mathcal{WC}_n: & 8n(1+n)h_1^2 + 2\big(c-n(3+2n)\big)h_1 + c(2n-1) = 0; \\ & (2n-1)(2n+1)h_2^2 + \big(c-n(6n+1)\big)h_2 + cn + 2n^2 = 0; \\ & 2(c+2n)h_3^2 - \big(n(2n+3)+c(4n+3)\big)h_3 + c(n+1)(2n+1) = 0; \\ & \text{where one of the solutions is for } \mathcal{WB}\text{, the other one for } \mathcal{WC}. \end{array} \tag{4.67}$$

[8]) For $n \leq 4$ $\mathcal{W}(2,3,4,5)$ truncates to $\mathcal{WA}_{n-1}$, for $n = 5$ the two algebras are isomorphic for the given values of c. We use therefore the symbol '$\lhd\!\rhd$'.

For this class of algebras we find the following truncations with generic $m > n$:

$$\mathrm{Orb}\,(\mathcal{WD}_m) \triangleright \mathrm{Orb}\,(\mathcal{WD}_n) \qquad \text{at} \quad c = -\frac{mn(3-4m-4n+4mn)}{m+n-1}, \tag{4.68}$$

$$\mathcal{WC}_m \triangleright \mathcal{WC}_n \qquad \text{at} \quad c = \frac{mn(3+2m+2n-4mn)}{1+m+n}, \tag{4.69}$$

$$\mathcal{WB}_m \triangleright \mathcal{WB}_n \qquad \text{at} \quad c = -\frac{(2m+n+2mn)(m+2n+2mn)}{m+n}, \tag{4.70}$$

$$\mathcal{WB}_m \triangleright \mathcal{WC}_n \qquad \text{at} \quad c = \frac{4mn(3+2m+2n)}{(1+2m-2n)(1-2m+2n)}. \tag{4.71}$$

Finally, for generic m and n, one has the following identifications where the larger algebra truncates to the smaller one:

$$\mathcal{WB}_m \triangleleft\!\triangleright \mathcal{WC}_n \qquad \text{at} \quad c = -\frac{2mn(3+2m+2n+4mn)}{1+2m+2n}, \tag{4.72}$$

$$\mathrm{Orb}\,(\mathcal{WD}_m) \triangleleft\!\triangleright \mathcal{WB}_n \qquad \text{at} \quad c = \frac{2mn(-3+4m-2n+4mn)}{1-2m-2n}, \tag{4.73}$$

$$\mathrm{Orb}\,(\mathcal{WD}_m) \triangleleft\!\triangleright \mathcal{WB}_n \qquad \text{at} \quad c = \frac{(n-m-2mn)(2mn-m-2n)}{m+n}, \tag{4.74}$$

$$\mathrm{Orb}\,(\mathcal{WD}_m) \triangleleft\!\triangleright \mathcal{WC}_n \qquad \text{at} \quad c = \frac{mn(3-4m+2n)}{(1-m+n)(1-2m+2n)}. \tag{4.75}$$

For special values of m and n one might find additional solutions.

Let us now discuss some interesting consequences of these identifications. Eq. (4.74) can be parametrized by the formula for the minimal models of $\mathcal{WB}_n$ (see appendix D) as $c_{\mathcal{B}_n}(2n, 2n+2(m-\frac{1}{2})+1)$. Thus, for half-integer $m = k+\frac{1}{2}$ $\mathcal{WB}_n$ truncates at $c_{\mathcal{B}_n}(2n, 2n+2k+1)$ to $\mathrm{Orb}\,(\mathcal{WD}_m) = \mathrm{Orb}\,(\mathcal{WB}(0,k))$. We can interpret $\mathrm{Orb}\,(\mathcal{WB}(0,k))$ as the unifying algebra of the $\mathcal{WB}_n$ models at $c_{\mathcal{B}_n}(2n, 2n+2k+1)$. Eq. (4.73) can be parametrized as $c_{\mathcal{B}_n}(2n-1+2m, 2n+1)$. We infer that $\mathcal{WB}_n$ truncates at these values to $\mathrm{Orb}\,(\mathcal{WD}_m)$ which serves as a unifying algebra for these $\mathcal{WB}_n$ minimal models. Using the realization of $\mathcal{WD}_m$ and $\mathrm{Orb}\,(\mathcal{WB}(0,m))$ in terms of $so(k)$-cosets these identifications can be summarized as follows:

$$\mathcal{WB}_n \cong \begin{cases} \frac{\widehat{so(k)}_\mu \oplus \widehat{so(k)}_1}{\widehat{so(k)}_{\mu+1}} \cong \mathrm{Orb}\left(\mathcal{WB}(0, \frac{k-1}{2})\right), & k \text{ odd} \\ \mathrm{Orb}\left(\frac{\widehat{so(k)}_\mu \oplus \widehat{so(k)}_1}{\widehat{so(k)}_{\mu+1}}\right) \cong \mathrm{Orb}\left(\mathcal{WD}_{\frac{k}{2}}\right), & k \text{ even} \end{cases}$$
$$\text{at} \quad \begin{cases} c = -\frac{(2kn+k-2n)(2kn-k-4n)}{2(k+2n)}, & \mu = -\frac{(2n-1)k-4n}{2n}, \\ c = \frac{kn(3+2n-2k-2kn)}{k+2n-1}, & \mu = -\frac{2n(k-2)}{2n+1}. \end{cases} \tag{4.76}$$

Note also that when replacing m by $-m$, eq. (4.75) is parametrized by $c_{\mathcal{C}_n}(m+n+1, 2m+2n+1)$ which can be interpreted as a truncation to the algebra $\mathrm{Orb}\,(\mathcal{WD}_{-m})$ (see below).

These interpretations are in agreement with the expectation gained from the study of the Kac determinant in the following subsection.

Interesting consequences follow from the fact that also the algebra $\mathcal{W}(2,4,6,8,10)$ which arises as the orbifold of the coset $\widehat{sl(2,\mathbb{R})}_k/\widehat{U(1)}$ (compare table 4) (hence the orbifold of the first unitary model of $\mathcal{WA}_{k-1}$) belongs to the class of $\mathcal{W}$-algebras of type $\mathcal{W}(2,4,\ldots)$ with

$$h_1 = \frac{2-c}{3}; \quad h_2 = \frac{c+1}{2\,(2-c)}; \quad h_3 = \frac{3\,c}{2\,(c+1)}. \tag{4.77}$$

Note that in contrast to this the orbifold of $\mathcal{WA}_{n-1}$ for c generic does not belong to this class, i.e. cannot be described by the general structure constants. Proceeding in the same way as for the previous truncations we find

$$\mathrm{Orb}\,(\mathcal{WD}_m) \triangleright \mathcal{W}(2,4,6,8,10) \cong \mathrm{Orb}\left(\frac{\widehat{sl(2,\mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad \begin{array}{l} c(m) = 2-3m \\ c(m) = \frac{4m-1}{2m+1} \\ c(m) = -\frac{2m}{2m-3}, \end{array} \tag{4.78}$$

$$\mathcal{WB}_m \triangleright \mathcal{W}(2,4,6,8,10) \cong \mathrm{Orb}\left(\frac{\widehat{sl(2,\mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad \begin{array}{l} c(m) = \frac{4m}{2m+3} \\ c(m) = -2m \\ c(m) = -\frac{m+2}{m-1}, \end{array} \tag{4.79}$$

$$\mathcal{WC}_m \triangleright \mathcal{W}(2,4,6,8,10) \cong \mathrm{Orb}\left(\frac{\widehat{sl(2,\mathbb{R})}_k}{\widehat{U(1)}}\right) \quad \text{at} \quad c(m) = -1-6m. \tag{4.80}$$

These truncations are compatible with the truncation of $\mathcal{WA}_{n-1}$ to $\mathcal{W}(2,3,4,5)$ (see eq. (4.65)) exactly at the c-values for the first unitary model of $\mathcal{WA}_{n-1}$. In particular, we find that the orbifold of the first unitary model of $\mathcal{WA}_{2n-1}$ is equal to the orbifold of the second unitary model of $\mathcal{WD}_{\frac{n}{2}}$ (this will be confirmed below using level-rank duality), and that the orbifold of the first unitary model of $\mathcal{WA}_{2n}$ is equivalent to a unitary model of $\mathcal{WB}_n$. For the example $n=2$, $c=\frac{8}{7}$ a detailed verification of this truncation $\mathrm{Orb}\,(\mathcal{W}(2,3,4,5)) \triangleright \mathcal{W}(2,4)$ is possible [Bl5]. Note that this model is probably the only unitary minimal model of $\mathcal{W}(2,4) \cong \mathcal{WB}_2$ [Ho2].

Truncations of Casimir $\mathcal{W}$-algebras and the Kac determinant

So far we have obtained indications for truncations of Casimir $\mathcal{W}$-algebras from a detailed study of structure constants. We will now provide further support for these truncations by inspection of the Kac determinant.

Let $\mathcal{L}_k$ be a simple Lie algebra of rank k over $\mathbb{C}$, Δ the set of its roots, ρ ($\rho^\vee$) the sum of its (dual) fundamental weights, h ($h^\vee$) its (dual) Coxeter number and $\mathcal{WL}_k$ the corresponding Casimir $\mathcal{W}$-algebra.

Then the Kac determinant of the vacuum Verma module $\mathcal{M}_N$ at level N related to $\mathcal{WL}_k$ is given by [BS]:

$$\det \mathcal{M}_N \sim \prod_{\beta\in\Delta} \prod_{\substack{mn\leq N \\ m,n\in\mathbb{N}}} \left((\alpha_+\rho + \alpha_-\rho^\vee, \beta) + \left(\tfrac{1}{2}(\beta,\beta)m\alpha_+ + n\alpha_-\right)\right)^{p_k(N-mn)} \tag{4.81}$$

where $p_k(x)$ is the number of partitions of x into k colours and the $\alpha_\pm$ are related to the central charge by

$$\alpha_+\alpha_- = -1, \qquad c = k - 12(\alpha_+\rho + \alpha_-\rho^\vee)^2. \tag{4.82}$$

At least one $\mathcal{W}$-algebra singular vector has to exist if a Casimir $\mathcal{W}$-algebra truncates at a certain value of the central charge. Therefore, the Kac determinant of the vacuum Verma module has to vanish at the level corresponding to the singular vector.

