Moduli Spaces of Unitary Conformal Field Theories

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Tag der Promotion:	

To my parents

Der Tanz

Ein Vierviertelschwein und eine Auftakteule
trafen sich im Schatten einer Säule,
die im Geiste ihres Schöpfers stand.
Und zum Spiel der Fiedelbogenpflanze
reichten sich die zwei zum Tanze
Fuß und Hand.

Und auf seinen dreien rosa Beinen
hüpfte das Vierviertelschwein graziös,
und die Auftakteul auf ihrem einen
wiegte rhythmisch ihr Gekrös.
Und der Schatten fiel,
und der Pflanze Spiel
klang verwirrend melodiös.

Doch des Schöpfers Hirn war nicht von Eisen,
und die Säule schwand, wie sie gekommen war,
und so mußte denn auch unser Paar
wieder in sein Nichts zurücke reisen.

Einen letzten Strich
tat der Geigerich –
und dann war nichts weiter zu beweisen.

Ch. Morgenstern, aus den Galgenliedern

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Chapter 1

Introduction

CONFORMAL QUANTUM FIELD THEORY has come into the attention of the Physics community more than fifteen years ago, though many important steps had been taken much earlier. For the fun of it, let us mention a few of its prominent outriders. For a survey on the emergence of conformal field theory see [Nah00a]. Already in 1910, Cunningham and Bateman had noticed conformal invariance of the Maxwell equations [Cun10, Bat10]. Bessel–Hagen determined the corresponding conserved quantities [BH21] as an application of Noether's methods [Noe18]. Pauli showed that the Dirac equation with vanishing mass is conformally invariant [Pau40]. In the context of quantum field theory, Thirring was the first to write down a consistent conformal quantum theory [Thi58], with a clarification by Johnson [Joh97], though neither of the two had taken notice of conformal invariance. Indeed, nowadays we know Thirring's moduli space as the continuous part of the moduli space of conformal field theories with central charge $c = 1$ which are associated to compactification of a boson on a circle.

It took until the mid 1980's that conformal quantum field theory made it into the spotlight of modern research. In their fundamental paper [BPZ84], Belavin, Polyakov and Zamolodchikov combined the representation theory of the Virasoro algebra – developed shortly before by Kac [Kac79] and by Feigin and Fuks [FF83] – with the idea of local operators and thereby constructed completely solvable conformal theories, the so–called minimal models. Since then, conformal field theories have found important applications in many branches of Physics. In statistical mechanical systems they describe second or higher order phase transitions [FQS84, FQS85a, Car89b, ZZ89], where good experimental data are at hand, too [BKU$^+$85, CFJW85, CUK86, KC84, MD85, PP87]. In solid state physics, they occur in two–dimensional phenomena like the Quantum–Hall–effect [FK91, CTZ93, CTZ94, CDTZ93, FV94], and in polymer physics [DS87]. There are integrable systems that can be described by Toda theories as well as hierarchies of partial differential equations, where conformal invariance plays a fundamental rôle [BFF$^+$90, BFO$^+$90, BG89, BCDS90]. In superstring theory, the modern and so far speculative attempt to unify the forces of Nature, conformal field theories describe possible string vacua, or theories at small string coupling constant [GSW87, GHMR85, GHMR86]. Though often buried beneath heaps of modern

terminology, even "brane physics" is intimately related to perceptions from conformal field theory. At least in the context of boundary conformal field theories, there can be no doubt about this [Car84, Car86a, Car86b, Car89a, RS98].
Apart from its applications in Physics, conformal field theory has developed to tempting mathematical beauty and among others has applications in topology [Kon, Wit89], in algebraic geometry (mirror symmetry) [GP90, COGP91, CLS90], and in the theory of lattices and representations of Kac–Moody algebras [Bor86, FLM88, DGM90a, DGM90b].
Even if we leave aside all that, conformal field theory is an interesting and challenging field of its own right. By now, it has been put onto a solid mathematical footing and can be studied independently from speculative notions that might come from path–integral methods, string theory, or M–theory. Naturally, a major question and steady driving force is the search for a complete classification of such theories, perhaps under certain additional constraints. Any result of this kind can yield direct implications on the fields mentioned above.
A classification, at least of rational conformal field theories, has only been achieved for those with central charge $c \leq 1$ up to now [BPZ84, FQS84, GKO85, GO85, GKO86, CIZ87b, CIZ87a, Gin88b, DVV88, Kir89, Flo93, Flo94], as well as their supersymmetric extensions [FQS84, FQS85b, GKO85, GKO86, BFK86, MSS89]. The situation becomes much more complicated for $c \geq 1$ due to the existence of a continuum of nonequivalent theories. In other words, the conformal quantum field theories with central charge $c \geq 1$ possess continuous parameters. The corresponding deformations can be understood on the basis of Schwinger's action principle [Sch58], and thus allow to talk about moduli spaces of such theories.
From the above, it is natural to raise the description of the moduli space $\widetilde{\mathcal{M}}^c$ of conformal field theories with fixed central charge c as a dominant question in the field. Ultimately, one would like to know $\widetilde{\mathcal{M}}^c$ as an algebraic space, the partition functions of theories in $\widetilde{\mathcal{M}}^c$ as functions on the moduli space and modular functions on the upper half plane, and an algorithm for the calculation of all operator product coefficients, depending again on $\widetilde{\mathcal{M}}^c$.
One can use the known results for $c = 1$ as a guiding line. The moduli space $\mathcal{M}^1$ of unitary conformal field theories possesses two continuous components, associated to a single boson compactified on a circle (the Thirring phase: see above), and $\mathbb{Z}_2$ orbifolds thereof (the Ashkin–Teller phase). These two branches of dimension one intersect in exactly one point. Moreover, there are three discrete "exceptional" points in the moduli space [Gin88b, DVV88, Kir89]. The partition function of each theory in $\mathcal{M}^1$ is known explicitly [Gin88b, DVV88, Kir89], the operator content, the generic W–algebra, and all operator product coefficients have been determined [Nah96].
Little is known for central charge $c > 1$, even if in general we restrict the discussion to the moduli space $\mathcal{M}^c$ of unitary conformal field theories with central charge c. An analogous study to the above was initiated for $c = 2$ in our publication [DW00]. Apart from the component of $\mathcal{M}^2$ associated to toroidal conformal field theories, we have found all 26 nonisolated nonexceptional components of the moduli space that can be obtained by an orbifold procedure from toroidal theories. This part of

the moduli space already exhibits a complicated graph like structure with a whole wealth of intersection points between the various branches. This picture gives little hope for a systematic study of theories with higher central charge, unless we impose further restrictions.
Here, supersymmetry comes to aid, which severely restricts the admissible representations of the Virasoro algebra in a given theory. The case of central charge $c = 3/2$ has already been studied [DGH88], though the well–known picture drawn there has turned out not to be quite correct (see below). The natural next step would be the classification of all unitary superconformal field theories with central charge $c = 3$. Some observations on this behalf are spread in different chapters of this thesis, and to obtain a conclusive result seems to be a matter of diligence, now.
In the present work though, we rather skip this step and directly proceed to the discussion of $N = (2, 2)$ superconformal field theories with central charge $c = 6$. In the context of superstring theory (see above), it is natural to impose some additional assumptions which lead to the study of the moduli space $\mathcal{M}$ of $N = (4, 4)$ superconformal field theories with central charge $c = 6$. The latter has already been identified with a high degree of plausibility [Nar86, Sei88, AM94], though a number of details remain to be clarified. $\mathcal{M}$ has two components, $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$, one 16–dimensional associated to the four–torus and one 80–dimensional associated to $K3$. The superconformal field theories in $\mathcal{M}^{tori}$ are well understood. One also understands some varieties of theories which belong to $\mathcal{M}^{K3}$, including about 30 isolated Gepner type models and varieties which contain orbifolds of theories in $\mathcal{M}^{tori}$. In the literature one can find statements concerning intersections of these subvarieties, but not all of them are correct. Indeed, their precise positions in $\mathcal{M}$ had not been studied at all up to now. One difficulty is due to the fact that the standard description of $\mathcal{M}^{tori}$ is based on the odd cohomology of the torus, which does not survive the orbifolding. As varieties of superconformal theories $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$ cannot intersect for trivial reasons. As ordinary conformal theories without $\mathbb{Z}_2$ grading intersections are possible and will be shown to occur.
We have tried to assemble the results of our studies in a pedagogical way. Therefore, this work also contains the author's reaction to her steady quest through the literature in search for digestible presentations of the mathematical background that she needed for a full understanding. Whenever gaps could not be filled in a satisfactory way by the literature, a paragraph or section has been included to do so. These parts of the present work might not be of poignant originality but should serve as recreational caesurae.
An example for such a reaction is our chapter 2. We choose not to repeat standard general facts about conformal field theories that can be extracted from the textbooks or well accessable review articles [Gin88a, Sch97, FMS96, Gab00]. On the other hand, much more elementary, as it seems, in the literature we did not find a conclusive shortlist of all properties obviously assumed in general for conformal field theories in our every day studies. Since axiomatic approaches to conformal field theories do not belong to our main research interest, not much effort is put into minimizing the list in section 2.1. However, it should serve as a stable and

honest basis for what follows. In view of our general interest in the moduli spaces of conformal field theories, we open the way for the study of local properties of such a moduli space by a discussion of deformations of conformal field theories in section 2.2.

In chapter 3 we very briefly introduce supersymmetry by listing the necessary properties of superconformal field theories and the corresponding extensions of the Virasoro algebra. In sections 3.1.1 and 3.2, respectively, we assemble properties of their representations that are needed later. Two main topics of this chapter are a detailed discussion of the elliptic genus in section 3.1.2 and the introduction of Gepner models and Gepner type models in section 3.1.3, without recourse to orbifold constructions. The latter approach, which is generally not found in the literature, appears to be much simpler than the traditional one.

Chapter 4 is devoted to the simplest and best understood examples of conformal field theories, the toroidal ones. We choose a slightly unfamiliar definition in section 4.1 but point out its equivalence to the well-known one in section 4.2. For later convenience, two and four dimensional toroidal theories are studied in more detail in sections 4.3 and 4.4. In particular, $SO(4,4)$ triality is discussed, which is essential for the determination of $\mathbb{Z}_M$ orbifolds within the moduli space of theories associated to $K3$. The last two sections of chapter 4 contain our results on rational conformal field theories: In section 4.5, we arrive at a characterization of rational toroidal conformal field theories and its geometric interpretation in terms of two dimensional tori with complex multiplication. Section 4.6 is devoted to the study of so–called singular varieties and their relation to rational toroidal theories.

From a conformal field theory with extra symmetries one can construct a new one by the so–called orbifold procedure, which chapter 5 is devoted to. Here, a vast supply of literature exists [DFMS87, Dix87, DHVW85, DHVW86, FV87], but the more reasonable it is to briefly summarize the most important features, as is done in section 5.1. Section 5.2 is concerned with the geometric understanding of orbifolds, which is only partly explained in the literature. Two special types of orbifold constructions are studied more closely in section 5.3–5.4, namely crystallographic orbifolds and orbifolds involving the spacetime fermion number operator. In particular, the corresponding one loop partition functions are determined in full generality. The generalized GSO projection and its properties are the topic of section 5.5. It turns out to be a useful tool for the classification of unitary superconformal field theories. Applied to tensor products of $N=(2,2)$ minimal models in section 5.6 it yields the standard construction of Gepner models and a new interpretation of tensor products of minimal models with $c=3$.

Chapter 6 is devoted to the study of the moduli space $\mathcal{M}^2$ of unitary conformal field theories with central charge $c=2$ that was mentioned above, already. Most of the results have been published in [DW00]. We give a classification of all nonisolated nonexceptional orbifold components of the moduli space in section 6.1 and find all their intersection points and lines in section 6.2. The latter part is very technical, but the general ideas of proof are summarized at the beginning of section 6.2 and may be best seen at work in sections 6.2.1 and 6.2.7. Theories obtained as tensor products of known models with central charge $c<2$, in particular the relation of our results to those on $c = 3/2$ superconformal field theories [DGH88] are discussed in section 6.3. We also correct the statements on multicritical points on the moduli space of $N = (1,1)$ superconformal field theories with $c = 3/2$ made in [DGH88]. Section 6.4 ends the chapter with a summary on the picture we have obtained so far.

Chapter 7 contains our main results, which concern the moduli space $\mathcal{M}$ of $N = (4,4)$ superconformal field theories with central charge $c = 6$. Most of them are accepted for publication in [NW] or will appear in [Wen]. In section 7.1 we briefly review the global description of $\mathcal{M}$ in an emended version as compared to the literature. Generic features of theories in $\mathcal{M}$, more precisely the determination of a generic part of partition functions, is the topic of section 7.2. In section 7.3 we discuss the component $\mathcal{M}^{K3}$ of $\mathcal{M}$ which consists of theories that are associated to $K3$. After in section 7.3.1 having recalled some of the necessary mathematical background and having presented its generalization to the situations we discuss later on, we determine the locations of $\mathbb{Z}_2$ and more generally $\mathbb{Z}_M$ orbifold conformal field theories within $\mathcal{M}^{K3}$, $M \in \{3,4,6\}$, in sections 7.3.2 and 7.3.4, respectively. The results enable us to discuss Nahm and Fourier–Mukai transforms from a purely conformal field theoretic point of view in section 7.3.3, such that we can prove T–duality and justify our global description of $\mathcal{M}$ without leaving this framework. Moreover, in section 7.3.5 we are able to explicitly locate certain orbifold conformal field theories in the subvariety of $\mathcal{M}$ that contains theories admitting a geometric interpretation on the Fermat quartic hypersurface. In section 7.3.6 we find the locations of the Gepner model $(2)^4$ and of some of its orbifolds within $\mathcal{M}$ by proving isomorphisms to nonlinear σ models. Again, the detailed proofs are quite technical, but the general idea is summarized in the first subsection. On first reading it should be no problem to skip the proofs of the subsequent theorems. As a result of our observations, we identify $(2)^4$ with one of the above orbifold models and thus are able to show that this Gepner model has a geometric interpretation with Fermat quartic target space. We find a meeting point of the moduli spaces of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold conformal field theories different from the one conjectured in [EOTY89]. In section 7.4 we close with a panoramic view of $\mathcal{M}$ that summarizes the results of chapter 7.

Chapter 8 contains a summary of the results obtained within this work and gives an outlook on those questions that arise from the picture we can draw so far.

Chapter 2

Moduli spaces of conformal field theories (CFTs)

The present chapter is an introductory one which serves to collect general properties our conformal field theories are assumed to have throughout this work (section 2.1). In particular, in view of our general interest in the moduli spaces of conformal field theories, we open the way for the study of local properties of such a moduli space by discussing deformations of conformal field theories in section 2.2.
We do not even try to present this chapter in a selfconsistent way. In particular, we expect the reader to be familiar with basic concepts of conformal and superconformal quantum field theory. For useful introductions, see [Gin88a, Sch97, FMS96, Gab00].

2.1 Unitary conformal field theories in general

We will not delve into the abyss of axiomatic definitions of conformal field theory; for definitions of vertex algebras and their relation to conformal field theory we refer to the literature, e.g. [Bor86, DLM98, FHL93, FLM88, GG00, Kac96, Zhu96]. However, it is definitely worthwhile to linger a bit and assemble a list of those assumptions, or rather properties, which we must use to discuss conformal field theories of the type we are interested in. We do not take care of possible overlaps or discuss minimality of our list. We follow unpublished ideas of Werner Nahm's [Nah00b] and modify the BPZ axioms [BPZ84, MS89] accordingly.

Property 1
A conformal field theory $\mathcal{C}$ (with unique vacuum) possesses a bigraded infinite dimensional VECTOR SPACE $\mathcal{H}$ OF STATES *over* $\mathbb{C}$,

$$\mathcal{H} = \bigoplus_{h,\overline{h}} V(h;\overline{h}).$$

Here, each $V(h;\overline{h})$ *is finite dimensional,* $\dim V(h;\overline{h}) = 0$ *if* $h < -h_0$ *or* $\overline{h} < -\overline{h}_0$ *for fixed* $h_0, \overline{h}_0 \in \mathbb{R}_0^+$, *and* $V(0;0) = \mathbb{C}$. *The unit element* $|0\rangle$ *in* $V(0;0) = \mathbb{C}$ *is called the* VACUUM (STATE). *The unit element of the dual* $(V(0;0))^*$ *is denoted* $\langle 0|$.

The subspaces

$$\mathcal{W} := \bigoplus_h V(h;0), \quad \overline{\mathcal{W}} := \bigoplus_{\overline{h}} V(0;\overline{h})$$

are called HOLOMORPHIC *and* ANTIHOLOMORPHIC W–ALGEBRAS*. *Moreover, the vector space $\mathcal{H}$ carries a real structure.*

We remark that from property 9 below it follows that the sets $\{h \mid \exists \overline{h} : V(h;\overline{h}) \neq \emptyset\} \subset \mathbb{R}$ and $\{\overline{h} \mid \exists h : V(h;\overline{h}) \neq \emptyset\} \subset \mathbb{R}$ do not have accumulation points. It is sometimes assumed that each $V(h;\overline{h})$ is a tensor product $V(h;\overline{h}) = V(h) \otimes \overline{V}(\overline{h})$, but this is not necessary for our approach. In order to avoid tedious repetitions, the left–right transformed analogue of some statement will often not be mentioned explicitly.
The basic ingredient to describe our conformal field theory is the notorious operator product expansion:

Property 2
On $\mathcal{H}$ we have an OPERATOR PRODUCT EXPANSION (OPE), *namely a map*

$$\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}\{z,\overline{z}\},$$

where $\mathcal{H}\{z,\overline{z}\}$ denotes the space of functions $f = f(z,\overline{z})$, $f : \mathbb{C} \to \overline{\mathcal{H}}$, that are real analytic on $\mathbb{C}^*$ and have the following behaviour around $z = 0$:

$$f(z,\overline{z}) = \sum_{r \in R, n \in \mathbb{Z}} a_{rn} z^{r+n} \overline{z}^r, \tag{2.1.1}$$

with countable $R \subset \mathbb{R}$, $a_{rn} \in \mathcal{H}$, and only finitely many nonzero coefficients a_{nr} with $r + n < 0$ or $r < 0$. The OPE is compatible with the gradings of $\mathcal{H}$, where the degree of $(z;\overline{z})$ is $(-1;-1)$. It is also compatible with the real structure on $\mathcal{H}$, i.e. in the OPE of real states all a_{rn} in (2.1.1) are real.

The assumption that only finitely many a_{nr} with $r + n < 0$ or $r < 0$ are nonzero is necessary for the existence of a partition function, which will be required in property 9. The OPE on $\mathcal{W}$ induces a map $\mathcal{W} \to (\mathrm{End}\, \mathcal{H})\,[z^{-1}, z]]$ which assigns a Laurent series $\Phi(z)$ with finitely many singular terms on each degree to every state $|\Phi\rangle \in \mathcal{W}$, such that $|\Phi\rangle = \Phi(0)|0\rangle$, and analogously on the right hand side (see property 7 below). This map is known as STATE–FIELD CORRESPONDENCE.
The properties of the OPE are now encoded in the properties of another basic structure of conformal field theories, the n point functions:

Property 3
Let $\mathcal{F}(z_1, \ldots, z_n)$ denote the space of maps from $\mathbb{C}^n$ with coordinates z_i to $\mathbb{C}$ that are real analytic outside partial diagonals and have a behaviour like (2.1.1) in each

*There is a slight confusion in the literature about the nomenclature of holomorphic states. They are frequently called CHIRAL STATES, a term that we reserve for elements of the chiral ring (see section 3.1.1). The structure of an algebra on $\mathcal{W}, \overline{\mathcal{W}}$ arises from the properties listed below.

singularity. For every $n \in \mathbb{N}$ we have a linear map

$$F_n : \mathcal{H}^{\otimes n} \to \mathcal{F}(z_1, \dots, z_n), \quad |\Phi^1\rangle \otimes \dots \otimes |\Phi^n\rangle \mapsto \langle 0|\Phi^1(z_1, \overline{z}_1) \cdots \Phi^n(z_n, \overline{z}_n)|0\rangle.$$

The right hand side above is called an n POINT FUNCTION.

The insertions $\Phi^i(z_i, \overline{z}_i)$ of an n point function are called FIELDS and can be interpreted as families of linear operators $\Phi^i(z_i, \overline{z}_i) : \mathcal{H} \to \overline{\mathcal{H}}$, parametrized by $z_i \in \mathbb{C}$ (see below). Then the state–field correspondence extends to the entire space of states $\mathcal{H}$.
The tricky part in the definition of a conformal field theory is to give axioms for the n point functions. We will impose further restrictions on them, so only a subset of $\mathcal{F}(z_1, \dots, z_n)$ occurs as set of n point functions of a given theory. It happens that only for the holomorphic fields we know exactly which subset is to be taken. For later convenience we include the case of a $\mathbb{Z}_2$ graded space of states $\mathcal{H}$ in the following property; all states in the present case are even, though.

Property 4
For fields $\Phi^j, j \in \{1, \dots, n\}$, of our theory which all are purely even or purely odd with respect to the $\mathbb{Z}_2$ grading

$$F_n(\Phi^1, \dots, \Phi^i, \Phi^{i+1}, \dots, \Phi^n) = \varepsilon F_n(\Phi^1, \dots, \Phi^{i+1}, \Phi^i, \dots \Phi^n)$$

with $\varepsilon = -1$ if both Φ^i, Φ^{i+1} are odd, and $\varepsilon = 1$ otherwise. Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{\otimes 2} \otimes \mathcal{H}^{\otimes k} & \longrightarrow & \mathcal{F}(z_0, \dots, z_{k+1}) \\ \downarrow & & \downarrow \\ \mathcal{H} \otimes \mathcal{H}^{\otimes k}\{z, \overline{z}\} & \longrightarrow & \mathcal{F}(z_1, \dots, z_{k+1})\{z, \overline{z}\}. \end{array}$$

Here, the horizontal arrows are given by the n point functions F_{k+2}, F_{k+1}, respectively, the left vertical arrow is the OPE in the first two factors $\mathcal{H}^{\otimes 2}$, and the right vertical arrow is the expansion around 0 with respect to $z = z_0 - z_1$.

We have now almost completed an axiomatic definition of the OPE and n point functions, apart from the following condition of closure:

Property 5
Given an $n+1$ point function as in property 3, for any fixed Φ^i, Φ^j the residues in $z_{ij} := z_i - z_j = 0$ (i.e. all coefficients of singular terms in the expansions around $z_{ij} = 0$) are n point functions that obey properties 3–5.

By the above, a conformal field theory actually is a representation of its OPE, where we use

Definition 2.1.1
A REPRESENTATION OF AN OPE $\mathcal{V} \otimes \mathcal{V} \to \mathcal{V}\{z, \overline{z}\}$ is a vector space of linear maps $F = \oplus_n F_n, F_n : \mathcal{V}^{\otimes n} \to \mathcal{F}(z_1, \dots, z_n)$ satisfying properties 4 and 5 above.

If $\mathcal{V}$ is $\mathbb{Z}_2$ graded and the OPE respects the grading, we modify the permutation invariance as stated in property 4 to define a FERMIONIC REPRESENTATION *of the OPE.*

Properties 3–5 show that we can view a conformal field theory as a representation of its OPE with the vector space of maps F given by multiples of

$$|\Phi^1\rangle \otimes \ldots \otimes |\Phi^n\rangle \longmapsto \langle 0|\Phi^1(z_1,\overline{z}_1)\cdots\Phi^n(z_n,\overline{z}_n)|0\rangle.$$

For later convenience we rather take a different viewpoint, namely interpret it as representation of the OPEs of its holomorphic and antiholomorphic W–algebras. To do so, we need to work a little harder. Firstly, for $f \in \mathcal{V}\{z,\overline{z}\}$ let $[f(z,\overline{z})]_{0,0} := a_{0,0}$ denote the constant coefficient of f in an expansion of type (2.1.1). Then for fields $\Phi^j, j\in\{1,2\}$, of dimensions $(h_j;\overline{h}_j)$ in our conformal field theory

$$(\Phi^1,\Phi^2) := \left[z^{h_1+h_2}\overline{z}^{\overline{h}_1+\overline{h}_2}\langle 0|\Phi^1(z,\overline{z})\Phi^2(0,0)|0\rangle\right]_{0,0}\delta_{h_1,h_2}\delta_{\overline{h}_1,\overline{h}_2}$$

defines a symmetric inner product on $\mathcal{H}$ by property 4. Together with the real structure on $\mathcal{H}$ it can be used to define a Hermitean product, since in property 3 we supposed compatibility of OPE and real structure. We assume the following

Property 6
The inner product $(\cdot,\cdot)$ is nondegenerate.

Note that we do not promote the prehilbert space $\mathcal{H}$ into a Hilbert space. From property 4 together with property 6 one can already deduce associativity of the OPE. Given $|\Phi\rangle \in \mathcal{H}$ we write $\langle\Phi| \in \mathcal{H}^*$ if

$$\forall\, |\Psi\rangle \in \mathcal{H}: \quad (\overline{\Phi},\Psi) = \langle\Phi|\Psi\rangle.$$

So far, we have not introduced the conformal part of the conformal field theory. Also, nothing has been said about the functional dependence of n point functions on the parameters of the fields. Both are fixed by

Property 7
The space of fields of a conformal field theory contains a real holomorphic field $T(z)$, $|T\rangle \in V(2;0)$ whose Fourier components L_n in $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{n-2}$ generate a left–handed VIRASORO ALGEBRA *with* CENTRAL CHARGE *c:*

$$[L_m,L_n] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m+n,0}.$$

T is HERMITEAN, *i.e. $(L_n)^\dagger = L_{-n}$ for all $n\in\mathbb{Z}$. Let $|\Phi\rangle \in V(h;\overline{h})$ denote a* PRIMARY STATE, *i.e. a lowest weight state of the left and the right handed Virasoro algebra, $L_m|\Phi\rangle = 0$ and $\overline{L}_m|\Phi\rangle = 0$ for $m<0$. Then $L_0|\Phi\rangle = h|\Phi\rangle$, and for any other $|\Psi^j\rangle \in \mathcal{H}, j\in\{1,\ldots,n\}$,*

$$\begin{aligned}&\frac{\partial}{\partial z}\langle 0|\Phi(z,\overline{z})\Psi^1(z_1,\overline{z}_1)\cdots\Psi^n(z_n,\overline{z}_n)|0\rangle\\ &\qquad = \langle 0|(L_1\Phi)(z,\overline{z})\Psi^1(z_1,\overline{z}_1)\cdots\Psi^n(z_n,\overline{z}_n)|0\rangle,\end{aligned}$$

and analogously on the right handed side. The left and right handed Virasoro algebras commute.
Generally, $(h;\overline{h})$ as above are called CONFORMAL DIMENSIONS *of a state in the theory.*

Property 7 can be weakened to the assumption that $\mathcal{H}$ is a representation of $sl(2,\mathbb{C})$, generated by $L_0, L_{\pm 1}$ as above. The vacuum state of property 1 is both holomorphic and antiholomorphic, so the corresponding field is constant. Hence $L_m|0\rangle = 0$ for $m \in \{0, \pm 1\}$, and $|0\rangle$ is the unique $sl(2,\mathbb{C}) \times sl(2,\mathbb{C})$ invariant state with $(h;\overline{h}) = (0;0)$.
Note that by property 7 one point functions of states in $V(h;\overline{h}) \neq V(0;0)$ vanish. Hence by property 2 the two point function $\langle 0|\Phi^1(z,\overline{z})\Phi^2(0,0)|0\rangle$ with $|\Phi^j\rangle \in V(h_j;\overline{h}_j)$ has a pole of type $z^{-h_1-h_2}\,\overline{z}^{-\overline{h}_1-\overline{h}_2}$ for $z \to \infty$, and

$$\langle 0|\Phi^1(z,\overline{z})\Phi^2(0,0)|0\rangle \sim z^{-h_1-h_2}\,\overline{z}^{-\overline{h}_1-\overline{h}_2} \in \mathcal{F}(z).$$

It now follows from (2.1.1) together with invariance under permutations (property 4) and property 6 that all states must have integer SPIN $h - \overline{h}$. We always label Fourier components of holomorphic fields by the energy, not by its negative, i.e. for $|\Phi\rangle \in V(h;0)$ we write $\Phi(z,\overline{z}) = \sum_{n\in\mathbb{Z}} \Phi_n z^{n-h}$, such that $|\Phi\rangle = \Phi_h|0\rangle$.
Property 7 shows that n point functions containing holomorphic fields will not depend on the corresponding antiholomorphic parameters. Then by property 2, the OPE of holomorphic $|\Phi^1\rangle, |\Phi^2\rangle \in \mathcal{W}$ is a rational function in z, since the singularity in $z = \infty$ is also at most a pole by the above. More generally, let $\Phi^j, j \in \{1,\dots,l\}$, denote holomorphic fields and $\Psi^j, j \in \{l+1,\dots,n\}$, any other fields of the theory. Then by property 4, iterated OPE shows that for any fixed $z_1,\dots,z_l,w_{l+1},\dots,w_n$, the n POINT FUNCTIONS

$$\langle 0|\Phi^1(z_1)\cdots\Phi^l(z_l)\Psi^{l+1}(w_{l+1},\overline{w}_{l+1})\cdots\Psi^n(w_n,\overline{w}_n)|0\rangle \tag{2.1.2}$$

are rational functions in the z_i which have a well defined Laurent series in $z_{ij} := z_i - z_j$. They can be continued to meromorphic functions on $\overline{\mathbb{C}}^l$ with poles only on partial diagonals or for some $z_i = \infty$. The space of such meromorphic functions is denoted $\mathbb{C}(z_1,\dots,z_l)$. Recall that rational functions on $\mathbb{C}$ are completely determined by their behaviour in the poles. An analogous condition holds for antiholomorphic fields Φ^j.
Now we see that by definition 2.1.1 our conformal field theory is a representation of the OPE of its holomorphic and antiholomorphic W–algebras: The vector space of maps is $\oplus_k \mathcal{H}^{\otimes k}$, and $F_{n-l} = |\Psi^{l+1}\rangle \otimes \ldots \otimes |\Psi^n\rangle \in \mathcal{H}^{\otimes n-l}$ acts on $\mathcal{W}^{\otimes l}$ by

$$F_{n-l} : |\Phi^1\rangle \otimes \ldots \otimes |\Phi^l\rangle \mapsto \langle 0|\Phi^1(z_1)\cdots\Phi^l(z_l)\Psi^{l+1}(w_{l+1},\overline{w}_{l+1})\cdots\Psi^n(w_n,\overline{w}_n)|0\rangle.$$

This is the standard example of a representation of an OPE and in fact the only one we will use.
Let us summarize what we have achieved up to now: A conformal field theory is a representation of its OPE, such that the n point functions as functions of

the parameters $z_1, \ldots, z_l$ of the holomorphic fields are meromorphic functions in $\mathbb{C}(z_1, \ldots, z_l)$. As functions of the parameters $w_{l+1}, \ldots, w_n$ of nonholomorphic fields, the n point functions are more complicated. The $Sl(2,\mathbb{C})$ invariance of property 7 defines a derivation on the space of n point functions which for parameters of holomorphic fields agrees with the usual derivation. Properties 2 on the OPE and 4 on its compatibility with n point functions show that no global monodromy exists for the corresponding connection. This property is generally referred to as PAIRWISE LOCALITY of the fields in the theory. For fermionic representations of the OPE, the monodromy may contribute at most a factor of -1.

All in all, our n point functions now are defined on $\overline{\mathbb{C}} \cong \mathbb{S}^2$, i.e. we are considering conformal field theories on the Riemann sphere. From property 7 it also follows that n point functions transform covariantly under conformal transformations. More precisely, let $\Phi^j, j \in \{1, \ldots, n\}$, denote primary fields in $\mathcal{C}$ of conformal dimensions $(h_j; \overline{h}_j)$. Let $\xi = \xi(\zeta)$ denote a global conformal transformation, and $\xi_j := \xi(z_j)$. Then

$$\begin{aligned} &\langle 0|\Phi^1(z_1,\overline{z}_1)\cdots\Phi^n(z_n,\overline{z}_n)|0\rangle \\ &\quad = \prod_{j=1}^n \left(\frac{d\xi}{dz}\right)^{h_j}_{|z=z_j} \left(\frac{d\overline{\xi}}{d\overline{z}}\right)^{\overline{h}_j}_{|\overline{z}=\overline{z}_j} \langle 0|\Phi^1(\xi_1,\overline{\xi}_1)\cdots\Phi^n(\xi_n,\overline{\xi}_n)|0\rangle. \end{aligned} \tag{2.1.3}$$

This condition already fixes the form of two and three point functions in the theory, because we can always find a conformal transformation that maps three points z_1, z_2, z_3 to $0, 1, \infty$, respectively. Two point functions vanish unless both fields have equal dimensions.

Given all properties of the n point functions listed so far, one can interpret them as matrix elements of the fields $\Phi(z,\overline{z})$. Their operator product $\Phi(z_1,\overline{z}_1)\cdots\Phi(z_n,\overline{z}_n)$ can also be defined for $|z_1| > \cdots > |z_n|$ ("RADIAL ORDERING") by iterated OPE. In particular, $\Phi(0,0)|0\rangle = |\Phi\rangle$. We have $\langle\Phi| = \lim_{\zeta\to\infty}\langle 0|(\Phi(\zeta,\overline{\zeta}))^\dagger$, and for real primary $|\Phi\rangle$ one checks with the help of (2.1.3)

$$(\Phi(z,\overline{z}))^\dagger\,\overline{z}^h z^{\overline{h}} = \left(\Phi(z,\overline{z})z^h\overline{z}^{\overline{h}}\right)^\dagger = \Phi\left(\frac{1}{\overline{z}},\frac{1}{z}\right)\overline{z}^{-h}z^{-\overline{h}}.$$

It follows that a real holomorphic field $\Phi(z)$ is Hermitean with respect to the inner product of property 6, iff $(\Phi(\overline{z}))^\dagger = \Phi(z)$, or equivalently $(\Phi_n)^\dagger = \Phi_{-n}$. Up to a sign, this is nothing but CPT invariance.

We now see that the representation of the OPE introduced above can also be viewed as the space of maps isomorphic to $\mathcal{W}^*\otimes\mathcal{W}$, and $F = \langle\Phi|\otimes|\Psi\rangle \in \mathcal{W}^*\otimes\mathcal{W}$ acts by

$$F_n : (\Phi^1, \ldots, \Phi^n) \mapsto \langle\Phi|\Phi^1(z_1,\overline{z}_1)\cdots\Phi^n(z_n,\overline{z}_n)|\Psi\rangle.$$

From property 6 together with property 7 it also follows that $c > 0$ and $\overline{c} > 0$.