Let us now concentrate on degenerate values of the central charge of the Casimir $\mathcal{W}$-algebra where one has $\alpha_+ = \frac{q}{\sqrt{pq}}$ and $\alpha_- = -\frac{p}{\sqrt{pq}}$ with $p,q \in \mathbb{N}$ such that

$$c_{\mathcal{L}_k}(p,q) = k - 12\frac{(q\,\rho - p\,\rho^\vee)^2}{pq}. \tag{4.83}$$

The condition that a singular vector occurs at level N implies that one of the non-embedded factors of the Kac determinant with $N = mn$ (m, n positive) is zero for a certain root β:

$$(\alpha_+\rho + \alpha_-\rho^\vee, \beta) + \left(\tfrac{1}{2}(\beta,\beta)m\alpha_+ + n\alpha_-\right) = 0. \tag{4.84}$$

Using the expressions for $\alpha_\pm$ this implies

$$\frac{p+q}{p-q} = \frac{\frac{1}{2}(\beta,\beta)m + n + (\rho,\beta) + (\rho^\vee,\beta)}{\frac{1}{2}(\beta,\beta)m - n + (\rho,\beta) - (\rho^\vee,\beta)}. \tag{4.85}$$

For given p and q the root leading to the lowest singular vector is the highest root of length 2 for all simple Lie algebras besides $\mathcal{C}_k$ where one has to take the sum of the simple roots (a table of these roots can be found in [Hum]). Choosing this root for β and parametrizing p and q as $p = h^\vee - 1 + r$, $q = h - 1 + s$ we obtain

$$\frac{r+s+h+h^\vee-2}{r-s-h+h^\vee} = \frac{m+n+h+h^\vee-2}{m-n-h+h^\vee}. \tag{4.86}$$

Obviously, $m = r$ and $n = s$ are solutions of this equation as long as $r + h^\vee \neq s + h$. If $r,s > 0$ this implies the existence of a $\mathcal{W}$-algebra singular vector at level $N = rs$ for central charge $c_{\mathcal{L}_k}(h^\vee - 1 + r, h - 1 + s)$. Note that for a given value of the central charge one has to choose minimal p, q such that $p \geq h^\vee$, $q \geq h$. This choice of not necessarily coprime integers ensures $r,s > 0$.

A truncation takes place if the singular vector at level $N = rs$ corresponds to a non-composite field and if with the vanishing of the simple field of dimension N also all other simple fields with dimension greater than N become null fields. A singular vector at $N = rs$ with N less than the maximal spin of the generating simple fields indicates a truncation of the Casimir algebra. We will now discuss examples where truncations can be obtained from this argument supporting the truncations evaluated above.

The Casimir algebra of type $\mathcal{WA}_n$ truncates for $c = c_{\mathcal{A}_n}(n+r, n+s)$ to an algebra of type $\mathcal{W}(2,3,\ldots,rs-1)$. This supports the identification (4.64) which corresponds to $r = 1$. Furthermore, for the kth unitary minimal model of $\mathcal{WA}_n$ this predicts a truncation to a

$\mathcal{W}(2,3,\ldots,k^2+3k+1)$ ($r=k+1$, $s=k+2$) which will be established below. From the free field realization [FL1] it follows that all non-composite fields of dimension greater than N vanish if the non-composite field of dimension N is a null field. Thus, for $\mathcal{WA}_n$ the only assumption which is not proven yet is that the $\mathcal{W}$-algebra singular vector at level $N = rs$ corresponds to a non-composite field.

The Casimir algebras related to the simple Lie algebras of type $\mathcal{B}_n$ and $\mathcal{C}_n$ truncate for $r{\cdot}s$ even to $\mathcal{W}$-algebras of type $\mathcal{W}(2,4,\ldots,rs-2)$ for $c=c_{\mathcal{B}_n}(2n-2+r,2n-1+s)$ and $c=c_{\mathcal{C}_n}(n+r,2n-1+s)$, respectively. This verifies eqs. (4.68)-(4.75). Furthermore, it implies the truncation of $\mathcal{WC}_n$ at $c_{\mathcal{C}_n}(m+n+1,2m+2n+1)$ to an algebra of type $\mathcal{W}(2,4,\ldots,2m(m+2))$. These unifying objects are in fact the algebras $\mathcal{WD}_{-m}$ as we will see below.

The $\mathbb{Z}_2$ orbifold of the $\mathcal{WD}_n$ Casimir algebra (see section 4.1.4) truncates for $c=c_{\mathcal{D}_n}(2n-3+r,2n-3+s)$ for $r{\cdot}s$ even to an algebra of type $\mathcal{W}(2,4,\ldots,rs-2)$. For the kth unitary minimal model of Orb$(\mathcal{WD}_n)$ ($s=r+1=k+2$) this predicts a unifying $\mathcal{W}$-algebra of type $\mathcal{W}(2,4,\ldots,k^2+3k)$. This identification can be verified applying a character argument to level-rank-duality. The case $k=2$ corresponds the second line of (4.78).

The truncations of Casimir $\mathcal{W}$-algebras indicated by the Kac determinant are summarized in the following table.

Casimir algebra	c	*truncated algebra*
$\mathcal{WA}_n$	$c_{\mathcal{A}_n}(n+r,n+s)$	$\mathcal{W}(2,\ldots,rs-1)$
$\mathcal{WB}_n$	$c_{\mathcal{B}_n}(2n-2+r,2n-1+s)$	$\mathcal{W}(2,4,\ldots,rs-2)$ for $r{\cdot}s$ even
$\mathcal{WC}_n$	$c_{\mathcal{C}_n}(n+r,2n-1+s)$	$\mathcal{W}(2,4,\ldots,rs-2)$ for $r{\cdot}s$ even
Orb$(\mathcal{WD}_n)$	$c_{\mathcal{D}_n}(2n-3+r,2n-3+s)$	$\mathcal{W}(2,4,\ldots,rs-2)$ for $r{\cdot}s$ even

Table 5: Truncations of Casimir $\mathcal{W}$-algebras

4.2.2 Coset realization of unifying $\mathcal{W}$-algebras and level-rank-duality

In the foregoing subsection we found unifying $\mathcal{W}$-algebras for the first unitary minimal models $c_{\mathcal{A}_{n-1}}(n+1,n+2)$ and the non-unitary minimal models $c_{\mathcal{A}_{n-1}}(n+1,n+3)$ of $\mathcal{WA}_{n-1}$. They have a realization as the cosets $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ and $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$. From the study of various truncations of Casimir algebras one expects infinitely many unifying $\mathcal{W}$-algebras. Our aim is to give for some of them a concrete realization in terms of cosets. In particular, we will generalize the above results for the non-unitary minimal models of $\mathcal{WA}_{n-1}$ with central charge $c_{\mathcal{A}_{n-1}}(n+k,n+r)$. We recall that we were able to show that the special $\mathcal{W}(2,4,6)$ is realized by the coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$ in section 4.1.3. In ref. [Bl5] we derived its minimal models from the representation theory of the coset. In fact, this algebra is the first member of a new series of unifying algebras – denoted as $\mathcal{WD}_{-m}$ – which are unifying objects for certain $\mathcal{WC}$ minimal models. Note that we anticipated this connection already in the previous section; compare the remarks made after eqs. (4.75, 4.76) and in the paragraphs preceding table 5. In the spirit of [Hf3] one can write down the minimal series of these algebras. Using character arguments it is possible to support an explicit coset realization based on the symplectic Lie algebras $sp(2m)$. Furthermore, we present coset realizations of unifying $\mathcal{W}$-algebras of some series of minimal models of the (Casimir) algebras $\mathcal{WB}$ and $\mathrm{Orb}(\mathcal{WD})$ proposed by the results of section 4.2.1. These relationships generalize level-rank-duality of coset pairs.

Unifying $\mathcal{W}$-algebras for the unitary minimal models of $\mathcal{WA}_n$

Due to the level-rank duality [BG,Al]

$$\frac{\widehat{sl(n)}_k \oplus \widehat{sl(n)}_1}{\widehat{sl(n)}_{k+1}} \cong \frac{\widehat{sl(k+1)}_n}{\widehat{sl(k)}_n \oplus \widehat{U(1)}} = \mathcal{CP}(k) \tag{4.87}$$

one expects that the symmetry algebra of the $\mathcal{CP}(k)$ model is a unifying $\mathcal{W}$-algebra for the kth unitary model of $\mathcal{WA}_{n-1}$. Note that the l.h.s. of (4.87) is defined for integer n and arbitrary k, whereas the r.h.s. is defined for integer k and general n. The isomorphism in (4.87) is valid iff k and n are both positive integers.

We will calculate the spin content of $\mathcal{CP}(k)$ using character techniques [9]. According to [BFH] we have to count the states in the complement of $\widehat{sl(k)}_n \oplus \widehat{U(1)}$ in $\widehat{sl(k+1)}_n$ that are invariant under $sl(k) \oplus U(1)$. This will be carried out for generic level n using the character argument. Let Δ^k be the root system of $sl(k+1)$ and let $\vec{\Theta}$ be in h^* (h is a Cartan subalgebra of the horizontal subalgebra $sl(k+1)$ of $\widehat{sl(k+1)}_n$). We will write $\vec{\Theta} = (\vec{\theta}, \Theta_k)$ where $\vec{\theta}$ corresponds to $sl(k)$ and Θ_k to $U(1)$. The counting function for the subspace of the vacuum module of $\widehat{sl(k+1)}_n$ that does not contain any states generated by $\widehat{sl(k)}_n \oplus \widehat{U(1)}$ is given by:

$$\frac{q^{-\frac{2k}{24}}}{\prod_{\vec{\alpha}\in\Delta^k\setminus\Delta^{k-1}} \prod_{n\geq 1}(1 - e^{2\pi i\,\vec{\Theta}\cdot\vec{\alpha}} q^n)} = \sum_{\vec{\lambda},m} C^0_{\vec{\lambda},m}(q)\ e^{2\pi i\,(\vec{\theta}\cdot\vec{\lambda} + \Theta_k m)}. \tag{4.88}$$

Since this module carries a representation of $sl(k) \oplus U(1)$ it can be decomposed into representations of $sl(k) \oplus U(1)$ which is indicated by the r.h.s. of (4.88). The $C^0_{\vec{\lambda},m}(q)$ are the string functions where $\vec{\lambda}$ and m are the weights of $sl(k)$ and $U(1)$ respectively that label the representation. Invariance under $U(1)$ is equivalent to $m = 0$. The part invariant under $sl(k)$ can be obtained by summation over its Weyl group. Thus, the vacuum character of the coset under consideration is given by the branching function $B^0_{0,0}(q)$ which can be written as the following sum over string functions:

$$B^0_{0,0}(q) = \sum_{w\in W} \epsilon(w)\, C^0_{-w(\rho)+\rho,0}(q). \tag{4.89}$$

where W is the Weyl group of $sl(k)$. The string functions can be obtained as follows. Inserting $z^2 = e^{2\pi i\,\vec{\Theta}\cdot\vec{\alpha}}$ with $\vec{\alpha}$ a positive root into (4.12), the left hand side of (4.88) can

[9]) The (unique) simple field with spin 3 has been calculated in [Vo].

be written in the form

$$\frac{q^{-\frac{2k}{24}}}{\prod_{\vec{\alpha}\in\Delta^k\setminus\Delta^{k-1}}\prod_{n\geq 1}(1-e^{2\pi i\,\vec{\Theta}\cdot\vec{\alpha}}q^n)} = \sum_{\{n_1,\dots,n_k\}\in\mathbb{Z}^k} \frac{\prod_{i=1}^k(\phi_{n_i}(q)-\phi_{n_i-1}(q))}{\eta^{2k}(q)}\; e^{2\pi i\,\vec{\Theta}\cdot\left(\sum_{i=1}^k n_i\vec{\alpha}_i\right)}. \tag{4.90}$$

Using an embedding of the roots of $sl(k+1)$ into $\mathbb{R}^k$ one obtains a map $(\vec{\lambda},m)\to(n_1,\dots,n_k)$. Now, comparison of the r.h.s. of eqs. (4.88) and (4.90) gives an explicit representation of the string functions in terms of the ϕ's. It is now not difficult to calculate the vacuum character of the $\mathcal{CP}(k)$ model for the first few values of k. In table 6 we present the results for $1\leq k\leq 4$ and give the spin content of the coset as well as a conjecture for general k (for $k=1$ compare section 4.1.1). The character of the vacuum Verma module freely generated by fields of dimension d_1, d_2, ..., d_n is denoted by $\chi_{d_1,d_2,\dots,d_n}$.

k	*vacuum character*
1	$B^0_{0,0}(q)=\eta(q)^{-2}\big(\phi_0(q)-q\,\phi_1(q)\big)$ $B^0_{0,0}(q)-\chi_{2,3,4,5}(q)=-2q^8-4q^9-9q^{10}+\mathcal{O}(q^{11})$
2	$B^0_{0,0}(q)=\eta(q)^{-4}\big((\phi_0(q)-q\,\phi_1(q))^2-(\phi_1(q)-\phi_0(q))^2\big)$ $B^0_{0,0}(q)-\chi_{2,\dots,11}(q)=-2q^{14}-6q^{15}-15q^{16}+\mathcal{O}(q^{17})$
3	$B^0_{0,0}(q)-\chi_{2,\dots,19}(q)=-2q^{22}-6q^{23}-17q^{24}+\mathcal{O}(q^{25})$
4	$B^0_{0,0}(q)-\chi_{2,\dots,29}(q)=-2q^{32}-6q^{33}+\mathcal{O}(q^{34})$
k	$B^0_{0,0}(q)-\chi_{2,\dots,k^2+3k+1}(q)=-2q^{k^2+3k+4}-6q^{k^2+3k+5}+\mathcal{O}(q^{k^2+3k+6})$

Table 6: Vacuum character for the $\mathcal{CP}(k)$-model

From the vacuum characters given in table 6 and the arguments presented above we conjecture that

$$\mathcal{CP}(k)=\frac{\widehat{sl(k+1)}_n}{\widehat{sl(k)}_n\oplus\widehat{U(1)}}\cong\mathcal{W}(2,3,\dots,k^2+3k+1). \tag{4.91}$$

This is compatible with the truncations predicted in the section 4.2.1 for the l.h.s. of (4.87).

Unifying $\mathcal{W}$-algebras for the non-unitary minimal models of $\mathcal{WA}_n$

Our aim is now to find a generalization of the result that $\mathcal{W}^{sl(3)}_{2,1}/\widehat{U(1)}$ is the unifying $\mathcal{W}$-algebra of the non-unitary $\mathcal{WA}_{n-1}$ minimal models $c_{\mathcal{A}_{n-1}}(n+1,n+3)$ to the unifying $\mathcal{W}$-algebras of the minimal models $c_{\mathcal{A}_{n-1}}(n+k,n+r)$. From the experience with the case $k=1,r=3$ (Polyakov-Bershadsky algebra) we expect that the unifying $\mathcal{W}$-algebras for all $\mathcal{WA}_{n-1}$ minimal models can be obtained by cosets of some Drinfeld-Sokolov reductions based on $sl(r)$ that have a $\widehat{sl(k)}\oplus\widehat{U(1)}$ Kac-Moody subalgebra [10]). Therefore, let us consider the algebras

$$\mathcal{W}^{sl(r)}_{r-k,1^k}=\mathcal{W}\left(1^{k^2},2,3,\dots,r-k,\left(\tfrac{r-k+1}{2}\right)^{2k}\right). \tag{4.92}$$

[10]) This remark is due to E. Ragoucy.

The k^2 currents form a $\widehat{sl(k)} \oplus \widehat{U(1)}$ Kac-Moody algebra, the fields of dimension $2, \ldots, r-k$ are singlets with respect to this Kac-Moody and the $2k$ fields of dimension $\frac{r-k+1}{2}$ are a Kac-Moody multiplet. More precisely, these $2k$ fields transform as two $U(1)$-charge conjugate defining representations of $sl(k)$. The conjecture is that

$$\frac{\mathcal{W}^{sl(r)}_{r-k,1^k}}{\widehat{sl(k)} \oplus \widehat{U(1)}} \cong \mathcal{WA}_{n-1} \qquad \text{at} \qquad c_{\mathcal{A}_{n-1}}(n+k, n+r). \tag{4.93}$$

Using the results from above (table 5) the conjecture (4.93) implies in particular

$$\frac{\mathcal{W}^{sl(r)}_{r-k,1^k}}{\widehat{sl(k)} \oplus \widehat{U(1)}} \cong \mathcal{W}(2,3,\ldots,(k+1)r+k) \tag{4.94}$$

with two generic null fields at dimension $(k+1)r+k+3$. Identifying $\mathcal{W}^{sl(r+1)}_{1^{r+1}}$ with the unconstrained $\widehat{sl(r+1)}$ Kac-Moody algebra, the case $r = k+1$ is identical to the $\mathcal{CP}(k)$ cosets discussed in the preceding subsection. The results for the $\mathcal{CP}(k)$ cosets also confirm the conjectured identities (4.93) and (4.94). One can also apply (4.93) to the case $k = 0$. In this case, the $sl(2)$ embedding is the principal one and there is no Kac-Moody subalgebra. Thus, the l.h.s. of (4.93) is just $\mathcal{WA}_{r-1}$ and we recover eq. (4.64).

For a first check let us investigate the case $k = 1$. Following [BFH] we obtain for the $\widehat{U(1)}$-coset of the $\mathcal{W}$-algebra $\mathcal{W}^{sl(r)}_{r-1,1} \cong \mathcal{W}(1,2,\ldots,r-1,\frac{r}{2},\frac{r}{2})$ (the spin $\frac{r}{2}$ fields have $U(1)$-charge ± 1 whereas all other simple fields are uncharged)

$$\frac{\mathcal{W}^{sl(r)}_{r-1,1}}{\widehat{U(1)}} \cong \mathcal{W}(2,3,\ldots,2r+1). \tag{4.95}$$

Comparing with table 5 we see that this algebra has indeed the spin content of the unifying algebra for the models $c_{\mathcal{A}_{n-1}}(n+1, n+r)$. To get a further confirmation we treat the case $r = 4$. We constructed explicitly the OPEs for the algebra $\mathcal{W}^{sl(4)}_{3,1} \cong \mathcal{W}(1,2,2,2,3)$. To check our conjecture we calculated the structure constant C_{33}^4 in the coset $\mathcal{W}(1,2,2,2,3)/\widehat{U(1)} \cong \mathcal{W}(2,3,4,5,6,7,8,9)$ yielding truncations of $\mathcal{WA}_{n-1}$ to $\mathcal{W}(2,3,4,5,6,7,8,9)$ for:

$$c_1(n) = -4\frac{(n-1)(2n-3)}{(n-3)(n-4)} \qquad c_2(n) = -4\frac{(n-1)(2n-1)}{(n+4)} \qquad c_3(n) = -\frac{(3n+1)(5n+3)}{(n+3)}. \tag{4.96}$$

The second truncation is exactly the truncation of the non-unitary models $c_{\mathcal{A}_{n-1}}(n+1, n+4)$ of $\mathcal{WA}_{n-1}$.

In order to perform a further check we consider the case $k = 2$. In this case we have according to (4.92) $\mathcal{W}^{sl(r)}_{r-2,1^2} = \mathcal{W}\left(1^4, 2, 3, \ldots, r-2, \left(\frac{r-1}{2}\right)^4\right)$. Denote the four fields of scale dimension $\frac{r-1}{2}$ by $W_1^\pm$ and $W_2^\pm$. The upper index of these fields refers to the $U(1)$-charge. $W_{1,2}^+$ and $W_{1,2}^-$ are two $sl(2)$-doublets. In the spirit of [BFH] one can see that the *classical* invariants under $sl(2) \oplus U(1)$ are generated by

$$S_{m,n} := \partial^m W_1^+ \partial^n W_1^- + \partial^m W_2^- \partial^n W_2^+ \tag{4.97a}$$

and the relations are generated by

$$\det \begin{pmatrix} S_{m_1,n_1} & S_{m_1,n_2} & S_{m_1,n_3} \\ S_{m_2,n_1} & S_{m_2,n_2} & S_{m_2,n_3} \\ S_{m_3,n_1} & S_{m_3,n_2} & S_{m_3,n_3} \end{pmatrix} = 0 \tag{4.97b}$$

with two sets of pairwise distinct integers $\{m_1, m_2, m_3\}$ and $\{n_1, n_2, n_3\}$. Following the argumentation of [BFH] it is now simple to determine the field content of the coset (4.94) for $k = 2$:

$$\frac{\mathcal{W}^{sl(r)}_{r-2,1^2}}{\widehat{sl(2)} \oplus \widehat{U(1)}} \cong \mathcal{W}(2, 3, \ldots, 3r + 2) \tag{4.98}$$

with two generic null fields at dimension $3r + 5$.

In order to perform further checks of (4.94) on the level of the characters we need a generalization of (4.13):

$$\frac{1}{\prod_{m \geq \Delta - 1}(1 - e^{2\pi i\theta} q^m) \prod_{m \geq \Delta}(1 - e^{-2\pi i\theta} q^m)} = \sum_{n \in \mathbb{Z}} \frac{\psi_n^\Delta(q)}{\prod_{m>0}(1 - q^m)^2} e^{2\pi i\theta n} \tag{4.99a}$$

with

$$\psi_n^\Delta(q) = \sum_{m \geq \Delta - 1} (-1)^{m-\Delta+1} q^{\frac{(m-\Delta+1)(m-\Delta+2)}{2} + nm} \prod_{\nu=m-\Delta+2}^{m+\Delta-1} (1 - q^\nu). \tag{4.99b}$$

Now one can compute the vacuum character of the coset (4.93) in complete analogy to the $\mathcal{CP}(k)$ models using (4.102), i.e. the only modification is that one has to substitute $\psi_n^\Delta(q)$ with $\Delta = \frac{r-k+1}{2}$ for $\phi_n(q)$. We have checked for $k = 3$, $4 \leq r \leq 8$ and $k = 4$, $5 \leq r \leq 8$ that this character argument is in agreement with (4.94).

We conclude this subsection with a remark concerning the algebras $\mathcal{WA}_{-n}$. In [Hf4] there has been collected evidence that these algebras are of type $\mathcal{W}\left(2, 3, \ldots, n^2 - 1\right)$ $(n > 2)$. Similar to the case of the $\mathcal{WD}_{-n}$-algebras discussed below, one can conjecture the coset realization $\mathcal{WA}_{-n-1} \cong \widehat{sl(n)}_\kappa \oplus \widehat{sl(n)}_{-1} / \widehat{sl(n)}_{\kappa-1}$.

Unifying $\mathcal{W}$-algebras for $\mathcal{WD}_n$: Level-rank-duality of $\widehat{so(n)}_k \oplus \widehat{so(n)}_1 / \widehat{so(n)}_{k+1}$

We consider now the impact of level-rank-duality on the coset $\widehat{so(n)}_k \oplus \widehat{so(n)}_1 / \widehat{so(n)}_{k+1}$. From known results in the literature [BG,Al] and arguments to be presented below, we conjecture that

$$(\mathrm{Orb})\left(\frac{\widehat{so(n)}_k \oplus \widehat{so(n)}_1}{\widehat{so(n)}_{k+1}}\right) \cong \mathrm{Orb}\left(\frac{\widehat{so(k+1)}_n}{\widehat{so(k)}_n}\right). \tag{4.100}$$

For n even, the coset on the l.h.s. realizes $\mathcal{WD}_{\frac{n}{2}}$ and one has to take an orbifold whereas for n odd, the coset already corresponds to the orbifold of $\mathcal{WB}(0, \frac{n-1}{2})$ and no additional projection has to be taken. Eq. (4.100) implies that the orbifold of the coset $\mathcal{W}$-algebra $\widehat{so(k+1)}_n/\widehat{so(k)}_n$ is a unifying $\mathcal{W}$-algebra for the orbifolds of the kth unitary minimal models of $\mathcal{WD}_{\frac{n}{2}}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ for n even or odd respectively. It should be clear to the reader that the methods used in [BG,Al] to derive level-rank-duality are insensitive to orbifolds and therefore they originally did not turn up in the equality (4.100). Because of the orbifolding procedure the coset $\widehat{so(k+1)}_n/\widehat{so(k)}_n$ is more difficult to deal with in full generality than those we discussed before. Therefore we will just look at the first three examples.