We now add a kind of "universality condition" for the OPE:

Property 8

If the space of states $\mathcal{H}$ together with the n point functions $\mathcal{H}^{\otimes n} \xrightarrow{F_n} \mathcal{F}(z_1, \ldots, z_n)$ satisfying properties 2–7 belong to a CONFORMAL FIELD THEORY *$\mathcal{C}$, then for any other space of states $\mathcal{H}'$ with n point functions F'_n satisfying properties 2–7 the following holds:*
Given a map $A : \mathcal{H} \to \mathcal{H}'$ which maps the Virasoro fields in $\mathcal{H}, \mathcal{H}'$ onto each other such that for all n point functions F'_n the map $F'_n \circ A^{\otimes n}$ is an n point function of $\mathcal{C}$, then A is an isomorphism which respects the gradings of $\mathcal{H}$ and $\mathcal{H}'$.

In this work, we shall consider UNITARY CONFORMAL FIELD THEORIES, i.e. we assume the inner product of property 6 to be positive definite and each representation of the Virasoro algebra in $\mathcal{H}$ to be unitary with respect to this inner product. Apart from the fact that the additional assumption of unitarity means a simplification, one can argue on physical grounds why this property is a sensible requisite:
In field theory, unitarity is equivalent to conservation of probability and therefore is fundamental. In statistical mechanical systems, unitarity is expressed by reflection positivity and can only be expected to hold for the associated effective theory, if at all it exists. By assuming unitarity, we also ensure that $\mathcal{H}$ splits into a direct sum of irreducible representations of the left and right handed Virasoro algebras.
Above, we have considered conformal field theories on the Riemann sphere $\mathbb{S}^2 \cong \overline{\mathbb{C}}$. We want to study conformal field theories on the torus, however, which are also defined on the sphere. Property 9 below will ensure that this is well defined. Our assumptions then suffice to give a consistent theory on Riemann surfaces of arbitrary genus [Son88]. So let us describe Euclidean theories on the torus with parameter σ in the upper half plane. The world sheet coordinates are called $\xi_0, \xi_1 \in [0,1]$, and $\xi = \xi_0 + \sigma\xi_1$ parametrizes the worldsheet torus $\Xi(\sigma)$. Imaginary and real part of ξ are interpreted as imaginary time and space coordinates on the worldsheet of a string, respectively. We frequently use RADIAL COORDINATES $z = e^{\xi_0 + \sigma\xi_1}, z \in Z$, to parametrize the worldsheet on an annulus $Z \subset \mathbb{C}^*$ and deliberately switch between the two parametrizations. Note that time ordering then translates into radial ordering.
We assume that in our theory the two point functions of all components of the energy momentum tensor are Lorentz invariant, conserved quantities. This leads to the condition that the left and right handed central charges of the theory agree [Gin88a, §3.1]. $c = \overline{c}$ can also be deduced from cancellation of local gravitational anomalies [AGW84], in other words is an assumption we need anyway if we want to generalize to a conformal field theory defined on Riemann surfaces of arbitrary genus.
The modes $(L_n)_{\Xi(\sigma)}$ of the Virasoro field on the torus $\Xi(\sigma)$ transform by

$$(L_n)_{\Xi(\sigma)} = L_n - \tfrac{c}{24}\delta_{n,0}, \quad (\overline{L}_n)_{\Xi(\sigma)} = \overline{L}_n - \tfrac{c}{24}\delta_{n,0}$$

to those on the annulus Z. On $\Xi(\sigma)$, imaginary time translation of $\mathrm{Im}(\xi)$ by its period $2\pi\sigma_2$ is accompanied with spatial translation of $\mathrm{Re}(\xi)$ by $2\pi\sigma_1$. Since in the operator formalism spin is measured by $P = (L_0 - \overline{L}_0)_{\Xi(\sigma)} = L_0 - \overline{L}_0$, and energy is measured by $H = (L_0 + \overline{L}_0)_{\Xi(\sigma)}$, the operator that describes propagation of states along $\Xi(\sigma)$ is

$$e^{-2\pi\sigma_2 H} e^{2\pi i \sigma_1 P} = q^{(L_0)_{\Xi(\sigma)}} \overline{q}^{(\overline{L}_0)_{\Xi(\sigma)}} = q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24} =: \mathcal{O}(q, \overline{q}),$$

where $q := e^{2\pi i \sigma}, \overline{q} := e^{-2\pi i \overline{\sigma}}$. The VACUUM CORRELATOR $\langle 0|\mathcal{O}(q,\overline{q})|0\rangle$ on the torus therefore is a trace over the entire space of states with respect to a graded basis, the ONE–LOOP PARTITION FUNCTION

$$Z(\sigma) = \mathrm{tr}_{\mathcal{H}}\, \mathcal{O}(q, \overline{q}) = (q\overline{q})^{-c/24}\, \mathrm{tr}_{\mathcal{H}}\, q^{L_0} \overline{q}^{\overline{L}_0}.$$

If the theory contains a left and a right handed $U(1)$ current J and $\overline{J}$, as for example is the case for $N = (2,2)$ superconformal field theories, left and right handed charges are denoted $(Q; \overline{Q})$. Then we use a $U(1)$ equivariant version of $\mathcal{O}$ and thus of the partition function. Namely*, with $y := e^{2\pi i z}, \overline{y} := e^{-2\pi i \overline{z}}$ we set $\mathcal{O}(q, y; \overline{q}, \overline{y}) := y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24}$ and

$$Z(\sigma, z) = \mathrm{tr}_{\mathcal{H}}\, \mathcal{O}(q, y; \overline{q}, \overline{y}) = (q\overline{q})^{-c/24}\, \mathrm{tr}_{\mathcal{H}}\, y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0} \overline{q}^{\overline{L}_0}. \tag{2.1.4}$$

As a final property, we now assume

Property 9

The n point functions on the torus, computed as traces, exist and are modular covariant. This in particular holds for the vacuum correlator (2.1.4), the partition function of the theory, which is also assumed to depend analytically on each of the parameters $h, \overline{h}, (Q, \overline{Q})$ that occur as eigenvalues of $L_0, \overline{L}_0, (J_0, \overline{J}_0)$.

Property 9 implies that the partition function (2.1.4) is assumed to be convergent for all values of σ in the upper halfplane $\mathbb{H} := \{\zeta \in \mathbb{C} \mid \mathrm{Im}\, \zeta > 0\}$. The latter restriction is a remnant of the radial ordering. Note that it follows from property 9 that the set of dimensions $(h; \overline{h})$ cannot have an accumulation point. If the set of representations of the holomorphic fields in the theory is finite, i.e. for RATIONAL CONFORMAL FIELD THEORIES (see definition 3.1.13), convergence of Z can also be deduced from a weaker finiteness condition [Zhu96].

From a conceptual point of view it would be favorable to deduce modular covariance of the correlation functions from first principles. This is obviously possible for theories described by functional integrals (once they are defined), since we have assumed property 8. For rational conformal field theories it has been proven in [Nah91, Zhu96] under some assumptions on the analytic properties of the operators in the theory. The proof can certainly be extended to the quasirational case, where the representations of the holomorphic W–algebra form a measurable set with respect to an appropriate measure. A general proof seems out of reach, though.

A main step in the proof [Nah91] of modular invariance for rational conformal field theories is the following: Firstly, Euclidean n point functions $Z^n_\gamma(\sigma)$ on the torus, or more precisely

$$Z^n_\gamma(\sigma)(\Phi^1, \ldots, \Phi^n)(z_1, \ldots, z_n) := \mathrm{tr}_{\mathcal{H}_\gamma} \left(\mathcal{O}(q, \overline{q}) \Phi^1(z_1, \overline{z}_1) \cdots \Phi^n(z_n, \overline{z}_n) \right),$$

*The coinciding notation z for the radial coordinate $z \in Z$ on one hand and the parameter measuring charge on the other should not lead to confusion, since the respective meaning is always clear from the context.

transform into one another under the modular group $PSL(2,\mathbb{Z})$. Denote by S,T the generators of $PSL(2,\mathbb{Z})$,

$$S,T \in PSL(2,\mathbb{Z}), \zeta \in \mathbb{H}: \qquad S: \zeta \mapsto -\frac{1}{\zeta}, \quad T: \zeta \mapsto \zeta + 1, \tag{2.1.5}$$

and by $\{Z^n_\gamma\}_{\gamma\in I}$ the set of such n point functions. Then

$$Z^n_\gamma(S\sigma) = \sum_{\gamma'\in I} \widehat{S}_{\gamma\gamma'} Z^n_{\gamma'}(\sigma), \quad Z^n_\gamma(T\sigma) = \sum_{\gamma'\in I} \widehat{T}_{\gamma\gamma'} Z^n_{\gamma'}(\sigma).$$

The matrices $\widehat{S}, \widehat{T}$ are constant, in particular independent of σ. This fact is tacitly used throughout the literature, though it is far from obvious a priori.
We will also have to *assume* that a moduli space of conformal field theories exists, this can not be derived from first principles. In particular, we assume the perturbation series in conformal field theory to converge. All data that describe the neighbourhood of a conformal field theory in the moduli space are encoded in the n point functions of the theory itself in terms of the perturbation series and depend analytically on the dimensions and charges of the fields in the theory. This is one reason for our additional assumptions in property 9.

2.2 Deformations of conformal field theories

Let us now discuss deformations of a unitary conformal field theory that preserve the infinite conformal symmetry and the central charge c. This is a delicate subject, since no axiomatic notion of such a deformation has been formulated whatsoever, and to develop one exceeds the scope of this thesis. However, the framework given in section 2.1 can be seen as a first step towards such a development. Namely, a conformal field theory is viewed as a particular representation of the OPE whose properties are encoded in the n point functions. In other words, a conformal field theory is a space of specific maps, a description that is close to algebraic geometers' notion of an algebraic variety. Once this interpretation is better understood, in particular once we know exactly what kind of maps we have to restrict our n point functions to, one can hope to formulate a consistent definition of deformations of conformal field theories similar to Kodaira–Spencer theory or more generally motivic approaches. The structures we will meet below indeed are reminiscent of these mathematical objects.
Here we will use L.P. Kadanoff's approach to deformation theory [Kad79, KB79] (see also [DVV87] for a very clear account on this subject). It is motivated by statistical mechanical methods and goes back to Schwinger's action principle [Sch58]. In the following, let $\{\mathcal{C}_\delta\}_{\delta\in\mathbb{R}}$ denote a smooth family of unitary conformal field theories with central charge c. By this we mean that the spaces of states of $\mathcal{C}_\delta$ form a vector bundle over $\{\delta|\delta\in\mathbb{R}\}$, such that n point functions are sections in C^∞–bundles with respect to δ. All quantum numbers of fields Φ_δ are assumed to depend analytically on δ. Omission of δ will always mean that we are discussing

the theory $\mathcal{C} = \mathcal{C}_0$. If $\mathcal{C}_\delta$ is governed by an action S_δ as proposed in Schwinger's action principle [Sch58], by our assumption that the data of a conformal field theory also encode the geometry of its neighbourhood in moduli space, we can find a field $\mathcal{D}(z,\overline{z})$ in the theory $\mathcal{C}$ which generates the deformation along $\mathcal{C}_\delta$. Namely, the partition function $Z_\delta(\sigma)$ of $\mathcal{C}_\delta$ is the path integral $\int e^{-S_\delta}$ over the space of solutions to the equations of motion given by S_δ, and

$$S_\delta = S - \delta \int_Z dz d\overline{z} \mathcal{D}(z,\overline{z}).$$

By dimensional analysis it is clear that $\mathcal{D}$ has to have conformal dimensions $(h;\overline{h}) = (1;1)$. Such fields are called MARGINAL. One can also deduce $h+\overline{h} = 2$ for $\mathcal{D}$ from the computation of the β function of the theory and the interpretation of the moduli space of conformal field theories as the set of fixed points under the renormalization group flow [Car87]. Since $h = \overline{h} = 1$, $\mathcal{D}$ cannot be a derivative of another field in the theory and so automatically is QUASIPRIMARY, i.e. $L_{-1}\mathcal{D} = 0, \overline{L}_{-1}\mathcal{D} = 0$. In order for $\mathcal{D}$ to generate a deformation of the theory, it must also stay marginal along the family $\mathcal{C}_\delta$. If this is true, $\mathcal{D}$ is called EXACTLY MARGINAL. One remark of caution is in place here: It seems tempting to directly interpret exactly marginal operators as tangent vectors of the moduli space of unitary conformal field theories with given central charge. Though this indeed is possible in almost all examples we discuss in this thesis, the general structure of conformal field theories does not provide this interpretation. Namely, the conditions for exact marginality worked out in conclusion 2.2.1 are not linear in $\mathcal{D}$. Therefore the set of marginal fields will not form a vector space in general, and the moduli space cannot be expected to be a manifold. If it is, on the other hand, by property 6 the two point functions of exactly marginal operators define a natural Riemannian metric on the moduli space, the ZAMOLODCHIKOV METRIC [Zam86].

To check whether $\mathcal{D}$ is exactly marginal we have to make sure that it does not change its conformal dimensions under deformation and does not mix with other $(1;1)$ fields. Conditions for exact marginality are read off from the generic form of the n point functions that is implied by (2.1.3). Without loss of generality we can assume that the set $\{\Phi^j\}$ of primary fields in $\mathcal{C}$ is orthonormal with respect to the Hermitean product of property 6, such that

$$\langle 0|\Phi^i(z,\overline{z})\Phi^j(0,0)|0\rangle = \frac{\delta_{ij}}{z^{2h_i}\overline{z}^{2\overline{h}_i}}.$$

We will now present the calculation [Kad79] for the change of dimensions $(h;\overline{h})$ of a primary field Φ in $\mathcal{C}$ under a deformation generated by $\mathcal{D}$. A dot will denote $\frac{d}{d\delta}_{|\delta=0}$. Then by the above formula

$$\frac{d}{d\delta}\left(\langle 0|\Phi(z,\overline{z})\Phi(0,0)|0\rangle^{\mathcal{C}_\delta}\right)_{|\delta=0} = -2\left(\dot{h}\ln z + \dot{\overline{h}}\ln\overline{z}\right) z^{-2h}\,\overline{z}^{-2\overline{h}} + \cdots$$

where we omit terms that have no logarithmic singularity in $z = 0$. On the other hand,

$$\langle 0|\Phi(z,\overline{z})\Phi(0,0)|0\rangle^{\mathcal{C}_\delta} = \langle 0|\Phi(z,\overline{z})\Phi(0,0)e^{\delta\int_Z dz d\overline{z}\mathcal{D}(z,\overline{z})}|0\rangle,$$

so up to renormalization

$$\frac{d}{d\delta}\left(\langle 0|\Phi(z,\overline{z})\Phi(0,0)|0\rangle^{\mathcal{C}_\delta}\right)_{|\delta=0} = \int dw d\overline{w} \langle 0|\Phi(z,\overline{z})\mathcal{D}(w,\overline{w})\Phi(0,0)|0\rangle.$$

We see that $\mathcal{D}$ must be Hermitean in order to preserve the real structure on $\mathcal{H}$ [Sch58]. The integral is logarithmically divergent in $w = z$ and $w = 0$, since $\mathcal{D}$ is actually associated to an infinite number of deformations which are all equivalent by wave function renormalization (see below) but are all summed over in the above integral. To regularize, we introduce a cutoff ε around the singularities by integration over $\widetilde{G}_{z,\varepsilon} := \{w \in \mathbb{C} | |w| > \varepsilon, |w - z| > \varepsilon\}$. Then we pick a particular renormalization (2.2.1) of $\mathcal{D}$, such that the dependence on the cutoff drops out. We now use (2.1.3) with $w(\zeta) = \frac{\zeta z}{z-\zeta}$ and denote by $G_{z,\varepsilon}$ the image of $\widetilde{G}_{z,\varepsilon}$, so for z large, ε small, $G_{z,\varepsilon} = \{w \in \mathbb{C} | \varepsilon < |w| < \frac{|z|^2}{\varepsilon}\}$. Then comparison of the logarithmic singularities in the above expressions gives

$$\dot{h} \ln z + \dot{\overline{h}} \ln \overline{z} = -\frac{1}{2} \int_{G_{z,\varepsilon}} dw d\overline{w} \;\; \langle\Phi| \, \frac{\mathcal{D}_{0,0}}{|w|^2} \, |\Phi\rangle = -\pi \langle\Phi|\mathcal{D}_{0,0}|\Phi\rangle \ln \frac{|z|^2}{\varepsilon^2}.$$

Here and in the following, given a marginal field Ψ we define operators $\Psi_{m,m}$ on $|\Phi\rangle \in \mathcal{H}$ for $m \in \mathbb{N}$ by

$$\Psi_{m,m}|\Phi\rangle = \lim_{z\to 0} |z|^{2-2m} \Psi(z,\overline{z})|\Phi\rangle.$$

Then analogously

$$\frac{d}{d\delta}\left(\langle 0|\mathcal{D}(z,\overline{z})\Psi(0,0)|0\rangle^{\mathcal{C}_\delta}\right)_{|\delta=0} = 2\pi \langle\mathcal{D}|\Psi_{0,0}|\mathcal{D}\rangle \ln \frac{|z|^2}{\varepsilon^2}.$$

As already mentioned, we can ignore the ε dependence in the above expressions if we use wave function renormalization, e.g. for $\Phi = \mathcal{D}$

$$\mathcal{D} \mapsto \mathcal{D} + \delta\pi \ln \varepsilon^2 \left(\langle\mathcal{D}|\mathcal{D}_{0,0}|\mathcal{D}\rangle\mathcal{D} + \langle\mathcal{D}|\Psi_{0,0}|\mathcal{D}\rangle\Psi\right). \tag{2.2.1}$$

If $\mathcal{D}$ is exactly marginal, its left and right dimensions must not change along $\mathcal{C}_\delta$, i.e. $\langle\mathcal{D}|\mathcal{D}_{0,0}|\mathcal{D}\rangle = \langle\mathcal{D}_{-1,-1}|\mathcal{D}_{0,0}|\mathcal{D}_{1,1}\rangle = 0$. Moreover, the wave function renormalization (2.2.1) must not mix $\mathcal{D}$ with other fields Ψ of dimensions $(1;1)$, i.e. $\langle\mathcal{D}|\Psi_{0,0}|\mathcal{D}\rangle = 0$ for all such fields. The conditions for the dimensions of $\mathcal{D}$ not to change to higher order deformation theory are now deduced by the same reasoning. In general, $n+2$ point functions will occur as nth order obstructions to exact marginality. We assume that renormalization works similarly to (2.2.1), but now apart from wave function renormalizations of type (2.2.1) mixing of the fields with $\mathbb{1}$ produces divergent terms that do not contribute to the variation of dimensions. This means that only the "connected part"

$$\langle\Phi|\mathcal{D}(w_1,\overline{w}_1)\cdots\mathcal{D}(w_k,\overline{w}_k)|\Phi\rangle_{conn}$$

of an n point function is to be taken into account [Car87]. It is determined by subtracting all contributions from the ordinary n point function that come from complete factorizations into n' point functions, $n' < n$, i.e. correspond to disconnected Feynman diagrams. Generalizing Kadanoff's result [Kad79] we therefore find:

Conclusion 2.2.1
If $\mathcal{D}$ is an operator in the conformal field theory $\mathcal{C}$ which generates a deformation of $\mathcal{C}$ that preserves conformal symmetry and the central charge, then $\mathcal{D}$ has conformal dimensions $(1;1)$. Moreover, for any other field Ψ with dimensions $(1;1)$ of the theory,

$$\langle \mathcal{D}|\Psi_{0,0}|\mathcal{D}\rangle = 0,$$

and for $k \in \mathbb{N}$ with

$$\begin{aligned} G_{z,\varepsilon} \;:=\; & \big\{(w_1,\dots,w_k) \in \mathbb{C}^k \mid \varepsilon|w_i+z| < |z|^2, \varepsilon|w_i+z| < |w_i z|, \\ & \quad \varepsilon|w_i+z||w_j+z| < |z|^2|w_i-w_j|\big\} \end{aligned}$$

the integral

$$\int_{G_{z,\varepsilon}} dw_1 d\overline{w}_1 \cdots dw_k d\overline{w}_k \langle \Phi|\mathcal{D}(w_1,\overline{w}_1)\cdots\mathcal{D}(w_k,\overline{w}_k)|\Phi\rangle_{conn}$$

for $\Phi = \mathcal{D}$ has no logarithmic singularity in $z = 0$. If the latter is true for an arbitrary primary field Φ of $\mathcal{C}$, Φ does not change its dimensions along $\mathcal{C}_\delta$.

As an example one can study any theory with left and right handed $U(1)$ currents $j, \overline{\jmath}$. Since the relevant expectation value vanishes for an odd number of insertions, by Wick's theorem it suffices to determine

$$F(w_1,w_2) := \langle j_{-1}|j(w_1)j(w_2)|j_1\rangle = \frac{1}{w_1^2} + \frac{1}{w_2^2} + \frac{1}{(w_1-w_2)^2}.$$

The relevant expression for the $(1;1)$ field $\mathcal{D} = j\overline{\jmath}$ then is

$$\begin{aligned} \textstyle\int_{G_{z,\varepsilon}} \; & dw_1 d\overline{w}_1 dw_2 d\overline{w}_2 \langle \mathcal{D}|\mathcal{D}(w_1,\overline{w}_1)\mathcal{D}(w_2,\overline{w}_2)|\mathcal{D}\rangle_{conn} \\ & = \int_{G_{z,\varepsilon}} dw_1 d\overline{w}_1 dw_2 d\overline{w}_2 \left(|F(w_1,w_2)|^2 - \frac{1}{|w_1|^4} - \frac{1}{|w_2|^4} - \frac{1}{|w_1-w_2|^4} \right) \\ & = 2\int_{G_{z,\varepsilon}} dw_1 d\overline{w}_1 dw_2 d\overline{w}_2 \,\mathrm{Re}\left(\frac{1}{w_1^2}\frac{1}{\overline{w}_2^2} + \frac{1}{(w_1-w_2)^2}\left(\frac{1}{\overline{w}_1^2} + \frac{1}{\overline{w}_2^2} \right) \right) \end{aligned}$$

which has no logarithmic singularity in z. This is true more generally, as long as F above is a real analytic function where all singularities are of type $\frac{1}{z^2}+$ regular terms [Car87]. Thus $\mathcal{D}$ is exactly marginal iff it does not mix with other $(1;1)$ fields.
We can now use conclusion 2.2.1 to find conditions for holomorphic primary fields Φ in $\mathcal{C}$ to remain holomorphic along $\mathcal{C}_\delta$. Since the three point function we have to compute by conclusion 2.2.1 contains a one point function of the antiholomorphic part of $\mathcal{D}$ which vanishes, holomorphicity is always maintained to first order deformation theory. This of course shows that already to understand the continuous part of the moduli space of unitary conformal field theories with $c = 1$

[DVV88, Gin88b] which is described by compactification of a single boson on a circle of radius R, we need second order deformation theory. These models are the one dimensional cases of bosonic toroidal conformal field theory that will be introduced in section 4.1. Anticipating this discussion, we remark that the space of states of such a theory is generated by a holomorphic field of dimensions $(1;0)$, i.e. a $U(1)$ current j as above, its antiholomorphic analog, and an infinite number of ground states $|m,n\rangle$, $m,n\in\mathbb{Z}$, which have charges $Q_{m,n},\overline{Q}_{m,n}=\frac{1}{\sqrt{2}}(mR\pm\frac{n}{R})$ with respect to the currents $j,\overline{\jmath}$ and dimensions $(h;\overline{h})=(\frac{Q^2}{2};\frac{\overline{Q}^2}{2})$. Holomorphic vertex operators therefore exist for rational $R^2=\frac{r}{s}$ with $h_t=t^2rs, t\in\mathbb{Z}$, and $|m,n\rangle=|ts,tr\rangle$. As to the change of their dimensions along the line parametrized by R,

$$\frac{\partial}{\partial R}h_t(R)_{|R=\frac{r}{s}}=0,\quad \frac{\partial^2}{\partial R^2}h_t(R)_{|R=\frac{r}{s}}=Q^2_{ts,tr}.$$

Clearly, the fact that none of the holomorphic vertex operators exists for generic R is visible only in second order deformation theory. The deformation of R is generated by $\mathcal{D}=j\overline{\jmath}$ as in our above example, and we may compute

$$\langle ts,tr|\mathcal{D}(w_1,\overline{w}_1)\mathcal{D}(w_2,\overline{w}_2)|ts,tr\rangle_{conn}=\frac{\frac{1}{2}Q^2_{ts,tr}}{w_1w_2(\overline{w}_1-\overline{w}_2)^2}.$$

For large $|z|$ we can simplify the relevant expression in conclusion 2.2.1 with respect to polar coordinates $w_j=r_je^{i\varphi_j}$ by performing the r_j integration over the domain $\{\varepsilon<r_j<\frac{|z|^2}{\varepsilon},|r_1-r_2|>\varepsilon\}$ to see that integration over $G_{z,\varepsilon}$ gives an $\ln|z|$ type singularity. This is in accord with our above observation that $|ts,tr\rangle$ does not remain holomorphic to second order deformation theory.
It might seem that also in general it is simple to derive conditions for holomorphic $|\Phi\rangle\in V(h;0)$ to remain holomorphic along $\mathcal{C}_\delta$. The relevant four point function

$$\langle 0|\Phi(z_1)\mathcal{D}(w_1,\overline{w}_1)\mathcal{D}(w_2,\overline{w}_2)\Phi(z_2)|0\rangle$$

behaves as $z_i^{-2h_i}$ for $z_i\to\infty$, and as $|w_i|^{-4}$ for $w_i\to\infty$. If we assume the OPE $\Phi(z_1)\mathcal{D}(w_1,\overline{w}_1)$ to have leading singularity of order S we find that the integrand of the crucial integral in conclusion 2.2.1 for second order deformation theory is

$$\frac{p_{2S}(w_1,w_2)}{w_1^Sw_2^S|w_1-w_2|^4}$$

with a homogeneous symmetric polynomial p_{2S} of degree $2S$. By the substitution $v_i:=\frac{z^2}{w_i+z}$ we see that second order perturbation theory is entirely encoded in the $\ln z$ type singularities of the integrals

$$I_a:=\int_{\substack{|v_i-z|>\varepsilon,|v_i|>\varepsilon,\\|v_i-v_j|>\varepsilon}}\frac{dv_1d\overline{v}_1dv_2d\overline{v}_2}{|v_1-v_2|^4}\left(\left(\frac{v_1}{v_2}\right)^a\left(\frac{v_2-z}{v_1-z}\right)^a+\left(\frac{v_2}{v_1}\right)^a\left(\frac{v_1-z}{v_2-z}\right)^a\right),$$

$a\in\mathbb{N}$, which we unfortunately have no closed formula for. Above, we showed that I_1 has an $\ln z$ type singularity.

We have also tried to apply conclusion 2.2.1 to a nontrivial example. Since we do not want to present any details, here we anticipate some properties of the relevant model that will be discussed later in this work. We consider the Gepner type model $(\widetilde{2})^4$ (see theorem 3.1.18) with $c = 6$. By theorem 7.3.29, it has a nonlinear σ model description as $\mathbb{Z}_2$ orbifold of a toroidal conformal field theory on the torus $T^4 = \mathbb{R}^4/\Lambda$ with lattice $\Lambda = D_4$. Now let $\mathcal{C}_\delta$ denote the deformation of $(\widetilde{2})^4$ corresponding to a simultaneous blow up of all 16 singularities of type $\mathbb{Z}_2$ in the orbifold $T^4/\mathbb{Z}_2$ (see [Cve87, CLO88] for a discussion of related questions from the string theoretic point of view, mainly on Calabi–Yau threefolds). With conclusion 2.2.1 we are able to prove that of the generic $su(2)_1^2$ Kac–Moody algebra that every $\mathbb{Z}_2$ orbifold of a superconformal toroidal theory with $c = 6$ has (7.3.17), only $su(2)_1 = su(2)^{susy}$ of section 7.1 survives along $\mathcal{C}_\delta$. This of course was to be expected, since $\mathcal{C}_\delta$ does not stay within the subspace of $\mathbb{Z}_2$ orbifold conformal field theories in the moduli space. On the other hand, using the fact that the energy momentum tensor of $(\widetilde{2})^4$ must remain holomorphic along $\mathcal{C}_\delta$, we can deduce that

$$I_2 - 2I_1$$

does *not* have $\ln z$ type singularities, but a direct analytic argument is lacking. Summarizing one may say that conclusion 2.2.1 seems not yet to be formulated in a way that is useful for applications.

We close this chapter by introducing an idea advocated in [Zam87a] for the determination of generic holomorphic fields in some moduli space of unitary conformal field theories. Assume that a quasiprimary field Φ, $|\Phi\rangle \in V(h;0)$, of $\mathcal{C}$ is deformed to a family Φ^δ of fields along $\mathcal{C}_\delta$. If Φ^δ is not holomorphic for $\delta \neq 0$, the family $\overline{L}_1\Phi^\delta$ has dimensions $(h^\delta;\overline{h}^\delta)$ with $\overline{h}^\delta > 1$. Then $\Psi := \lim_{\delta\to 0} \overline{L}_1\Phi^\delta \in V(h;1)$, but clearly $\Psi \neq \overline{L}_1\Phi = 0$. Moreover, Ψ is quasiprimary, so the irreducible representation of the Virasoro algebra, which $|\Phi^\delta\rangle$ is a lowest weight state of, must split into smaller ones at $\delta = 0$. It follows that $\dim V(h;0) - \dim \lim_{\delta\to 0}(\overline{L}_1 V_\delta \cap V(h;1))$ fields are generically holomorphic with dimensions $(h;0)$ along $\mathcal{C}_\delta$. In particular, there must exist generic holomorphic fields in $V(h;0)$ if $\dim V(h;0) > \dim V(h;1)$. Another signal for the existence of generic holomorphic fields is a jump in the number of quasiprimary fields in $V(h^\delta;1)$ at $\delta = 0$.

These conditions suffice to show that in the $\mathbb{Z}_2$ orbifold component of the moduli space of unitary conformal field theories with $c = 1$ there is a generic part $V(4;0)$ of the space of states. Together with the energy momentum tensor it generates the generic W–algebra $\mathcal{W}(2,4)$ of this branch of the moduli space [Nah96].

As we will see in section 7.2, Zamolodchikov's method unfortunately does not give any insight for the case of conformal field theories on $K3$.

Chapter 3

Superconformal field theories (SCFTs)

We shall begin this chapter with a list of further properties that are presumed if we work with superconformal field theories. Section 3.1 is devoted to $N = (2,2)$ superconformal field theories. In section 3.1.1 we discuss chiral rings and the spectral flow. Section 3.1.2 contains a more detailed discussion of the elliptic genus, whereas section 3.1.3 introduces an important class of (rational!) $N = (2,2)$ superconformal field theories, namely the minimal models, Gepner models, and Gepner type models. In section 3.2 we briefly discuss properties of $N = (4,4)$ superconformal field theories we will need in chapter 7. More precisely, we consider $N = (2,2)$ superconformal field theories, where the superconformal algebra is linearly extended to the $N = (4,4)$ superconformal algebra studied by T. Eguchi and A. Taormina [ET87, ET88a, ET88b, ET88c, Tao90]. In section 3.2.1 we present a free field realization of this algebra, whereas in section 3.2.2 we list the characters of its irreducible representations and give some of their properties found in joint work with Anne Taormina.
Let us now turn to the axioms of superconformal field theories, again following unpublished ideas of Werner Nahm's [Nah].

Property 10
An $N = (1,1)$ superconformal field theory $\mathcal{C}$ possesses a bigraded infinite dimensional vector space $\mathcal{H}$ of states as in property 1 of section 2.1 which is also $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded,

$$\mathcal{H} = \mathcal{H}_b \oplus \mathcal{H}_f, \quad \mathcal{H}_k = \mathcal{H}_k^{NS} \oplus \mathcal{H}_k^{R}, k \in \{b, f\}.$$

$\mathcal{H}_b$ contains the purely BOSONIC STATES, *and $\mathcal{H}_f$ the purely* FERMIONIC STATES.
NS stands for NEVEU–SCHWARZ SECTOR, *and R for* RAMOND SECTOR.
$\mathcal{H}$ carries a real structure which induces a real structure on each of the sectors $\mathcal{H}_k^{\mathcal{S}}, k \in \{b, f\}, \mathcal{S} \in \{NS, R\}$.

It is should be understood that the latter assumption is equivalent to the requirement that each sector is invariant under conjugation.
If the conformal field theory arises from a string theory, the Neveu–Schwarz sector corresponds to spacetime bosons, and the Ramond sector to spacetime fermions.

We will return to this issue in the context of spectral flow in section 3.1.1.
An $N = (1,1)$ superconformal field theory is best understood as an extension of an ordinary conformal field theory as defined in section 2.1:

Property 11
The bosonic part $\mathcal{H}_b = \mathcal{H}_b^{NS} \oplus \mathcal{H}_b^R$ of the space of states is itself the space of states of a conformal field theory. Its OPE is compatible with the $\mathbb{Z}_2$ grading of $\mathcal{H}_b$, where Neveu–Schwarz states are regarded as even, and Ramond states as odd.

With property 11 one can show that $\mathcal{H}_b^R$ generates the entire space $\mathcal{H}_b$ by OPE, i.e. $\mathcal{H}_b^R \otimes \mathcal{H}_b^R \to \mathcal{H}_b^{NS}\{z,\overline{z}\}$, and the fields in the image of the OPE are in $1:1$ correspondence to states in $\mathcal{H}_b^{NS}$. Abbreviating this, we simply say that FUSION reads $[\mathcal{H}_b^R] \times [\mathcal{H}_b^R] = [\mathcal{H}_b^{NS}]$. Namely, $[\mathcal{H}_b^R] \times [\mathcal{H}_b^R]$ contains the vacuum state since $\mathcal{H}_b^R$ is invariant under conjugation. Then the assertion follows from the fact that $\mathcal{H}_b^R$ is a representation of $\mathcal{H}_b^{NS}$ and property 8. As to the other sectors contained in a superconformal field theory by property 10, the state–field correspondence (see the discussion below (2.1.3)) extends to $\mathcal{H}_f^{NS}$:

Property 12
For each state $|\Psi\rangle \in \mathcal{H}_f^{NS}$ there is a field $\Psi(z,\overline{z})$ such that $\Psi(0,0)|0\rangle = |\Psi\rangle$. The OPE of property 2 extends to $\mathcal{H}_f^{NS}$ and is compatible with the $\mathbb{Z}_2$ grading on $\mathcal{H}^{NS} = \mathcal{H}_b^{NS} \oplus H_f^{NS}$, where bosonic states are even and fermionic ones are odd.
The Neveu–Schwarz sector $\mathcal{H}^{NS}$ of the space of states is a fermionic representation of the OPE of $\mathcal{H}_b^{NS}$ (definition 2.1.1). The generators of the left and right Virasoro algebras act as in property 7, extending the n point functions to functions of n complex parameters as explained in section 2.1. Moreover, fusion is given by $[\mathcal{H}_f^{NS}] \times [\mathcal{H}_f^{NS}] = [\mathcal{H}_b^{NS}]$.