<u>**k = 1**</u>: One observes that $\widehat{so(2)}_n/\widehat{so(1)}_n \cong \widehat{U(1)}$ and that the first unitary minimal models of $\mathcal{WD}_{\frac{n}{2}}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ all have $c = 1$. Therefore, the RCFTs related to $k = 1$ boil down to the classification of $c = 1$ theories [Gi2,Kir]. The r.h.s. of (4.100) is given by the orbifold branch of the $c = 1$ RCFTs [Gi2] leading to a symmetry algebra of type $\mathcal{W}(2, 4, \frac{d}{2})$ with $d \in \mathbb{N}$. Notice that at least the first members of the series $\mathcal{WD}_{\frac{n}{2}}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ truncate to an algebra of type $\mathcal{W}(2, 4, \frac{n}{2})$ for $c = 1$ [Eh3]. The $\mathbb{Z}_2$ orbifold of these algebras is again a $\mathcal{W}$-algebra of type $\mathcal{W}(2, 4, \frac{d}{2})$. However, the spin content of this algebra depends on the level n in (4.100), i.e. $d = d(n)$. In this respect the case $k = 1$ is different from the cases $k > 1$.

<u>**k = 2**</u>: For the unifying $\mathcal{W}$-algebra of the second unitary minimal models we can make use of the fact that $\widehat{so(3)}_n/\widehat{so(2)}_n \cong \widehat{sl(2, \mathbb{R})}_{2n}/\widehat{U(1)}$. This coset has been extensively discussed in section 4.1.1. In particular, we know that the set of spins of the generators is not a subset of those of the Casimir algebra $\mathcal{WD}_{\frac{n}{2}}$. Therefore, we definitely have to take the $\mathbb{Z}_2$ orbifold on the r.h.s. of (4.100). This orbifold has been argued in section 4.1.4 to lead to a $\mathcal{W}(2, 4, 6, 8, 10)$. Note that the structure constant C_{44}^4 of the orbifold (see appendix B) and that of $\mathcal{WD}_{\frac{n}{2}}$ are indeed equal for $k = 2n$, i.e. $c = \frac{2n-1}{n+1}$ (compare eq. (4.78)). Since for $k = 2$ the minimal models on the r.h.s. of (4.100) are orbifolds of $\mathbb{Z}_{2n}$ parafermions, one can easily look at the first few minimal models on both sides of (4.100) (for the relation between $\mathbb{Z}_{2n}$ parafermions and $\mathcal{WD}_{\frac{n}{2}}$ and $\mathcal{WB}(0, \frac{n-1}{2})$ see also [FL2]). One observes that already the $\mathbb{Z}_{2n}$ parafermionic models contain more fields than the second unitary minimal models of $\mathcal{WD}_{\frac{n}{2}}$. This is a first argument that we also have to take a $\mathbb{Z}_2$ orbifold of the l.h.s. of (4.100). Furthermore, one can also examine the structure constants of the $\mathcal{W}(2, 4, 4, 6)$ that corresponds to $\mathcal{WD}_4$. From their explicit expressions [K3] one concludes that for no positive value of the central charge c any of the two fields with conformal dimension 4 drops out. This is a further strong argument to take the $\mathbb{Z}_2$ orbifold also on the l.h.s. of (4.100).

<u>**k = 3**</u>: Finally we look at the third unitary minimal models. Observing that $\widehat{so(4)}_n/\widehat{so(3)}_n \cong \widehat{sl(2, \mathbb{R})}_n \oplus \widehat{sl(2, \mathbb{R})}_n/\widehat{sl(2, \mathbb{R})}_{2n}$ leads us to a coset which we discussed in section 4.1.3. Like for $k = 2$ we know that the set of spins of the generators is not a subset of those of $\mathcal{WD}_{\frac{n}{2}}$ showing once again that a $\mathbb{Z}_2$ orbifold has to be taken on the r.h.s. of (4.100). In section 4.1.4 we argued that this orbifold leads to an algebra of type $\mathcal{W}(2, 4, 6, 8, 10, 12, 14, 16, 18)$. Again, we can check equality of the structure constants C_{44}^4 of $\mathcal{WD}_{\frac{n}{2}}$ with (4.40) at $\kappa = \mu = n$, i.e. $c = 3n^2/((n+2)(n+1))$.

From these examples we conjecture for the spin content of $\mathrm{Orb}\left(\widehat{so(k+1)}_n/\widehat{so(k)}_n\right)$:

$$\mathrm{Orb}\left(\frac{\widehat{so(k+1)}_n}{\widehat{so(k)}_n}\right) \cong \mathcal{W}(2,4,\ldots,k(k+3)). \tag{4.101}$$

This spin content of the coset agrees with the expectation from the study of the Kac-determinant (compare the paragraph preceding table 5).

Unifying $\mathcal{W}$-algebras of special $\mathcal{WC}_n$ minimal models: $\mathcal{WD}_{-k}$

In section 4.2.1 we proposed unifying $\mathcal{W}$-algebras for the minimal models of the Casimir algebras $\mathcal{WC}_n$ with central charge $c_{\mathcal{C}_n}(n+k+1, 2n+2k+1)$. From the study of the Kac-determinant one expects the following spin content for those unifying algebras

$$\mathcal{WD}_{-k} \cong \mathcal{W}(2,4,\ldots,2k(k+2)). \tag{4.102}$$

Recall that for the central charge equal to $c_{\mathcal{C}_n}(n+k+1, 2n+2k+1)$ a truncation $\mathrm{Orb}(\mathcal{WD}_l) \lhd\!\rhd \mathcal{WC}_n$ takes place if one formally replaces k by $-l$. Originally, the term '$\mathcal{WD}_{-k}$' has been introduced in [Hf3] for the special $\mathcal{W}(2,4,6) \cong \mathcal{WD}_{-1}$ (discussed in section 4.1.3) because it is possible to describe the structure constants of this algebra by the general formulae for the orbifold of $\mathcal{WD}_k$ if one inserts $k=-1$. In section 4.1.3 a coset realization has been given wherefrom the highest weights of the irreducible representations can be derived in a straightforward manner [Bl5,Bl6]. With this data one can check that this $\mathcal{W}(2,4,6) \cong \widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$ unifies the minimal models $c_{\mathcal{C}_n}(n+2, 2n+3)$ of the $\mathcal{WC}_n$ Casimir algebras.

Our aim is to give an explicit coset realization for the general case $\mathcal{WD}_{-k} \cong \mathcal{W}(2,4,\ldots,2k(k+2))$. Therefore, one has to make sense of $\mathcal{D}_{-n} = so(-2n)$. Indeed negative-dimensional groups $SU(-n), SO(-2n), Sp(-2n)$ have been introduced in the context of representation theory of the classical groups $SU(n)$, $SO(2n)$, $Sp(2n)$ (see e.g. [Du]). There exist striking relations in representation theory which can be explained in a more natural way by 'analytic continuations' in n. For example the dimension formula of an irreducible representation of $SO(2n)$ equals up to a sign the dimension formula of $Sp(2n)$ for the transposed Young tableau upon the substitution $n \to -n$. Furthermore, the pth order Casimir of $SO(2n)$ in the totally antisymmetric rank-r tensor representation equals up to a sign the pth order Casimir of $Sp(2n)$ in the totally symmetric rank-r tensor representation upon the substitution $n \to -n$ [Du]. These relations arise naturally if one *defines* the negative-dimensional groups via $SO(-2n) \cong \overline{Sp(2n)}$ and $Sp(-2n) \cong \overline{SO(2n)}$. The overbar means the interchange of symmetrization and antisymmetrization.

We conclude that $\mathcal{D}_{-n} = so(-2n)$ is related to $\mathcal{C}_n = sp(2n)$. From the Sugawara central charge one can establish the identification $\widehat{so(-2n)}_\kappa \leftrightarrow \widehat{sp(2n)}_{-\frac{\kappa}{2}}$. This leads us to the study of the general coset $\widehat{sp(2n)}_\kappa \oplus \widehat{sp(2n)}_\mu/\widehat{sp(2n)}_{\kappa+\mu}$. The central charges for the minimal models read

$$c_{\kappa,\mu}(n) = \frac{n\mu\kappa(2n+1)(2n+2+\mu+\kappa)}{(\mu+\kappa+n+1)(\mu+n+1)(\kappa+n+1)}. \tag{4.103}$$

Specification to $\mu = -\frac{1}{2}$ yields the formula

$$c(n,\kappa) = -\frac{\kappa n(2\kappa+4n+3)}{(\kappa+n+1)(2\kappa+2n+1)}. \tag{4.104}$$

This coincides with the $c_{\mathcal{C}_\kappa}(\kappa+n+1, 2\kappa+2n+1)$ minimal models of the $\mathcal{WC}_\kappa$ Casimir algebras. Once again, looking at formula (4.75) (this equation describes the truncation of $\mathcal{WC}_n$ to $\mathcal{WD}_m$) and substituting m by $-m$ we recover eq. (4.104). These facts strongly indicate that the $\mathcal{WD}_{-n}$ algebras can be realized as the diagonal $\widehat{sp(2n)}$ coset. Further confirmation for this relationship comes from the comparison of the coupling constant C_{44}^4 of $\mathcal{WD}_{-n}$ (obtained by analytic continuation of the coupling constant of $\mathcal{WD}_n$) with C_{44}^4 of the diagonal coset (see below).

The next step to be carried out is the determination of the vacuum character of the diagonal $\widehat{sp(2n)}$ coset. As before, it is useful to study the coset $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$ first, since the former one can be viewed as a deformation of it. To determine the vacuum character of $\widehat{sp(2n)}_{-\frac{1}{2}}$ one uses the realization of $\widehat{sp(2n)}_{-\frac{1}{2}}$ by n commuting (β,γ) systems. This is due to the fact that the vacuum character of $\widehat{sp(2n)}_{-\frac{1}{2}}$ is freely generated in terms of the (β,γ) systems which is not true for the canonical choice in terms of the currents themselves. We start with n bosonic ghost-antighost fields (β_i,γ_i) satisfying the OPEs

$$\beta_i(z)\ \gamma_i(w) = (z-w)^{-1} + reg., \tag{4.105}$$

and define the currents $H_i = \beta_i\,\gamma_i$ satisfying the OPEs

$$H_i(z)\ H_j(w) = -\delta_{ij}(z-w)^{-2} + reg.. \tag{4.106}$$

The ghost fields β_i, γ_i have charge ± 1 with respect to the currents H_i:

$$H_i(z)\ \gamma_j(w) = \frac{\delta_{ij}\gamma_j(w)}{z-w} + reg., \qquad H_i(z)\ \beta_j(w) = -\frac{\delta_{ij}\beta_j(w)}{z-w} + reg.. \tag{4.107}$$

The realization of $\widehat{sp(2n)}_{-\frac{1}{2}}$ in the Cartan-Weyl basis $\{H_i(z), E_\alpha(z)\}$ is given by [11])

root vector α	*current* $E_\alpha(z)$
$2e_i$	$\gamma_i\gamma_i$
$-2e_i$	$-\beta_i\beta_i$
e_i+e_j	$\gamma_i\gamma_j$
$-e_i-e_j$	$-\beta_i\beta_j$
e_i-e_j	$\gamma_i\beta_j$
$-e_i+e_j$	$\beta_i\gamma_j$

[11]) If $\{e_1,\ldots,e_n\}$ are orthonormal unit vectors in the standard euclidean $\mathbb{R}^n$, the root system of $\mathcal{C}_n = sp(2n)$ is realized by the vectors $\pm 2e_i$ and $\pm(e_i \pm e_j)$.

The branching functions of the coset $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$ are determined by the decomposition of the highest weight modules $\chi_{\vec{\Lambda}}(q,\vec{\theta})$ of $\widehat{sp(2n)}_{-\frac{1}{2}}$ into highest weight modules $\chi_{\vec{\lambda}}(\vec{\theta})$ of $sp(2n)$:

$$\chi_{\vec{\Lambda}}(q,\vec{\theta}) = \sum_{\vec{\lambda}} b_{\vec{\lambda}}^{\vec{\Lambda}}(q)\ \chi_{\vec{\lambda}}(\vec{\theta}). \tag{4.108}$$

For the vacuum character one has the identity ($\kappa = -\frac{1}{2}$) (see section 2.1):

$$b_0^{\kappa\Lambda_0}(q) = \sum_{w\in W} \epsilon(w) c^{\kappa\Lambda_0}_{w\star\rho+\kappa\Lambda_0}(q) q^{\frac{1}{2\kappa}|w\star\rho|^2} \tag{4.109}$$

with $\rho = \sum_{i=1}^n (n-i+1)e_i$. We have to determine the string functions $c^{\kappa\Lambda_0}_{w\star\rho+\kappa\Lambda_0}(q)$. One obtains from (4.13a) with $z^2 = e^{2\pi i\theta}q^{\frac{1}{2}}$ the following identity

$$\frac{1}{\prod_{n\geq 0}(1-e^{2\pi i\theta}q^{n+\frac{1}{2}})(1-e^{-2\pi i\theta}q^{n+\frac{1}{2}})} = \sum_{m\in\mathbb{Z}} \frac{q^{\frac{m}{2}}\phi_m(q)e^{2\pi i\,\theta m}}{\prod_{n\geq 1}(1-q^n)^2}. \tag{4.110}$$

Using the realization of $\widehat{sp(2n)}_{-\frac{1}{2}}$ in terms of the (β,γ) systems one obtains for the vacuum character of $\widehat{sp(2n)}_{-\frac{1}{2}}$:

$$\frac{q^{\frac{n}{24}}}{\prod_{i=1}^{n}\prod_{k\geq 0}(1-e^{2\pi i\,\theta_i}q^{k+\frac{1}{2}})(1-e^{-2\pi i\,\theta_i}q^{k+\frac{1}{2}})} = \sum_{\vec{\lambda}\in\mathbb{Z}^n} \frac{q^{\frac{3n}{24}}}{\eta^{2n}(q)} q^{\frac{\lambda_1+\ldots+\lambda_n}{2}} \phi_{\lambda_1}(q)\ldots\phi_{\lambda_n}(q)e^{2\pi i\,\vec{\theta}\cdot\vec{\lambda}}. \tag{4.111}$$

Thus the string functions of $\widehat{sp(2n)}_{-\frac{1}{2}}$ are given by

$$c^{-\frac{1}{2}\Lambda_0}_{\vec{\lambda}-\frac{1}{2}\Lambda_0}(q)\ q^{-|\vec{\lambda}|^2} = \frac{q^{\frac{3n}{24}}}{\eta^{2n}(q)} q^{\frac{\lambda_1+\ldots+\lambda_n}{2}} \phi_{\lambda_1}(q)\ldots\phi_{\lambda_n}(q). \tag{4.112}$$

Inserting this into eq. (4.109) yields the explicit form of the vacuum character of the $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$ coset. In table 7 below we present the first few examples and give a conjecture for the spin content in the general case. The spin content of the coset is indeed compatible with the truncations of the $\mathcal{WC}_m$ Casimir algebras for $c_{\mathcal{C}_m}(m+n+1, 2m+2n+1)$.

n	*vacuum character*
1	$b_0^0(q) = \eta(q)^{-2}\left(\phi_0(q) - q\,\phi_2(q)\right)$ $b_0^0(q) - \chi_{2,4,6}(q) = -q^{11} - 2q^{12} - 3q^{13} + \mathcal{O}(q^{14})$
2	$b_0^0(q) = \eta(q)^{-4}\left(\phi_0^2 - \phi_0\phi_2 - q\phi_1^2 + 2q^2\phi_3\phi_1 - q^2\phi_4\phi_0 + q^3\phi_4\phi_2 - q^3\phi_3^2\right)$ $b_0^0(q) - \chi_{2,4,6,\ldots,16}(q) = -q^{21} - 2q^{22} + \mathcal{O}(q^{23})$
3	$b_0^0(q) - \chi_{2,4,6,\ldots,30}(q) = -q^{35} - 2q^{36} + \mathcal{O}(q^{38})$
4	$b_0^0(q) - \chi_{2,4,6,\ldots,48}(q) = -q^{53} - 2q^{54} + \mathcal{O}(q^{55})$
n	$b_0^0(q) - \chi_{2,\ldots,2n(n+2)}(q) = -q^{2n(n+2)+5} - 2q^{2n(n+2)+6} + \mathcal{O}(q^{2n(n+2)+7})$

Table 7: Vacuum character for $\widehat{sp(2n)}_{-\frac{1}{2}}/sp(2n)$

Minimal models of $\mathcal{WD}_{-m}$

The minimal models of $\mathcal{WD}_{-1} \cong \mathcal{W}(2,4,6)$ could be deduced in [Hf3] with the help of some known examples [Eh3] from those of the $\mathcal{WD}_m$-algebras being in agreement with the results from the direct calculation of the conformal dimensions using the coset realization $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$ [Bl5].

We will now try to continue the minimal models of $\mathcal{WD}_m$ to negative values of m beyond $m=-1$. The central charges of the minimal models of the $\mathcal{WD}_m$-algebras are given by

$$c_{\mathcal{D}_m}(p,q) = m\left(1 - 2(m-1)(2m-1)\frac{(p-q)^2}{pq}\right) \tag{4.113}$$

and following [Hf3] we have to make certain assumptions for the allowed values in the case of $\mathcal{WD}_{-m}$: $q = p+1; p = 2n+2x+1$ odd. The value $n=0$ corresponds to the trivial model at $c=0$ implying $x=m$. This leads to the following ansatz for the central charge of the minimal models of $\mathcal{WD}_{-m}$

$$c_{\mathcal{D}_{-m}}(n) = -\frac{mn(3+4m+2n)}{(1+m+n)(1+2m+2n)}. \tag{4.114}$$

This central charge equals the one of the $\widehat{sp(2n)}$ cosets eq. (4.104). The dimensions of the highest weights can be obtained as follows: we start from the dimensions for the $\mathcal{WD}_m$-algebra [LF] [12])

$$h(p, q=p+1) = \frac{\left(\sum_{r=1}^m (x_r+1)\omega_r\right)^2 - \left(\sum_{r=1}^m \omega_r\right)^2}{2p(p+1)}, \tag{4.115}$$

where $x_r = l_r(p+1) - l'_r p - 1$ with positive integers l_r and l'_r. The fundamental weights obey

$$\begin{array}{ll} \omega_r \cdot \omega_s = r & (1 \le r \le s \le m-2) \\ \omega_r \cdot \omega_{m-1} = \omega_r \cdot \omega_m = \frac{r}{2} & (1 \le r \le m-2) \\ \omega_{m-1}\cdot\omega_m = \frac{m-2}{4} & \omega_{m-1}^2 = \omega_m^2 = \frac{m}{4}. \end{array} \tag{4.116}$$

[12]) The formula in [LF] contains a misprint.

The experience with the algebra $\mathcal{W}(2,4,6)$ tells us that we have to extend the summation in (4.115) to infinity and take $l_r = l'_r = 1$ (i.e. $x_r = 0$) for all $r > n$. Therefore we obtain by inserting the fundamental weights eq. (4.116) into eq. (4.115) and replacing $m \to -m$, $p \to 2n + 2m + 1$

$$h_{\vec{l},\vec{l}'}(n) = \frac{\sum_{r=1}^{n} r(x_r - 2m - 1 - r)x_r + 2\sum_{r<s} r x_r x_s}{4(n+m+1)(2n+2m+1)}. \tag{4.117}$$

On l_r, l'_r we have to impose the additional constraints

$$\sum_{r=1}^{n}(l_r + l'_r - 2) \leq m + 1\,, \qquad 1 \leq l_r, l'_r \leq m + 1. \tag{4.118}$$

Using eq. (D.2) in appendix D for the minimal models of the Casimir algebras we have checked in a few cases that the minimal models of $\mathcal{WD}_{-n}$ coincide with those minimal models of $\mathcal{WC}_m$ at the central charge $c_{\mathcal{C}_m}(m+n+1, 2m+2n+1)$.

From the realization of $\mathcal{WD}_{-m}$ in terms of the diagonal $\widehat{sp(2m)}$ coset it should be possible to check these formulae in a more direct fashion using representation theory of Kac-Moody algebras.

Science shall never find that formula by which its necessary character could be proved.
Actually science itself might cease if we were to find the clue to the secret.
Baron Lorand Von Eötvös

5 Summary and Outlook

In the first part of this thesis we proved a generalized Verlinde formula for fermionic RCFTs by showing that the fusion algebras deduced from this formula coincide with those obtained from the corresponding bosonic projections applying the ordinary Verlinde formula and 'simple current' arguments. Using this generalized Verlinde formula we were able to calculate the fusion algebras of several fermionic RCFTs. We found the important fact that in the fermionic case the S-matrix is in general neither unitary nor symmetric but obeys the equation $S^\dagger HS = H$ with $H = MD^{-1}$. Here, M is a diagonal matrix encoding the multiplicities of the HWRs of the theory, whereas D is a diagonal matrix defined in terms of the orbit lengths under the action of the 'simple current' in the fusion algebra of the bosonic projection (cf. section 3.1). In a concrete example we have shown that it is not possible to avoid non-trivial D-matrices by suitable extensions of the fusion algebras. There is strong evidence that this holds in general. As a consequence we made the observation that in the fusion product of two conjugate Ramond fields of the fermionic RCFT the identity may occur with coefficient one or *two*. Our examples show that one has to weaken the axioms of fusion algebras for fermionic RCFTs allowing more general fusion charge conjugation matrices $\mathcal{N}_{ij}^{1} = (MD)_{i,j}$ being different from the 'usual' charge conjugation $C = S^2$. It will be interesting to know if these results can be confirmed with the direct methods for the computation of the fusion algebra [Na2]. In addition, we considered the representation theory of the horizontal subalgebra on the highest weights in the Ramond sector for fermionic $\mathcal{W}$-algebras. For fermionic $\mathcal{W}(2,\delta)$-algebras the dimensions of the irreducible representations are directly encoded in the fusion algebra of the corresponding rational model. However, this result is not valid in general as one can demonstrate in the case of fermionic $\mathcal{W}$-algebras with more than two generators. The concrete examples treated so far include fermionic $\mathcal{W}(2,\delta)$-algebras, the minimal models of the $N = 1$ super Virasoro algebra and $N = 1$ $\mathcal{SW}$-algebras with two generators.
Using 'simple current' arguments it is possible to define in a very natural way the fusion product of twisted fields in the case of bosonic $\mathcal{W}(2,\delta)$-algebras admitting an outer $\mathbb{Z}_2$-automorphism. Unfortunately, it is not apparent how to write down a generalized Verlinde formula in these cases but this point should be pursued in the future. Another point of interest is a study of fusion algebras of $N = 2$ supersymmetric theories which we did not consider here.
It should also be of interest to use the generalized Verlinde formula to set up a classification program for fusion algebras of fermionic theories in complete analogy to the treatment of the bosonic case [CPR,Eh4]. Especially it would be very interesting to know whether fermionic RCFTs exist which cannot be obtained by a $\mathbb{Z}_2$ 'simple current' extension of its bosonic projection.