We remark that due to the additional signs in fermionic representations of the OPE, states $|\Psi\rangle \in \mathcal{H}_f^{NS}$ have half integer spin $h - \overline{h}$. If Ψ is a holomorphic fermionic field of dimension h in the Neveu–Schwarz sector, then its mode expansion is $\Psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Psi_r z^{r-h}$.
The remaining sector of the space of states in property 10 is now constructed by fusion from the others:

Property 13
The OPEs on $\mathcal{H}_b$ and $\mathcal{H}^{NS}$ can be extended to a nonlocal OPE $\mathcal{H}_f^{NS} \otimes \mathcal{H}_b^R \to \mathcal{H}_f^R\{\{z,\overline{z}\}\}$, where $\mathcal{H}\{\{z,\overline{z}\}\}$ denotes the space of functions $f = f(z,\overline{z})$, $f : \mathbb{C} \to \overline{\mathcal{H}}$, that are real analytic on $\mathbb{C}^*$ and have the following behaviour around $z = 0$:

$$f(z,\overline{z}) = \sum_{r\in R, n\in\mathbb{Z}} a_{rn} z^{r+n-\frac{1}{2}} \overline{z}^r,$$

with countable $R \subset \mathbb{R}$, $a_{rn} \in \mathcal{H}$, and only finitely many singular terms. Properties 3–7 can be adjusted accordingly to n point functions that contain arbitrary fields in $\mathcal{H}$. In particular, the action of fermionic Neveu–Schwarz fields on bosonic Ramond states is well defined and maps them surjectively into the fourth sector of the space of states: $[\mathcal{H}_f^{NS}] \times [\mathcal{H}_b^R] = [\mathcal{H}_f^R]$.

By our assumptions fermionic fields act on the entire Ramond sector. Consistency of the OPE shows that a fermionic holomorphic field Ψ of (half integer!) dimension h on this sector is represented by the operators Ψ_n in its mode expansion $\Psi(z) = \sum_{n\in\mathbb{Z}} \Psi_n z^{n-h}$. A fermionic field Ψ is never single valued in the Ramond sector: $\Psi(e^{2\pi i}z, e^{-2\pi i}\overline{z}) = -\Psi(z,\overline{z})$.
By properties 1–13 we have actually defined a FERMIONIC CONFORMAL FIELD THEORY. In general, the entire set of states in $\mathcal{H}$ is not pairwise local (rather SEMILOCAL, i.e. local up to signs), and the partition function of the fermionic conformal field theory is defined to be that of its bosonic subtheory with space of states $\mathcal{H}_b$. This is a conformal field theory of its own right by the properties listed in section 2.1. We now introduce the FERMION NUMBER OPERATOR $(-1)^F$ to be the unitary operator with eigenvalues ± 1 that commutes with bosonic fields and anticommutes with fermionic ones and such that $(-1)^F|0\rangle = |0\rangle$. Then we can decompose partition functions of superconformal field theories into four parts, with $z = 0$ if no $U(1)$ currents $J, \overline{J}$ exist:

$$\begin{aligned}
Z &= \tfrac{1}{2}\left(Z_{NS} + Z_{\widetilde{NS}} + Z_R + Z_{\widetilde{R}}\right),\\
Z_{NS}(\sigma,z) &:= \mathrm{tr}_{NS}\left[q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right],\\
Z_{\widetilde{NS}}(\sigma,z) &:= \mathrm{tr}_{NS}\left[(-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right],\\
Z_R(\sigma,z) &:= \mathrm{tr}_R\left[q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right],\\
Z_{\widetilde{R}}(\sigma,z) &:= \mathrm{tr}_R\left[(-1)^F q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}y^{J_0}\overline{y}^{\overline{J}_0}\right].
\end{aligned} \tag{3.1}$$

Recall that the transition from worldsheet coordinates (ξ_0,ξ_1) to radial coordinates $(z,\overline{z})$ exchanges twisted and untwisted boundary conditions on the fermions. In terms of the worldsheet torus $\Xi(\sigma)$, the insertion of $(-1)^F$ in the above vacuum correlators results in a trace over the sector where fermionic fields obey *untwisted* boundary conditions in direction of imaginary time. Modular transformations act on the worldsheet torus and can be interpreted as permuting the fermionic boundary conditions. In particular, $Z_{\widetilde{R}}$ is the trace over fermionic fields with untwisted boundary conditions on both worldsheet coordinates and so transforms modular covariantly on its own.
The simplest and quite instructive example for a fermionic conformal field theory is the ISING MODEL which describes a single Majorana fermion ψ and its antiholomorphic counterpart $\overline{\psi}$. The ground states of irreducible representations of the holomorphic W–algebra are denoted $|0\rangle, |\psi\rangle, |\overline{\psi}\rangle, |\varepsilon\rangle, |\sigma\rangle, |\mu\rangle$ and have conformal dimensions $(h;\overline{h})$ equal to $(0;0), (\frac{1}{2};0), (0;\frac{1}{2}), (\frac{1}{2};\frac{1}{2}), (\frac{1}{16};\frac{1}{16}), (\frac{1}{16};\frac{1}{16})$, respectively. The bosonic ground states are $|0\rangle, |\varepsilon\rangle, |\sigma\rangle$. From the fusion rules (see e.g. [Gin88a, (7.20)]) one reads that the four sectors of property 10 are given by

$$\begin{aligned}
\mathcal{H}_b^{NS} &= \mathrm{span}_{\mathbb{C}}(|0\rangle, |\varepsilon\rangle), & \mathcal{H}_b^{R} &= \mathrm{span}_{\mathbb{C}}(|\sigma\rangle),\\
\mathcal{H}_f^{NS} &= \mathrm{span}_{\mathbb{C}}(|\psi\rangle, |\overline{\psi}\rangle), & \mathcal{H}_f^{R} &= \mathrm{span}_{\mathbb{C}}(|\mu\rangle).
\end{aligned} \tag{3.2}$$

Note that $\mathcal{H}_b^R \cong \mathcal{H}_f^R$, but fusion is governed by the fact that the OPE must be compatible with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading: $[|\mu\rangle] \times [|\mu\rangle] = [|0\rangle] + [|\varepsilon\rangle], [|\mu\rangle] \times [|\sigma\rangle] = [|\psi\rangle] + [|\overline{\psi}\rangle]$. The partition function of the Ising model reads

$$Z_{Ising}(\sigma) = \frac{1}{2}\left(\left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right| + \left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right| + \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right|\right). \tag{3.3}$$

Here, $\vartheta_j(\sigma, z), j \in \{1, \ldots, 4\}$, are the classical Jacobi theta functions (see appendix A), and $\eta(\sigma)$ is the Dedekind eta function. For ease of notation we will write $\eta = \eta(\sigma), \vartheta_j(z) = \vartheta_j(\sigma, z)$, and $\vartheta_j = \vartheta_j(\sigma, 0)$ in the following.
The Ising model of course is not an $N = (1, 1)$ superconformal field theory. It fails to obey our last property:

Property 14
The Neveu–Schwarz sector $\mathcal{H}^{NS}$ contains an $N = (1, 1)$ superconformal algebra, i.e. apart from the bosonic Virasoro field $T(z)$ there is a fermionic holomorphic field $G(z)$ of dimension $h = 3/2$, such that $G(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r z^{r-3/2}$, and the operators G_r satisfy

$$\begin{aligned} [L_m, G_r] &= \left(r - \tfrac{m}{2}\right) G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \tfrac{c}{3}\left(r^2 - \tfrac{1}{4}\right)\delta_{r+s,0}. \end{aligned}$$

G is Hermitean, i.e. $(G(\overline{z}))^\dagger = G(z)$ or equivalently $(G_r)^\dagger = G_{-r}$, and there is an analogous field $\overline{G}$ on the right handed side.

To end the axiomatic part of this thesis, we remark that a superconformal field theory is not uniquely defined by its bosonic sector $\mathcal{H}_b$. We will meet examples of this phenomenon in section 6.3, and remarks 7.3.26, 7.3.30.

3.1 $N = (2, 2)$ Superconformal field theories

By definition, an $N = (2, 2)$ superconformal field theory $\mathcal{C}$ is an $N = (1, 1)$ superconformal field theory, where the supercurrent G of property 14 splits into two fields $G = \frac{1}{\sqrt{2}}(G^+ + G^-)$ in the Neveu–Schwarz sector such that $(G^+(\overline{z}))^\dagger = G^-(z)$. Moreover,

$$\begin{aligned} [L_m, G_r^\pm] &= \left(r - \tfrac{m}{2}\right) G_{m+r}^\pm, \\ \{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0, \\ \{G_r^+, G_s^-\} &= 2L_{r+s} + (s - r)J_{r+s} + \tfrac{c}{3}\left(r^2 - \tfrac{1}{4}\right)\delta_{r+s,0}, \\ [L_m, J_n] &= mJ_{m+n}, \quad [J_m, G_r^\pm] = \pm G_{m+r}^\pm, \quad [J_m, J_n] = \tfrac{c}{3} n \delta_{m+n,0}, \end{aligned} \tag{3.1.1}$$

so there are fields in $\mathcal{C}$ that realize the linear extension of the $N = (1, 1)$ superconformal algebra to the $N = (2, 2)$ superconformal algebra. The moding is given by $m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$ in the Neveu–Schwarz and $m, n, r, s \in \mathbb{Z}$ in the Ramond sector. We do not discuss the twisted $N = (2, 2)$ superconformal algebra [CF75].

The $U(1)$ currents $(J;\overline{J})$ measure the charges $(Q;\overline{Q})$. It is of considerable importance that the $U(1)$ charges can be decoupled from any field Φ of the theory. To do so, we bosonize $J,\overline{J}$ to $J = i\sqrt{\frac{c}{3}}\partial H, \overline{J} = i\sqrt{\frac{c}{3}}\partial\overline{H}$. Then any field Φ with charges $(Q;\overline{Q})$ can be written as

$$\Phi = :e^{i\sqrt{\frac{3}{c}}(QH-\overline{QH})}P(J;\overline{J}): \widetilde{\Phi}, \tag{3.1.2}$$

where up to a cocycle factor $P(J;\overline{J})$ is a polynomial in $J,\overline{J}$, and its derivatives, and $\widetilde{\Phi}$ commutes with $J,\overline{J}$, and $H,\overline{H}$ [Sen86, Sen87, DFMS87].
To discuss moduli spaces of $N = (2,2)$ superconformal field theories, we have to consider deformations of such theories which preserve the supersymmetry. In section 2.2 we explained that by standard conjectures a deformation of a conformal field theory $\mathcal{C}$ is generated by some field $\mathcal{D}(z,\overline{z})$ in $\mathcal{C}$. In order to preserve conformal symmetry and the central charge, $\mathcal{D}(z,\overline{z})$ must be exactly marginal. Using the BRST formalism one can show that the deformation generated by a field $\mathcal{D}(z,\overline{z})$ of dimensions $(h;\overline{h}) = (1;1)$ preserves $N = (1,1)$ supersymmetry iff it is the top component of an $N = (1,1)$ multiplet [FMS86]. By Kadanoff's criterion (conclusion 2.2.1) one can check that in case that the $N = (1,1)$ superconformal algebra is extended to an $N = (2,2)$ superconformal algebra, such fields are automatically exactly marginal [Dix87] and preserve the $N = (2,2)$ superconformal algebra. Thus we find

Conjecture 3.1.1
Deformations of an $N = (2,2)$ superconformal field theory $\mathcal{C}$ that preserve the superconformal structure correspond to the fields $d(z,\overline{z})$ of dimensions $(h;\overline{h}) = (\frac{1}{2};\frac{1}{2})$ and charges $|Q| = |\overline{Q}| = 1$ in $\mathcal{C}$. Application of the appropriate supercurrents $G^{\pm}$ produces an exactly marginal operator $\mathcal{D}(z,\overline{z})$ that generates the deformation as discussed in section 2.2.

In particular, we see that the moduli space of $N = (2,2)$ superconformal field theories appears to have a well defined tangent space isomorphic to the vector space spanned by fields with quantum numbers $h = \overline{h} = \frac{1}{2}$ and $|Q| = |\overline{Q}| = 1$. If no further supersymmetry is present, this decomposes into a product of spaces corresponding to $(\frac{1}{2};\frac{1}{2})$ fields with $Q = \overline{Q}$ or $Q = -\overline{Q}$, respectively. The moduli space then locally decomposes into a product, too.

3.1.1 Chiral rings and spectral flow

To discuss chiral rings, we take a short detour to the study of irreducible unitary representations of the $N = 2$ superconformal algebra (3.1.1). Here, the central charge c is assumed to take arbitrary values. At special values, namely for the minimal series, different character formulas hold, see e.g. section 3.1.3. A generic representation with lowest weight state $|\Phi\rangle$ of dimension h and charge Q can be built as Fock space representation by acting with creation operators $L_m, m > 0, J_m, m > 0, G_r^{\pm}, r > 0$ on $|\Phi\rangle$. The generic character in the NS sector therefore is given by [BFK86]

$$\begin{aligned}\chi^{NS}_{h,Q}(\sigma,z) &= q^h y^Q \chi^{NS}_{gen}(\sigma,z),\\ \chi^{NS}_{gen}(\sigma,z) &:= q^{-\frac{c}{24}}\prod_{n=1}^{\infty}(1-q^n)^{-2}\left(1+yq^{n-\frac{1}{2}}\right)\left(1+y^{-1}q^{n-\frac{1}{2}}\right)=\frac{q^{\frac{1}{8}-\frac{c}{24}}}{\eta^2}\frac{\vartheta_3(z)}{\eta}.\end{aligned} \tag{3.1.3}$$

The above is the correct character, as long as no NULL VECTORS $|\Psi\rangle \in \mathcal{H} : \forall\, |\Psi'\rangle \in \mathcal{H}\ \langle\Psi'|\Psi\rangle = 0$ occur, which is true generically. Unitarity imposes restrictions on the possible values of (h,Q), and precisely at the unitarity bound it may happen that additional null vectors appear. The simplest example of this phenomenon arises from the observation that by (3.1.1), in the NS sector,

$$\{G^+_{\pm\frac{1}{2}}, G^-_{\mp\frac{1}{2}}\} = 2L_0 \mp J_0,$$

which is a nonnegative operator since $(G^+_r)^\dagger = G^-_{-r}$. In other words, $h \geq \frac{|Q|}{2}$ in the NS sector, and precisely at the bound $h = \frac{|Q|}{2}$ the Fock space representation built on $|\Phi\rangle$ will contain additional null vectors and split into smaller irreducible representations. States with $h = \pm\frac{Q}{2}$ are called CHIRAL or ANTICHIRAL, respectively, and are automatically primary. In general, the domain of unitarity in the (h,Q) plane is surrounded by a convex polygonal curve. If (h,Q) hits the unitary bound in a corner which can correspond to a double or triple intersection point of edges, additional null vectors do occur. The characters then are

$$\begin{aligned} \chi^{II,NS}_{h,Q}(\sigma,z) &= \frac{q^h y^Q}{1+y^{\operatorname{sign} m}q^{|m|-\frac{1}{2}}}\,\chi^{NS}_{gen}(\sigma,z)\\ \text{or}\quad \chi^{III,NS}_{h,Q}(\sigma,z) &= \frac{q^h y^Q(1-q)}{\left(1+y^{\operatorname{sign} m}q^{|m|-\frac{1}{2}}\right)\left(1+y^{\operatorname{sign} m}q^{|m|+\frac{1}{2}}\right)}\,\chi^{NS}_{gen}(\sigma,z)\end{aligned} \tag{3.1.4}$$

for a double or a triple intersection point, respectively, where $m \in \mathbb{Z}$ [Dob87]. The corresponding representations are called MASSLESS as opposed to the MASSIVE generic representations (3.1.3) above. We see that for general chiral or antichiral states $m = \pm 1$ in the first formula of (3.1.4), corresponding to a double point. On the other hand, the same reasoning as for $G^\pm_{1/2}$ now applied to $G^\pm_{3/2}$ leads to the bound $h \leq \frac{c}{6}$ for chiral and antichiral states. If $h = \frac{c}{6}$ for an (anti–)chiral state, the second line in (3.1.4) applies, again with $m = \pm 1$. Note that by (3.1.2) there can be at most one chiral and one antichiral state $|\Phi\rangle$ with $h = \frac{c}{6}$ since all parts of Φ that commute with H must have vanishing dimension.

It is not hard to see that the OPE of chiral fields Φ^1, Φ^2 cannot contain singular terms and that their normal ordered product $\lim_{z\to 0}\Phi^1(z)\Phi^2(0)$ is a chiral field again. Hence the normal ordered product defines a multiplication on the sets of (anti–)chiral fields, turning them into rings, the so–called CHIRAL and ANTICHIRAL RINGS. Recalling left and right handedness, we find four interesting rings in every unitary conformal field theory $\mathcal{C}$, denoted by $(c,c), (a,c), (c,a), (a,a)$, respectively. In fact, since $h, \overline{h} \leq \frac{c}{6}$ and dimensions may not have an accumulation point by property 9 of section 2.1, these rings are finite.

Since the vacuum character of the $N = 2$ superconformal algebra (3.1.1) is

$$\chi_0(\sigma, z) = \frac{1 - q}{\left(1 + yq^{\frac{1}{2}}\right)\left(1 + y^{-1}q^{\frac{1}{2}}\right)}\,\chi_{gen}^{NS}(\sigma, z),$$

it is easy to compute quantum dimensions for generic unitary representations:

Definition 3.1.2
Let $\{\chi_i\}_{i\in I}$ denote the set of characters of the holomorphic W–algebra $\mathcal{W}$ of a unitary conformal field theory $\mathcal{C}$. The unique irreducible representation of $\mathcal{W}$ containing the vacuum is called VACUUM REPRESENTATION, *and its character is denoted χ_0. Then the representation with label i has* QUANTUM DIMENSION

$$i \in I: \quad d_i := \lim_{\sigma\to 0} \frac{\chi_i(\sigma)}{\chi_0(\sigma)}.$$

By definition, the quantum dimension of the vacuum representation is 1, and the same is true for the generic representation in a triple point at the unitarity bound. All other generic quantum dimensions are ∞, so quantum dimensions appear not to be a good means to characterize generic irreducible unitary representations of the $N = 2$ superconformal algebra, and in particular the (anti–) chiral rings. It proves useful [Nah] to introduce RELATIVE QUANTUM DIMENSIONS

$$d_i' := \lim_{\sigma\to 0} \frac{\chi_i(\sigma)}{\chi_{gen}(\sigma)}.$$

Now, the generic massive representations have relative quantum dimension 1, and massless representations have relative quantum dimension $\frac{1}{2}$ or 0, if they correspond to a double or triple point, respectively. Since we assume all data of the conformal field theory to depend analytically on parameters of deformation, this observation in particular proves the following standard folk–lore on chiral rings:

Theorem 3.1.3
The number of fields with given left and right charges in the $(c, c), (a, c), (c, a), (a, a)$ rings of unitary $N = (2, 2)$ superconformal field theories with fixed central charge is constant over generic points of each component of the moduli space. It can only increase in nongeneric points.

Since by conjecture 3.1.1 the fields with $h = \overline{h} = \frac{1}{2}$ in the direct sum of these rings are in $1 : 1$ correspondence with tangent vectors of the moduli space, theorem 3.1.3 in particular implies that a tangent sheaf to the moduli space can indeed be defined.
Above, we have concentrated on the NS sector of our superconformal field theory. An analogous situation of course occurs in the Ramond sector, where the expectation value of $\{G_0^+, G_0^-\}$ imposes the bound $h \geq \frac{c}{24}$ on all states. At the unitarity bound we find the RAMOND GROUND STATES, and the analog of theorem 3.1.3 applies to the set of Ramond ground states in an $N = (2, 2)$ superconformal field theory.

The similarity of the two sectors is not an accident; if our superconformal field theory arises from string theory the structures are isomorphic, in fact. The reason is that (3.1.1) actually describes a one parameter family of $N=2$ superconformal algebras, if we allow $r \in \mathbb{R}$ for the moding of $G_r^{\pm}$. More precisely, for every $\vartheta \in [-\frac{1}{2}, \frac{1}{2})$ the modes $L_n^{\vartheta}, J_n^{\vartheta}, G_r^{\vartheta}$,

$$L_n^{\vartheta} := L_n + \vartheta J_n + \frac{c}{6}\vartheta^2 \delta_{n,0}, \quad J_n^{\vartheta} := J_n + \frac{c}{3}\vartheta\delta_{n,0}, \quad G_r^{\vartheta} := G_{r\pm\vartheta}^{\pm} \tag{3.1.5}$$

generate an $N=2$ superconformal algebra. $\vartheta = -\frac{1}{2}, \vartheta = 0$ give the representations in the NS and the R sector in (3.1.1). Hence each representation of the $N=2$ superconformal algebra in the Ramond sector with character χ^R can be obtained from a representation in the Neveu–Schwarz sector with character χ^{NS} such that

$$\chi^R(\sigma, z) = q^{\frac{c}{24}} y^{\frac{c}{6}} \chi^{NS}(\sigma, z + \frac{\sigma}{2}). \tag{3.1.6}$$

(3.1.5) is generated by a $U(1)$ gauge transformation, which for $\vartheta = \pm\frac{1}{2}$ is known as SPECTRAL FLOW with operator

$$U_{\vartheta} := :e^{i\sqrt{\frac{c}{3}}\vartheta H}:, \quad \vartheta \in \left\{\pm\tfrac{1}{2}\right\}, \tag{3.1.7}$$

where we have bosonized the $U(1)$ current $J = i\sqrt{\frac{c}{3}}\partial H$ as for (3.1.2). If the vertex operators $U_{\pm\frac{1}{2}}$ of (3.1.7) are realized as fields in our theory, they act on the irreducible representations of the Virasoro algebra contained in $\mathcal{C}$ by transforming NS representations into R ones and vice versa. Since $[U_{\frac{1}{2}}] \times [U_{-\frac{1}{2}}] = [\mathbb{1}]$, this is an isomorphism, and because in the context of string theory states in the NS and R sectors correspond to spacetime bosons and fermions, respectively, spectral flow gives the action of SPACETIME SUPERSYMMETRY on the conformal field theory. In order not to destroy the left–right coupling we generally use the left–right symmetric combinations of spectral flows. Then spectral flow acts on a state with quantum numbers $(h, Q\,; \overline{h}, \overline{Q})$ in our theory by

$$(h, Q\,; \overline{h}, \overline{Q}) \quad \stackrel{U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}}{\longmapsto} \quad \left(h \pm \tfrac{Q}{2} + \tfrac{c}{24}, Q \pm \tfrac{c}{6}\,; \overline{h} \pm \tfrac{\overline{Q}}{2} + \tfrac{c}{24}, \overline{Q} \pm \tfrac{c}{6}\right). \tag{3.1.8}$$

In particular, $U_{\frac{1}{2}}\overline{U}_{\frac{1}{2}}$ maps antichiral states to Ramond ground states and Ramond ground states to chiral states, and a ring structure isomorphic to the (c, c) ring can be imposed on the set of Ramond ground states of our theory. Moreover, (c, c) and (a, a) rings are isomorphic, and the (c, a) ring is isomorphic to the (a, c) ring.
By our $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading in the space of states of a superconformal field theory, the operators of spectral flow must belong to states in the Ramond sector. By locality it now follows from (3.1.2) that the operators of spectral flow can only belong to the theory if the charges $(Q; \overline{Q})$ of all states in $\mathcal{H}$ obey $Q - \overline{Q} \in \mathbb{Z}$, and $Q - \overline{Q}$ is even for those in $\mathcal{H}_b$. Vice versa, if this condition on the charges holds for all states of a superconformal field theory $\mathcal{C}$, then the operator of spectral flow is semilocal to all fields in the theory and local exactly to the bosonic ones. Hence the representation

of the OPE of our theory $\mathcal{C}$ can be extended to the corresponding representation of the $N = 2$ superconformal algebra built on $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$. In property 8, however, we have assumed that $\mathcal{C}$ cannot be contained in another consistent theory with the same central charge. It follows that the spectral flow fields are already realized as fields in $\mathcal{C}$.

By the above, $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$ are realized as fields in our theory iff the unitary operator $e^{i\pi(J_0-\overline{J}_0)}$ only has eigenvalues ± 1 on states of our theory. Since

$$e^{i\pi J_0}G^{\pm}(z)e^{-i\pi J_0} = -G^{\pm}(z),$$

and the vacuum has $(h,Q;\overline{h},\overline{Q}) = (0,0;0,0)$, we can then set $(-1)^F = e^{i\pi(J_0-\overline{J}_0)}$, and from (3.1.8) we see that $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$ are bosonic. It is a simple task to write down the relations between the four parts of the partition function introduced in (3.1). Summarizing, we have

Theorem 3.1.4
An $N = (2,2)$ superconformal field theory is invariant under (left–right symmetric) spectral flow, iff all states in the theory have charges $(Q;\overline{Q})$ with $Q - \overline{Q} \in \mathbb{Z}$ such that they are bosonic iff $Q - \overline{Q}$ is even. In this case, the fields that generate the spectral flows are realized as fields in the theory. Moreover, given the Neveu–Schwarz part Z_{NS} already the entire partition function of the theory can be determined:

$$\begin{array}{rcll} Z_R(\sigma,z) &=& (q\overline{q})^{\frac{c}{24}}(y\overline{y})^{\frac{c}{6}}Z_{NS}(\sigma,z+\frac{\sigma}{2}), & \\ Z_{\widetilde{NS}}(\sigma,z) &=& Z_{NS}(\sigma,z+\frac{1}{2}), & Z_{\widetilde{R}}(\sigma,z) = Z_R(\sigma,z+\frac{1}{2}). \end{array} \tag{3.1.9}$$

Finally we remark that by [Dix87] the $N = (1,1)$ superconformal algebra of a theory that is invariant under spectral flow is automatically extended to the $N = (2,2)$ superconformal algebra (3.1.1).

3.1.2 Witten index and elliptic genus

In theorem 3.1.3 we observed that the number of fields with given charges in the (c,c) ring of a unitary $N = (2,2)$ superconformal field theory is fixed over generic points of the moduli space. The numbers may increase over nongeneric points, however, and therefore do not give good invariants on the moduli space. That one can construct a combination of characters, the elliptic genus, which remains invariant on each irreducible component of the moduli space of $N = (2,2)$ superconformal field theories was first pointed out by Witten and will be the object of this section. We assume that our theories are invariant under spectral flow throughout this section, and set $(-1)^F = e^{\pi i(J_0-\overline{J}_0)}$.

Let us start by introducing the Witten index, a special case of the elliptic genus which is also defined for every character of a representation of the $N = 2$ superconformal algebra. Consider such a representation in the Ramond sector with space of states $\mathcal{H}_i$ and character χ_i^R. If for $|\Phi\rangle \in \mathcal{H}_i$ we have $G_0^+|\Phi\rangle \neq 0$, then by (3.1.1)

$(|\Phi\rangle, G_0^+|\Phi\rangle)$ form a pair of states with identical dimensions but charge shifted by 1. The same reasoning applies to G_0^-. On the other hand, as was discussed in section 3.1.1, $G_0^+|\Phi\rangle = G_0^-|\Phi\rangle = 0$ iff $|\Phi\rangle$ is a Ramond ground state, i.e. obeys $h = \frac{c}{24}$. Hence the expression

$$I := \mathrm{tr}_{\mathcal{H}_i}\, e^{\pi i J_0} q^{L_0 - \frac{c}{24}} = \chi_i^R(\sigma, z = \tfrac{1}{2}) \tag{3.1.10}$$

counts Ramond ground states, weighted by their charge, and in particular is a constant. This motivates

Definition 3.1.5
The WITTEN INDEX I *of a representation of the* $N = 2$ *superconformal algebra in the Ramond sector with character* χ_i^R *is defined by (3.1.10). For a Neveu–Schwarz sector representation with character* χ_i^{NS}*, one has to apply spectral flow (3.1.6) to define*

$$I := e^{\frac{2\pi i c}{12}} q^{\frac{c}{24}} \chi_i^{NS}(\sigma, z = \frac{\sigma+1}{2}).$$

Let us compute the Witten indices for the generic representations discussed in section 3.1.1: From (3.1.3) we directly read $I_{gen} = 0$ for generic representations without null vectors; for representations at the unitary bound corresponding to double points, (3.1.4) shows

$$I_{h,Q}^{II} = \lim_{\zeta \to -1} \frac{e^{\frac{2\pi i c}{12}} q^{h+\frac{Q}{2}} \zeta^Q}{1 + (\zeta q^{\frac{1}{2}})^{\mathrm{sign}\, m} q^{|m| - \frac{1}{2}}} \prod_{n=1}^{\infty} \frac{(1+\zeta q^n)(1+\zeta^{-1} q^{n-1})}{(1-q^n)^2}.$$

Hence $I_{h,Q}^{II}$ will vanish unless $m = -1$, in which case we are discussing a representation built on an antichiral state, since the Witten index is a constant. By analogous reasoning for triple points

$$I^{II}_{-\frac{Q}{2},Q} = I^{III}_{-\frac{Q}{2},Q} = e^{2\pi i (\frac{Q}{2} + \frac{c}{12})},$$

and vanishing Witten indices everywhere else. Let us now define the elliptic genus as it occurs in conformal field theory:

Definition 3.1.6
The CONFORMAL FIELD THEORETIC ELLIPTIC GENUS *of a unitary* $N = (2,2)$ *superconformal field theory* $\mathcal{C}$ *with Ramond sector of states* $\mathcal{H}^R$ *is given by*

$$\mathcal{E}(\sigma, z) := \mathrm{tr}_{\mathcal{H}^R} (-1)^F y^{J_0} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}.$$

From the above, the Witten index of a conformal field theory is just its elliptic genus evaluated at $z = 0$. The most important properties of the elliptic genus for our purposes are summarized by

Theorem 3.1.7
The conformal field theoretic elliptic genus is independent of $\overline{q}$ *and transforms covariantly under conformal transformations of* σ*, i.e.*

$$\mathcal{E}(\sigma+1, z) = \mathcal{E}(\sigma, z), \quad \mathcal{E}(-\frac{1}{\sigma}, \frac{z}{\sigma}) = e^{2\pi i \frac{c}{6} \frac{z^2}{\sigma}} \mathcal{E}(\sigma, z).$$

It is an invariant on each irreducible component of the moduli space of $N = (2,2)$ superconformal field theories with given central charge c.

Proof:
Modular covariance of $\mathcal{E}$ follows from the fact that the $\widetilde{R}$ part $Z_{\widetilde{R}}$ of the partition function in (3.1) itself transforms covariantly under modular transformations, and by definition 3.1.6

$$\mathcal{E}(\sigma, z) = Z_{\widetilde{R}}(\sigma, z\,;\,\overline{\sigma}, \overline{z} = 0). \tag{3.1.11}$$

Let us decompose the space of states $\mathcal{H}^R$ into tensor products of left and right handed representations of the $N = (2,2)$ superconformal algebra:

$$\mathcal{H}^R = \bigoplus_{(j,j^*)\in J} \mathcal{H}^R_j \otimes \mathcal{H}^R_{j^*}$$

with corresponding characters $\chi^R_j, \chi^R_{j^*}$. Then by definitions 3.1.6 and 3.1.5,

$$\mathcal{E}(\sigma, z) = \sum_{(j,j^*)\in J} \chi^R_j(\sigma, z + \tfrac{1}{2})\chi^R_{j^*}(\overline{\sigma}, \overline{z} = \tfrac{1}{2}) = \sum_{(j,j^*)\in J} \chi^R_j(\sigma, z + \tfrac{1}{2}) I_{j^*}, \tag{3.1.12}$$

which is independent of $\overline{q}$. We now insert the generic characters of $N = 2$ representations of section 3.1.1 in (3.1.12) as well as their Witten indices that were computed above. Hence the only representations that contribute to the elliptic genus are those which on the right hand side are built on a Ramond ground state. The quantum numbers $(\overline{h} = \frac{c}{24}, \overline{Q})$ cannot change over generic points of the moduli space by theorem 3.1.3 since spectral flow was assumed to act as isomorphism on each theory. Since $Q - \overline{Q} \in \mathbb{Z}$ and $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$, neither can the quantum numbers (h, Q) change. If on the other hand an additional right handed representation becomes massless in a nongeneric point of the moduli space, a massive representation splits into a number of (three) massless ones, whose Witten indices by the very definition 3.1.5 still add up to zero. It follows that the net effect on the elliptic genus is also zero if a massive representation hits the unitarity bound. $\square$

Let us briefly comment on the rôle of the elliptic genus in theoretical physics. In [Wit94], Witten pointed out that if the conjectured Landau–Ginzburg description of minimal models [Mar89, VW89] were true, this should be directly visible in an agreement of their elliptic genera. The conjectured agreement was proven shortly after in [FY93]. Witten had in particular rederived the free field realization [FGMP91, FGLS92] of the $N = 2$ superconformal algebra in Landau–Ginzburg language, relating the elliptic genus to the index of the supercharge G, and the $U(1)$ symmetry of the $N = 2$ superconformal field theory to the R symmetry of the Landau–Ginzburg model. The latter observation is also a basic tool in the proof [FY93]. The interpretation of the elliptic genus as loop space index of a loop space Dirac operator obtained from a supercharge G had become of importance in the context of string theory even before [SW86, Wit85, AKMW87, Wit87]. It fits well into the standard conjectures on chiral rings, based on [LVW89]. Namely, suppose that in an $N = (2,2)$ superconformal field theory $\mathcal{C}$ with central charge

$c = 3d/2, d/2 \in \mathbb{N}$, only integer and half integer charges appear both for the left and the right movers. Then the (c,c) ring of $\mathcal{C}$ is strongly believed to be isomorphic to the cohomology of a Calabi–Yau manifold of complex dimension d. More precisely, if $\mathcal{C}$ has a nonlinear sigma model interpretation on some Calabi–Yau manifold X, then the (c,c) ring is isomorphic to $H^{*,*}(X,\mathbb{C})$. Indeed, the anticommutation relations (3.1.1) of the supercharges $G^{\pm}$ directly show that we can build a G^{+}-cohomology on the (c,c) ring [LVW89], and countless evidence in favor of the conjecture has been collected up to now. In section 5.2.2 we will show that even if the assumption of left and right handed (half) integer charges is dropped, one may find geometric interpretations for the chiral rings. As to the elliptic genus, for a lot of superconformal field theories that satisfy the above assumptions on charges and central charge the conformal field theoretic elliptic genus agrees with the geometric elliptic genus of the associated Calabi–Yau manifold [EOTY89, KYY94] with

Definition 3.1.8 [Zag86, Tau89, Hir88, Wit88]
Let X denote an (almost) complex compact manifold of complex dimension $\frac{d}{2}$, and T its tangent bundle. For an arbitrary parameter t and a vector bundle E one denotes $\Lambda_t E := \sum_i t^i \Lambda^i E, S_t E := \sum_i t^i S^i E$, where $\Lambda^i E, S^i E$ are the antisymmetric and the symmetric tensor products of i copies of E. Then

$$\mathbb{E}_{q,y} := \bigotimes_{n=0}^{\infty} \Lambda_{-yq^{n-1}} T^* \otimes \bigotimes_{n=0}^{\infty} \Lambda_{-y^{-1}q^n} T \otimes \bigotimes_{n=0}^{\infty} S_{q^n} T^* \otimes \bigotimes_{n=0}^{\infty} S_{q^n} T,$$

and the GEOMETRIC ELLIPTIC GENUS *of X is*

$$\mathcal{E}_X(\sigma, z) := y^{-\frac{d}{4}} \int_X ch(\mathbb{E}_{q,y}) Td(T).$$

Apart from the fact that the geometric elliptic genus shares all properties of the conformal field theoretic elliptic genus and in particular is an invariant on the moduli space of Calabi–Yau manifolds of given topology, we note the following very useful

Theorem 3.1.9
The geometric elliptic genus $\mathcal{E}_X$ of a Calabi–Yau manifold X takes the following special values:

$$\begin{aligned} \mathcal{E}_X(\sigma, z=0) &= \chi(X), \quad \mathcal{E}_X(\sigma, z=\tfrac{1}{2}) = \sigma(X) + \mathcal{O}(q), \\ q^{\frac{d}{4}} \mathcal{E}_X(\sigma, z=\tfrac{\sigma+1}{2}) &= \widehat{A}(X) + \mathcal{O}(q), \end{aligned}$$

where $\chi(X), \sigma(X), \widehat{A}(X)$ denote the Euler characteristic, the signature, and the $\widehat{A}$ genus of X, well known indices of geometric Dirac operators defined on X.

As to the cases that will become of importance in chapter 7, we remark

Theorem 3.1.10
The elliptic genus of a complex two torus vanishes, and the elliptic genus of a $K3$ surface X is

$$\begin{aligned}\mathcal{E}_{X=K3}(\sigma,z) &= 24\left(\frac{\vartheta_3(z)}{\vartheta_3}\right)^2 - 2\frac{\vartheta_4^4-\vartheta_2^4}{\eta^4}\left(\frac{\vartheta_1(z)}{\eta}\right)^2\\ &= \frac{2}{\eta^6}\left(\vartheta_2(z)^2\vartheta_3^2\vartheta_4^2+\vartheta_3(z)^2\vartheta_2^2\vartheta_4^2+\vartheta_4(z)^2\vartheta_2^2\vartheta_3^2\right).\end{aligned}$$

Theorem 3.1.10 can be easily proven by making use of the properties of theta functions as well as theorem 3.1.9 with known values for $\chi(X)$ and $\widehat{A}(X)$; we extensively use the theta function formulas listed in appendix A.
Despite the intriguing geometric interpretation of the conformal field theoretic elliptic genus, in the proof of theorem 3.1.7 there was no need to make use of any additional assumptions concerning left and right charges in $\mathcal{C}$, nor of the loop space index of a supercharge. For later convenience and the sake of completeness we list some basic properties of the conformal field theoretic elliptic genus that accordingly hold for the geometric elliptic genus as well:

Theorem 3.1.11
Consider an $N = (2,2)$ superconformal field theory with central charge $c = 3d/2, d/2 \in \mathbb{N}$, which is invariant under spectral flow and such that all left and right charges in the Ramond sector are integer or half integer, depending on c being even or odd. Then the conformal field theoretic elliptic genus is a theta function of degree $n = \frac{c}{3}$ and characteristic $(0,0;-2\pi in, b), e^b = q^{-\frac{c}{6}}$, (definition A1.1) for every fixed parameter $\sigma \in \mathbb{H}$.

Proof:
$\mathcal{E}(\sigma,z+1) = \mathcal{E}(\sigma,z)$ follows directly from our assumption on the charges in $\mathcal{C}$ together with (3.1.12) and

$$\chi^R_{gen}(\sigma,z) = \frac{y^{\frac{c}{6}-\frac{1}{2}}}{\eta^2}\frac{\vartheta_2(z)}{\eta},$$

as obtained from (3.1.3) together with (3.1.6). On the other hand, spectral flow was assumed to provide an isomorphism on the space of states $\mathcal{H}$ of $\mathcal{C}$. Twofold spectral flow then gives an isomorphism of $\mathcal{H}^R$ which in particular leaves the partition function invariant. Theorem 3.1.4 applied to (3.1.11) then shows

$$\mathcal{E}(\sigma,z+\sigma) = q^{-\frac{c}{6}}y^{-\frac{c}{3}}\mathcal{E}(\sigma,z).$$

$\square$

By theorem 3.1.11 and the general properties of theta functions discussed in appendix A we can fix the elliptic genus once we know it for $\frac{c}{3} = \frac{d}{2}$ special values of z. Theorem 3.1.9 then becomes very useful. It is also worthwhile mentioning that for the cases discussed in theorem 3.1.11 the elliptic genus is already entirely fixed

by its leading order behaviour, or more precisely [FY93] by the central charge of $\mathcal{C}$ and the limit $\lim_{\sigma\to i\infty}\mathcal{E}(\sigma,z)$. In other words, knowledge of the Ramond ground states and their charges fixes the elliptic genus, which by theorem 3.1.7 is an important invariant on the respective component of the moduli space of $N=(2,2)$ superconformal field theories.
We will now state a kind of "inverse" of theorem 3.1.10 which will prove very useful for the discussion of the moduli space of $N=(4,4)$ superconformal field theories with central charge $c=6$ in chapter 7:

Theorem 3.1.12
Suppose that $\mathcal{C}$ is an $N=(2,2)$ superconformal field theory with central charge $c=6$ which is invariant under spectral flow and such that all left and right charges are integer. Then the conformal field theoretic elliptic genus $\mathcal{E}$ of $\mathcal{C}$ is an integer multiple of $\frac{1}{2}\mathcal{E}_{X=K3}$, where $\mathcal{E}_{X=K3}$ is the geometric elliptic genus of the $K3$ surface given in theorem 3.1.10.

Proof:
From theorem 3.1.11 we know that $\mathcal{E}$ is a theta function of degree $n=2$ with characteristic $(0,0;-2\pi in,b), e^b=q^{-1}$. By the general properties of theta functions (see appendix A), the space $\mathcal{T}_2(q^{-1})$ of such functions is twodimensional, and $\vartheta_1(z)^2,\vartheta_3(z)^2\in\mathcal{T}_2(q^{-1})$ are linearly independent. Therefore,

$$\mathcal{E}(\sigma,z)=a(\sigma)\left(\frac{\vartheta_3(z)}{\vartheta_3}\right)^2+b(\sigma)\frac{(\vartheta_1(z))^2}{\eta^6}.$$

Since the Witten index $\mathcal{E}(\sigma,z=0)$ of $\mathcal{C}$ must be constant in σ, we know that a is actually independent of σ, which justifies the ansatz

$$\mathcal{E}(\sigma,z)=a\mathcal{E}_{X=K3}(\sigma,z)+f(\sigma)\frac{(\vartheta_1(z))^2}{\eta^6}.$$

The leading order term in $\frac{(\vartheta_1(z))^2}{\eta^6}$ is 1. Hence f is holomorphic in $q=0$ since this is so for $\mathcal{E}$ by definition 3.1.6 together with the bound $h\geq\frac{c}{24}$ in the Ramond sector that was discussed below theorem 3.1.3. Since $\mathcal{E}$ by theorem 3.1.7 transforms covariantly under conformal transformations of σ, we find that $f(\sigma+1)=f(\sigma)$ and $f(-\frac{1}{\sigma})=\sigma^2 f(\sigma)$. Thus f is a modular form of weight two. By the theory of modular forms $f\equiv 0$ is the only such function. The assertion of the theorem now follows from the fact that $\mathcal{E}$ must have integer coefficients by construction. $\square$

3.1.3 Minimal, Gepner and Gepner type models

In this section we introduce some of the most important examples of unitary $N=(2,2)$ superconformal field theories. For central charge $c<1$, unitary conformal field theories may only occur at discrete values of c. The corresponding superconformal field theories are the MINIMAL MODELS (k), $k\in\mathbb{N}$ [VPZ86, BFK86, ZF86, Qiu87] which have central charges $c=3k/(k+2)$. To construct the model

(k) we may start from a $\mathbb{Z}_k$ parafermion theory and add a free bosonic field. More precisely, (k) is the coset model

$$\frac{SU(2)_k \otimes U(1)_2}{U(1)_{k+2,diag}} \,. \tag{3.1.13}$$

The primary fields are denoted by $\Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z})$, where $l \in \{0,\ldots,k\}$ is twice the spin of the corresponding field in the affine $SU(2)_k$. As a matter of convenience we have tacitly specialized to the diagonal invariant by imposing $l = \overline{l}$. The remaining quantum numbers $m, \overline{m} \in \mathbb{Z}_{2(k+2)}$ and $s, \overline{s} \in \mathbb{Z}_4$ label the representations of $U(1)_{k+2,diag}$ and $U(1)_2$ in the decomposition (3.1.13), respectively, and must obey $l \equiv m + s \equiv \overline{m} + \overline{s}$ (2). Here, the fields with even (odd) s create states in the left handed Neveu-Schwarz (Ramond) sector, and analogously for $\overline{s}$ and the right handed sectors. Moreover, the identification

$$\Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z}) \sim \Phi^{k-l}_{m+k+2,s+2;\overline{m}+k+2,\overline{s}+2}(z,\overline{z}) \tag{3.1.14}$$

holds. By (3.1.13), the corresponding characters $X^l_{m,s;\overline{m},\overline{s}}$ can be obtained from the level k string functions $c^l_j, l \in \{0,\ldots,k\}, j \in \mathbb{Z}_{2k}$, of $SU(2)_k$ and classical theta functions $\Theta_{a,b}, a \in \mathbb{Z}_{2b}$, of level $b = 2k(k+2)$ by [Gep88, RY87, Qiu87]

$$\begin{aligned} X^l_{m,s;\overline{m},\overline{s}}(\sigma,z) &= \chi^l_{m,s}(\sigma,z)\cdot\chi^l_{\overline{m},\overline{s}}(\overline{\sigma},\overline{z}), \\ \chi^l_{m,s}(\sigma,z) &= \sum_{j=1}^{k} c^l_{4j+s-m}(\sigma)\Theta_{2m-(k+2)(4j+s),2k(k+2)}\left(\sigma, \frac{z}{k+2}\right). \end{aligned} \tag{3.1.15}$$

Modular transformations act by

$$\begin{aligned} \chi^l_{m,s}(\sigma+1,z) &= exp\left[2\pi i\left(\frac{l(l+2)-m^2}{4(k+2)}+\frac{s^2}{8}-\frac{c}{24}\right)\right]\chi^l_{m,s}(\sigma,z) \\ \chi^l_{m,s}\left(-\tfrac{1}{\sigma},\tfrac{z}{\sigma}\right) &= \kappa(k)\sum_{l',m',s'}\sin\left(\frac{\pi(l+1)(l'+1)}{k+2}\right)e^{\pi i\frac{mm'}{(k+2)}}e^{-\pi i\frac{ss'}{2}}\chi^{l'}_{m',s'}(\sigma,z), \end{aligned} \tag{3.1.16}$$

where $\kappa(k)$ is a constant depending only on k and the summation runs over $l' \in \{0,\ldots,k\}, m' \in \{-k-1,\ldots,k+2\}, s' \in \{-1,\ldots,2\}, l'+m'+s' \equiv 0$ (2).
Let $\psi^l_{m,s}$ denote a lowest weight state in the irreducible representation of the $N = 2$ superconformal algebra with character $\chi^l_{m,s}$. Conformal dimension and charge of $\psi^l_{m,s}$ then are

$$h^l_{m,s} = \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8} \bmod 1, \quad Q^l_{m,s} = \frac{m}{k+2} - \frac{s}{2} \bmod 2. \tag{3.1.17}$$

The above values for $h^l_{m,s}$ hold without mod 1 in the domain $|m-s| \le l$; if (3.1.14) does not suffice to transform into this domain, choose a representative with $m - s = l - 2$ if it exists, otherwise the one with $m - s = l + 2$, use $h^l_{m,s}$ as in

(3.1.17), but add 1. The value for $Q^l_{m,s}$ is taken mod 2 such that $|Q^l_{m,s}| \leq 1$. The fusion rules are

$$\left[\psi^l_{m,s}\right] \times \left[\psi^{l'}_{m',s'}\right] = \sum_{\substack{\bar{l}=|l-l'| \\ \bar{l}\equiv l+l'(2)}}^{\min(l+l',2k-l-l')} \left[\psi^{\bar{l}}_{m+m',s+s'}\right]. \tag{3.1.18}$$

Note that by (3.1.17) and (3.1.18) the operators $U_{\pm\frac{1}{2}}, \overline{U}_{\pm\frac{1}{2}}$ of left and right handed spectral flow (3.1.7) are associated to the fields $\Phi^0_{\pm1,\pm1;0,0} = \psi^0_{\pm1,\pm1}$ and $\Phi^0_{0,0;\pm1,\pm1} = \overline{\psi^0_{\pm1,\pm1}}$, respectively. To fix which fields are bosonic or fermionic, we use [FKS92, (4.5)],

$$\Phi^{l_1}_{m_1,s_1;\overline{m}_1,\overline{s}_1} \otimes \Phi^{l_2}_{m_2,s_2;\overline{m}_2,\overline{s}_2} = (-1)^{\frac{1}{4}(s_1-\overline{s}_1)(s_2-\overline{s}_2)} \Phi^{l_2}_{m_2,s_2;\overline{m}_2,\overline{s}_2} \otimes \Phi^{l_1}_{m_1,s_1;\overline{m}_1,\overline{s}_1}. \tag{3.1.19}$$

The minimal model (k) contains only states of the form $\Phi^l_{m,s;m,\overline{s}}$, $s - \overline{s} \equiv 0 \bmod 2$, so by construction $Q - \overline{Q} \in \mathbb{Z}$ for all states in the theory and $Q - \overline{Q} \in 2\mathbb{Z}$ exactly for the bosonic ones. Hence theorem 3.1.4 shows that the minimal model (k) is invariant under spectral flow $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$.
The NS-part of our modular invariant partition function is now given by

$$Z_{NS}(\sigma, z) = \tfrac{1}{2} \sum_{\substack{l=0,\dots,k \\ m=-k-1,\dots,k+2, \\ l+m\equiv 0(2)}} \left(\chi^{l,0}_m(\sigma,z) + \chi^{l,2}_m(\sigma,z)\right)\left(\chi^{l,0}_m(\overline{\sigma},\overline{z}) + \chi^{l,2}_m(\overline{\sigma},\overline{z})\right), \tag{3.1.20}$$

and the entire partition function is obtained by the flows (3.1.9).
In the case $k = 2$ which we mostly employ in this work, the parafermion algebra is nothing but the algebra satisfied by the Majorana fermion ψ of the Ising model (3.3). By inspection of the charge lattice one may confirm that the minimal model (2) can readily be constructed by tensoring the Ising model with the theory which describes a bosonic field φ compactified on a circle of radius $R = 2$ (see section 2.2 or 4.1). The primary fields decompose as

$$\begin{aligned} \Phi^l_{m,s;\overline{m},\overline{s}}(z,\overline{z}) &= \Xi^l_{m-s;\overline{m}-\overline{s}}(z,\overline{z})\; e^{\frac{i}{2\sqrt{2}}(-m+2s)\varphi}(z)\; e^{\frac{i}{2\sqrt{2}}(-\overline{m}+2\overline{s})\overline{\varphi}}(\overline{z}) \\ \Xi^0_{j;\overline{\jmath}}(z,\overline{z}) &= \Xi^2_{j\pm2;\overline{\jmath}\pm2}(z,\overline{z}) \;=\; \xi^0_j(z)\xi^0_{\overline{\jmath}}(\overline{z}), \quad \xi^0_0 = \mathbb{1},\; \xi^0_2 = \psi, \end{aligned} \tag{3.1.21}$$

and $\Xi^1_{1,1} = \Xi^1_{-1,-1}$, $\Xi^1_{1,-1} = \Xi^1_{-1,1}$ denote the ground states $|\sigma\rangle, |\mu\rangle$ of the two $h = \overline{h} = \frac{1}{16}$ representations of the Ising model. Indeed, the level 2 string functions are obtained from the characters of lowest weight representations in the Ising model by dividing by the Dedekind eta function.
Let us now turn to the discussion of Gepner models. We do not choose the usual approach which uses the orbifold construction (see section 5.6), but rather a definition based on the FLOW INVARIANT ORBIT TECHNIQUE of [EOTY89], the heart of which is Gepner's original β METHOD [Gep87, Gep88]. The technique can be generalized to obtain various Gepner type models, as we shall see below. In short, we will describe a method to pick a number of tensor products of $N = 2$ irreducible

representations with characters (3.1.15) and (a priori) arbitrary left–right coupling, such that the corresponding theory is well defined in the sense of our properties 1–14.

We begin by recalling some basic facts about rational conformal field theories. These theories are studied more intensively in section 4.5, so the discussion will be brief here.

Definition 3.1.13

Let $\mathcal{C}$ denote a unitary conformal field theory. Its (anti–)holomorphic W–algebra is RATIONAL, *if it possesses only a finite number of irreducible representations, which will be denoted by $\mathcal{H}_i, \overline{\mathcal{H}_{\bar\imath}}$ with characters $\chi_i, \chi_{\bar\imath}$ in the following, $i \in I, \bar\imath \in \overline{I}$, and $I, \overline{I}$ some index sets. An (anti–)holomorphic W–algebra is* QUASIRATIONAL, *if all its quantum dimensions (definition 3.1.2) are finite, or analogously for right-movers. The conformal field theory $\mathcal{C}$ is* (QUASI–)RATIONAL, *if its holomorphic and antiholomorphic W–algebras are (quasi–)rational.*

For our construction we will make use of the Verlinde formula:

Theorem 3.1.14 (Verlinde formula [Ver88, MS88, TUY89, Fal94])

Let $\mathcal{W}$ denote the W–algebra of a rational conformal field theory $\mathcal{C}$ with notations as in definition 3.1.13. $\mathcal{S}$ denotes the S–matrix of $\mathcal{C}$, defined by

$$\chi_i(S\sigma) = \sum_{j\in I} \mathcal{S}_{ij}\chi_j(\sigma)$$

and S as in (2.1.5). Then $\mathcal{S}$ is unitary with $\mathcal{S} = \mathcal{S}^T$, and $\mathcal{S}^2 = C$ is the charge conjugation which assigns the complex conjugate to each state. Moreover, $\mathcal{S}^* = C\mathcal{S} = \mathcal{S}C$, and $\mathcal{S}$ diagonalizes the fusion matrices N_i, where $(N_i)_j^k = N_{ij}^k$ and $[\mathcal{H}_i] \times [\mathcal{H}_j] = \sum_k N_{ij}^k [\mathcal{H}_k]$. We have

$$N_{ijk} = \sum_{l\in I} C_{kl} N_{ij}^l = \sum_{l\in I} \frac{\mathcal{S}_{il}\mathcal{S}_{jl}\mathcal{S}_{kl}}{\mathcal{S}_{0l}}.$$

It is now easy to see that the quantum dimensions (definition 3.1.2) for rational W–algebras can be calculated by $d_i = \frac{\mathcal{S}_{i0}}{\mathcal{S}_{00}}$. Since fusion in a well defined conformal field theory must have finite coefficients, every rational W–algebra is quasirational. By theorem 3.1.14 the first step to find all rational unitary $N = (2,2)$ superconformal field theories with holomorphic W–algebra $\mathcal{W}$ and isomorphic antiholomorphic W–algebra is to determine all matrices M with $M_{00} = 1$ such that the partition function

$$Z(\sigma, z) = \sum_{i,j\in I} M_{ij}\chi_i(\sigma, z)\chi_j(\overline{\sigma}, \overline{z})$$

is modular invariant. For the $N = 2$ superconformal minimal models this has been achieved by T. Gannon [Gan97], who in particular showed that the standard–lore of an ADE classification of these matrices [FST97, CZ97] is at best "one–to–many". We will mainly use the diagonal invariant $M_{ij} = \delta_{ij}$ which always works by theorem 3.1.14, but will be more general wherever possible. It should be noted

that the above only gives a procedure to classify all modular invariant partition functions built from a known set of characters; whether consistent theories with these partitions functions exist remains an open question.
An important property of the S–matrix that is deduced from the Verlinde formula is the fact that no entry in its first row may vanish: $\mathcal{S}_{0i} \geq \mathcal{S}_{00} > 0$ (use $N_{i0}^j = \delta_i^j$ in theorem 3.1.14). So if we are able to split $\chi_0(-\frac{1}{\sigma}, \frac{z}{\sigma})$ into a sum of characters $\chi_i(\sigma, z), i \in I$, of irreducible representations of the W–algebra $\mathcal{W}$ of $\mathcal{C}$, we are sure to determine all irreducible representations of $\mathcal{W}$ in $\mathcal{C}$. This is possible for tensor products of representations occurring in minimal models, if $\mathcal{W}$ contains the $N = 2$ superconformal algebras of the tensor factors, even with arbitrary left–right coupling: The representations in general will be sums of tensor products, and in $\chi_0(-\frac{1}{\sigma}, \frac{z}{\sigma})$ we only need to follow the orbit of a single tensor product under the action of the W–algebra. We remark that by a tensor product of $N = (2,2)$ superconformal field theories we always mean the FERMIONIC TENSOR PRODUCT, i.e. we tensorize sector by sector in order to have a well defined $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading for the resulting theory. On the level of partition functions this amounts to multiplying the four parts $Z_{NS}, Z_{\widetilde{NS}}, Z_R, Z_{\widetilde{R}}$ separately.

Definition 3.1.15
Pick positive integers $k_i, i \in \{1, \dots, r\}$, such that $c := \sum_{i=1}^r \frac{3k_i}{k_i+2} = 3d/2, d/2 \in \mathbb{N}$. By $\mathcal{W}_{Gepner}$ we denote the algebra of holomorphic fields generated by the $N = 2$ superconformal algebras of each factor theory in the tensor product $(k_1) \otimes \dots \otimes (k_r)$ of $N = 2$ minimal models together with the combined left–handed operators of twofold spectral flow,

$$U_1 = \bigotimes_{i=1}^r \Phi_{2,2;0,0}^0.$$

Note that by (3.1.7) this field has charge $\frac{c}{3} \in \mathbb{N}$, which ensures that the fields in $\mathcal{W}_{Gepner}$ are pairwise local. $\overline{\mathcal{W}}_{Gepner}$ denotes the analogous algebra on the right hand side. The diagonal sums $J, T, G^\pm$ of the fields which generate the $N = 2$ superconformal algebras of the factor theories in $(k_1) \otimes \dots \otimes (k_r)$ generate an $N = 2$ superconformal algebra $\mathcal{A} \subset \mathcal{W}_{Gepner}$ with central charge c. Let χ_0 denote the character of the representation $\mathcal{W}_{Gepner}$ of $\mathcal{A}$, and decompose $\chi_0(-\frac{1}{\sigma}, \frac{z}{\sigma})$ into a sum of characters $\chi_i(\sigma, z), i \in I$, of irreducible representations $\mathcal{H}_i$ of $\mathcal{W}_{Gepner}$ by following $\mathcal{W}_{Gepner}$ orbits of products of characters (3.1.15) as described above. Then the space of states of the GEPNER MODEL *$(k_1) \cdots (k_r)$ is $\mathcal{H} := \oplus_{i \in I} \mathcal{H}_i \otimes \overline{\mathcal{H}}_i$, and its partition function is given by*

$$Z(\sigma, z) = \sum_{i \in I} \chi_i(\sigma, z) \chi_i(\overline{\sigma}, \overline{z}).$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading is induced by that of the minimal model factors, where U_1 is bosonic iff $\frac{c}{3}$ is even.

Before we explain how to use the above construction in practice, let us work out the most important properties of Gepner models:

Theorem 3.1.16
Let $\mathcal{C}$ denote a Gepner model with central charge c. Then $\mathcal{C}$ is a rational $N=(2,2)$ superconformal field theory. The states in $\mathcal{C}$ have only integer left and right charges in the Neveu–Schwarz sector. In the Ramond sector, left and right charges are all integer or all half integer depending on $\frac{c}{3}$ being even or odd. $\mathcal{C}$ is invariant under spectral flow.

Proof:
If we can show that $\mathcal{C}$ is a well defined conformal field theory, the first assertion follows directly from definition 3.1.15. To do so, we need to ensure pairwise (semi–) locality between the representations $\mathcal{H}_i \otimes \overline{\mathcal{H}}_i$. By construction, $\mathcal{H}_0 \otimes \overline{\mathcal{H}}_0$ is local to all $\mathcal{H}_i \otimes \overline{\mathcal{H}}_i$. Moreover, each representation $\mathcal{H}_i \otimes \overline{\mathcal{H}}_i$ contains states that belong to the tensor product $(k_1) \otimes \cdots \otimes (k_r)$ of minimal models, which itself is a well defined superconformal field theory. By associativity of the OPE, pairwise (semi–) locality on the entire space of states $\mathcal{H}$ follows.
Recall from (3.1.7) that the Ramond field $U_1 \in \mathcal{W}_{Gepner}$ has charge $\frac{c}{3} \in \mathbb{N}$ and that by (3.1.2) states $|\Phi\rangle \in \mathcal{H}$ are (semi–) local to U_1 iff they have (half) integer charge. Therefore, all left and right charges must belong to $\frac{1}{2}\mathbb{Z}$. In the Neveu–Schwarz sector charges must be integer, whereas in the Ramond sector we have half integer charges iff $\frac{c}{3}$ is odd. This proves the second assertion of the theorem. The last one follows directly from theorem 3.1.4, since we have just proven that all states in $\mathcal{C}$ have charges $(Q;\overline{Q})$ with $Q-\overline{Q} \in \mathbb{Z}$. That $Q-\overline{Q} \in 2\mathbb{Z}$ exactly for bosonic fields again follows directly from the fact that this is true for the minimal models, and U_1 was assumed to be bosonic iff its charge $\frac{c}{3}$ is even. □

Theorem 3.1.16 together with property 8 shows that our definition 3.1.15 of Gepner models indeed agrees with the standard one [Gep87, Gep88, GVW89]. There will be more to say about this in section 5.6. To accomplish Gepner's entire construction of a consistent theory of superstrings in $10-d$ dimensions we actually would firstly have to take into account $8-d$ additional free superfields representing flat $(10-d)$-dimensional Minkowski space in light-cone gauge, secondly perform the GSO projection onto odd integer left and right charges (see section 5.5) and thirdly convert the resulting theory into a heterotic one. However, at the stage described above we have constructed a consistent conformal field theory with central charge $c=3d/2$, which for $d=4$ is associated to a $K3$ surface or a torus (see chapter 7), so we may and will omit these last three steps of Gepner's construction. Moreover, Gepner models satisfy the assumptions of theorem 3.1.11, so the conformal field theoretic elliptic genus of a Gepner model is already determined by its leading order terms. In particular, comparison of a graded basis of the (c,c) ring of a Gepner model with the cohomology of a Calabi–Yau manifold suffices to prove agreement of the elliptic genera. This agreement has been shown for all Gepner models with $c=3,6,9$ in [FKSS90].
Our definition 3.1.15 implies a simple technique to compute the characters of W–algebra representations for Gepner models as well as the partition function, as introduced in [EOTY89]. We will describe how to determine the Neveu–Schwarz part Z_{NS} of the partition function of $(k_1)\cdots(k_r)$. Theorem 3.1.16 shows that the

other parts can by obtained with the flows (3.1.9). For explicit examples, see [Tao90] or [EOTY89], where the models $(2)^4$ and $(1)^6$ are discussed. We will carry out the slightly more general calculation for a Gepner type model in theorem 3.1.18 below.
Definition 3.1.15 directly shows how to determine the NS part of the vacuum character of the Gepner model $(k_1)\cdots(k_r)$, which we denote NS_0, following [EOTY89] (then $\chi_0 = \frac{1}{2}(NS_0 + \widetilde{NS_0})$). One has to take the product of the NS vacuum characters $\chi^0_{0,0} + \chi^0_{0,2}$ of the minimal models $(k_1), \dots, (k_r)$, and add on the images under twofold spectral flow U_1. Since by the fusion rules (3.1.18) U_1 is a simple current, this is easy to calculate. We have determined the vacuum characters of all Gepner models with $c = 6$ numerically by this technique. The results are listed in appendix C. The simple structure of the W–algebra of $(k_1)\cdots(k_r)$ also implies that each of the characters χ_i of its irreducible representations in the Gepner model has the form of a tensor product of minimal model characters plus its images under U_1. Since Z_{NS} corresponds to the trace over fermions with antiperiodic boundary conditions on the worldsheet torus in both space and imaginary time directions, Z_{NS} transforms covariantly under the S–transform. To determine the other NS characters of the theory, it therefore suffices to consider $NS_0(-\frac{1}{\sigma}, \frac{z}{\sigma})$. Its splitting into expressions of the type ($\otimes_{i=1}^r$ (minimal model character) + U_1 orbit) defines the first row of a matrix $(s_{jk})_{jk}$,

$$NS_j(-\frac{1}{\sigma}, \frac{z}{\sigma}) = e^{\frac{iz^2}{2\pi\sigma}} \sum_{k\in I} s_{jk} NS_k(\sigma, z),$$

where the FLOW INVARIANT ORBITS NS_j [EOTY89] are normalized such that at least one of the tensor products of minimal model characters has coefficient 1 and all of the coefficients are integer. This normalization ignores combinatorial factors needed to actually determine the characters χ_i, since by permutation symmetries of the factors each NS_j appears with some multiplicity D_j. In particular, the matrix $(s_{ij})_{ij}$ is not to be confused with an S–matrix $(\mathcal{S}_{ij})_{ij}$, it is not symmetric. The prefactors are necessary to symmetrize s, i.e. $\mathcal{S}_{ij} = \frac{s_{ij}}{D_j} = \frac{s_{ji}}{D_i}$ for all $i, j \in I$. Since $D_0 = 1$, one finds $D_i = \frac{s_{0i}}{s_{i0}}$. For the explicit calculations it is easier to determine D_i by a combinatorial argument: $M := \mathrm{lcm}\,\{k_j + 2, j = 1, \dots, r\}$ is the length of a standard orbit of the operator U_1, since by (3.1.18) $U_1^M = \mathbb{1}$. It in particular is the length (the number of summands of minimal model character's tensor product type) of NS_0. Let l_i denote the length of the orbit NS_i, and κ_i the number of nontrivial permutations of factors of a summand of NS_i with coefficient 1. By construction, κ_i is independent of the choice of the summand. Then

$$D_i := \kappa_i \frac{M}{l_i}, \qquad Z_{NS}(\sigma, z) = \sum_{i\in I} D_i NS_i(\sigma, z) NS_i(\overline{\sigma}, \overline{z}).$$

Definition 3.1.15 cries for a generalization to more general W–algebras:

Definition 3.1.17
Pick positive integers $k_1, \dots, k_r$ as in definition 3.1.15. Let $\mathcal{W}$ denote an extension of $\mathcal{W}_{Gepner}$ by a number of (left handed) operators taken from $\otimes_{j=1}^r SU(2)_{k_j} \otimes$

$U(1)_2/U(1)_{k_j+2,diag}$ as in (3.1.13) but ignoring the restrictions on left–right couplings in the minimal models. Make sure, however, that all fields in $\mathcal{W}$ are pairwise local. On the right handed side, let $\overline{\mathcal{W}}$ denote the corresponding isomorphic algebra. The character of the representation $\mathcal{W}$ of the $N=2$ superconformal algebra $\mathcal{A} \subset \mathcal{W}$ is denoted χ_0. The irreducible representations of $\mathcal{W}$ whose characters $\chi_i, i \in I$, occur in $\chi_0(-\frac{1}{\sigma}, \frac{z}{\sigma})$ are denoted $\mathcal{H}_i$. Then the Gepner type model *$(\widetilde{k_1})\cdots(\widetilde{k_r})$ has space of states $\oplus_{i\in I}\mathcal{H}_i \otimes \overline{\mathcal{H}}_i$ and partition function*

$$Z(\sigma, z) = \sum_{i\in I} \chi_i(\sigma, z)\chi_i(\overline{\sigma}, \overline{z}).$$

That Gepner type models give well defined unitary rational $N=(2,2)$ superconformal field theories can be seen in exactly the same fashion as for the Gepner models in theorem 3.1.16. Gepner models constructed from other than the A–invariant for the coupling of $l, \bar{l}$ in the minimal model factors can now be obtained as Gepner type models. In section 5.6 we will see that those Gepner type models where $\mathcal{W} \supset \mathcal{W}_{Gepner}$ is an extension by simple currents can also be interpreted as orbifolds of ordinary Gepner models. The condition of pairwise locality of fields in $\mathcal{W}$ translates into the level matching conditions for the orbifold construction.
The flow invariant orbit technique applies literally to our generalization of the Gepner models, as we will now show for an example, namely the model $(\widetilde{2})^4$ discussed in theorem 7.3.29.