In the second part of this thesis we focussed on quantum $\mathcal{W}$-algebras belonging to the new

class of deformable $\mathcal{W}$-algebras with infinitely non-freely generated classical limits. We collected evidence that these algebras are *finitely* non-freely generated on the quantum level. Quantum $\mathcal{W}$-algebras of this type occur naturally in cosets and orbifolds of $\mathcal{W}$-algebras.
By explicit calculation of the operator product expansions we provided evidence that the coset algebras $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ and $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$ are finitely generated on the quantum level. This is supported by the study of the vacuum character. The mechanism which prohibits the additional generators in the quantum case is a cancellation due to normal ordering of the classical relations. Normal ordering of the classical relations between the classical generators involves quantum correction terms which allow to express the 'higher' generators in terms of normal ordered products of a finite number of 'lower' generators and their derivatives. From comparison of structure constants we obtained that $\widehat{sl(2,\mathbb{R})}/\widehat{U(1)}$ as well as the isomorphic coset $SVIR(N=2)/\widehat{U(1)}$ provide a realization of the special non-freely generated $\mathcal{W}(2,3,4,5)$ found earlier by direct construction [Hf2]. In addition we put some effort on the analysis of the general coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_\mu / \widehat{sl(2,\mathbb{R})}_{\kappa+\mu}$. Apart from a discussion of the spin content for several different values of the central charge where the coset suffers a truncation we presented the self-coupling constant of the unique primary spin 4 field. An important byproduct is the observation that the fourth special solution of $\mathcal{W}(2,4,6)$ [KW] is realized as the diagonal coset $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}} / \widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$.
Furthermore, we investigated orbifolds of quantum $\mathcal{W}$-algebras in a purely algebraic manner. They behave similarly to the coset algebras discussed above, i.e. they belong to the class of finitely non-freely generated $\mathcal{W}$-algebras. They occur also naturally in the context of unifying $\mathcal{W}$-algebras. We stress the important observation that in contrast to orbifolds of classical Casimir $\mathcal{W}$-algebras the orbifold of a quantum Casimir $\mathcal{W}$-algebra does not lead to a Casimir subalgebra.
In examples we discussed some properties of the vacuum preserving algebra (VPA) and the Bowcock-Watts (BW) classical limit of non-freely generated quantum $\mathcal{W}$-algebras [BW]. In contrast to algebras in the Drinfeld-Sokolov class the VPA does not yield a finite-dimensional Lie-algebra. Analogously, the classical limits are not finitely generated any more.

In the last part we turned to a detailed discussion of unifying $\mathcal{W}$-algebras. Firstly, we have shown the existence of many identifications between $\mathcal{W}$-algebras using a particular parametrization of the structure constants. The identifications are closely related to truncations of $\mathcal{W}$-algebras. We predicted various truncations of Casimir $\mathcal{W}$-algebras which could be confirmed and generalized by inspection of the Kac determinant.

For some of the unifying $\mathcal{W}$-algebras predicted by these truncations we were able to present coset realizations thereby showing that unifying $\mathcal{W}$-algebras belong in general to the class of non-freely generated $\mathcal{W}$-algebras. A part of these coset realizations can also be inferred from level-rank duality of dual coset pairs (see e.g. [BS,BG,Al]). For the Casimir algebras $\mathcal{WA}_{n-1}$ we conjectured coset realizations for all unifying $\mathcal{W}$-algebras predicted by the studies of the Kac determinant. Further important examples are the $\mathcal{WD}_{-n}$ algebras which arise from the symplectic cosets $\widehat{sp(2n)}_\kappa \oplus \widehat{sp(2n)}_{-\frac{1}{2}} / \widehat{sp(2n)}_{\kappa-\frac{1}{2}}$. They are unifying algebras for $\mathcal{WC}$ minimal models. In passing we mention that in [Hf4] analogously to $\mathcal{WD}_{-n}$ the existence of the algebras $\mathcal{WA}_{-n} \cong \mathcal{W}(2,\ldots,n^2-1)$ $(n>2)$ with the coset realization

$\widehat{sl(n)}_\kappa \oplus \widehat{sl(n)}_{-1}/\widehat{sl(n)}_{\kappa-1}$ has been conjectured. The coset realizations of unifying algebras known to us are collected in the table 8. We stress that the existence of all unifying $\mathcal{W}$-algebras predicted above is not yet firmly established.

Casimir algebra	*central charge* c	*coset realization of unifying algebra*	*dimensions of simple fields*	*dimension of first null field*
$\mathcal{WA}_{n-1}$	$c_{\mathcal{A}_{n-1}}(n+k,n+r)$	$\frac{\mathcal{W}^{sl(r)}_{r-k,1^k}}{\widehat{sl(k)}\oplus\widehat{U(1)}}$	$2,3,\ldots,kr+r+k$	$kr+r+k+3$
$\mathcal{WB}_n$	$c_{\mathcal{B}_n}(2n+k-1,2n+1)$ $c_{\mathcal{B}_n}(2n,2n+k)$	$(\mathrm{Orb})\left(\frac{\widehat{so(k)}_\kappa\oplus\widehat{so(k)}_1}{\widehat{so(k)}_{\kappa+1}}\right)$	$2,4,\ldots,2k$	$2k+4$
$\mathcal{WC}_n$	$c_{\mathcal{C}_n}(n+k+1,2n+2k+1)$	$\frac{\widehat{sp(2k)}_n\oplus\widehat{sp(2k)}_{-\frac{1}{2}}}{\widehat{sp(2k)}_{n-\frac{1}{2}}}$	$2,4,\ldots,2k^2+4k$	$2k^2+4k+5$
$\mathrm{Orb}\,(\mathcal{WD}_n)$	$c_{\mathcal{D}_n}(2n+k-2,2n+k-1)$	$\mathrm{Orb}\left(\frac{\widehat{so(k+1)}_{2n}}{\widehat{so(k)}_{2n}}\right)$	$2,4,\ldots,k^2+3k$	k^2+3k+4

Table 8: Coset realizations of unifying $\mathcal{W}$-algebras

The coset realization of unifying $\mathcal{W}$-algebras gives a so far unknown coset realization for some of the minimal models of the non-simply laced Casimir $\mathcal{W}$-algebras $\mathcal{WB}_n$ and $\mathcal{WC}_n$. A further interesting observation concerning minimal models of diagonal cosets is the following conjecture: At least one of the levels κ, μ has to be an integer if $\hat{g}_\kappa \oplus \hat{g}_\mu/\hat{g}_{\kappa+\mu}$ has a minimal model. Note that the only counterexample to this conjecture we know of (presented in [ACT]) is incorrect [Bl5].

There are several interesting open questions to answer in the future. Firstly, we collected evidence for the conjecture that all minimal models of unifying $\mathcal{W}$-algebras arise from identifications with minimal models of Casimir $\mathcal{W}$-algebras. If this turns out to be true unifying $\mathcal{W}$-algebras do not give rise to new minimal models and may therefore be irrelevant for the classification of rational conformal field theories. Nevertheless these unifying structures provide us with new insights into conformal field theory. For example, this gives a unified approach to the conformally invariant second order phase transition of $\mathbb{Z}_n$ spin quantum chains. They all share the same $\mathcal{W}(2,3,4,5)$ symmetry algebra and the growth of the number of states with energy is always bounded by the number of partitions into two colours. Secondly, one could address the question whether all unifying $\mathcal{W}$-algebras can be realized as cosets of quantum DS reductions. Finally, the representation theory of unifying $\mathcal{W}$-algebras and quantum DS reductions related to nonprincipal $sl(2)$ embeddings has not been worked out so far. However, in the case of $\widehat{sl(2,\mathbb{R})}_\kappa/\widehat{U(1)}$ and $\widehat{sl(2,\mathbb{R})}_\kappa \oplus \widehat{sl(2,\mathbb{R})}_{-\frac{1}{2}}/\widehat{sl(2,\mathbb{R})}_{\kappa-\frac{1}{2}}$ [Bl5,Bl6] one can see that all minimal models arise from identifications with Casimir algebras. It is interesting to know whether all minimal models of all unifying $\mathcal{W}$-algebras are minimal models of the corresponding Casimir algebras. If this should be true, one could hope to reconstruct the representation theory of $\mathcal{W}$-algebras obtained by DS reduction for a nonprincipal embedding using the fact that the coset by the Kac-Moody subalgebra that survives the reduction is a unifying $\mathcal{W}$-algebra for Casimir $\mathcal{W}$-algebras.

In the sciences, we are now uniquely privileged to sit side by side with the giants on whose shoulders we stand.
Gerald Holton

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Appendix A: Fusion algebras of $\mathcal{W}(2,4)$ at $c=-11,-\frac{11}{14}$

In this appendix we give the fusion algebras of the bosonic projection of the $N=1$ super Virasoro algebra for the central charges $c=-11$ and $c=-\frac{11}{14}$. As shown in section 4.1.4 the orbifold yields an algebra of type $\mathcal{W}(2,4,6)$ for generic values of the central charge. However, for these two values (among others) the orbifold truncates to $\mathcal{W}(2,4)$. This enables us to calculate the S-matrix in these cases and herewith the fusion algebra.