Theorem 3.1.18
By $(\widetilde{2})^4$ we denote the Gepner type model obtained by enhancing $\mathcal{W}_{Gepner}$ of $(2)^4$ with the simple current

$$\widetilde{J_{12}} := \Phi^0_{4,2;0,0} \otimes \Phi^0_{4,2;0,0} \otimes \Phi^0_{0,0;0,0} \otimes \Phi^0_{0,0;0,0}$$

and all currents obtained by permutations of factors of $\widetilde{J_{12}}$. The flow invariant orbits of $(\widetilde{2})^4$ are

$$\begin{aligned}
NS_0(\sigma,z) &= \frac{1}{4}\left(\left(\frac{\vartheta_3}{\eta}\right)^4 + \left(\frac{\vartheta_4}{\eta}\right)^4\right)\left(\frac{\vartheta_3(z)}{\eta}\right)^2 + \frac{1}{2}\frac{\vartheta_3^2\vartheta_4^2}{\eta^4}\left(\frac{\vartheta_4(z)}{\eta}\right)^2,\\
NS_1(\sigma,z) &= \frac{1}{8}\left(\frac{\vartheta_2}{\eta}\right)^4\left(\frac{\vartheta_3(z)}{\eta}\right)^2 - \frac{1}{4}\frac{\vartheta_2^2\vartheta_4^2}{\eta^4}\left(\frac{\vartheta_1(z)}{\vartheta_3}\right)^2 = NS_2(\sigma,z),\\
NS_3(\sigma,z) &= \frac{1}{16}\left(\frac{\vartheta_2}{\eta}\right)^4\left(\frac{\vartheta_3(z)}{\eta}\right)^2 = NS_4(\sigma,z) = NS_5(\sigma,z),
\end{aligned}$$

and the NS part of its partition function is

$$Z_{NS} = |NS_0|^2 + 8|NS_1|^2 + 12|NS_2|^2 + 16|NS_3|^2 + 48|NS_4|^2 + 96|NS_5|^2.$$

Proof:
The relevant characters of the minimal model (2) can be computed with the general formula (3.1.15):

$$\begin{aligned}
A &:= (\chi^0_{0,0}+\chi^0_{0,2})(\sigma,z) &&= \frac{1}{2\eta}\left(\sqrt{\frac{\vartheta_3}{\eta}}\vartheta_3(2\sigma,z)+\sqrt{\frac{\vartheta_4}{\eta}}\vartheta_4(2\sigma,z)\right),\\
B &:= (\chi^0_{-2,0}+\chi^0_{-2,2})(\sigma,z) &&= \frac{1}{2\eta}\left(\sqrt{\frac{\vartheta_3}{\eta}}\vartheta_2(2\sigma,z)+\sqrt{\frac{\vartheta_4}{\eta}}i\vartheta_1(2\sigma,z)\right),\\
C &:= (\chi^0_{4,0}+\chi^0_{4,2})(\sigma,z) &&= \frac{1}{2\eta}\left(\sqrt{\frac{\vartheta_3}{\eta}}\vartheta_3(2\sigma,z)-\sqrt{\frac{\vartheta_4}{\eta}}\vartheta_4(2\sigma,z)\right),\\
D &:= (\chi^0_{2,0}+\chi^0_{2,2})(\sigma,z) &&= \frac{1}{2\eta}\left(\sqrt{\frac{\vartheta_3}{\eta}}\vartheta_2(2\sigma,z)-\sqrt{\frac{\vartheta_4}{\eta}}i\vartheta_1(2\sigma,z)\right),\\
E &:= (\chi^1_{1,0}+\chi^1_{1,2})(\sigma,z) &&= \frac{q^{\frac{1}{16}}y^{\frac{1}{4}}}{2\eta}\sqrt{\frac{\vartheta_2}{\eta}}\vartheta_3(2\sigma,z+\frac{\sigma}{2}),\\
F &:= (\chi^1_{3,0}+\chi^1_{3,2})(\sigma,z) &&= \frac{q^{\frac{1}{16}}y^{\frac{1}{4}}}{2\eta}\sqrt{\frac{\vartheta_2}{\eta}}\vartheta_2(2\sigma,z+\frac{\sigma}{2}).
\end{aligned}$$

From (3.1.18) we read that U_1 acts by $A\mapsto B\mapsto C\mapsto D\mapsto A, E\leftrightarrow F$, and the $\widetilde{J}$ act by $A\leftrightarrow C, B\leftrightarrow D$ on two factors. Thus the flow invariant orbit technique gives

$$NS_0 = A^4+B^4+C^4+D^4+6(A^2C^2+B^2D^2).$$

From (3.1.16) we find the following S–transforms:

$$\begin{aligned}
A &\mapsto 2\sqrt{2}(A+B+C+D)-4(E+F),\\
B &\mapsto 2\sqrt{2}(A-B+C-D)+4i(E-F),\\
C &\mapsto 2\sqrt{2}(A+B+C+D)+4(E+F),\\
D &\mapsto 2\sqrt{2}(A-B+C-D)-4i(E-F),
\end{aligned}$$

so the other flow invariant orbits read

$$\begin{aligned}
NS_1 &= E^4+F^4\\
NS_2 &= A^2B^2+B^2C^2+C^2D^2+D^2A^2+4ABCD\\
NS_3 &= A^3C+AC^3+B^3D+BD^3\\
NS_4 &= AB^2C+ACD^2+A^2BD+BC^2D\\
NS_5 &= E^2F^2.
\end{aligned}$$

The rest of the proof is pure theta function gymnastics with the help of appendix A. The correct prefactors D_i for the $|NS_i|^2$ in the partition function are obtained from $D_i=\kappa_i\frac{16}{l_i}$ with $(\kappa_i)=(1,1,6,4,12,6), (l_i)=(16,2,8,4,4,1)$. □

3.2 $N = (4,4)$ Superconformal field theories

In chapter 7 we will discuss superconformal field theories with central charge $c = 6$. More precisely, we will concentrate on theories which possess at least $N = (2,2)$ supersymmetry, and make the additional assumption that the four operators of spectral flow (3.1.7), i.e. $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}, U_{\pm\frac{1}{2}}\overline{U}_{\mp\frac{1}{2}}$, are realized as fields of the theory. Because $c = 6$, the operators $U_{\pm 1}$ of twofold lefthanded spectral flow have conformal dimensions $(h, \overline{h}) = (1,0)$ and enhance the generic $u(1)$ Kac-Moody algebra of our theory to $su(2)_1$. The $N = (2,2)$ superconformal algebra is enhanced to an $N = (4,4)$ superconformal algebra, a special case of the Ademollo et al algebra [ABD+76]. More generally, this $N = 4$ superconformal algebra has been studied by T. Eguchi and A. Taormina [ET87, ET88a, ET88b, ET88c, Tao90]. It exists for values $c = 6k$ of the central charge, where $k \in \mathbb{N}$ is the level of the affine $su(2)_k$ subalgebra. The latter is generated by $J^3, J^\pm$ with

$$
\begin{array}{rclrcl}
[L_m, J^3_n] &=& nJ^3_{m+n}, & [L_m, J^\pm_n] &=& nJ^\pm_{m+n}, \\
[2J^3_m, 2J^3_n] &=& 2kn\delta_{m+n,0}, & [J^3_m, J^\pm_n] &=& \pm J^\pm_{m+n}, \\
[J^+_m, J^-_n] &=& kn\delta_{m+n,0} + 2J^3_{m+n}, & [J^\pm_m, J^\pm_n] &=& 0.
\end{array}
\tag{3.2.1}
$$

On comparison with (3.1.1) we find $J = 2J^3$ for the $U(1)$ current J of the $N = 2$ superconformal algebra with central charge $c = 6k$. The disagreement in standard normalizations is a steady source of confusion. In this work, charges will always be measured with respect to the $U(1)$ current J of the $N = 2$ subalgebra. If for level $k = 1$ we bosonize the $U(1)$ current $J = i\sqrt{2}\partial H$ as for (3.1.2), we find $J^\pm = e^{\pm i\sqrt{2}H}$. The $N = 4$ superconformal algebra contains four fermionic fields $G^\pm, G'^\pm$, and the full algebra is given by

$$
\begin{array}{rclrcl}
\{G^\pm_r, G'^\mp_s\} &=& 2(r-s)J^\pm_{r+s}, & \{G^\pm_r, G'^\pm_s\} &=& 0, \\
\{G^{(\prime)+}_r, G^{(\prime)-}_s\} &=& 2L_{r+s} & \pm \quad 2(s-r)J^3_{r+s} &+& 2k\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\
[L_m, G^\pm_r] &=& \left(r - \frac{m}{2}\right)G^\pm_{m+r}, & [L_m, G'^\pm_r] &=& \left(r - \frac{m}{2}\right)G'^\pm_{m+r}, \\
[J^3_m, G^\pm_r] &=& \pm\frac{1}{2}G^\pm_{m+r}, & [J^3_m, G'^\pm_r] &=& \mp\frac{1}{2}G'^\pm_{m+r}, \\
[J^\pm_m, G^\mp_r] &=& \pm G'^\mp_{m+r}, & [J^\pm_m, G^\pm_r] &=& 0, \\
[J^\pm_m, G'^\pm_r] &=& \mp G^\pm_{m+r}, & [J^\pm_m, G'^\mp_r] &=& 0.
\end{array}
\tag{3.2.2}
$$

Thus we see that the above $N = 4$ superconformal algebra with central charge $c = 6k$ contains an $N = 2$ superconformal algebra with same central charge generated by $J, G^\pm, T$.

3.2.1 Free field realization

In this section we present a free field realization of the $N = 4$ superconformal algebra (3.2.2) for $k = 1$. We need four Abelian currents $j^l, l \in \{1, \ldots, 4\}$, and four Majorana fermions $\psi^l, l \in \{1, \ldots, 4\}$. Normalizations are chosen such that for

$\psi_\pm^{(l)} := \frac{1}{\sqrt{2}}\left(\psi^{2l-1} + \psi^{2l}\right), j_\pm^{(l)} := \frac{1}{\sqrt{2}}\left(j^{2l-1} + j^{2l}\right), l \in \{1,2\}$, we have the OPEs

$$\psi_+^{(k)}(z)\psi_-^{(l)}(w) \sim \frac{\delta^{kl}}{z-w}, \quad j_+^{(k)}(z)j_-^{(l)}(w) \sim \frac{\delta^{kl}}{(z-w)^2}.$$

Then the following fields realize the $N = 4$ superconformal algebra (3.2.2):

$$\begin{aligned} J^3 &= \tfrac{1}{2}\left(:\psi_+^{(1)}\psi_-^{(1)}: + :\psi_+^{(2)}\psi_-^{(2)}:\right), & J^\pm &= \pm :\psi_\pm^{(1)}\psi_\pm^{(2)}:, \\ G^\pm &= \sqrt{2}\left(:\psi_\pm^{(1)}j_\mp^{(1)}: + :\psi_\pm^{(2)}j_\mp^{(2)}:\right), & G'^\pm &= \sqrt{2}\left(:\psi_\mp^{(1)}j_\mp^{(2)}: - :\psi_\mp^{(2)}j_\mp^{(1)}:\right), \end{aligned} \tag{3.2.3}$$
$$T = :j_+^{(1)}j_-^{(1)}: + :j_+^{(2)}j_-^{(2)}: + \tfrac{1}{2}\left(:\partial\psi_+^{(1)}\psi_-^{(1)}: + :\partial\psi_-^{(1)}\psi_+^{(1)}: + :\partial\psi_+^{(2)}\psi_-^{(2)}: + :\partial\psi_-^{(2)}\psi_+^{(2)}:\right).$$

From the above fields we can build an additional affine $su(2)_1$ Kac–Moody algebra:

$$A = \tfrac{1}{2}\left(:\psi_+^{(1)}\psi_-^{(1)}: - :\psi_+^{(2)}\psi_-^{(2)}:\right), \quad A^\pm = \pm :\psi_\pm^{(1)}\psi_\mp^{(2)}:. \tag{3.2.4}$$

Our notation is chosen such that the Dirac fermions $\psi_\pm^{(k)}$ have charges ± 1 with respect to the $U(1)$ current $J = 2J^3$ of the $N = 2$ subalgebra. Moreover, for $k \in \{1,2\}$,

$$G^\pm(z)\psi_\mp^{(k)}(w) \sim \frac{\sqrt{2}}{z-w} j_\mp^{(k)}(w), \quad G'^\pm(z)\psi_\pm^{(k)}(w) \sim \frac{\sqrt{2}\varepsilon_{kl}}{z-w} j_\mp^{(l)}(w).$$

3.2.2 Characters of $N = 4$ irreducible representations

The irreducible representations of the $N = 4$ superconformal algebra at $c = 6$ with $su(2)_1$ current algebra have been discussed in [ET87, ET88a, ET88b, ET88c, Tao90]. We give a short summary of the features which will be needed in this thesis. States are labelled by their conformal dimension h as well as their charge Q with respect to a Cartan torus of $su(2)_1$. Note that any two Cartan tori are related by conjugation, hence the spectrum does not depend on the choice of this torus. The irreducible representations are determined by their lowest weight values of (h, Q). Similarly to the $N = 2$ case (section 3.1.1), there are two types of unitary irreducible representations, MASSIVE and MASSLESS ones, where the latter are those at the unitarity bound. The structure of null vectors is more complicated in this case [ET87], and apart from the vacuum representation with $(h, Q) = (0, 0)$, the lowest weight states of massless representations are labelled by $(h, Q) = (\frac{1}{2}, \pm 1)$ in the Neveu-Schwarz sector and by $(h, Q) = (\frac{1}{4}, \pm 1)$ or $(h, Q) = (\frac{1}{4}, 0)$ in the Ramond sector. By [ET88c, (18)-(23)], the characters in the Neveu–Schwarz sector are

$$\begin{aligned} ch_{0,0}^{NS}(\sigma, z) &:= \sum_{m\in\mathbb{Z}} q^{m^2/2-1/8}\, y^m \frac{yq^{m-1/2} - 1}{1 + yq^{m-1/2}} \frac{\vartheta_3(z)}{\eta^3}, \\ ch_{1/2,1/2}^{NS}(\sigma, z) &:= \sum_{m\in\mathbb{Z}} q^{m^2/2-1/8}\, y^m \frac{1}{1 + yq^{m-1/2}} \frac{\vartheta_3(z)}{\eta^3}, \\ ch_{h,0}^{NS}(\sigma, z) &:= \frac{q^{h-1/8}}{\eta}\left(\frac{\vartheta_3(z)}{\eta}\right)^2, \quad h > 0. \end{aligned} \tag{3.2.5}$$

The transformations into the other sectors as usual are obtained by the flows (3.1.9). Note that

$$\lim_{h\to 0} ch^{NS}_{h,0} = ch^{NS}_{0,0} + 2ch^{NS}_{1/2,1/2}, \tag{3.2.6}$$

so a massive representation splits into three massless ones if it hits the unitarity bound $h \to 0$. We compute the WITTEN INDEX (definition 3.1.5) for each of these representations using (A1.2) and (3.2.6):

$$\begin{aligned}
I_{h,0} &= 0,\\
I_{1/2,1/2} &= -\lim_{\zeta\to -1}\left(\sum_{m\in\mathbb{Z}} q^{m^2/2+m/2}\zeta^m \frac{1+\zeta^{-1}}{1+\zeta q^m}\prod_{n=1}^{\infty}\frac{(1+\zeta q^n)(1+\zeta^{-1}q^n)}{(1-q^n)^2}\right) = 1,\\
I_{0,0} &= I_{h,0} - 2I_{1/2,1/2} = -2.
\end{aligned} \tag{3.2.7}$$

Hence the massless representations are exactly those with nonvanishing Witten index, and only the massive ones can be deformed continuously with respect to the value of h. The characters (3.2.5) are closely related to the MORDELL FUNCTIONS

$$\begin{aligned}
h_1(\sigma) &:= \frac{1}{\eta\vartheta_4}\sum_{m\in\mathbb{Z}}\frac{(-1)^m q^{m^2/2-1/8}}{1-q^{m-1/2}}, \quad h_2(\sigma) := \frac{1}{\eta\vartheta_2}\sum_{m\in\mathbb{Z}}\frac{q^{m^2/2+m/2}}{1-q^m},\\
h_3(\sigma) &:= \frac{1}{\eta\vartheta_3}\sum_{m\in\mathbb{Z}}\frac{q^{m^2/2-1/8}}{1+q^{m-1/2}},
\end{aligned}$$

as we will show below. Note that h_1 and h_3 are exchanged if we replace $q^{1/2}$ by $-q^{1/2}$ in $q^{1/6}h_k$. On expanding h_3 we find a formal power series in $q^{1/6}$,

$$h_3 = q^{1/3}\left(2 - 6q^{1/2} + 16q - 34q^{3/2} + 72q^2 - 138q^{5/2} \pm \cdots\right)$$

with even (odd) powers of $q^{1/6}$ corresponding to positive (negative) coefficients. A proof of this observation for the even powers follows from equation (3.2.12) below. Mordell gave the following relation between h_1, h_2, h_3 [Mor33, p. 347]:

$$\frac{1}{\eta}\int_{-\infty}^{\infty} d\alpha\, \frac{q^{\alpha^2/2}}{2\cosh\pi\alpha} = h_3(\sigma) + h_3(-\frac{1}{\sigma}) = h_1(\sigma) + h_2(-\frac{1}{\sigma}). \tag{3.2.8}$$

From (3.2.5) and (3.2.8) we directly find

$$\begin{aligned}
ch^{NS}_{0,0}(\sigma,0) &= \left(\frac{q^{-1/8}}{\eta} - 2h_3\right)\left(\frac{\vartheta_3}{\eta}\right)^2, \quad & ch^{NS}_{1/2,1/2}(\sigma,0) &= h_3\left(\frac{\vartheta_3}{\eta}\right)^2,\\
ch^{NS}_{0,0}(\sigma,\tfrac{1}{2}) &= \left(\frac{q^{-1/8}}{\eta} - 2h_1\right)\left(\frac{\vartheta_4}{\eta}\right)^2, \quad & ch^{NS}_{1/2,1/2}(\sigma,\tfrac{1}{2}) &= h_1\left(\frac{\vartheta_4}{\eta}\right)^2.
\end{aligned} \tag{3.2.9}$$

Because the $N=4$ superconformal algebra contains $su(2)_1$ we can rewrite the characters (3.2.5) in terms of the characters of $su(2)_1$,

$$\begin{aligned}
\chi_1^0(\sigma,z) &= \frac{1}{\eta}\sum_{n=-\infty}^{\infty} q^{n^2}y^{2n} &&= \frac{\vartheta_3(2\sigma,2z)}{\eta(\sigma)},\\
\chi_1^{1/2}(\sigma,z) &= \frac{1}{\eta}\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}y^{2n+1} &&= \frac{\vartheta_2(2\sigma,2z)}{\eta(\sigma)}.
\end{aligned}$$

Both χ_1^l are theta functions of degree $n=2$ and characteristic $(0,0;-2\pi in,b), e^b = q^{-1}$, (see appendix A), and so are $\vartheta_1(z)^2$ and $\vartheta_3(z)^2$ (A1.2). Because the space $\mathcal{T}_2(q^{-1})$ of such theta functions is two dimensional this means that the characters can be written as linear combinations of $\vartheta_1(z)^2$ and $\vartheta_3(z)^2$ with coefficients that only depend on σ, not on z. From (3.2.7) and (3.2.9) we therefore find

$$\begin{aligned}
ch_{0,0}^{NS}(\sigma,z) &= 2\left(\frac{\vartheta_1(z)}{\vartheta_3}\right)^2 + \left(\frac{q^{-1/8}}{\eta} - 2h_3\right)\left(\frac{\vartheta_3(z)}{\eta}\right)^2,\\
ch_{1/2,1/2}^{NS}(\sigma,z) &= -\left(\frac{\vartheta_1(z)}{\vartheta_3}\right)^2 + h_3\left(\frac{\vartheta_3(z)}{\eta}\right)^2.
\end{aligned} \tag{3.2.10}$$

By (A1.2) we thus have*

$$\begin{aligned}
&\vartheta_4(z)^2 ch_{1/2,1/2}^{NS}(\sigma,z) - \vartheta_3(z)^2 ch_{1/2,1/2}^{NS}(\sigma,z+\tfrac{1}{2})\\
&\qquad = \frac{\vartheta_3(z)^2\vartheta_2(z)^2}{\vartheta_3^2} - \frac{\vartheta_4(z)^2\vartheta_1(z)^2}{\vartheta_3^2} \overset{(A3.4)}{=} \vartheta_2(2z)\vartheta_2.
\end{aligned} \tag{3.2.11}$$

If we insert $z=0$ and use (3.2.9) we obtain

$$h_3 - h_1 = \frac{\eta^2}{\vartheta_4^2}\frac{\vartheta_2^2}{\vartheta_3^2} = \vartheta_2^4\frac{\eta^2}{\vartheta_2^2\vartheta_3^2\vartheta_4^2} = \frac{1}{4}\left(\frac{\vartheta_2}{\eta}\right)^4, \tag{3.2.12}$$

where we have used (A2.1). Note that this means that we have total control over the modular properties of the part of h_3 given by even powers of $q^{1/6}$. By (3.2.8) we also have

$$h_2 - h_3 = \frac{1}{4}\left(\frac{\vartheta_4}{\eta}\right)^4, \quad h_2 - h_1 = \frac{1}{4}\left(\frac{\vartheta_3}{\eta}\right)^4.$$

*The following observations were obtained in joint work with Anne Taormina, and the last equality of (3.2.11) was achieved by comparison with results of Sander Zwegers.

Chapter 4

Toroidal conformal field theories

This chapter is mainly devoted to toroidal conformal field theories. Their definition and general properties are discussed in sections 4.1 and 4.2. The moduli spaces of toroidal conformal field theories in the two and four dimensional cases are discussed separately in sections 4.3 and 4.4. In section 4.5, we give a characterization of rational toroidal conformal field theories and its geometric interpretation in terms of two dimensional tori with complex multiplication. Section 4.6 is devoted to the study of so–called singular varieties and their relation to rational conformal field theories.

4.1 Toroidal theories in arbitrary dimensions

We take a somewhat unusual viewpoint by abstractly defining a (bosonic) toroidal conformal field theory as follows*:

Definition 4.1.1
A unitary conformal field theory with central charge $c = d \in \mathbb{N}$ is called TOROIDAL CONFORMAL FIELD THEORY*, if its holomorphic and antiholomorphic W–algebras contain a $u(1)_1^d$ affine Kac–Moody algebra each. A choice of Abelian currents generating these algebras will always be denoted $j^1, \dots, j^d; \overline{\jmath}^1, \dots, \overline{\jmath}^d$ if it is normalized to*

$$j^k(z)\, j^l(w) \sim \frac{\delta^{kl}}{(z-w)^2}. \tag{4.1.1}$$

Charges with respect to $(j;\overline{\jmath}) := (j^1, \dots, j^d; \overline{\jmath}^1, \dots, \overline{\jmath}^d)$ are denoted $p = (p_l; p_r)$.

Generally, let $V[p]$ denote a primary field with charge p in a toroidal conformal field theory $\mathcal{C}$. Since by associativity of the OPE charges transform additively under fusion, we have $[V[p]] \times [V[p']] = [V[p+p']]$. By the Sugawara form of the energy momentum tensor of $\mathcal{C}$, for any choice of generators as in (4.1.1)

$$T(z) = \tfrac{1}{2} \sum_{k=1}^{d} :j^k j^k: (z), \quad \overline{T}(\overline{z}) = \tfrac{1}{2} \sum_{k=1}^{d} :\overline{\jmath}^k \overline{\jmath}^k: (\overline{z}). \tag{4.1.2}$$

*In section 4.2, we will explain the relation to the standard point of view, from which it will also become clear why these theories are called TOROIDAL.

Thus $V[p]$ has dimensions $(h;\overline{h}) = (\frac{p_l^2}{2};\frac{p_r^2}{2})$, and the operator product expansion for primary fields $V[p], V[p']$ in $\mathcal{C}$ is given by

$$V[p](z,\overline{z})\,V[p'](w,\overline{w}) \sim c_{p,p'}(z-w)^{p_l p_l'}(\overline{z}-\overline{w})^{p_r p_r'}\,V[p+p'](w,\overline{w})+\cdots \quad (4.1.3)$$

for some coefficients $c_{p,p'} \in \mathbb{C}$, where the dots replace higher order terms in $(z-w)$ or $(\overline{z}-\overline{w})$. Note that on the circle $(z-w)(\overline{z}-\overline{w}) = 1$, from which we have analytically continued our theory after Wick rotating to a Euclidean theory on the torus, the operator product expansion (4.1.3) is entirely determined by

$$p\cdot p' = (p_l;p_r)\cdot(p_l';p_r') := p_l p_l' - p_r p_r', \quad (4.1.4)$$

which thus is the natural scalar product on the set of charge vectors. The form (4.1.4) in particular is independent of the choice of generators $j^k,\overline{\jmath}^k$. Since the conjugate of $V[p]$ is $V[-p]$, together with (4.1.3) this shows that the set of charges occurring in a toroidal conformal field theory $\mathcal{C}$ form a lattice, the CHARGE LATTICE $\Gamma \subset \mathbb{R}^{2d}$. In particular, every charge $p \in \Gamma$ must appear with multiplicity one, since otherwise by fusing $[V_k[p]]\times[V_k[-p]] = [\mathbb{1}_k]$ we find two fields $\mathbb{1}_1, \mathbb{1}_2$ with vanishing left and right dimensions in contradiction to uniqueness of the vacuum (property 1 in section 2.1).

If we use coordinates $\alpha := (p_l+p_r)/\sqrt{2}, \beta := (p_l-p_r)/\sqrt{2}$, then $\Gamma \subset (\mathbb{R}^d)^* \oplus \mathbb{R}^d$ with scalar product

$$(\alpha,\beta)\cdot(\alpha',\beta') = \alpha\beta' + \alpha'\beta. \quad (4.1.5)$$

There is a maximal positive definite d-plane given by $\alpha = \beta$ or equivalently $p_r = 0$ in $(\mathbb{R}^d)^* \oplus \mathbb{R}^d = \mathbb{R}^{d,d}$. In particular, the WORLDSHEET PARITY TRANSFORMATION acts by $(p_l;p_r) \mapsto (p_r;p_l)$ and interchanges this d-plane with its orthogonal complement ($p_l = 0$), plus inducing a sign change of the bilinear form (4.1.5) on $\mathbb{R}^{d,d}$. Rotations $O(d)\times O(d)$ in these d-planes relate different choices of generators $j^k,\overline{\jmath}^k$ obeying (4.1.1).

By (4.1.3) and (4.1.4) the condition of locality together with that of integer spin $h-\overline{h}$ for all fields in $\mathcal{C}$ enforces the charge lattice Γ to be an even integer lattice. Moreover, the irreducible representations of the operator product expansion of $\mathcal{C}$ which are local to all vertex operators $V[p], p \in \Gamma$, in the theory are labelled by Γ^*/Γ. We claim $\Gamma^*/\Gamma = \{0\}$, i.e. that Γ is self–dual. To see this, let $\mathcal{A}$ denote the algebra generated by the Abelian currents of a toroidal conformal field theory $\mathcal{C}$ with charge lattice Γ. By (4.1.2), $\mathcal{A}$ in particular contains the left and right moving Virasoro algebras of our theory. Moreover, the irreducible representations $\mathcal{H}_p$ of $\mathcal{A}$ are labelled by the charge $p = (p_l;p_r) \in \Gamma$ of their lowest weight state $|p\rangle \in \mathcal{H}_p$. Action of the Fourier modes of the Abelian currents in $\mathcal{A}$ gives the standard Fock space representation, so the characters of the irreducible representations of $\mathcal{A}$ are

$$p\in\Gamma: \quad \chi_p(\sigma) = \mathrm{tr}_{\mathcal{H}_p}\, q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}} = \frac{q^{\frac{p_l^2}{2}-\frac{d}{24}}\,\overline{q}^{\frac{p_r^2}{2}-\frac{d}{24}}}{(\prod_n(1-q^n)(1-\overline{q}^n))^d} = \frac{1}{|\eta|^{2d}}q^{\frac{p_l^2}{2}}\,\overline{q}^{\frac{p_r^2}{2}}. \quad (4.1.6)$$

The entire partition function of $\mathcal{C}$ therefore is

$$Z_\Gamma(\sigma) = \frac{1}{|\eta|^{2d}} \sum_{p=(p_l;p_r)\in\Gamma} q^{\frac{p_l^2}{2}} \,\overline{q}^{\frac{p_r^2}{2}}. \tag{4.1.7}$$

Z_Γ must be modular invariant by property 9 of section 2.1, which shows $\Gamma^* = \Gamma$. Summarizing, we can associate an even self–dual lattice Γ of signature (d, d) to every toroidal conformal field theory $\mathcal{C}$. The converse is true as well [CENT85, Nar86]:

Theorem 4.1.2
A toroidal conformal field theory with central charge $c = d \in \mathbb{N}$ is uniquely determined by its charge lattice Γ, an even, self–dual lattice with signature (d, d). The moduli space $\mathcal{M}_d^{Narain}$ of such theories agrees with the moduli space of even self–dual lattices with signature (d, d):

$$\mathcal{M}_d^{Narain} = O(d) \times O(d) \backslash O(d, d)/O(\Gamma^{d,d}), \tag{4.1.8}$$

where $\Gamma^{d,d}$ denotes the standard even self–dual lattice of signature (d, d) and $O(\Gamma^{d,d})$ its automorphism group. The group $O(d)\times O(d)$ describes rotations in the maximal positive definite d-plane $\{(\alpha, \alpha) \in \mathbb{R}^{d,d}\}$ and its orthogonal complement.
The Zamolodchikov metric on $\mathcal{M}_d^{Narain}$ is the group invariant one.

Proof:
Given an even, self-dual lattice $\Gamma \subset \mathbb{R}^{d,d}$, one can construct a toroidal conformal field theory with charge lattice Γ by the vertex operator or a nonlinear sigma model construction [Nar86], as we will see in section 4.2. Therefore, to prove the above form of $\mathcal{M}_d^{Narain}$ it remains to be shown that Γ determines a toroidal conformal field theory uniquely. Note that Γ uniquely determines the operator product expansions (4.1.3) of primary fields. Therefore, given two consistent sets $c^k_{\alpha,\beta}, k \in \{1, 2\}$ of coefficients we must arrange $c^1_{\alpha,\beta} = c^2_{\alpha,\beta}$ for all $\alpha, \beta \in \Gamma$ by normalizing the primary fields appropriately. In other words, we must find constants $d_\gamma \in \mathbb{R}$ for any $\gamma \in \Gamma$ such that $\forall\, \alpha, \beta \in \Gamma : c^2_{\alpha,\beta} = d_\alpha d_\beta c^1_{\alpha,\beta}$. This is possible, because having fixed $d_\alpha, d_\beta, d_\gamma, d_\delta \in \mathbb{R}$ such that

$$c^2_{\alpha,\beta} = d_\alpha d_\beta c^1_{\alpha,\beta},\ c^2_{\alpha,\gamma} = d_\alpha d_\gamma c^1_{\alpha,\gamma},\ c^2_{\alpha,\delta} = d_\alpha d_\delta c^1_{\alpha,\delta},\ c^2_{\beta,\gamma} = d_\beta d_\gamma c^1_{\beta,\gamma}$$

for four nonzero two point functions $c^k_{\alpha,\beta}, c^k_{\alpha,\gamma}, c^k_{\alpha,\delta}, c^k_{\beta,\delta}$, by the crossing symmetries

$$\frac{c^1_{\alpha,\beta} c^1_{\gamma,\delta}}{c^1_{\alpha,\gamma} c^1_{\beta,\delta}} = \frac{c^2_{\alpha,\beta} c^2_{\gamma,\delta}}{c^2_{\alpha,\gamma} c^2_{\beta,\delta}} \quad \text{and} \quad \frac{c^1_{\alpha,\gamma} c^1_{\beta,\delta}}{c^1_{\alpha,\delta} c^1_{\beta,\gamma}} = \frac{c^2_{\alpha,\gamma} c^2_{\beta,\delta}}{c^2_{\alpha,\delta} c^2_{\beta,\gamma}}$$

etc. we automatically have $c^2_{\gamma,\delta} = d_\gamma d_\delta c^1_{\gamma,\delta}$ and $c^2_{\beta,\delta} = d_\beta d_\delta c^1_{\beta,\delta}$. If more than two of the six two point functions vanish, then by similar arguments the normalization of one of the primaries is independent of the three others and a consistent choice of $d_\alpha, d_\beta, d_\gamma, d_\delta \in \mathbb{R}$ is therefore possible, too.
To determine the Zamolodchikov metric on $\mathcal{M}_d^{Narain}$, note that deformations of toroidal conformal field theories are given by the operators $\mathcal{O}_{m,n} = j^m \overline{j}^n, m, n \in$

$\{1,\dots,d\}$. Each $\mathcal{O}_{m,n}$ is integrable marginal to all orders, which is easily checked by Kadanoff's criterion (conclusion 2.2.1) together with Cardy's trick [Car87] as discussed in section 2.2. The operator product expansion with any primary field $V[p], p\in\Gamma$, shows that $\mathcal{O}_{m,n}$ acts as the generator

$$X_{m,n} = p_r^m \frac{\partial}{\partial p_l^n} + p_l^n \frac{\partial}{\partial p_r^m} \tag{4.1.9}$$

of $O(d,d)$. Since the Zamolodchikov metric is given by the two point correlators of $\mathcal{O}_{m,n}$ [Zam86], the last statement of the theorem follows immediately. □

Definition 4.1.3
An $N=(1,1)$ superconformal field theory $\mathcal{C}$ with central charge $c=3d/2, d\in\mathbb{N}$ is called TOROIDAL SUPERCONFORMAL THEORY, *if the following holds:*
$\mathcal{C}=\mathcal{C}_b\otimes\mathcal{C}_f$, where $\mathcal{C}_b$ is a toroidal conformal field theory with central charge d and $\mathcal{C}_f$ is a fermionic conformal field theory as introduced at the beginning of chapter 3. Each generator j of the left handed $u(1)_1^d$ current algebra in $\mathcal{C}_b$ has a superpartner ψ in $\mathcal{C}_f$ of dimensions $(h;\overline{h})=(\frac{1}{2};0)$, and analogously on the righthanded side.

One now directly deduces the following

Theorem 4.1.4
Assume $\mathcal{C}=\mathcal{C}_b\otimes\mathcal{C}_f$ to be an $N=(1,1)$ toroidal superconformal field theory as in definition 4.1.3. If $j^1,\dots,j^d$ are Abelian currents with (4.1.1), then their superpartners $\psi^1,\dots,\psi^d$ are d free Majorana fermions,

$$\psi^k(z)\psi^l(w) \sim \frac{\delta^{kl}}{z-w},$$

with coupled spin structures. The supercharge of $\mathcal{C}$ is given by $G=\sum_k :\psi^k j^k:$, and the energy momentum tensor is $T(z)=\frac{1}{2}\sum_k :j^k j^k: +\frac{1}{2}\sum_k :\psi^k\partial\psi^k:$.
If d is even, then $\mathcal{C}$ also carries $N=(2,2)$ supersymmetry. We define

$$\psi_\pm^{(k)} := \tfrac{1}{\sqrt{2}}\left(\psi^{2k-1}\pm i\psi^{2k}\right), \quad j_\pm^{(k)} := \tfrac{1}{\sqrt{2}}\left(j^{2k-1}\pm ij^{2k}\right), \quad k\in\{1,\dots,d/2\}, \tag{4.1.10}$$

and

$$J := \sum_{k=1}^{d/2} :\psi_+^{(k)}\psi_-^{(k)}:, \quad G^\pm := \sqrt{2}\sum_{k=1}^{d/2} :\psi_\pm^{(k)} j_\mp^{(k)}:$$

to obtain the $U(1)$ current J and the supercharges $G^\pm$ of the superconformal algebra.
The moduli spaces of toroidal superconformal field theories are given by $\mathcal{M}_d^{Narain}$ as defined in (4.1.8), as well.

To make contact to our axiomatic approach to superconformal field theories of section 3 denote by $\mathcal{H}^S_{k,Ising}, S\in\{NS,R\}, k\in\{b,f\}$, the respective sectors of the Ising model (3.2) and by $\mathcal{H}_{tor}$ the space of states of $\mathcal{C}_b$. Then for the toroidal superconformal field theory $\mathcal{C}$ we have $\mathcal{H}^S_k=\mathcal{H}_{tor}\otimes\left(\mathcal{H}^S_{k,Ising}\right)^{\otimes d}$.

Note that for $d=4$ as explained in section 3.2, the $N=2$ superconformal algebra is enhanced to an $N=4$ superconformal algebra. The free field realization (3.2.3) is in exact agreement with (4.1.10).
Since a free Dirac fermion $\psi_\pm^{(k)}$ in the Neveu-Schwarz (Ramond) sector has half integer (integer) modes and charge ± 1 with respect to the $U(1)$ current J given in theorem 4.1.4, the partition function of a toroidal superconformal field theory with charge lattice Γ is

$$\begin{aligned} Z_\Gamma^{NS}(\sigma,z) &= Z_\Gamma(\sigma)\cdot\left|\frac{\vartheta_3(z)}{\eta}\right|^d, & Z_\Gamma^{\widetilde{NS}}(\sigma,z) &= Z_\Gamma(\sigma)\cdot\left|\frac{\vartheta_4(z)}{\eta}\right|^d,\\ Z_\Gamma^{R}(\sigma,z) &= Z_\Gamma(\sigma)\cdot\left|\frac{\vartheta_2(z)}{\eta}\right|^d, & Z_\Gamma^{\widetilde{R}}(\sigma,z) &= Z_\Gamma(\sigma)\cdot\left|\frac{\vartheta_1(z)}{\eta}\right|^d. \end{aligned} \tag{4.1.11}$$

Z_Γ is defined in (4.1.7), and if d is odd, set $z=0$.
By (4.1.11), $N=(2,2)$ superconformal toroidal theories are invariant under spectral flow (3.1.8), so (3.1.9) holds.

4.2 Nonlinear sigma models on tori

In this section we give the relation between the abstract definitions 4.1.1 and 4.1.3 of a toroidal (super–)conformal field theory $\mathcal{C}$ and Narain's original construction [CENT85, Nar86]. Let us study the data that are encoded in the charge lattice Γ of $\mathcal{C}$. Pick a maximal nullplane $Y\subset\mathbb{R}^{d,d}=(\mathbb{R}^d)^*\oplus\mathbb{R}^d$ such that $Y\cap\Gamma\subset\Gamma$ is a sublattice of rank d. Apply an $O(d)\times O(d)$ transformation to fix the relative position of the left and right handed bases of currents $\{j^1,\dots,j^d\}$ and $\{\overline{\jmath}^1,\dots,\overline{\jmath}^d\}$, respectively, such that the equation of this plane becomes $\beta=0$. One factor of $O(d)\times O(d)$ that rotates one of the planes $\{(\alpha,\pm\alpha)\}$ suffices to do so. Now put $Y\cap\Gamma=(\Lambda^*;0)$, where Λ^* denotes the dual of a lattice $\Lambda\subset\mathbb{R}^d$. Next choose a dual nullplane Y^0 such that $Y\oplus Y^0=\mathbb{R}^{d,d}$ and $Y^0\cap\Gamma\subset\Gamma$ is a lattice of rank d, too. Existence of Y^0 can be shown by a Gram type algorithm. Then $Y^0=\{(-\widetilde{B}\beta,\beta)\mid\beta\in\mathbb{R}^d\}$ for some skew matrix $\widetilde{B}\in Skew(d):=Skew(d\times d,\mathbb{R})$. We set $B:=\Lambda^T\widetilde{B}\Lambda$, then different choices of Y^0 merely correspond to translations of B by integral matrices. Thus B can be viewed as an element of $Skew(d)/Skew(d\times d;\mathbb{Z})$, and the choice of parameters (Λ,B) only depends on the choice of Y. Altogether, we can now use $(\Lambda,B)\in O(d)\backslash Gl(d)\times Skew(d)$ to parametrize toroidal conformal field theories. Here we identify $\Lambda\in GL(d)$ with the image of $\mathbb{Z}^d$ under Λ. Note that the choice of coordinates by $A\in O(d)$, $A\mapsto A\Lambda$, corresponds to the action of the diagonal of $O(d)\times O(d)$ in (4.1.8). Explicitly we find

$$\begin{aligned} \Gamma=\Gamma(\Lambda,B) &= \Big\{(p_l(\mu,\lambda);p_r(\mu,\lambda)):=\\ &\quad \tfrac{1}{\sqrt{2}}\Big(\mu-\widetilde{B}\lambda+\lambda;\mu-\widetilde{B}\lambda-\lambda\Big)\Big|\,(\mu,\lambda)\in\Lambda^*\oplus\Lambda\Big\}. \end{aligned} \tag{4.2.1}$$

With respect to coordinates (α, β) as in (4.1.5), we define

$$\begin{aligned} S^-: O(d)\backslash GL(d) \times Skew(d) &\longrightarrow O(d) \times O(d)\backslash O(d,d) \quad =: \quad \mathcal{T}^{d,d}, \\ S^-(\Lambda, B) &= \begin{pmatrix} (\Lambda^T)^{-1} & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} \mathbb{1} & -B \\ 0 & \mathbb{1} \end{pmatrix}. \end{aligned} \tag{4.2.2}$$

Then (4.2.2) is an isometry, and $\Gamma = \Gamma(\Lambda, B)$ is the image of the standard lattice $\Gamma^{d,d}$ under $S^-(\Lambda, B)$. $\mathcal{T}^{d,d}$ is the Teichmüller space of $\mathcal{M}_d^{Narain}$.
Recall that a $u(1)$ current gives the standard generator of a translation. Since the charges with respect to the $2d$ real Abelian currents of a toroidal conformal field theory $\mathcal{C}$ with central charge $c = d \in \mathbb{N}$ are quantized to be contained in the charge lattice $\Gamma = \Gamma(\Lambda, B)$, these currents must correspond to translations in a compact manifold of real dimension d. Now it will be easy to see that $\mathcal{C}$ has a nonlinear sigma model description with target space $T^d = \mathbb{R}^d/\Lambda$ and B–field B.
In general, a NONLINEAR SIGMA MODEL on a compact target manifold X assigns an action to any twocycle on X. This action is the sum of the area of the cycle for a given Ricci flat metric plus the image of the cycle under a cohomology element $B \in H^2(X, \mathbb{R})$. Since integer shifts of the action are irrelevant, the physically relevant B-field is the projection of B to $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. Thus the parameter space of nonlinear sigma models has the form $\{\textit{Ricci flat metrics}\} \times \{B - \textit{fields}\}$.
The action for the (bosonic) nonlinear σ–model on the torus $T^d = \mathbb{R}^d/\Lambda$, which describes d (non–single valued) massless scalar fields $\Phi^k: Z \to T^d, k \in \{1, \ldots, d\}$, therefore is

$$S_{bos} = \frac{1}{2\pi} \int_Z dz d\overline{z}\; (G_{kl} + B_{kl}) \partial\Phi^k(z, \overline{z}) \overline{\partial}\Phi^l(z, \overline{z}), \tag{4.2.3}$$

where we have set $\alpha' = 1$ by choosing a unit of length. Here, the constant symmetric tensor $G_{kl} = \langle \lambda_k, \lambda_l \rangle$ defines the flat metric on T^d with respect to real coordinates $t^1, \ldots, t^d$ along the onecycles $\lambda_1, \ldots, \lambda_d$ which generate $\Lambda \cong H_1(T^d, \mathbb{Z})$. The antisymmetric tensor $B_{kl} = -B_{lk}$ determines the B-field flux $B = \sum_{kl} B_{kl} dt^k \wedge dt^l \in H^2(T^d, \mathbb{R})$ through any twocycle in T^d. Actually, $B \in \check{H}^2(T^d, U(1)) \cong H^2(\Lambda, U(1)) = Skew(d)/Skew(d \times d; \mathbb{Z}) \cong H^2(T^d, \mathbb{R})/H^2(T^d, \mathbb{Z})$ as above. The B–field degrees of freedom allow for nontrivial phases in the action of the operator $\check{T}_\delta$ of translation by lattice vectors $\lambda_0 \in \Lambda, \delta := \frac{1}{\sqrt{2}}(\lambda_0; -\lambda_0) \in \Gamma(\Lambda, 0)$: On the sector of the space of states built on $V[p], p = p(\mu, \lambda) \in \Gamma(\Lambda, B)$, $\check{T}_\delta$ acts by multiplication with $\varepsilon(\lambda, \lambda_0) := e^{2\pi i \delta \cdot p} = e^{-2\pi i \langle B\lambda, \lambda_0 \rangle}$. Note that ε defines a COCYCLE

$$\begin{aligned} H^2(G, U(1)) \ni \varepsilon \quad &\Longleftrightarrow \quad \varepsilon: G \times G \to U(1), \\ \forall\, g, g_1, g_2, h \in G: \quad \varepsilon(g_1 g_2, h) &= \varepsilon(g_1, h)\varepsilon(g_2, h), \\ \varepsilon(g, h)\varepsilon(h, g) &= \mathbb{1}, \\ \varepsilon(g, g) &= \mathbb{1}, \end{aligned} \tag{4.2.4}$$

with discrete G and $G = \Lambda$ in the present case. This observation due to C. Vafa [Vaf86] is generalized in the context of orbifold conformal field theories, see sections 5.1 and 5.2.

Ricci flatness of G together with $dB = 0$ ensures conformal invariance of the quantum field theory governed by (4.2.3). Recall that on a torus every Ricci flat metric is flat [Bes87, 4.50]. Thus the d^2 parameters (G_{kl}, B_{kl}) in (4.2.3) indeed exhaust the parameters of a nonlinear sigma model. On the other hand, since every positive definite symmetric matrix $G \in Gl(d)$ possesses a Cholesky decomposition $G = \Lambda^T\Lambda$ with uniquely determined $\Lambda \in O(d)\backslash Gl(d)$, the space $\mathcal{T}^{d,d}$ given in (4.2.2) indeed agrees with the Teichmüller space of toroidal nonlinear sigma models.
For the supersymmetric nonlinear sigma model on T^d we also have to introduce d fermionic fields $\Psi^k : Z \to T^d, k \in \{1, \dots, d\}$, and the action is given by

$$S = S_{bos} + S_{ferm}, \quad S_{ferm} = \frac{1}{2\pi} \int_Z dz d\overline{z}\; G_{kl} \Psi^k \not{D} \overline{\Psi}^l(z, \overline{z}). \tag{4.2.5}$$

Each Φ^k in (4.2.3) decomposes into a left- and a rightmoving part $\Phi^k(z, \overline{z}) = \frac{1}{2}\left(\varphi^k(z) + \overline{\varphi}^k(\overline{z})\right), k \in \{1, \dots, d\}$, and analogously for the Ψ^k in (4.2.5), $\Psi^k(z, \overline{z}) = \frac{1}{2}\left(\psi^k(z) + \overline{\psi}^k(\overline{z})\right)$. The fields $j^k = i\partial\varphi^k$ are the Abelian left handed $u(1)$ currents of the theory as in definition 4.1.1, and the ψ^k are their superpartners as in theorem 4.1.4. With the expressions for $\psi^{(k)}_\pm, j^{(k)}_\pm$ in (4.1.10) the generators of the $N = (1,1)$ or $N = (2,2)$ superconformal algebra now are read off directly from the action (4.2.5), in accordance with theorem 4.1.4.
Note that the ψ^k correspond to the covariantly constant spinors dt^k in the Clifford algebra of $T^*(T^d)$ which trivialize the spin bundle of the torus. The use of $\psi^{(k)}_\pm, j^{(k)}_\pm$ induces a choice of complex coordinates and thus of a complex structure on T^d, in other words, we have an isomorphism $T^d \cong \mathbb{C}^{d/2}/\Lambda$. Let us shortly digress to make the above remark $\Lambda \cong H_1(T^d, \mathbb{Z})$ more precise: If $dz_1, \dots, dz_{d/2}$ is a basis of $H^{1,0}(T^d, \mathbb{C})$, then

$$H_1(X, \mathbb{R}) \ni \quad A \longmapsto \left(\int_A dz_1, \dots, \int_A dz_{d/2} \right) \quad \in \mathbb{C}^{d/2}$$

defines an isomorphism $(\int dz)$. Set $L := (\int dz)(H_1(T^d, \mathbb{Z}))$, then $\mathbb{C}^{d/2}/L$ is the Albanese torus of T^d and in particular $\mathbb{C}^{d/2}/\Lambda \cong T^d \cong \mathbb{C}^{d/2}/L$ [BPdV84, p.35], thus $\Lambda \cong H_1(T^d, \mathbb{Z})$, as remarked before. This observation basically coincides with the following

Theorem 4.2.1 (Torelli theorem for complex tori [BPdV84, I.14.2])
Let $T, \widetilde{T}$ denote two complex tori of same dimension, and $\alpha : H^1(T, \mathbb{Z}) \to H^1(\widetilde{T}, \mathbb{Z})$ an isomorphism, such that its $\mathbb{C}$–linear extension maps $H^{1,0}(T)$ isomorphically onto $H^{1,0}(\widetilde{T})$. Then α is induced by an isomorphism from $\widetilde{T}$ to T.

The above remark as well as theorem 4.2.1 will be tacitly used in the following. We will view $H^{1,0}(T^d) \subset H^1(T^d, \mathbb{R}) \cong \mathbb{R}^{d/2,d/2}$ as fixed reference $\frac{d}{2}$–plane, which is isotropic with respect to the intersection pairing. The relative position of $\Lambda^* \cong H^1(T^d, \mathbb{Z})$ to $H^{1,0}(T^d)$ then determines the particular torus uniquely.
Summarizing, we have seen that every toroidal (super–)conformal field theory $\mathcal{C}$ in $\mathcal{M}^{Narain}_d$ with charge lattice $\Gamma = \Gamma(\Lambda, B)$ can be realized as a nonlinear sigma model

on the (complex) torus $T^d = \mathbb{R}^d/\Lambda$ with B–field B. We call a pair of parameters $(\Lambda, B) \in O(d)\backslash Gl(d) \times Skew(d)/Skew(d \times d;\mathbb{Z})$ a GEOMETRIC INTERPRETATION of $\mathcal{C}$ and write $\mathcal{C} = \mathcal{T}(\Lambda, B)$. By abuse of notation $\mathcal{T}(\Lambda, B)$ may denote a bosonic conformal or a superconformal toroidal field theory, depending on the context.
The ground states $|p\rangle$ for $p \in \Gamma(\Lambda, B)$ can now be parametrized by pairs $(\mu, \lambda) \in \Lambda^* \oplus \Lambda$ such that $p = p(\mu, \lambda)$ as in (4.2.1). Then, $|p\rangle$ describes a string with momentum μ and winding mode λ around the corresponding torus cycles. Note that spin is measured by $P = L_0 - \overline{L}_0$ and has value $\mu\lambda \in \mathbb{Z}$ for $|p\rangle$, independently of the B–field. A B-field flux through a twocycle that $|p\rangle$ winds around only has influence on the energy $h + \overline{h} = \frac{p_l^2}{2} + \frac{p_r^2}{2}$ of $|p\rangle$.
The determination of (Λ, B) depends on the choice of a nullplane Y, so a given conformal field theory will have infinitely many geometric interpretations. By theorem 4.1.2 we know $\mathcal{M}_d^{Narain} = \mathcal{T}^{d,d}/O(\Gamma^{d,d})$. As can be deduced from results in [LP81, Nik80b], $O(\Gamma^{d,d})$ acts transitively on maximal isotropic primitive sublattices of $\Gamma^{d,d}$. Therefore, the set of geometric interpretations of $\mathcal{C}$ is an $O(\Gamma^{d,d})$ orbit in $\mathcal{T}^{d,d}$. In particular, (4.2.2) shows that lattice automorphisms of Λ and integral shifts of B give isomorphic theories. Together with the automorphism which interchanges the nullplanes Y and Y^0 that fix the geometric interpretation for a given theory, these symmetries generate a subgroup $O^+(\Gamma^{d,d}) \subset O(\Gamma^{d,d})$ of index 2. The latter symmetry is induced by torus T–duality (see section 7.3.3). In general, the space of maximal positive definite subspaces of a metric space W has two components, and $O^+(W)$ denotes the subgroup of elements of $O(W)$ which do not interchange these components. Note that for positive definite W we have $SO(W) = O^+(W)$. For a lattice $\Gamma \subset W$ we put $O^+(\Gamma) := O(\Gamma) \cap O^+(W)$. The space

$$\mathcal{M}_d^{tori} := SO(d) \times O(d)\backslash O^+(d,d)/O^+(\Gamma^{d,d}) \cong \mathcal{T}^{d,d}/O^+(\Gamma^{d,d}) \tag{4.2.6}$$

thus is a double cover of $\mathcal{M}_d^{Narain}$. It will be relevant for the description of $\mathbb{Z}_2$ orbifold conformal field theories on $K3$ in section 7.3.2. To obtain $\mathcal{M}_d^{Narain}$, we identify toroidal nonlinear sigma models which are related by target space orientation change and by theorem 4.1.2 indeed define isomorphic toroidal conformal field theories. Note that as can be seen from (4.2.1) worldsheet parity change transforms $\mathcal{T}(\Lambda, B)$ into $\mathcal{T}(\Lambda, -B)$, an inequivalent theory for generic values of B.
We now come to a concept which is of major importance in the context of Calabi-Yau compactification and nonlinear σ models, namely the idea of LARGE VOLUME LIMIT. A precise notion is necessary of how to associate a unique geometric interpretation to a theory described by an even self dual lattice Γ when parameters of volume go to infinity. Intuitively, because of the uniqueness condition, this should describe the limit where all the radii of the torus in this particular geometric interpretation are large. Because in the charge lattice (4.2.1) $\lambda \in \Lambda$ and $\mu \in \Lambda^*$ are interpreted as winding and momentum modes, the corresponding nullplane Y should have the property

$$\begin{aligned} Y \cap \Gamma &= \operatorname{span}_{\mathbb{Z}} \left\{ \tfrac{1}{\sqrt{2}}(\mu;\mu) \in \Gamma \,\middle|\, \|\mu\|^2 \ll 1 \right\} \\ &\subset \operatorname{span}_{\mathbb{Z}} \left\{ (p_l;p_r) \in \Gamma \,\middle|\, \|p_l\|^2 \ll 1, \|p_r\|^2 \ll 1 \right\} =: \widetilde{\Gamma}. \end{aligned} \tag{4.2.7}$$

Because $\|p_l\|^2 - \|p_r\|^2 \in \mathbb{Z}$, for $(p_l; p_r) \in \widetilde{\Gamma}$ we have $\|p_l\|^2 = \|p_r\|^2$. This shows $Y \cap \Gamma = \widetilde{\Gamma}$ because any $(p_l; p_r) \notin Y^\perp = Y$ must have large components. Moreover, if a maximal isotropic plane Y as in (4.2.7) exists, then it is uniquely defined, thus yielding a sensible notion of large volume limit. Large volume and small volume limits are interchanged by T–duality.

4.3 Toroidal theories in two dimensions

Since in chapter 6 we will discuss our results on the classification of unitary conformal field theories with central charge $c = 2$, let us study the moduli space of two–dimensional (bosonic) toroidal theories in greater detail. See also [DVV87] for a thorough account on this subject.
Since $H^{1,0}(T^2)$ is spanned by one holomorphic differential form dz only, the isomorphism $\Lambda \cong H_1(T^2, \mathbb{Z})$ is defined by the PERIODS $\int_A dz, \int_B dz$ with respect to a symplectic basis (A, B) of $H_1(T^2, \mathbb{Z})$. The complex parameter $\tau := \int_B dz / \int_A dz \in \mathbb{H}$, is the only entry of the NORMALIZED PERIOD MATRIX [BPdV84, p.38] and thus defines the complex structure of T^2. On the other hand, since every metric on a two–dimensional torus is Kähler, and $\dim_{\mathbb{R}} H^2(T^2, \mathbb{R}) = 1$, the volume ρ_2 of T^2 already specifies the Kähler class of the metric G. This means that we can group the four real parameters G_{kl}, B_{kl} of the theory into two complex ones, $\tau, \rho \in \mathbb{H}$, by

$$\tau = \tau_1 + i\tau_2 := \frac{G_{12}}{G_{22}} + i\frac{\sqrt{\det(G_{kl})}}{G_{22}}, \quad \rho = \rho_1 + i\rho_2 := B_{12} + i\sqrt{\det(G_{kl})}. \quad (4.3.1)$$

Here τ is the image of $\Lambda \in Gl(2)$ under the natural projection $Gl(2) \to Sl(2)^{(\tau)} \cong \mathbb{H}$. If $O(2,2;\mathbb{R}) \ni \Gamma(\Lambda, B) \mapsto (\tau, \rho)$, then $\rho \in \mathbb{H} \cong Sl(2)^{(\rho)}$, where $Sl(2)^{(\rho)}$ is the commutant of $Sl(2)^{(\tau)}$ in $O(2,2;\mathbb{R})$. The decompositon of the Teichmüller space of $\mathcal{M}_2^{Narain}$ into $\mathbb{H} \times \mathbb{H}$ also follows from our remark below conjecture 3.1.1. Namely, theories in $\mathcal{M}_2^{Narain}$ generically have $N = (2,2)$ supersymmetry, and the tangent space decomposes into two factors corresponding to $(\frac{1}{2}; \frac{1}{2})$ fields with $Q = \overline{Q} = \pm 1$ and $Q = -\overline{Q} = \pm 1$, respectively.
Standard generators λ_1, λ_2 of Λ are given by

$$\lambda_1 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \text{ and } \widetilde{B} = \frac{\rho_1}{\rho_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.3.2)$$

By (4.3.1) for $\lambda = m_2\lambda_1 + m_1\lambda_2 \in \Lambda$ and $\mu = n_2\mu_1 + n_1\mu_2 \in \Lambda^*$ as above (4.2.1) reads

$$p_{\substack{l\\r}} = \frac{1}{\sqrt{2\tau_2\rho_2}} \left\{ \begin{pmatrix} n_2\tau_2 \\ -n_2\tau_1 + n_1 \end{pmatrix} + \rho_1 \begin{pmatrix} m_1\tau_2 \\ -m_2 - m_1\tau_1 \end{pmatrix} \pm \rho_2 \begin{pmatrix} m_2 + m_1\tau_1 \\ m_1\tau_2 \end{pmatrix} \right\}. \quad (4.3.3)$$

Instead of $V[p]$ with $p = (p_l; p_r)(\mu, \lambda)$ as in (4.3.3) we also write $|m_1, m_2, n_1, n_2\rangle$. If $\Gamma = \Gamma(\Lambda, B)$, and (Λ, B) is related to (τ, ρ) by (4.3.1), for the partition function (4.1.7) we write

$$Z(\tau_1, \tau_2, \rho_1, \rho_2) := Z_\Gamma(\sigma) = \frac{1}{|\eta|^4} \sum_{\mu \in \Lambda^*, \lambda \in \Lambda} q^{\frac{1}{2}(p_l(\mu,\lambda))^2} \overline{q}^{\frac{1}{2}(p_r(\mu,\lambda))^2}. \quad (4.3.4)$$

In terms of the new parameters (τ,ρ) the duality group $O(\Gamma^{2,2})$ in (4.1.8) translates into the group generated by $PSL(2,\mathbb{Z})\times PSL(2,\mathbb{Z})$, which acts by Möbius transformations on each factor of $\mathbb{H}\times\mathbb{H}$, and the dualities

$$U,V:\mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\mathbb{H},\quad U(\tau,\rho):=(\rho,\tau),\ V(\tau,\rho):=(-\overline{\tau},-\overline{\rho}). \tag{4.3.5}$$

In terms of the parameters (τ,ρ) the moduli space (4.1.8) therefore is

$$\mathcal{M}_2^{Narain}=\left(\mathbb{H}/PSL(2,\mathbb{Z})\times\mathbb{H}/PSL(2,\mathbb{Z})\right)/(\mathbb{Z}_2\times\mathbb{Z}_2). \tag{4.3.6}$$

By the above interpretation of τ and ρ the duality U interchanges complex and (complexified) Kähler structure of T^2 and is known as MIRROR SYMMETRY. In view of our comment about the decomposition of the Teichmüller space into $\mathbb{H}\times\mathbb{H}$ it corresponds to the transformation $(Q;\overline{Q})\mapsto(Q;-\overline{Q})$, indeed mirror symmetry [GP90], as already mentioned in [Dix87] and [LVW89] (see theorem 5.6.3). Compared to the description (4.1.8) of the moduli space by equivalence classes of lattices, V corresponds to conjugation by $\mathrm{diag}(-1,1,-1,1)$ on $O(2)\times O(2)\backslash O(2,2;\mathbb{R})$ which is target space orientation change. Now world sheet parity $(\Lambda,B)\mapsto(\Lambda,-B)$ is given by $(\tau,\rho)\mapsto(\tau,-\overline{\rho})$, not a duality symmetry.
Using either (4.1.9) or (4.3.3) it is not hard to see that the Zamolodchikov metric on $\mathcal{M}_2^{Narain}$ is induced by the product of hyperbolic metrics on each of the factors $\mathbb{H}$ in (4.3.6).
We now show that the partition function (4.3.4) obeys a remarkable triality symmetry:

Theorem 4.3.1 [DVV87, Dij95]
Consider a toroidal conformal field theory with central charge $c=2$, which is specified by parameters $(\tau,\rho)\in\mathbb{H}\times\mathbb{H}$ for the target space T^2. As usual, the modulus of the world sheet torus is σ. Then the partition function obeys the identity

$$|\eta(\sigma)|^4\,Z(\tau,\rho)(\sigma)=|\eta(\tau)|^4\,Z(\rho,\sigma)(\tau).$$

Proof:
Since from (4.3.3) it is obvious that the partition function $Z(\tau,\rho)$ is invariant under mirror symmetry $U:\tau\leftrightarrow\rho$, it will suffice to show that $|\eta(\sigma)|^4Z(\tau,\rho)(\sigma)$ is invariant under $\tau\leftrightarrow\sigma$ as well. This will follow from a rederivation of the expression for the partition function in the path integral formalism* (compare with [Gin88a, §8.1]). We start from the bosonic action (4.2.3) and give the solutions of the classical equations of motion $\partial\overline{\partial}\Phi_0=0$:

$$\Phi_0^{e_1,e_2}(z,\overline{z})=\frac{1}{2i\sigma_2}\sqrt{\frac{\rho_2}{\tau_2}}\left(e_1(z-\overline{z})+e_2(\overline{z}\sigma-z\overline{\sigma})\right),\quad e_k\in\mathbb{Z}\oplus\mathbb{Z}\tau.$$

The corresponding contributions to the path integral are computed from (4.2.3),

$$S(e_1,e_2)=\frac{\pi}{\sigma_2\tau_2}\left\{\rho|e_1+e_2\overline{\sigma}|^2-\overline{\rho}|e_1+e_2\sigma|^2\right\}.$$

*We thank N. Obers for this hint.

Therefore with $q_\rho := e^{2\pi i\rho}, \overline{q}_\rho := e^{-2\pi i\overline{\rho}}$ we have

$$|\eta(\sigma)|^4 Z(\tau,\rho)(\sigma) = \sum_{e_1,e_2\in\mathbb{Z}\oplus\mathbb{Z}\tau} q_\rho^{\frac{|e_1+e_2\overline{\sigma}|^2}{2\sigma_2\tau_2}} \, \overline{q}_\rho^{\frac{|e_1+e_2\sigma|^2}{2\sigma_2\tau_2}},$$

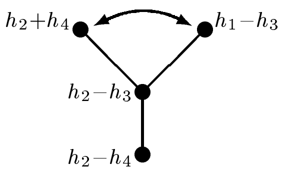
which is obviously invariant under $\tau \leftrightarrow \sigma$. $\square$

The above triality should be remembered carefully: The expression $|\eta|^4 Z$ gives the spectrum of $\frac{1}{2}$ BPS states; theorem 4.3.1 asserts that for them worldsheet and target space tori are interchangeable.

4.4 Toroidal theories in four dimensions

In chapter 7 we will discuss the moduli space of superconformal field theories with central charge $c = 6$. There, an appropriate description of toroidal superconformal field theories will be needed. In particular, for the discussion of orbifold conformal field theories in sections 7.3.2 and 7.3.4 the description given in sections 4.1 and 4.2 in terms of the first cohomology group of the torus will prove not appropriate, since $H^1(T^4,\mathbb{R})$ never survives the orbifold procedure. Therefore the aim of this section is to give an alternative description of the moduli space of toroidal superconformal field theories in dimension $d = 4$.
Here, a version of the celebrated D_4 TRIALITY comes to aid. Triality considerations have a long history in superstring and supergravity theories, see for example [Sha80, Cur82, GO86]. Concerning recent work, as communicated to us by N. Obers, $SO(4,4)$ is crucial in the conjectured duality between heterotic strings on the four–torus and type IIA on $K3$ [OP00a, KOP00]. In connection with the calculation of $G(\mathbb{Z})$ invariant string theory amplitudes one can use triality to write down new identities for Eisenstein series [OP00a, OP00b]. The results of this section, however, have been obtained by Werner Nahm and are accepted for publication in our joint work [NW].
In addition to its defining representation, the double cover of the group $SO^+(d,d)$ also has half-spinor representations, namely its images in $SO^+(H^{odd}(T,\mathbb{R}))$ and in $SO^+(H^{even}(T,\mathbb{R}))$. For $d = 4$ one has the obvious isomorphism $SO^+(4,4) \cong SO^+(H^{odd}(T,\mathbb{R}))$, which together with $SO^+(4,4) \cong SO^+(H^{even}(T,\mathbb{R}))$ yields D_4 triality [LM89, I.8]. Indeed, for $Spin(4,4)$ representations on $\mathbb{R}^{4,4}$ there is the same triality relation as for $Spin(8)$ representations on $\mathbb{R}^8$, i.e. an $\mathcal{S}_3$ permuting the vector representation, the chiral and the antichiral Weyl spinor representation. By $h_1,\dots,h_4$ we denote generators of the Cartan subalgebra of $so(4,4)$. We will now construct an automorphism of $\mathcal{T}^{4,4}$ which is induced by the triality automorphism that acts on the Cartan subalgebra by

$$h_1 \mapsto \tfrac{1}{2}(h_1+h_2+h_3+h_4), \quad h_2 \mapsto \tfrac{1}{2}(h_1+h_2-h_3-h_4),$$
$$h_3 \mapsto \tfrac{1}{2}(h_1-h_2+h_3-h_4), \quad h_4 \mapsto \tfrac{1}{2}(h_1-h_2-h_3+h_4):$$



Since $V := |\det\Lambda|$ is the volume of the torus $T = \mathbb{R}^d/\Lambda$, we can decompose $O(4)\backslash GL(4) \cong SO(4)\backslash SL(4) \times \mathbb{R}^+$ by $\Lambda \mapsto (\Lambda_0, V)$. Now let $T_{\Lambda_0} = \mathbb{R}^4/\Lambda_0$, where Λ_0 is a lattice of determinant 1 and is viewed as element of $SL(4)$. Consider the induced representation ρ of $SL(4)$ on the exterior product $\Lambda^2(\mathbb{R}^4)$ which defines an isomorphism $\Lambda^2(\Lambda_0) \cong H_2(T_{\Lambda_0}, \mathbb{Z})$ for every $\Lambda_0 \in SL(4)$. Because ρ commutes with the action of the Hodge star operator $*$ and $*^2 = \mathbb{1}$ on twoforms, $SL(4)$ is actually represented by $SO^+(3,3)$. In terms of coordinates as in (4.2.2) and with $\Lambda = V^{1/4}\Lambda_0 = (\lambda_1, \ldots, \lambda_4)$ we can write

$$\begin{aligned} \rho\,(\Lambda_0) \quad &= \quad V^{-1/2}\,(\lambda_1\wedge\lambda_2\,,\,\lambda_1\wedge\lambda_3\,,\,\lambda_1\wedge\lambda_4\,,\,\lambda_3\wedge\lambda_4\,,\,\lambda_4\wedge\lambda_2\,,\,\lambda_2\wedge\lambda_3) \\ &\qquad\qquad \in SO^+\left(H_2(T,\mathbb{R})\right) \cong SO^+(3,3). \end{aligned} \tag{4.4.1}$$

From (4.2.2) one checks that we can choose h_i such that it generates dilations of the radius R_i of our torus T^4/Λ in direction of the generator λ_i of Λ. Since $\exp(\vartheta h_i)$ scales $V^{\pm 1/2}$ by $e^{\pm\vartheta/2}$, it is easy to see that

$$S^-(\Lambda,0) \longmapsto S^+(\Lambda,0) = \left(\begin{array}{c|c|c} V^{1/2} & 0 & 0 \\ \hline 0 & \rho(\Lambda_0) & 0 \\ \hline 0 & 0 & V^{-1/2} \end{array}\right)$$

is induced by the above triality automorphism. It will now suffice to extend this to an automorphism of $\mathcal{T}^{4,4}$. To do so, we use $Skew(4) \cong \mathbb{R}^{3,3}$ which will simply be written $Skew(4) \ni B \mapsto b \in \mathbb{R}^{3,3}$ in the following. Then,

$$S^-(\Lambda,B) \longmapsto S^+(\Lambda,B) = \left(\begin{array}{c|c|c} V^{1/2} & 0 & 0 \\ \hline 0 & \rho(\Lambda_0) & 0 \\ \hline 0 & 0 & V^{-1/2} \end{array}\right) \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline b & \mathbb{1} & 0 \\ \hline -\frac{\|B\|^2}{2} & -b^T & 1 \end{array}\right) \tag{4.4.2}$$

is the seeked map. By (4.4.2) the geometric interpretation of a superconformal field theory is translated from a description in terms of the lattice of the underlying torus, i.e. in terms of $\Lambda \cong H_1(T_\Lambda, \mathbb{Z})$, to a description in terms of $H_2(T_\Lambda, \mathbb{Z}) \cong \Lambda^2(\Lambda)$. This translation will be essential for understanding the embedding of orbifold conformal field theories in the moduli space of theories associated to $K3$ that will be discussed in sections 7.3.2 and 7.3.4.
We remark that $SL(4)/\mathbb{Z}_2 \cong SO^+(3,3)$ and $SO(4) \cong SO(3) \times SO(3)/\mathbb{Z}_2$ show $SO(4)\backslash SL(4) \cong \mathcal{T}^{3,3}$ such that all in all the isomorphism ρ in (4.4.1) gives

$$\mathcal{T}^{4,4} \overset{(4.2.2)}{\cong} O(4)\backslash GL(4) \times Skew(4) \cong \mathcal{T}^{3,3} \times \mathbb{R}^+ \times \mathbb{R}^{3,3}. \tag{4.4.3}$$

This isomorphism can be generalized, as we will see in (7.1.5).
Recall from section 4.2 that we obtain the $2:1$ cover $\mathcal{M}_4^{tori}$ of the moduli space $\mathcal{M}_4^{Narain}$ if we do not identify theories that are related by target space orientation

change, and that world sheet parity is not a symmetry of toroidal conformal field theories. Consider the four–fold cover $SO(4) \times SO(4) \backslash SO^+(4,4)/SO^+(\Gamma^{4,4})$ of $\mathcal{M}_4^{Narain}$. The transformations which exchange its sheets are given by target space orientation change and T–duality, as can be read off from equation (4.2.1). Now the outer automorphisms of $SO^+(4,4)$ related to worldsheet parity and target space orientation are interchanged by the triality automorphism (4.4.2). This shows that we must use $\mathcal{M}_4^{tori}$ rather than $\mathcal{M}_4^{Narain}$, i.e. keep target space orientation for the description of the orbifold conformal field theories within the moduli space of theories with $c = 6$ in sections 7.3.2 and 7.3.4. We also want to keep the left–right distinction in the conformal field theory. Torus T–duality just yields a reparametrization of the theory and should be divided out of the moduli space.