For $\mathcal{W}(2,4)$ at $c=-11$ the effective central charge $\tilde{c}$ is equal to 1, so that this model belongs to the parabolic $\mathcal{W}(2,\delta)$-algebras which have been studied by Flohr [Fl1]. Using the explicit form of the S-matrix one is able to calculate the fusion algebra of this model [Fl2,Fl1]. There exist 10 HWRs of $\mathcal{W}(2,4)$ at $c=-11$: $h\in\{0,\frac{3}{2},-\frac{1}{3},\frac{1}{6},-\frac{1}{2},\tilde{0},-\frac{1}{8},-\frac{3}{8},-\frac{11}{24},\frac{13}{24}\}$. The fusion algebra is given by:

$$
\begin{aligned}
&[\Phi_{\frac{3}{2}}][\Phi_{\frac{3}{2}}]=[\Phi_0] \qquad [\Phi_{\frac{3}{2}}][\Phi_{-\frac{1}{3}}]=[\Phi_{\frac{1}{6}}] \qquad [\Phi_{\frac{3}{2}}][\Phi_{-\frac{1}{2}}]=[\Phi_{\tilde{0}}]\\
&[\Phi_{\frac{3}{2}}][\Phi_{-\frac{1}{8}}]=[\Phi_{-\frac{1}{8}}] \qquad [\Phi_{\frac{3}{2}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{3}{8}}] \qquad [\Phi_{\frac{3}{2}}][\Phi_{-\frac{11}{24}}]=[\Phi_{\frac{13}{24}}]\\[1ex]
&[\Phi_{-\frac{1}{3}}][\Phi_{-\frac{1}{3}}]=[\Phi_0]+[\Phi_{-\frac{1}{3}}]+[\Phi_{-\frac{1}{2}}] \qquad [\Phi_{-\frac{1}{3}}][\Phi_{\frac{1}{6}}]=[\Phi_{\frac{3}{2}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{\tilde{0}}]\\
&[\Phi_{-\frac{1}{3}}][\Phi_{-\frac{1}{2}}]=[\Phi_{-\frac{1}{3}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{1}{3}}][\Phi_{\tilde{0}}]=[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}]\\
&[\Phi_{-\frac{1}{3}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}] \qquad [\Phi_{-\frac{1}{3}}][\Phi_{-\frac{1}{8}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]\\
&[\Phi_{-\frac{1}{3}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{\frac{13}{24}}] \qquad [\Phi_{-\frac{1}{3}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]\\[1ex]
&[\Phi_{\frac{1}{6}}][\Phi_{\frac{1}{6}}]=[\Phi_0]+[\Phi_{-\frac{1}{3}}]+[\Phi_{-\frac{1}{2}}] \qquad [\Phi_{\frac{1}{6}}][\Phi_{-\frac{1}{2}}]=[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}]\\
&[\Phi_{\frac{1}{6}}][\Phi_{\tilde{0}}]=[\Phi_{-\frac{1}{3}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{\frac{1}{6}}][\Phi_{-\frac{1}{8}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]\\
&[\Phi_{\frac{1}{6}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}] \qquad [\Phi_{\frac{1}{6}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]\\
&[\Phi_{\frac{1}{6}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{\frac{13}{24}}]\\[1ex]
&[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{2}}]=[\Phi_0]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{1}{2}}][\Phi_{-\frac{1}{8}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}]\\
&[\Phi_{-\frac{1}{2}}][\Phi_{\tilde{0}}]=[\Phi_{\frac{3}{2}}]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{1}{2}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]\\
&[\Phi_{-\frac{1}{2}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{1}{8}}]+2[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}] \qquad [\Phi_{-\frac{1}{2}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{\frac{13}{24}}]\\[1ex]
&[\Phi_{\tilde{0}}][\Phi_{\tilde{0}}]=[\Phi_0]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{\tilde{0}}][\Phi_{-\frac{1}{8}}]=[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}]\\
&[\Phi_{\tilde{0}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{1}{8}}]+2[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]+[\Phi_{\frac{13}{24}}] \qquad [\Phi_{\tilde{0}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{\frac{13}{24}}]\\
&[\Phi_{\tilde{0}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{1}{8}}]+[\Phi_{-\frac{3}{8}}]+[\Phi_{-\frac{11}{24}}]\\[1ex]
&[\Phi_{-\frac{1}{8}}][\Phi_{-\frac{1}{8}}]=[\Phi_0]+[\Phi_{\frac{3}{2}}]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}] \qquad [\Phi_{-\frac{1}{8}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}]\\
&[\Phi_{-\frac{1}{8}}][\Phi_{-\frac{3}{8}}]=[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{1}{8}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}]\\[1ex]
&[\Phi_{-\frac{3}{8}}][\Phi_{-\frac{3}{8}}]=[\Phi_0]+[\Phi_{\frac{3}{2}}]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+2[\Phi_{-\frac{1}{2}}]+2[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{11}{24}}][\Phi_{\frac{13}{24}}]=[\Phi_{\frac{3}{2}}]+[\Phi_{-\frac{1}{3}}]+[\Phi_{\tilde{0}}]\\
&[\Phi_{-\frac{3}{8}}][\Phi_{\frac{13}{24}}]=[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{-\frac{11}{24}}][\Phi_{-\frac{11}{24}}]=[\Phi_0]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]\\
&[\Phi_{-\frac{3}{8}}][\Phi_{-\frac{11}{24}}]=[\Phi_{-\frac{1}{3}}]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]+[\Phi_{\tilde{0}}] \qquad [\Phi_{\frac{13}{24}}][\Phi_{\frac{13}{24}}]=[\Phi_0]+[\Phi_{\frac{1}{6}}]+[\Phi_{-\frac{1}{2}}]
\end{aligned}
\tag{A.1}
$$

Note that $[\Phi_{\frac{3}{2}}]$ is a 'simple current'. Exploiting this fact one obtains the fusion algebra of the supersymmetric model (see section 3.4).

We turn now to the second case $c = -\frac{11}{14}$. This value of the central charge is contained in the minimal series of the Virasoro algebra and can be parametrized as $c = c_{Vir}(7, 12)$. Our previous calculations in [Eh1] showed that $\mathcal{W}(2,4)$ has a rational model at this value of c and that the $\mathcal{W}$-characters diagonalize the modular invariant partition function (E_6, A_6) [CIZ] given by:

$$Z = \sum_{s=1}^{3} \left(\mid \chi_{1,s} + \chi_{7,s} \mid^2 + \mid \chi_{4,s} + \chi_{8,s} \mid^2 + \mid \chi_{5,s} + \chi_{11,s} \mid^2 \right).$$

Hence the $\mathcal{W}(2,4)$ characters read

$$\chi_s^{\mathcal{W},1} = \chi_{1,s} + \chi_{7,s}, \quad \chi_s^{\mathcal{W},2} = \chi_{4,s} + \chi_{8,s}, \quad \chi_s^{\mathcal{W},3} = \chi_{5,s} + \chi_{11,s}, \quad s = 1, 2, 3$$

and the $\mathcal{W}(2,4)$-minimal model has the HWRs $h \in \{0, \frac{3}{2}, \frac{2}{7}, \frac{11}{14}, -\frac{1}{14}, \frac{3}{7}, \frac{11}{16}, -\frac{3}{112}, \frac{13}{112}\}$. Using the S-matrix for Virasoro minimal models and the ordinary Verlinde formula one arrives – after performing the appropriate change of basis – at the following fusion algebra for the $\mathcal{W}(2,4)$-minimal model $c = -\frac{11}{14}$:

$$\begin{array}{lll}
[\Phi_{\frac{3}{2}}][\Phi_{\frac{3}{2}}] = [\Phi_0] & [\Phi_{\frac{3}{2}}][\Phi_{\frac{2}{7}}] = [\Phi_{\frac{11}{14}}] & [\Phi_{\frac{3}{2}}][\Phi_{-\frac{1}{14}}] = [\Phi_{\frac{3}{7}}] \\
[\Phi_{\frac{3}{2}}][\Phi_{\frac{11}{16}}] = [\Phi_{\frac{11}{16}}] & [\Phi_{\frac{3}{2}}][\Phi_{-\frac{3}{112}}] = [\Phi_{-\frac{3}{112}}] & [\Phi_{\frac{3}{2}}][\Phi_{\frac{13}{112}}] = [\Phi_{\frac{13}{112}}]
\end{array}$$

$$\begin{array}{ll}
[\Phi_{\frac{2}{7}}][\Phi_{\frac{2}{7}}] = [\Phi_0] + [\Phi_{\frac{3}{7}}] & [\Phi_{\frac{2}{7}}][\Phi_{\frac{11}{14}}] = [\Phi_{\frac{3}{2}}] + [\Phi_{-\frac{1}{14}}] \\
[\Phi_{\frac{2}{7}}][\Phi_{-\frac{1}{14}}] = [\Phi_{\frac{11}{14}}] + [\Phi_{-\frac{1}{14}}] & [\Phi_{\frac{2}{7}}][\Phi_{\frac{3}{7}}] = [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{3}{7}}] \\
[\Phi_{\frac{2}{7}}][\Phi_{\frac{11}{16}}] = [\Phi_{-\frac{3}{112}}] & [\Phi_{\frac{2}{7}}][\Phi_{-\frac{3}{112}}] = [\Phi_{\frac{11}{16}}] + [\Phi_{\frac{13}{112}}] \\
[\Phi_{\frac{2}{7}}][\Phi_{\frac{13}{112}}] = [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}] &
\end{array}$$

$$\begin{array}{ll}
[\Phi_{\frac{11}{14}}][\Phi_{\frac{11}{14}}] = [\Phi_0] + [\Phi_{\frac{3}{7}}] & [\Phi_{\frac{11}{14}}][\Phi_{-\frac{1}{14}}] = [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{3}{7}}] \\
[\Phi_{\frac{11}{14}}][\Phi_{\frac{3}{7}}] = [\Phi_{\frac{11}{14}}] + [\Phi_{-\frac{1}{14}}] & [\Phi_{\frac{11}{14}}][\Phi_{\frac{11}{16}}] = [\Phi_{-\frac{3}{112}}] \\
[\Phi_{\frac{11}{14}}][\Phi_{-\frac{3}{112}}] = [\Phi_{\frac{11}{16}}] + [\Phi_{\frac{13}{112}}] & [\Phi_{\frac{11}{14}}][\Phi_{\frac{13}{112}}] = [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}]
\end{array}$$

$$\begin{array}{ll}
[\Phi_{-\frac{1}{14}}][\Phi_{-\frac{1}{14}}] = [\Phi_0] + [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{3}{7}}] & [\Phi_{-\frac{1}{14}}][\Phi_{\frac{3}{7}}] = [\Phi_{\frac{3}{2}}] + [\Phi_{\frac{11}{14}}] + [\Phi_{-\frac{1}{14}}] \\
[\Phi_{-\frac{1}{14}}][\Phi_{\frac{11}{16}}] = [\Phi_{\frac{13}{112}}] & [\Phi_{-\frac{1}{14}}][\Phi_{-\frac{3}{112}}] = [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}] \\
[\Phi_{-\frac{1}{14}}][\Phi_{\frac{13}{112}}] = [\Phi_{\frac{11}{16}}] + [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}] &
\end{array}$$

$$\begin{array}{ll}
[\Phi_{\frac{3}{7}}][\Phi_{\frac{3}{7}}] = [\Phi_0] + [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{3}{7}}] & [\Phi_{\frac{3}{7}}][\Phi_{\frac{11}{16}}] = [\Phi_{\frac{13}{112}}] \\
[\Phi_{\frac{3}{7}}][\Phi_{-\frac{3}{112}}] = [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}] & [\Phi_{\frac{3}{7}}][\Phi_{\frac{13}{112}}] = [\Phi_{\frac{11}{16}}] + [\Phi_{-\frac{3}{112}}] + [\Phi_{\frac{13}{112}}]
\end{array}$$

$$\begin{array}{ll}
[\Phi_{\frac{11}{16}}][\Phi_{\frac{11}{16}}] = [\Phi_0] + [\Phi_{\frac{3}{2}}] & [\Phi_{\frac{11}{16}}][\Phi_{-\frac{3}{112}}] = [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{11}{14}}] \\
[\Phi_{\frac{11}{16}}][\Phi_{\frac{13}{112}}] = [\Phi_{-\frac{1}{14}}] + [\Phi_{\frac{3}{7}}] &
\end{array}$$

$$\begin{array}{ll}
[\Phi_{-\frac{3}{112}}][\Phi_{-\frac{3}{112}}] = [\Phi_0] + [\Phi_{\frac{3}{2}}] + [\Phi_{-\frac{1}{14}}] + [\Phi_{\frac{3}{7}}] & [\Phi_{-\frac{3}{112}}][\Phi_{\frac{13}{112}}] = [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{11}{14}}] + [\Phi_{-\frac{1}{14}}] + [\Phi_{\frac{3}{7}}] \\
[\Phi_{\frac{13}{112}}][\Phi_{\frac{13}{112}}] = [\Phi_0] + [\Phi_{\frac{3}{2}}] + [\Phi_{\frac{2}{7}}] + [\Phi_{\frac{11}{14}}] + [\Phi_{-\frac{1}{14}}] + [\Phi_{\frac{3}{7}}] &
\end{array} \tag{A.2}$$