4.5 Rational conformal field theories and CM tori

In this section we discuss the notion of RATIONAL CONFORMAL FIELD THEORY and give necessary and sufficient conditions for a toroidal conformal field theory to be rational. Rational conformal field theories are completely understood for central charges $c < 1$ since the work of Belavin, Polyakov and Zamolodchikov's, namely their construction of the minimal series [BPZ84]. The situation is rather different for $c \geq 1$, which has been the object of intensive study so far. One starting point was the work of E. Verlinde's, who proposed his celebrated VERLINDE FORMULA [Ver88] (theorem 3.1.14), the proof of which has promoted renowned mathematical results [TUY89, Fal94]. G. Anderson and G. Moore [AM88] and independently C. Vafa [Vaf88] proved that rational conformal field theories have rational central charge and all fields must have rational dimensions. Whether these conditions are sufficient remains an open problem; for toroidal conformal field theories they are, as we will also see below. Steps towards a classification of rational conformal field theories have been taken by S. Mukhi, A. Sen, and collaborators (S.D. Mathur, S. Panda) in the line of thought of Verlinde's formula [MMS88, MMS89, MPS89]: Characters of such theories are classified by employing the restrictions imposed by modular invariance as well as the usual conditions on their coefficients and normalizations. In principle, a classification of all rational conformal field theories with a given finite number of characters and zeroes of the Wronskian in the interior of the moduli space is possible by these methods. However, they do not lead to a conceptually new perception of the meaning of rationality. A different viewpoint is taken by T. Gannon and A. Coste who approach the problem from a study of affine Kac–Moody algebras. In [Gan97], a classification of all possible modular invariant partition functions for the algebras $\widehat{u(1)^m}$ and $su(2) \oplus u(1)^m$ is presented which in particular corrects the standard lore of an ADE classification for $N = 2$ superconformal minimal models (see also section 3.1.3). Properties of characters of rational conformal field theories are discussed in [CG99]. One should note that already the discussion of unitary conformal field theories with central charge $c = 1$

shows that the condition of rationality probably must be generalized in order to make use of this concept in a general setting. Steps in this direction have been taken in [Nah94, Nah96]; this, however, is not the object of the present section.

The definition of rationality is diverse in the literature (see, e.g., the appendix of [HMV88]), in this work we follow [Nah96] and [Gan97] and use definition 3.1.13. If $\mathcal{C}$ is a toroidal conformal field theory $\mathcal{C} = \mathcal{T}(\Lambda, B)$ with space of states $\mathcal{H}$ (see definition 4.1.1), then by our discussion in section 4.1 a lowest weight vector $|p\rangle \in \mathcal{H}$ of an irreducible representation of $\mathcal{W}$ with charge $p \in \Gamma(\Lambda, B)$ has conjugate $|-p\rangle \in \mathcal{H}$. We use $\pm p \in I$ as labels for the corresponding representations. Since $[V[p]] \times [V[-p]] = [\mathbb{1}]$, and quantum dimensions d_i (definition 3.1.2) transform multiplicatively under fusion, $d_p \cdot d_{-p} = 1$. On the other hand, fusion of a lowest weight vector with itself shows $d_i^2 \geq 1 \forall\, i \in I$, so all quantum dimension in $\mathcal{C}$ are equal to 1.

For the rest of this section let $\mathcal{C}$ denote a toroidal conformal field theory with charge lattice $\Gamma = \Gamma(\Lambda, B), \Lambda \subset \mathbb{R}^d$, and central charge $c = d$. Its holomorphic W–algebra is $\mathcal{W}$, and we write $\mathcal{W} = \mathcal{W}(h_1, \ldots, h_N)$ if $\mathcal{W}$ is generated by N holomorphic fields of dimensions $h_1, \ldots, h_N$. By definition 4.1.3 of toroidal superconformal field theories and definition 3.1.13 of rational conformal field theories, to discuss rational toroidal theories it will be sufficient to restrict ourselves to purely bosonic theories. Holomorphic and antiholomorphic vertex operators are characterized by the condition $p_r = 0$ or $p_l = 0$, respectively, and thus by (4.2.1) are parametrized by lattices

$$\Lambda^0_{\substack{l\\r}} := \left\{ (\mu, \lambda) \in \Lambda^* \oplus \Lambda \mid \mu = (\widetilde{B} \pm \mathbb{1})\lambda \right\}$$

(recall from section 4.2 that $\widetilde{B} = (\Lambda^T)^{-1} B \Lambda^{-1}$). Their images in the charge lattices are denoted Γ^0_l and Γ^0_r, i.e.

$$\Gamma^0_{\substack{l\\r}} := \left\{ \pm\sqrt{2}\lambda \mid \lambda \in \Lambda :\ \mu = (\widetilde{B} \pm \mathbb{1})\lambda \in \Lambda^* \right\}. \tag{4.5.1}$$

Note that Γ^0_l and Γ^0_r are even integer lattices, and that for any $(\mu, \lambda), (\mu', \lambda') \in \Lambda^0_l$ or Λ^0_r we have $2\langle \lambda, \lambda' \rangle \in \mathbb{Z}$. By $\Gamma_{l,r}$ we denote the projection of Γ onto left and right moving parts:

$$\Gamma_{\substack{l\\r}} := \left\{ \tfrac{1}{\sqrt{2}}(\mu - \widetilde{B}\lambda \pm \lambda) \mid (\mu, \lambda) \in \Lambda^* \oplus \Lambda \right\}.$$

The following lemma will directly lead to a first formulation of a criterion for rationality of our conformal field theory. That condition iii. below is equivalent to rationality of the corresponding conformal field theory was already mentioned in [HMV88], but with restriction to vanishing B-field.*

*Indeed, the author did enjoy proving this not completely trivial statement!

Lemma 4.5.1
The following conditions on the lattice Λ are equivalent:

i. rk $\Lambda_l^0 = d$.

ii. rk $\Lambda_r^0 = d$.

iii. $G := \Lambda^T\Lambda \in Gl(d,\mathbb{Q})$, *and* $B \in Skew(d) \cap \mathrm{Mat}(d,\mathbb{Q})$.

iv. $S^-(\Lambda,B)^T S^-(\Lambda,B) \in Gl(2d,\mathbb{Q})$.

v. $(\Gamma_l)^* = \Gamma_l^0$ *and* $(\Gamma_r)^* = \Gamma_r^0$.

Proof:
<u>i.⇒iii.</u>
Denote by $(\mu_k,\lambda_k), k \in \{1,\dots,d\}$, generators of the lattice Λ_l^0. Let $L, M \in \mathrm{Mat}(d,\mathbb{Z}) \cap Gl(d,\mathbb{Q})$ denote the matrices defined by

$$(\lambda_1,\dots,\lambda_d) = \Lambda L, \quad (\mu_1,\dots,\mu_d) = (\Lambda^T)^{-1}M.$$

Then, by definition (4.5.1) of Λ_l^0, we have

$$(\Lambda^T)^{-1}M = (\widetilde{B}+\mathbb{1})\Lambda L. \tag{4.5.2}$$

Thus for any $(\mu,\lambda),(\mu',\lambda') \in \Lambda_l^0$ we have $\langle\lambda,(\widetilde{B}+\mathbb{1})\lambda'\rangle = \langle\lambda,\mu'\rangle \in \mathbb{Z}$. Moreover, $2\langle\lambda_i,\lambda_j\rangle \in \mathbb{Z}$ for any i,j, since the corresponding vertex operators are pairwise local, and $\Lambda L \in Gl(d,\mathbb{R})$. Hence

$$Q := (\Lambda L)^{-1}(\widetilde{B}+\mathbb{1})\Lambda L \stackrel{(4.5.2)}{=} (\Lambda L)^{-1}(\Lambda^T)^{-1}M \quad \in Gl(d,\mathbb{Q}). \tag{4.5.3}$$

Moreover,

$$G := \Lambda^T\Lambda = MQ^{-1}L^{-1} \quad \in Gl(d,\mathbb{Q}),$$

the first assertion in iii. On the other hand (4.5.3) together with $L \in Gl(d,\mathbb{Q})$ shows

$$B = \Lambda^T\widetilde{B}\Lambda = GLQL^{-1} - G \quad \in Gl(d,\mathbb{Q}),$$

our second assertion in iii.
<u>ii.⇒iii.</u>
is shown by replacing Γ_l^0 by Γ_r^0 and $(\widetilde{B}+\mathbb{1})$ by $(\widetilde{B}-\mathbb{1})$ everywhere in the above argument.
<u>iii.⇒i.</u>
Given $G = \Lambda^T\Lambda \in Gl(d,\mathbb{Q})$ and $B \in \mathrm{Mat}(d,\mathbb{Q})$, we can choose a diagonal matrix $L \in \mathrm{Mat}(d,\mathbb{Z}) \cap Gl(d,\mathbb{Q})$ such that

$$M := (B+G)L \quad \in \mathrm{Mat}(d,\mathbb{Z}). \tag{4.5.4}$$

Since B is antisymmetric we see that $B+G = B+\Lambda^T\Lambda$ is nonsingular, so also $M \in Gl(d,\mathbb{Q})$. But (4.5.4) is equivalent to (4.5.2), so the columns of $(\Lambda^T)^{-1}M$ and ΛL each define d linearly independent vectors $\mu_1,\dots,\mu_d \in \Lambda^*$ and $\lambda_1,\dots,\lambda_d \in \Lambda$ satisfisfying $\mu_k = (\widetilde{B}+\mathbb{1})\lambda_k$. Thus we have found d linearly independent vectors $(\mu_k,\lambda_k) \in \Lambda_l^0$ proving rk $\Lambda_l^0 = d$.

iii.⇒ii.
is shown by replacing Λ_l^0 by Λ_r^0 and $(\widetilde{B}+\mathbb{1})$ by $(\widetilde{B}-\mathbb{1})$ everywhere in the above argument.
iii.⇔iv. is clear by (4.2.2).
i.⇒v.
With the above notation, we have d linearly independent vectors $(\mu_k, \lambda_k), k \in \{1,\dots,d\}$, generating the lattice Λ_l^0, so $\sqrt{2}\lambda_k, k \in \{1,\dots,d\}$, are generators of Γ_l^0. For any $p = \frac{1}{\sqrt{2}}(\mu - \widetilde{B}\lambda + \lambda) \in \Gamma_l$, where $\mu \in \Lambda^*, \lambda \in \Lambda$, we find

$$\begin{aligned} \langle \sqrt{2}\lambda_k, p\rangle &= \langle \lambda_k, \mu - \widetilde{B}\lambda + \lambda\rangle \\ &= \langle \lambda_k, \mu\rangle + \langle \widetilde{B}\lambda_k, \lambda\rangle + \langle \mu_k - \widetilde{B}\lambda_k, \lambda\rangle \;=\; \langle \lambda_k, \mu\rangle + \langle \mu_k, \lambda\rangle \in \mathbb{Z}. \end{aligned}$$

This shows $\Gamma_l^0 \subset (\Gamma_l)^*$.
On the other hand, $v \in (\Gamma_l)^*$ means

$$\forall\, \mu \in \Lambda^*, \forall\, \lambda \in \Lambda : \quad \frac{1}{\sqrt{2}}\langle \mu - (\widetilde{B} - \mathbb{1})\lambda, v\rangle \in \mathbb{Z}. \tag{4.5.5}$$

So in particular from setting $\lambda = 0$ we see that $v = \sqrt{2}\lambda_0$, $\lambda_0 \in \Lambda$. Then for $\mu_0 := (\widetilde{B} + \mathbb{1})\lambda_0$ setting $\mu = 0$ in (4.5.5) shows

$$\forall\, \lambda \in \Lambda : \quad \langle \mu_0, \lambda\rangle = \langle (\widetilde{B} + \mathbb{1})\lambda_0, \lambda\rangle = \frac{1}{\sqrt{2}}\langle v, -(\widetilde{B} - \mathbb{1})\lambda\rangle \in \mathbb{Z},$$

i.e. $\mu_0 \in \Lambda^*$. But then $v = \frac{1}{\sqrt{2}}(\mu_0 - (\widetilde{B} - \mathbb{1})\lambda_0) \in \Gamma_l^0$, our first assertion in v.
The second assertion is proven by replacing Γ_l^0 by Γ_r^0 and $(\widetilde{B} \pm \mathbb{1})$ by $(\widetilde{B} \mp \mathbb{1})$ everywhere in the above argument, which is possible since we have already shown that i.⇒ii.
v.⇒i.
is clear since $\mathrm{rk}\, \Gamma_l^0 = \mathrm{rk}\, (\Gamma_l)^* = d$, thus also $\mathrm{rk}\, \Lambda_l^0 = d$. □

By definition 4.2.2, $\Gamma = \Gamma(\Lambda, B)$ is the image of the standard lattice $\Gamma^{d,d}$ under $S^-(\Lambda, B)$. Thus $S^-(\Lambda, B)^T S^-(\Lambda, B)$ is the metric matrix of Γ, if this is understood as lattice in $\mathbb{R}^{2d}$ with the standard Euclidean metric.

Theorem 4.5.2
Let $\mathcal{C} = \mathcal{T}(\Lambda, B)$ denote a toroidal conformal field theory with central charge $c = d$. Then $\mathcal{C}$ is a rational conformal field theory iff $G := \Lambda^T\Lambda \in Gl(d, \mathbb{Q})$ and $B \in Skew(d) \cap \mathrm{Mat}(d, \mathbb{Q})$ (or equivalently $S^-(\Lambda, B)^T S^-(\Lambda, B) \in Gl(2d, \mathbb{Q})$, by lemma 4.5.1). $\mathcal{C}$ then has W-algebra $\mathcal{W}(\underbrace{1,\dots,1}_{d}, |\lambda_1|^2, |\lambda_1|^2, \dots, |\lambda_d|^2, |\lambda_d|^2)$, where $\lambda_k, k \in \{1,\dots,d\}$, denote generators of Λ_l^0.
In particular, rational conformal field theories are dense in $\mathcal{M}_d^{Narain}$.

Proof:
Let $j^i, \overline{\jmath}^i, i \in \{1,\dots,d\}$, denote the left and right moving Abelian currents of $\mathcal{C}$ as in definition 4.1.1, and $\sqrt{2}\lambda_k, \sqrt{2}\,\overline{\lambda_k}$ the generators of Γ_l^0 and Γ_r^0, respectively, where

$k \in \{1, \ldots, K\}, \overline{k} \in \{1, \ldots, \overline{K}\}$. The holomorphic W–algebra $\mathcal{W}$ of $\mathcal{C}$ is generated by the j^i and the vertex operators $V[\pm\sqrt{2}\lambda_k]$ with charges $\pm\sqrt{2}\lambda_k$, and analogously for the antiholomorphic W–algebra. Recall from the discussion of section 4.1 that a lowest weight vector $|p\rangle$ of an irreducible representation of $\mathcal{W}$ with charge $p \in \Gamma$ and $p_l \in \Gamma_l$ with respect to j has dimension $h = \frac{p_l^2}{2}$, and that fusion is given by $[V[\pm\sqrt{2}\lambda_k]] \times [V[p]] = [V[p \pm \sqrt{2}\lambda_k]]$. This shows that the characters of irreducible representations of $\mathcal{W}$ are

$$\chi_p^L(\sigma) := \frac{1}{\eta(\sigma)^d} \sum_{\gamma \in L} q^{\frac{1}{2}(p+\gamma)^2}, \tag{4.5.6}$$

where $L = \Gamma_l^0$ and $p \in L^*$ (compare to (4.1.6)). Analogously we see that the characters of irreducible representations of $\overline{\mathcal{W}}$ are given by the same formula (4.5.6), now setting $L = \Gamma_r^0$. With $\Lambda^0 := \Lambda_r^0 + \Lambda_l^0 \subset \Lambda^* \oplus \Lambda$ the irreducible representations of $\mathcal{W}$ and $\overline{\mathcal{W}}$ are labelled by $p_l(\mu_0, \lambda_0)$ and $p_r(\mu_0, \lambda_0)$ respectively, where $(\mu_0, \lambda_0) \in (\Lambda^* \oplus \Lambda)/\Lambda^0$. We see that $\mathcal{W}$ and $\overline{\mathcal{W}}$ are rational, iff Λ^0 has maximal rank $2d$, i.e. rk $\Lambda_l^0 =$ rk $\Lambda_r^0 = d$ or equivalently $S^-(\Lambda, B)^T S^-(\Lambda, B) \in Gl(2d, \mathbb{Q})$ by lemma 4.5.1, which completes the proof of the theorem. □

We remark that the partition function (4.1.7) of a rational toroidal theory decomposes as follows:

$$\begin{aligned} Z_\Gamma(\sigma) &= \frac{1}{|\eta(\sigma)|^{2d}} \sum_{(\lambda^0,\mu^0)\in(\Lambda^*\oplus\Lambda)/\Lambda^0} \; \sum_{\lambda\in\Gamma_l^0,\, \overline{\lambda}\in\Gamma_r^0} q^{\frac{1}{2}(p_l(\mu_0,\lambda_0)+\lambda)^2} \overline{q}^{\frac{1}{2}(p_r(\mu_0,\lambda_0)+\overline{\lambda})^2} \\ &= \sum_{(\lambda^0,\mu^0)\in(\Lambda^*\oplus\Lambda)/\Lambda^0} \chi_{p_l(\mu_0,\lambda_0)}^{\Gamma_l^0}(\sigma) \left(\chi_{p_r(\mu_0,\lambda_0)}^{\Gamma_r^0}\right)^* (\overline{\sigma}). \end{aligned}$$

As a result of theorem 4.5.2 we see that a toroidal conformal field theory is rational iff the extension of the operator product expansion to parameters $(z, \overline{z}) \in \mathbb{C}^2$ is represented on a Riemann surface with finitely many sheets. Equivalently, all dimensions of fields in the theory must be rational.
We now wish to make contact to the geometry of the underlying torus in a toroidal conformal field theory. To do so, we need the following

Definition 4.5.3
A complex torus $T^d = \mathbb{C}^{d/2}/\Lambda$ has COMPLEX MULTIPLICATION, *iff there is an $\alpha \in \mathbb{C} - \mathbb{R}$ such that $\alpha\Lambda \subset \Lambda$. The torus T^d then is called* CM TORUS.

It is easy to see that we can define a complex structure on $T^2 = \mathbb{R}^2/\Lambda$ such that T^2 is a CM torus, iff $v\Lambda^T\Lambda \in Gl(2, \mathbb{Q})$ for some $v \in \mathbb{R}$. Equivalently, T^2 with modulus τ is a CM torus, iff $\tau \in \mathbb{Q}[\sqrt{-D}]$ for some $D \in \mathbb{N}$. For later convenience we also introduce the notion of rational CM tori:

Definition 4.5.4
A twotorus $T^2 = \mathbb{R}^2/\Lambda$ is a RATIONAL CM TORUS, *if $\Lambda^T\Lambda \in Gl(2, \mathbb{Q})$, or equivalently T^2 has modulus $\tau \in \mathbb{H}$ and volume $V \in \mathbb{R}^+$ such that $V^2 \in \mathbb{Q}$ and $\tau \in \mathbb{Q}[iV]$.*

We can now formulate a geometric criterion for rationality of a toroidal conformal field theory as follows:

Theorem 4.5.5
Let $\mathcal{C} = \mathcal{T}(\Lambda, B)$ denote a toroidal conformal field theory with even integer central charge $c = d$. Then $\mathcal{C}$ is a rational conformal field theory iff $B \in Skew(d) \cap \mathrm{Mat}(d, \mathbb{Q})$ and $T^d = \mathbb{R}^d/\Lambda$ possesses a finite cover which is the product of $\frac{d}{2}$ rational CM tori.

Proof:
By theorem 4.5.2 it remains to be shown that $G := \Lambda^T\Lambda \in Gl(d, \mathbb{Q})$ iff we can find $M \in Gl(d, \mathbb{Q})$ such that for $\Lambda = \Lambda^0 M$ we have $\Lambda^0 = \mathrm{diag}(\Lambda_1^0, \ldots, \Lambda_{d/2}^0)$, $\Lambda_k^0 \in Gl(2, \mathbb{R})$, and $T_k^2 := \mathbb{R}^2/\Lambda_k^0$ is a rational CM torus for $k \in \{1, \ldots, \frac{d}{2}\}$.
Firstly, assume $G \in Gl(d, \mathbb{Q})$. We pick generators of Λ such that the corresponding matrix Λ is upper triangular. Then the pairwise scalar products of the vectors l_k in $\Lambda^T = (l_1, \ldots, l_d)$ are all rational. Therefore, a Gram type algorithm can be used to construct $M' \in Gl(d, \mathbb{Q})$, such that $\Lambda^0 = \Lambda M'$ obeys $\Lambda^0 = \mathrm{diag}(\Lambda_1^0, \ldots, \Lambda_{d/2}^0)$,

$$\Lambda_k^0 = \begin{pmatrix} d_k & d_k\tau_1^{(k)} \\ 0 & d_k\tau_2^{(k)} \end{pmatrix}, \quad (\Lambda_k^0)^T\Lambda_k^0 \in Gl(2, \mathbb{Q}).$$

In other words, $T_k^2 := \mathbb{R}^2/\Lambda_k^0$ is a rational CM torus for all $k \in \{1, \ldots, \frac{d}{2}\}$, and the above condition is satisfied for $M^{-1} = M'$.
Vice versa, if $\Lambda = \Lambda^0 M$ with Λ^0 as above, in particular $(\Lambda^0)^T\Lambda^0 \in Gl(d, \mathbb{Q})$, so $G = \Lambda^T\Lambda = M^T(\Lambda^0)^T\Lambda^0 M \in Gl(d, \mathbb{Q})$. □

From theorem 4.5.2 and with (4.3.2) it is straightforward to directly deduce

Corollary 4.5.6
Let $\mathcal{C} = \mathcal{T}(\Lambda, B)$ denote a toroidal conformal field theory with central charge $c = 2$. Then $\mathcal{C}$ is a rational conformal field theory iff it has parameters $(\tau, \rho) \in \mathbb{H} \times \mathbb{H}$ such that $\tau, \rho \in \mathbb{Q}[\sqrt{-D}]$ for some $D \in \mathbb{N}$.

By the results of theorem 4.5.5 and corollary 4.5.6 the twodimensional case seems to be the only one where the notion of CM tori is indeed adequate to give a criterion for rationality of toroidal conformal field theories. Namely, in theorem 4.5.5 we necessarily have to make use of the fairly clumsy notion of rational CM tori. In particular, for even $d \geq 4$, $\mathcal{T}(\Lambda, 0)$ with $\Lambda = \oplus_{k=1}^{d/2}\Lambda_k^0$, Λ_k^0 corresponding to rational CM tori with *different* fields $\mathbb{Q}[\sqrt{-D_k}]$ is a rational theory, but the torus $\mathbb{R}^d/\Lambda$ is *not* a CM torus.

4.6 Singular varieties and rational conformal field theories

This section is devoted to the discussion of rational toroidal superconformal field theories with central charge $c = 6$. The general result of theorem 4.5.2 shows that these theories are dense in the moduli space of toroidal superconformal field theories. Therefore, one can expect the criterion of rationality for a toroidal conformal field theory to be related to the criterion of the underlying torus being singular or having complex multiplication [KN]. The result of theorem 4.6.7 below in comparison to theorem 4.5.2 is less encouraging, however, since it shows that the notion of a rational conformal field theory is much coarser than that of singular tori as well as CM tori. In any case, the idea is a good excuse to introduce some notions from complex geometry that we will need later as well as some of the standard lore in string theory. For more details, see [GH78, BPdV84], where most of the definitions below are taken from.

In the following, let X denote a complex manifold with $\dim_{\mathbb{C}} X = \frac{d}{2}$. We assume that a complex structure has been chosen, in particular we have a decomposition of the de Rham cohomology of X into $H^r(X) = H^r(X,\mathbb{C}) = \oplus_{p+q=r} H^{p,q}(X)$. On X we have the exponential sequence of sheaves,

$$0 \longrightarrow \mathbb{Z}_X \stackrel{j}{\longrightarrow} \mathcal{O}_X \stackrel{exp}{\longrightarrow} \mathcal{O}_X^* \longrightarrow 0.$$

The corresponding long exact sequence is very frequently used:

$$\begin{array}{rcccl} \cdots \to H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & H^1(X,\mathcal{O}_X^*) \longrightarrow \\ & \stackrel{\delta}{\longrightarrow} & H^2(X,\mathbb{Z}) & \stackrel{j^*}{\longrightarrow} & H^2(X,\mathcal{O}_X) \longrightarrow \cdots \end{array}$$

Here, $H^1(X,\mathcal{O}_X^*)$ is naturally isomorphic to the set of holomorphic line bundles on X which gains the structure of a group if equipped with the tensor product as multiplication. Then for a holomorphic line bundle $\mathcal{L} \in H^1(X,\mathcal{O}_X^*)$ one finds $\delta(\mathcal{L}) = c_1(\mathcal{L})$ [Hir66, Th.4.3.1]. Moreover, j^* is the PERIOD MAP.

Definition 4.6.1
The group of holomorphic line bundles on X is called PICARD GROUP *and denoted $Pic(X) \cong H^1(X,\mathcal{O}_X^*)$. The set of algebraic cocycles $NS(X) := \mathrm{Im}(\delta) = \ker(j^*) \subset H^2(X,\mathbb{Z})$ is called* NÉRON–SEVERI GROUP, *its rank $\rho(X) := \mathrm{rk}\, NS(X)$ is the* PICARD NUMBER *of X.*

Note that $NS(X) = Pic(X)/Pic^0(X)$, where $Pic^0(X) := \ker(\delta)$ is also known as PICARD TORUS or JACOBIAN of X and parametrizes flat vector bundles on X, or equivalently divisors which are algebraically equivalent to zero.

Definition 4.6.2
Suppose that $\dim_{\mathbb{C}} X = 2$, then $p_g := \dim_{\mathbb{C}} H^2(X,\mathcal{O}_X)$ is the GEOMETRIC GENUS *of X. Consider $H_X := H^2(X,\mathbb{Z})/(torsion)$, which is a Euclidean lattice. Let $\overline{NS(X)}$ denote the image of $NS(X)$ in H_X, a primitive sublattice. Its orthogonal*

complement $T_X := \overline{NS(X)}^\perp \cap H_X$ *is the* TRANSCENDENTAL LATTICE *parametrizing transcendental cocycles on* X.

The case of main interest to us is much simpler than the general one described above. Namely, in chapter 7 we will discuss moduli spaces of certain superconformal field theories with central charge $c = 6$. Though a clean proof is lacking, it is generally assumed that the additional assumptions we will make there ensure that all theories in the moduli space admit a nonlinear sigma model description by a generalization of the construction introduced in section 4.2. In general, suppose that an $N = (2, 2)$ superconformal field theory $\mathcal{C}$ has a nonlinear sigma model description on a compact manifold X. The condition of conformal invariance of $\mathcal{C}$ implies that X must carry a Ricci flat Einstein metric. $N = (2, 2)$ supersymmetry enforces X to be a complex Kähler manifold [Zum79]. Altogether, we can restrict our attention to compact Kähler manifolds with $c_1(X) = 0$, also known as CALABI–YAU MANIFOLDS*. Since central charge $c = 6 = 3d/2$ corresponds to complex dimension $d/2 = \dim_{\mathbb{C}} X = 2$, there are only two topologically distinct manifolds to be discussed. Namely, if the FIRST BETTI NUMBER $b_1(X) = \dim_{\mathbb{R}} H^1(X, \mathbb{R})$ of X vanishes, X is a $K3$–SURFACE by definition [Bes87, p. 365]. If $b_1(X) \neq 0$, the Poincaré–Hopf theorem [BGV92, Th.1.56] shows $\chi(X) = 0$, so X is a complex torus. In both cases the geometric genus (definition 4.6.2) is $p_g = 1$, and $H^2(X, \mathbb{Z})$ is torsion free. Then $NS(X) \cong H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is also known as PICARD LATTICE. In the $K3$ case, $NS(X) \cong Pic(X)$.
If X is an algebraic complex surface, $\overline{NS(X)}$ has signature $(1, \rho(X) - 1)$, and the transcendental lattice T_X has signature $(2p_g, h^{1,1}(X) - \rho(X))$. In particular, $\rho(X) \leq h^{1,1}(X)$. Complex surfaces where the latter inequality becomes an equality are of special interest:

Definition 4.6.3
A compact complex surface X *is called* SINGULAR†, *iff its Picard number* $\rho(X)$ *is maximal, in other words* $\rho(X) = h^{1,1}(X)$. *Equivalently,* X *is singular iff the complex structure given by the twoplane* $H^{2,0}(X) \oplus \overline{H^{2,0}(X)} \subset H^2(X)$ *is defined over* $\mathbb{Q}$, *i.e. real and imaginary parts of a generator of* $H^{2,0}(X)$ *span a twoplane* $\Omega = T_X \otimes \mathbb{R} \subset H^2(X, \mathbb{R})$ *generated by elements of* $H^2(X, \mathbb{Z})$.

We remark a first relation between singular complex surfaces and the notion of CM tori as introduced in definition 4.5.3. Namely, if X is a singular Calabi–Yau twofold, then $C_X := H^2(X, \mathcal{O}_X)/j^* H^2(X, \mathbb{Z})$ has the structure of a CM torus of complex dimension 1. For singular fourtori we have

Theorem 4.6.4
If X *is a singular torus and* $\dim_{\mathbb{C}} X = 2$, *then* X *is a CM torus.*

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*Some authors, especially in the physics literature, add the condition $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$ to the definition of Calabi–Yau manifolds. Since we understand the name as a reference to the celebrated Calabi–Yau theorem [Yau78] we will not follow this fashion.

†We thank Noriko Yui and Yasuhiro Goto for drawing our attention to the relevant literature concerning singular $K3$ surfaces.

Proof:
By [SM74, Th.4.1], X is singular iff $X = C_1 \times C_2$, where C_1, C_2 are mutually isogenous elliptic curves with complex multiplication. Equivalently, $X = \mathbb{R}^4/\Lambda$, where

$$\Lambda = v\Lambda^0, \quad \Lambda^0 = \left(\begin{array}{cc|cc} 1 & \tau_1 & \multicolumn{2}{c}{0} \\ 0 & \tau_2 & & \\ \hline \multicolumn{2}{c|}{0} & d & d\tau_1' \\ & & 0 & d\tau_2' \end{array}\right), \quad \begin{array}{l} d^2, \tau_1^{(\prime)}, (\tau_2^{(\prime)})^2 \in \mathbb{Q}, \\ \mathbb{Q}[i\tau_2] = \mathbb{Q}[i\tau_2']. \end{array} \tag{4.6.1}$$

In particular, one can find $A^{(\prime)}, B^{(\prime)}, C^{(\prime)}, D^{(\prime)} \in \mathbb{Z}$ such that $\alpha = A + B\tau = A' + B'\tau'$, $\tau^{(\prime)} = \tau_1^{(\prime)} + i\tau_2^{(\prime)}$, and $(A^{(\prime)} + B^{(\prime)}\tau^{(\prime)})\tau^{(\prime)} = C^{(\prime)} + D^{(\prime)}\tau^{(\prime)}$, i.e. $\alpha\Lambda \subset \Lambda$ and X is a CM torus. $\square$

We stress that the product $C_1 \times C_2$ of two arbitrary CM tori need not be singular by theorem 4.6.4. This can also be checked directly from (4.6.1).
For $K3$ surfaces, the following theorem is a simple consequence of the proof of the density theorem [BPdV84, Cor.VIII.8.5]:

Theorem 4.6.5
Singular complex structures form a dense subset of the moduli space of complex structures on $K3$.

On the other hand, we are in the happy situation that the singular Calabi–Yau twofolds are completely classified:

Theorem 4.6.6
Singular Abelian varieties of complex dimension 2 are uniquely determined by the quadratic form on their transcendental lattice. In other words, they are in $1:1$ correspondence to positive definite even quadratic forms modulo $SL(2,\mathbb{Z})$ conjugation [SM74, Th.3.1]. The same holds for singular $K3$ surfaces [SI77, Th.4].