Appendix B: The simple fields with spin 3, 4, 5 of $sl(\widehat{2,\mathbb{R}})/\widehat{U(1)}$

$$W_3 = -6k^2 J^+\partial J^- - 12kJ^oJ^+J^- + 4J^oJ^oJ^o + 6k^2\partial J^+J^- + 6k\partial J^oJ^o + k^2\partial^2 J^o$$

$$W_4 = 48k(3+2k)(17+16k)^{-1}\big(6(5-6k)k^2J^+J^+J^-J^- + 12k^2\left(1+k+k^2\right)J^+\partial^2J^- + 12k^2(11+5k)J^oJ^+\partial J^- + 12k(6+11k)J^oJ^oJ^+J^- - 3(6+11k)J^oJ^oJ^oJ^o - 12k^2(11+5k)J^o\partial J^+J^- + 12k^2\left(-8+3k-3k^2\right)\partial J^+\partial J^- + 12k^2(-5+6k)\partial J^oJ^+J^- - 6k(6+11k)\partial J^oJ^oJ^o + 3k\left(6-5k^2\right)\partial J^o\partial J^o + 12(-3+k)(-2+k)k^2\partial^2J^+J^- - 12k\left(1+k+k^2\right)\partial^2J^oJ^o - k^2\left(1+k+k^2\right)\partial^3J^o\big)$$

$$W_5 = -288k^2(3+2k)(4+3k)(107+64k)^{-1}\big(60(7-10k)k^3J^+J^+\partial J^-J^- + 20k^3\left(5+3k+k^2\right)J^+\partial^3J^- + 120(7-10k)k^2J^oJ^+J^+J^-J^- + 60k^2\left(8+19k+3k^2\right)J^oJ^+\partial^2J^- + 60k^2(64+17k)J^oJ^oJ^+\partial J^- + 120k(12+19k)J^oJ^oJ^oJ^+J^- - 24(12+19k)J^oJ^oJ^oJ^oJ^o - 60k^2(64+17k)J^oJ^o\partial J^+J^- - 480k^2\left(5+3k+k^2\right)J^o\partial J^+\partial J^- + 180(k-4)(k-3)k^2J^o\partial^2J^+J^- + 60k^3(10k-7)\partial J^+J^+J^-J^- + 60k^3\left(4k-2k^2-17\right)\partial J^+\partial^2J^- + 180k^2\left(3k^2-4\right)\partial J^oJ^+\partial J^- + 240k^2(10k-7)\partial J^oJ^oJ^+J^- - 60k(12+19k)\partial J^oJ^oJ^oJ^o + 60k^2\left(14k-11k^2-12\right)\partial J^o\partial J^+J^- + 180k\left(4-3k^2\right)\partial J^o\partial J^oJ^o + 120(k-4)(k-3)k^3\partial^2J^+\partial J^- + 60k^2\left(8-7k+4k^2\right)\partial^2J^oJ^+J^- - 30k\left(16+12k+7k^2\right)\partial^2J^oJ^oJ^o + 30k^2\left(4-3k^2\right)\partial^2J^o\partial J^o + 20(3-k)(k-4)k^3\partial^3J^+J^- - 20k^2\left(5+3k+k^2\right)\partial^3J^oJ^o - k^3\left(5+3k+k^2\right)\partial^4J^o\big)$$

$$d_{3,3} = 48(k-2)(k-1)k^3(4+3k)$$

$$d_{4,4} = 331776(k-3)(k-2)(k-1)k^6(1+2k)(3+2k)^2(4+3k)(17+16k)^{-1}$$

$$d_{5,5} = 477757440(k-4)(k-3)(k-2)(k-1)k^9(1+2k)(3+2k)^2(4+3k)^2(8+5k)(107+64k)^{-1}$$

$$\widetilde{C}_{4\,4}^{4} = 10368k^3(3+2k)\left(4k^3-15k^2-33k-4\right)(17+16k)^{-1}$$

$$\widetilde{C}_{4\,5}^{5} = 25920k^3(3+2k)(4+3k)\left(32k^3-236k^2-535k-125\right)\big((17+16k)(107+64k)\big)^{-1}.$$

Appendix C: Some structure constants of $\mathcal{W}_{2,1}^{sl(3)}/\widehat{U(1)}$

$$d_{3,3} = -(1+k)^2(3+k)(1+2k)(9+4k)(3+2k)^{-1}$$

$$d_{4,4} = 48k^2(1+k)^2(3+k)^4(1+2k)(5+3k)(9+4k)(12+5k)(3+2k)^{-2}\left(18-19k-15k^2\right)^{-1}$$

$$d_{5,5} = 480k^2(1+k)^2(3+k)^6(2k-1)(1+2k)(5+2k)^2(5+3k)(9+4k)(12+5k)(3+2k)^{-3}\left(50+5k-7k^2\right)^{-1}$$

$$d_{6,6} = \frac{1600k^2(1+k)^2(3+k)^8(-1+2k)(1+2k)(5+2k)^2(5+3k)(9+4k)(12+5k)}{(3+2k)^4\left(11+2k-k^2\right)\left(-50-5k+7k^2\right)^2}\times \left(-57600+25260k+67829k^2-3738k^3-19182k^4+2540k^5+4809k^6+882k^7\right)$$

$$d_6 = \frac{80k^2(1+k)^2(3+k)^6(-1+2k)(1+2k)(5+2k)^2(5+3k)(9+4k)(12+5k)}{3(3+2k)^3\left(11+2k-k^2\right)\left(7k^2-5k-50\right)\left(33+46k+21k^2\right)^{-1}}$$

$$\widetilde{C}_{4\,4}^{4} = 24(3+k)^2\left(216+186k+823k^2+1184k^3+573k^4+90k^5\right)\big((3+2k)\left(-18+19k+15k^2\right)\big)^{-1}$$

$$\widetilde{C}^{6}_{4\,4} = \tfrac{4}{5} \qquad \widetilde{C}^{7}_{4\,5} = \tfrac{2}{3}$$

$$\widetilde{C}^{5}_{4\,5} = \frac{60(3+k)^2\left(-10800-1740k-2408k^2-14301k^3-5041k^4+4947k^5+3233k^6+510k^7\right)}{(3+2k)\,(-50-5k+7k^2)\,(-18+19k+15k^2)}$$

$$\widetilde{C}^{4}_{4\,6} = \tfrac{160}{3}\,(3+k)^4\,(2k-1)\,(5+2k)^2\left(15k^2+19k-18\right)(3+2k)^{-2}\left((k^2-2k-11)(7k^2-5k-50)\right)^{-1} \times \left(576-132k-143k^2+485k^3+351k^4+63k^5\right)$$

$$\widetilde{C}^{6}_{4\,6} = \tfrac{4}{3}(3+k)^2\Big((3+2k)\left(k^2-2k-11\right)\left(7k^2-5k-50\right)\left(15k^2+9k-18\right)\Big)^{-1} (10254600+3254580k-4772094k^2+5168399k^3+6444501k^4- 267576k^5-1247560k^6+114411k^7+233577k^8+39690k^9).$$

Appendix D: Minimal models of Casimir $\mathcal{W}$-algebras

Let $\mathcal{K}$ be a simple Lie algebra of rank l over $\mathbb{C}$. The rational models of the Casimir $\mathcal{W}$-algebra related to this Lie algebra have central charge

$$c_{\mathcal{K}}(p,q) = l - \tfrac{12}{pq}(q\,\rho - p\,\rho^\vee)^2 \quad p,q \text{ coprime}, \quad h^\vee \le p \quad h \le q \tag{D.1}$$

where p and q have to be chosen minimal, h ($h^\vee$) denotes the (dual) Coxeter number of $\mathcal{K}$ and ρ ($\rho^\vee$) denotes the sum of its (dual) fundamental weights λ_i ($\lambda_i^\vee$). The conformal dimensions of the minimal model are given by [FKW]:

$$h_{\lambda,\nu^\vee} = \tfrac{1}{2pq}\left((q\lambda - p\nu^\vee)^2 - (q\rho - p\rho^\vee)^2\right) \tag{D.2}$$

where λ ($\nu^\vee$) lies in the (dual) weight lattice so that $\lambda = \sum_{i=1}^{l} l_i\lambda_i$, $\nu^\vee = \sum_{i=1}^{l} l_i^\vee \lambda_i^\vee$. λ and $\nu^\vee$ have to satisfy $\sum_{i=1}^{l} l_i m_i \le p-1$, $\sum_{i=1}^{l} l_i^\vee m_i^\vee \le q-1$ where m_i are the normalized components of the highest root ψ in the directions of the simple roots α_i, i.e. $\frac{\psi}{\psi^2} = \sum_{i=1}^{l} m_i \frac{\alpha_i}{\alpha_i^2}$. $m_i^\vee$ is given by $m_i^\vee = \frac{2}{\alpha_i^2} m_i$. Note that the set of conformal dimensions given by this condition has a symmetry so that all conformal dimensions of the minimal model occur with the same multiplicity in it (in the nonsimply laced cases the multiplicity is just 2). For more details see [FKW,Bo,FL2,GO].

Lie algebra	(m_i)	$(m_i^\vee)$
$\mathcal{A}_l$	$(1,\ldots,1)$	$(1,\ldots,1)$
$\mathcal{B}_l$	$(1,2,\ldots,2,1)$	$(1,2,\ldots,2)$
$\mathcal{C}_l$	$(1,\ldots,1)$	$(2,\ldots,2,1)$
$\mathcal{D}_l$	$(1,2,\ldots,2,1,1)$	$(1,2,\ldots,2,1,1)$
$\mathcal{E}_6$	$(1,2,2,3,2,1)$	$(1,2,2,3,2,1)$
$\mathcal{E}_7$	$(2,2,3,4,3,2,1)$	$(2,2,3,4,3,2,1)$
$\mathcal{E}_8$	$(2,3,4,6,5,4,3,2)$	$(2,3,4,6,5,4,3,2)$
$\mathcal{F}_4$	$(1,2,3,2)$	$(2,4,3,2)$
$\mathcal{G}_2$	$(2,1)$	$(2,3)$

Table 9: Values of $m_i, m_i^\vee$ for all simple Lie algebras [GO].

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