To get acquainted with structure and terminology it is a good exercise to compute the transcendental lattices for a couple of singular tori. We list the results for three examples that will also be needed later on. In general, we define a real fourtorus T^4 by specifying the lattice Λ such that $T^4 = \mathbb{R}^4/\Lambda$, or equivalently giving the flat metric $G = \Lambda^T\Lambda$ on T^4. We use the following *convention* to fix a complex structure on T^4 compatible with the hyperkähler structure given by G (see section 7.1): Let $x_1, \dots, x_4$ denote real coordinates on $T^4 = \mathbb{R}^4/\Lambda$, then $\lambda_k = \sum_l \Lambda_{lk} dx_l$ are generators of Λ. As described in definition 4.6.3, we can fix the complex structure of T^4 by specifying a twoplane $\Omega \subset H^2(X, \mathbb{R})$. Here, Ω is spanned by $dx_1 \wedge dx_3 + dx_4 \wedge dx_2, dx_1 \wedge dx_4 + dx_2 \wedge dx_3$, and the Kähler form is a multiple of $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. The quadratic form on the transcendental lattice T_X for $X = T^4$ is denoted by T_X as well. The root lattice of a simply laced Lie algebra A_n, D_n, E_n is denoted by the same letter. Then we find*:

$$\begin{array}{rclrclrcl} X &=& T^2\times T^2, & T^2 &=& \mathbb{C}/\langle 1,i\rangle: & T_X &=& \begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix},\\ X &=& \mathbb{R}^4/\Lambda, & \Lambda &=& D_4: & T_X &=& \begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix},\\ X &=& \mathbb{R}^4/\Lambda, & \Lambda^T &=& D_4: & T_X &=& \begin{pmatrix} 4 & 0\\ 0 & 4\end{pmatrix}. \end{array} \tag{4.6.2}$$

Although by theorem 4.6.6 the first two tori in (4.6.2) carry the same complex structure, they are not equal as Einstein manifolds. They differ by their Kähler form, if we stick to the conventions introduced above.
Let us now turn to the declared main object of this section, namely the relation between rational toroidal superconformal field theories and singular tori. We would like to prove a theorem giving a criterion for rationality in terms of singular or CM fourtori instead of rational CM twotori as in theorem 4.5.5. Note that the product X of twodimensional CM tori with moduli $\tau^{(1)},\dots,\tau^{(d/2)}$ is itself a CM torus iff $\tau^{(k)}\in\mathbb{Q}[\sqrt{-D}]$ for all $k\in\{1,\dots,\frac{d}{2}\}$ and fixed $D\in\mathbb{N}$. For $d=4$, this is also equivalent to X being singular. Therefore, by theorem 4.5.5, rationality of a toroidal conformal field theory for even $d\geq 4$ is a much weaker condition than that for the geometric interpretation to give a singular torus with possible restrictions on Kähler form and volume. But vice versa we have:

Theorem 4.6.7
Suppose that $T^4=\mathbb{R}^4/\Lambda$ is a singular torus with Kähler form $\omega=\delta\omega_0$ and $\omega_0\in H^2(T^4,\mathbb{Z})\otimes\mathbb{Q}, \delta^2\in\mathbb{Q}$. In other words, the threeplane $\Sigma\subset H^2(T^4,\mathbb{R})$ spanned by the twoplane Ω of definition 4.6.3 and ω is defined over $\mathbb{Q}$. Then there is an $\varepsilon\in\mathbb{R}$ with $\varepsilon^4\in\mathbb{Q}$, such that $\mathcal{T}(\varepsilon V\Lambda, B)$ is a rational superconformal field theory for all $V\in\mathbb{R}$ with $V^2\in\mathbb{Q}$ and all $B\in Skew(4)\cap\mathrm{Mat}(4,\mathbb{Q})$.

Proof:
By theorem 4.5.5 we need to show $\varepsilon^2\Lambda^T\Lambda\in Gl(4,\mathbb{Q})$ for some $\varepsilon\in\mathbb{R}$, $\varepsilon^4\in\mathbb{Q}$. Since T^4 is singular by assumption, we can suppose Λ to have the form (4.6.1). But then, using the conventions introduced above, the associated Kähler form is

$$\omega=\frac{v^2}{\sqrt{2}}\left(d^2\tau_2'\lambda_1\wedge\lambda_2+\tau_2\lambda_3\wedge\lambda_4\right)=\frac{v^2}{\sqrt{2}\tau_2}\omega_0,\quad \tau_2^2\in\mathbb{Q},$$

where $\omega_0\in H^2(X,\mathbb{Z})\otimes\mathbb{Q}$. Hence by the assumptions in the theorem we may choose $\varepsilon:=v$ to prove the above assertion. □

We remark that by the very description of the complex structure on a four–torus X the notion of X to be singular is naturally formulated in terms of the second cohomology. On the other hand, the general criterion for a toroidal conformal field theory to be rational (theorem 4.5.2) is given in terms of the first cohomology. The

*In our coordinates $D_4=\{x\in\mathbb{Z}^4\mid\sum_{i=1}^4 x_i\equiv 0\,(2)\}$ and $D_4^*=\mathbb{Z}^4+(\mathbb{Z}+1/2)^4$.

link between the two descriptions is the triality automorphism (4.4.2). Comparison of theorem 4.6.7 to the stronger result in theorem 4.5.2 shows that triality is not compatible with the notion of rationality. By the use of (4.6.1) and the comment below it is easy to explicitly construct counterexamples to an inverse of theorem 4.6.7.

Chapter 5

Orbifold conformal field theories

This chapter is devoted to orbifold conformal field theories. We briefly review their general construction and properties in sections 5.1 and 5.2, where the latter is basically concerned with the geometric understanding of orbifolds. For good introductions to the subject see also [DFMS87, Dix87, DHVW85, DHVW86, FV87]. The geometric interpretation of $\mathbb{Z}_M$ orbifolds of toroidal superconformal field theories with $c = 3$ is not found in the literature. Two special types of orbifold constructions are studied more closely in section 5.3–5.4, namely crystallographic orbifolds and orbifolds involving the spacetime fermion number operator. The generalized GSO projection and its properties are the topic of section 5.5. It is applied to tensor products of $N = (2, 2)$ minimal models in section 5.6 to yield the standard construction of Gepner models.

5.1 The orbifold construction

The orbifold construction is a general method to build a new conformal field theory $\mathcal{C}/G$ by modding out a finite symmetry group G of a conformal field theory $\mathcal{C}$. G is always assumed to be compatible with the conformal symmetry, i.e. leave the energy momentum tensor of $\mathcal{C}$ invariant. The theories $\mathcal{C}$ and $\mathcal{C}/G$ share those states of $\mathcal{C}$ which are invariant under G. In particular, they have the same central charge c. The so–called UNTWISTED SECTOR $\mathcal{H}^G$ of the space of states of $\mathcal{C}/G$ is therefore obtained from the space of states $\mathcal{H}$ of $\mathcal{C}$ by projecting with the operator $P := \frac{1}{|G|} \sum\limits_{g\in G} g$. We employ the shorthand notation

$$ {}_{g}\underset{\mathbb{1}}{\Box} := \mathrm{tr}_{\mathcal{H}} \left(g q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) $$

to write the untwisted sector partition function as

$$ Z^G := \mathrm{tr}_{\mathcal{H}} \left(P q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) = \frac{1}{|G|} \sum_{g\in G} {}_{g}\underset{\mathbb{1}}{\Box} . $$

One can interpret the box ${}_{g}\underset{\mathbb{1}}{\Box}$ as representing the worldsheet torus $\Xi(\sigma)$ of the original theory $\mathcal{C}$. The group element attached to either period of $\Xi(\sigma)$ denotes the effect of a displacement along the corresponding torus period. In other words, if $\mathcal{H}^g$ denotes the space of states corresponding to fields φ which on the worldsheet of the original theory are twisted by a g action under time translation, $\varphi(\xi_0+1,\xi_1) = g\varphi(\xi_0,\xi_1), \varphi(\xi_0,\xi_1+1) = \varphi(\xi_0,\xi_1)$, then

$$g\underset{\mathbb{1}}{\square} = \mathrm{tr}_{\mathcal{H}^g}\left(q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{c}{24}}\right).$$

For nontrivial G, Z^G cannot be modular invariant by property 8 of section 2.1, since $\mathcal{H}^G$ contains only a subset of representations of the operator product expansion of the modular invariant theory $\mathcal{C}$. To construct a new theory, we must add so–called TWISTED SECTORS $\mathcal{H}_f$, $f \in G$, corresponding to fields which are only well defined on the world sheet of the original theory up to the action of a nontrivial element $f \in G$:

$$|\varphi\rangle \in \mathcal{H}_f \;:\quad \varphi(\xi_0+1,\xi_1) = \varphi(\xi_0,\xi_1), \quad \varphi(\xi_0,\xi_1+1) = f\varphi(\xi_0,\xi_1). \tag{5.1.1}$$

The field Σ_f corresponding to a twisted ground state $|\Sigma_f\rangle$ in $\mathcal{H}_f$ introduces a cut from 0 to $\sigma \sim 0$ on the worldsheet torus $\Xi(\sigma)$ to establish the transformation property (5.1.1), i.e. the correct MONODROMY. The representation of the OPE with elements of $\mathcal{H}^G$ is given by the induced representation on the twisted sector. Correlation functions for $\mathbb{Z}_M$ twist fields have been studied in great detail in [HV87, DFMS87, ADGN88].

Whether the theory constructed by blindly adding twisted sectors will be consistent is not clear in general: We must not destroy locality. It is sufficient to check locality on the twisted ground states, i.e. ensure that for f of order m the field Σ_f has spin $h - \overline{h} \equiv 0 \bmod \frac{1}{m}$ such that one can project onto an integer spin and G invariant ground state, and moreover that twist fields have pairwise local OPE. These conditions are known as LEVEL MATCHING CONDITIONS [DHVW86, Vaf86]. They ensure existence of physical states in $\mathcal{H}_f$ and can also be interpreted as constraints from modular invariance under transformations of $\Xi(\sigma)$ that leave the respective boundary conditions invariant [Wit85], see also [Dix87]. We do not go into details, since by the above level matching conditions are trivially obeyed for SYMMETRIC ORBIFOLDS. These are orbifolds $\mathcal{C}/G$ where in an appropriate geometric interpretation G acts on left and right movers by the same representation, and therefore $h = \overline{h}$ and pairwise locality for twisted ground states follow directly. Though asymmetric orbifold conformal field theories are an interesting issue to study [NSV87], this exceeds the scope of the present thesis. In section 6.1.2 we will see that at least for the determination of nonexceptional irreducible components of the moduli space of unitary conformal field theories with $c = 2$ the asymmetric orbifolds are not needed.

We note that φ as in (5.1.1) also obeys

$$\forall\, g \in G: \quad g\varphi(\xi_0,\xi_1+1) = (gfg^{-1})g\varphi(\xi_0,\xi_1), \tag{5.1.2}$$

so $\forall\, g \in G : \;\; \mathcal{H}_f \cong \mathcal{H}_{gfg^{-1}}$. In the twisted sector $\mathcal{H}_f$ we again project onto group invariant states. Since for $[g,f] \neq 0$ by (5.1.2) we identify $\varphi, |\varphi\rangle \in \mathcal{H}_f$ with

$g\varphi, |g\varphi\rangle \in \mathcal{H}_{gfg^{-1}}$, to avoid overcounting of the twisted fields the projector with the correct prefactor is $P_f := \frac{1}{|G|} \sum_{g\in G:\, [g,f]=0} g$ (see also [Dix87]). We again use the shorthand

$${}_{g}\underset{f}{\square} := \mathrm{tr}_{\mathcal{H}_f}\left(g q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right)$$

to write the twisted sector partition function as

$$\widetilde{Z} = \sum_{f\in G, f\neq \mathbb{1}} \mathrm{tr}_{\mathcal{H}_f}\left(P_f q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) = \frac{1}{|G|} \sum_{\substack{g,f\in G,\\ f\neq\mathbb{1}, [g,f]=0}} {}_{g}\underset{f}{\square}.$$

If we interpret the boxes as representing the worldsheet torus $\Xi(\sigma)$ of $\mathcal{C}$, the restriction of the sum to commuting elements of G gives tribute to the relation $ABA^{-1}B^{-1} = [0]$ for $A, B \in H_1(\Xi(\sigma), \mathbb{Z})$. The total orbifold partition function is

$$Z_{G-orb} = \sum_{f\in G} \mathrm{tr}_{\mathcal{H}_f}\left(P_f q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) = \frac{1}{|G|} \sum_{\substack{g,f\in G,\\ [g,f]=0}} {}_{g}\underset{f}{\square}, \tag{5.1.3}$$

where we set $\mathcal{H}_{\mathbb{1}} := \mathcal{H}$ and $P_{\mathbb{1}} := P$. Modular invariance is most easily checked by using the worldsheet interpretation of the boxes ${}_{g}\underset{f}{\square}$ as representing the trace over fields φ on $\Xi(\sigma)$ such that $\varphi(\xi_0 + 1, \xi_1) = g\varphi(\xi_0, \xi_1)$ and $\varphi(\xi_0, \xi_1 + 1) = f\varphi(\xi_0, \xi_1)$. Namely,

$${}_{g}\underset{f}{\square}\left(-\frac{1}{\sigma}\right) = {}_{f}\underset{g}{\square}(\sigma)\,, \quad {}_{g}\underset{f}{\square}(\sigma + 1) = {}_{f\circ g}\underset{f}{\square}(\sigma), \tag{5.1.4}$$

from which modular invariance of (5.1.3) follows directly. Vice versa, these equations simplify the calculation of (5.1.3).

In the simplest example of the orbifold construction, the SHIFT ORBIFOLD of a toroidal theory $\mathcal{C} = \mathcal{T}(\Lambda, B)$, $\Lambda \subset \mathbb{R}^d$, G is generated by a shift $T_\Delta = e^{2\pi i p\cdot\Delta}$ on the charge lattice $\Gamma = \Gamma(\Lambda, B)$. In order for G to be finite, we have to assume $D\Delta \in \Gamma$ for some $D \in \mathbb{N}$, and then $G \cong \mathbb{Z}_D$. Set

$$\begin{aligned} \Gamma^m_{l\Delta} &:= \left\{ p \in \Gamma + l\Delta \mid p\cdot\Delta \in \mathbb{Z} + \tfrac{m}{D} \right\}, \quad l, m \in \{0, \ldots, D-1\}, \\ Z_{\Gamma^m_{l\Delta}} &:= \frac{1}{|\eta|^{2d}} \sum_{p=(p_l;p_r)\in\Gamma^m_{l\Delta}} q^{\frac{p_l^2}{2}} \overline{q}^{\frac{p_r^2}{2}}, \end{aligned} \tag{5.1.5}$$

then

$${}_{T_\Delta}\underset{\mathbb{1}}{\square} = \sum_{m=0}^{D-1} e^{\frac{2\pi i m}{D}} Z_{\Gamma^m_0}\, \frac{1}{2} \sum_{i=1}^{4} \left| \frac{\vartheta_i(z)}{\eta} \right|^d$$

(ignore the z-dependent sum in the discussion of bosonic toroidal theories). By (5.1.4), more generally

$${}_{T^k_\Delta}\underset{T^l_\Delta}{\square} = \sum_{m=0}^{D-1} e^{\frac{2\pi i k m}{D}} Z_{\Gamma^m_{l\Delta}}\, \frac{1}{2} \sum_{i=1}^{4} \left| \frac{\vartheta_i(z)}{\eta} \right|^d.$$

Thus if $\Delta = (\Delta_l; \Delta_r)$, twisted ground states have dimensions $(h; \overline{h}) = (\frac{(m\Delta_l)^2}{2}, \frac{(m\Delta_r)^2}{2})$, so in this case level matching conditions have the simple form $m^2\Delta^2 \in 2\mathbb{Z}$ for some $m \in \{1, \ldots, D-1\}$. Note that the shift orbifold of a toroidal theory $\mathcal{C}$ is again a toroidal theory by definition 4.1.1, since $G = \operatorname{span}(T_\Delta)$ leaves invariant the $u(1)$ currents j^k of $\mathcal{C}$. It has charge lattice

$$\widetilde{\Gamma} = \{p \in \Gamma + l\Delta \mid l \in \{0, \ldots, D-1\}, p \cdot \Delta \in \mathbb{Z}\}. \tag{5.1.6}$$

Let us return to the general discussion of orbifold conformal field theories. A priori, ${}_{g}\underset{f}{\square}$ is only defined up to a phase $\varepsilon(g, f)$, because the same is true for the action of $g \in G$ on a twisted ground state of $\mathcal{H}_f$. Only if $g = f^k$ for some $k \in \mathbb{Z}$, the phase is fixed by (5.1.4), and for all other boxes the choice is restricted by modular invariance to obey (4.2.4). For closed modular orbits in the twisted sector there remains an arbitrariness of the phase they contribute with. Here, conjugate subgroups must account with the same phase in order for the representation of G on the twisted sector to be consistent with (5.1.2). The remaining phase ambiguity is known as DISCRETE TORSION [Vaf86]. By the above it does not occur for orbifolds by cyclic groups and in general is parametrized by the group cohomology $H^2(G, U(1))$, see (4.2.4). In modern language it is interpreted as additional discrete degree of freedom of the B–field on the orbifold, if a geometric interpretation is at hand, see section 5.2. Discrete torsion will become relevant in the discussion of crystallographic orbifolds in section 5.3.

For non Abelian G, (5.1.3) can be written as sum over Abelian subgroups of G with overcounted terms subtracted off. To do so in general, we call a subgroup $H \subset G$ MAXIMAL ABELIAN if there is no Abelian $G' \subset G$ such that $H \subsetneqq G'$. We also introduce multiplicities $n_{H'} := \#\{H \subset G \text{ maximal Abelian} \mid H' \subsetneqq H \text{ maximal}\}$ and find

$$Z_{G-orb} = \frac{1}{|G|}\left(\sum_{\substack{H \subset G \text{ max.} \\ \text{Abelian}}} |H| Z_{H-orb} - \sum_{\substack{H' \subset G:\, \exists H \subset G \\ \text{max. Abelian, } H' \subsetneqq H}} (n_{H'} - 1) Z_{H'-orb} \right). \tag{5.1.7}$$

An important feature of the orbifold construction is the fact that the modding out by a solvable group G may be inverted by another G-orbifold [Gin88a, §8.5]: First note that for any normal subgroup $H \subset G$ and $\mathcal{C}' = \mathcal{C}/H$ we have $\mathcal{C}/G = \mathcal{C}'/(G/H)$. Therefore we only have to check that cyclic orbifolds can be inverted. If γ is a generator of $G = \mathbb{Z}_M$, then we can define an action of γ on the twisted states in $\mathcal{H}_{\gamma^m}$ by multiplication with $e^{2\pi i m/M}$. With (5.1.3) one now easily checks $(\mathcal{C}/G)/G = \mathcal{C}$.

5.2 The geometric orbifold construction

If a conformal field theory $\mathcal{C}$ possesses a geometric interpretation in terms of a nonlinear σ model on some manifold Y, we can expect the orbifold conformal field theory $\mathcal{C}/G$ to have a geometric interpretation on the (generically) singular variety Y/G. Geometric objects of this type were first studied mathematically by I. Satake and called V-MANIFOLDS [Sat57]. Nowadays, the more intuitive notion ORBIFOLD is also used in the mathematics literature for a singular variety all of whose singularities are quotient singularities.
The B–field of the orbifold theory is said to take values in $\check{H}^2(Y/G, U(1))$, which may have torsion [Asp99]. It is not clear to us how this is to be interpreted in general, but for finite G and $X := \widetilde{Y/G}$ the nongeometric degrees of freedom of the orbifold contain the "classical" B–field degrees of freedom in $H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$, and a discrete part $H^2(G, U(1))$. The latter accounts for discrete torsion [Vaf86], as mentioned above. Note that toroidal conformal field theories as described in section 4.2 are already an example of geometrically motivated orbifold conformal field theories. Here, $Y = \mathbb{R}^d$, and $G = \Lambda \subset \mathbb{R}^d$ is a lattice and acts by translations, $X = T^d = \mathbb{R}^d/\Lambda$. In this case from $\pi_1(Y) = 0$ one shows by the Cartan–Leray spectral sequence $\check{H}^2(Y/G, U(1)) \cong H^2(Y, U(1))^G \oplus H^2(G, U(1))$, where $H^2(Y, U(1)) = \{0\}$, and we get the well known B–field degrees of freedom $H^2(\Lambda, U(1)) \cong H^2(T^d,\mathbb{R})/H^2(T^d,\mathbb{Z})$.
A closer look at the literature reveals that basically the only nonlinear σ models which are fully understood are the toroidal conformal field theories and orbifolds thereof. Everything we need in order to give geometric interpretations of orbifold conformal field theories therefore has already been said in section 4.2. We will assume $\mathcal{C} = \mathcal{T}(\Lambda, B)$ in the following, a toroidal (super–)conformal field theory with central charge $c = d$ or $c = 3d/2$, respectively. In particular, we have d (non–single valued) massless scalar fields $\varphi^k : Z \to T^d = \mathbb{R}^d/\Lambda, k \in \{1, \dots, d\}$, which describe the embedding of the string worldsheet in T^d. Let G denote a symmetry of the conformal field theory which is induced by a geometric symmetry of T^d. By this we mean that G acts on $\mathbb{R}^d$ by some representation which in the following is simply denoted $g \in G, v \in \mathbb{R}^d : v \mapsto g.v$. This induces a left–right symmetric action on the $u(1)$ currents of the toroidal theory and thus on the charge lattice, in the notations of definition 4.1.1 and (4.2.1)

$$g : \left\{ \begin{array}{lcl} (j;\overline{\jmath}) & \mapsto & (g.j; g.\overline{\jmath}) \\ (p_l;p_r) & \mapsto & (g.p_l; g.p_r) \end{array} \right.$$

Since G is supposed to act as symmetry of the conformal field theory $\mathcal{C}$, on Λ it must act as lattice automorphism. Vice versa, by (4.2.1) an automorphism g of the lattice Λ induces a symmetry on the conformal field theory $\mathcal{T}(\Lambda, B)$ iff $B = \Lambda^T \widetilde{B} \Lambda$,

$$g^{-1}\widetilde{B}g = \widetilde{B} \quad \text{in} \quad Skew(d \times d, \mathbb{R}/\mathbb{Z}). \tag{5.2.1}$$

In particular, any automorphism of Λ induces a symmetry on $\mathcal{T}(v\Lambda, 0)$ for all $v \in \mathbb{R}^+$, but other values of B may be possible as well. In section 5.3 we will discuss all possible B–field values for every geometric symmetry of a twotorus.

Now let us study the corresponding orbifold conformal field theory $\mathcal{C}/G$ on $X = T^d/G$. In the twisted sector, for $|\varphi\rangle \in \mathcal{H}_f, f \in G - \{\mathbb{1}\}$, by (5.1.1) we find that $\alpha_0 := \varphi(z=0)$ is a fixed point of f. The φ^k then describe strings that wind around the fixed point $\alpha_0 \in X$ but are not closed on the covering torus T^d. So if f has J fixed points on T^d, then $\mathcal{H}_f$ decomposes into J isomorphic copies of spaces $\mathcal{H}_f^{(j)}$, $j \in \{1, \cdots, J\}$.

5.2.1 $\mathbb{Z}_M$ orbifold conformal field theories

The case $G = \mathbb{Z}_M$, d even, is particularly simple and shall now be discussed in detail. We assume $T^d = \times_{k=1}^{d/2} T^2_{(k)}$ (not necessarily orthogonal), such that $T^2_{(k)}$ is $\mathbb{Z}_M$ symmetric. Since the rotational symmetries of twodimensional lattices are well known, this restricts the possible values of M to $M \in \{2,3,4,6\}$. Let γ denote a generator of $\mathbb{Z}_M$. Then the standard $\mathbb{Z}_M$ action on $j_\pm^{(k)} = i\partial\varphi_\pm^{(k)}$ as in (4.1.10) is given by

$$\gamma : j_\pm^{(k)} \mapsto e^{\pm(-1)^k 2\pi i/M} j_\pm^{(k)}. \tag{5.2.2}$$

It induces a $\mathbb{Z}_M$ symmetry of the conformal field theory $\mathcal{C} = \mathcal{T}(\Lambda, B)$ iff (5.2.1) holds for $g = \gamma$ with respect to this action. Assuming that, we can construct $\mathcal{C}/\mathbb{Z}_M$.

In the twisted sector $\mathcal{H}_f$ for f of order m the $\varphi_\pm^{(k)}$ have mode expansion

$$\varphi_\pm^{(k)}(z) = \alpha_{\pm,0}^{(k)} + i \sum_{n \in \mathbb{Z} \pm 1/m} \frac{1}{n} \alpha_{\pm,n}^{(k)} z^n. \tag{5.2.3}$$

Hence the corresponding twisted ground state for a bosonic conformal field theory has dimensions

$$h = \overline{h} = \frac{d}{4}\frac{1}{m}\left(1 - \frac{1}{m}\right) \tag{5.2.4}$$

(see also [Dix87]).

It is now straightforward to construct the partition functions for $\mathbb{Z}_M$ orbifolds of toroidal conformal theories. This has been achieved in joint work with Werner Nahm. The formulation of theorems 5.2.1 and 5.2.2 is a generalization of corresponding formulas in [EOTY89]. Sayipjamal Dulat has checked the special case of $\mathbb{Z}_M$ orbifolds of toroidal superconformal theories with $c = 3$ [Dul00].

Let us give the general construction: Recall that the box ${}^{\gamma^l}\underset{\gamma^m}{\square}$ in (5.1.3) is interpreted as trace of $q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - c/24}$ over fields defined on $\Xi(\sigma)$ up to a γ^l, γ^m action:

$$\begin{array}{rcl} \varphi(\xi_0+1,\xi_1) &=& \gamma^l\varphi(\xi_0,\xi_1)+\lambda_0 \\ \varphi(\xi_0,\xi_1+1) &=& \gamma^m\varphi(\xi_0,\xi_1)+\lambda_1 \end{array}, \quad \lambda_0, \lambda_1 \in \Lambda. \tag{5.2.5}$$

For $(l,m) \neq (0,0)$ we see that φ may not have winding or momentum modes.

Therefore by (5.2.2) and (5.2.4) for a bosonic theory with $c = d$

$$\begin{aligned} \gamma^l \underset{\gamma^m}{\Box}(\sigma) &\sim (q\overline{q})^{-\frac{d}{24}+\frac{d}{4}\frac{m}{M}(1-\frac{m}{M})} \left| \prod_{n=1}^{\infty} \left(1 - e^{\frac{2\pi il}{M}} q^{n-1+\frac{m}{M}}\right) \left(1 - e^{-\frac{2\pi il}{M}} q^{n-\frac{m}{M}}\right) \right|^{-d} \\ &= (q\overline{q})^{-\frac{d}{4}\left(\frac{m}{M}\right)^2} \left| \frac{\eta(\sigma)}{\vartheta_1\left(\sigma, \frac{m}{M}\sigma + \frac{l}{M}\right)} \right|^d \quad =: \quad b_{l,m}(\sigma). \end{aligned} \tag{5.2.6}$$

It remains to determine the multiplicity $n_{l,m}$, the right hand side of (5.2.6) will occur with in the partition function. It is given by the number of different fixed points α which are in accord with (5.2.5), i.e.

$$\alpha = \gamma^l \alpha + \lambda_0, \quad \alpha = \gamma^m \alpha + \lambda_1.$$

Thus,

$$\begin{aligned} n_{l,m} &= \left| \left[\left((1-\gamma^l)^{-1}\Lambda\right) \cap \left((1-\gamma^m)^{-1}\Lambda\right) \right] / \Lambda \right|^{d/2} \text{ for } lm \neq 0, \\ n_{l,0} = n_{0,l} &= \left| \left((1-\gamma^l)^{-1}\Lambda\right) / \Lambda \right|^{d/2} = \left(\det(1-\gamma^l)\right)^{d/2} = \left(2\sin\tfrac{\pi l}{M}\right)^{d/2}. \end{aligned} \tag{5.2.7}$$

Clearly, $n_{l,0}^{2/d} = n_{0,l}^{2/d}$ is the number of $\mathbb{Z}_m \cong \text{span}(\gamma^l)$ fixed points on T^2, namely $4, 3, 2, 1$ for $m = 2, 3, 4, 6$. From (5.2.6) and (5.2.7) we find

Theorem 5.2.1
Suppose that $\mathcal{C} = \mathcal{T}(\Lambda, B)$, $B = \Lambda^T \widetilde{B} \Lambda$, is a bosonic toroidal conformal field theory with even $c = d$ such that $T^d = \mathbb{R}^d/\Lambda = \times_{k=1}^{d/2} T^2_{(k)}$, and $T^2_{(k)}$ is $\mathbb{Z}_M$ symmetric, and $g^{-1}\widetilde{B}g = \widetilde{B}$ in $Skew(d \times d, \mathbb{R}/\mathbb{Z})$ for all $g \in \mathbb{Z}_M$. Then the orbifold partition function of $\mathcal{C}/\mathbb{Z}_M$ for the $\mathbb{Z}_M$ action induced by (5.2.2) is

$$Z^{\Gamma(\Lambda,B)}_{\mathbb{Z}_M - orb}(\sigma) = \frac{1}{M} \left(Z_{\Gamma(\Lambda,B)}(\sigma) + \sum_{\substack{l,m \in \{0,\ldots,M-1\}, \\ (l,m)\neq(0,0)}} n_{l,m} b_{l,m}(\sigma) \right),$$

where $Z_{\Gamma(\Lambda,B)}$ is the partition function (4.1.7) of $\mathcal{C}$, $b_{l,m}(\sigma)$ was defined in (5.2.6), and $n_{l,m}$ is given by (5.2.7) or equivalently

$$\begin{aligned} \forall\ l, m \in \{0, \ldots, M-1\}, (l,m) \neq (0,0): \quad n_{l,0} &= n_{0,l} = \left(2\sin\tfrac{\pi l}{M}\right)^{d/2}, \\ n_{l,m} &= n_{l,l+m} = n_{m,M-l}. \end{aligned}$$

Modular invariance of the above partition function follows directly from our definition of the $n_{l,m}$ and the properties of theta functions given in appendix A.
We now turn to the $N = (2,2)$ supersymmetric case (4.2.5), where also the fermions have to be twisted. By (4.1.10) we must extend the action (5.2.2) on $j^{(k)}_\pm$ identically to $\psi^{(k)}_\pm$, if we want to preserve supersymmetry. In the sector twisted by γ^l, analogously to (5.2.4) the twisted Ramond ground state for a single Dirac fermion

$\psi_\pm^{(k)}$ has dimensions $h = \overline{h} = \frac{1}{2}\frac{l}{M}\left(\frac{l}{M} - 1\right) + \frac{1}{8}$. Its charge follows from bosonization: The Dirac fermion can be described as boson compactified on a circle of radius $R = \sqrt{2}$ [Gin88a, §8.2]. The $\mathbb{Z}_M$ action translates into a shift orbifold in this language, showing that twisted ground states are given by vertex operators with $h = \overline{h} = Q^2/2 = \overline{Q}^2/2$ in the Neveu–Schwarz sector. Thus $h = \overline{h} = \frac{1}{2}\left(\frac{l}{M}\right)^2$ and $Q = \overline{Q} = (-1)^k \frac{l}{M}$. The sign $(-1)^k$ gives tribute to the fact that $\psi_\pm^{(k)}$ and $\psi_\pm^{(k+1)}$ are twisted in opposite directions (5.2.2). Generalizing $b_{l,m}$ in (5.2.6), for $(l,m) \neq (0,0)$ we now define

$$\begin{aligned}
f_{l,m}^{odd,NS}(\sigma,z) &:= (y\overline{y})^{\frac{m}{M}} \left| \frac{\vartheta_3\left(\sigma, z + \frac{m}{M}\sigma + \frac{l}{M}\right)}{\vartheta_1\left(\sigma, \frac{m}{M}\sigma + \frac{l}{M}\right)} \right|^2, \\
f_{l,m}^{ev,NS}(\sigma,z) &:= \left| \frac{\vartheta_3\left(\sigma, z + \frac{m}{M}\sigma + \frac{l}{M}\right)\vartheta_3\left(\sigma, z - \frac{m}{M}\sigma - \frac{l}{M}\right)}{\vartheta_1\left(\sigma, \frac{m}{M}\sigma + \frac{l}{M}\right)^2} \right|^2, \qquad (5.2.8) \\
f_{l,m}^{NS}(\sigma,z) &:= \begin{cases} \left(f_{l,m}^{ev,NS}(\sigma,z)\right)^{\frac{d}{4}} & \text{if } d \equiv 0(4) \\ f_{l,m}^{odd,NS}(\sigma,z)\left(f_{l,m}^{ev,NS}(\sigma,z)\right)^{\frac{d-2}{4}} & \text{if } d \equiv 2(4). \end{cases}
\end{aligned}$$

Note that the $U(1)$ current J of $\mathcal{C} = \mathcal{T}(\Lambda, B)$ is invariant under the $\mathbb{Z}_M$ action; it follows that so are the operators $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$ of spectral flow (3.1.7), i.e. $\mathcal{C}/\mathbb{Z}_M$ is invariant under spectral flow by theorem 3.1.4. Then we find

Theorem 5.2.2
Suppose that $\mathcal{C} = \mathcal{T}(\Lambda, B)$, $B = \Lambda^T \widetilde{B} \Lambda$, is a toroidal $N = (2,2)$ superconformal field theory with $c = 3d/2$ such that $T^d = \mathbb{R}^d/\Lambda = \times_{k=1}^{d/2} T^2_{(k)}$, and $T^2_{(k)}$ is $\mathbb{Z}_M$ symmetric, and $g^{-1}\widetilde{B}g = \widetilde{B}$ in $Skew(d \times d, \mathbb{R}/\mathbb{Z})$ for all $g \in \mathbb{Z}_M$. Then $\mathcal{C}/\mathbb{Z}_M$ with the $\mathbb{Z}_M$ action induced by (5.2.2) is an $N = (2,2)$ superconformal field theory, and the NS part of its partition function is

$$Z_{\mathbb{Z}_M-orb}^{\Gamma(\Lambda,B),NS}(\sigma,z) = \frac{1}{M}\left(Z_{\Gamma(\Lambda,B)}^{NS}(\sigma,z) + \sum_{\substack{l,m\in\{0,\dots,M-1\},\\ (l,m)\neq(0,0)}} n_{l,m} f_{l,m}^{NS}(\sigma,z) \right).$$

Here $Z_{\Gamma(\Lambda,B)}^{NS}$ is the NS part of the partition function (4.1.11) of $\mathcal{C}$, $f_{l,m}^{NS}(\sigma,z)$ was defined in (5.2.8), and $n_{l,m}$ is given in theorem 5.2.1.
The orbifold conformal field theory is invariant under spectral flow, and the entire partition function is obtained from the above by the flows (3.1.9). Effectively, in the definition of $f_{l,m}^{NS}$ above ϑ_3 must be replaced by $\vartheta_4, \vartheta_2, \vartheta_1$ to obtain the $\widetilde{NS}, R, \widetilde{R}$ parts of the partition function, respectively.

Unfortunately, the formula in theorem 5.2.2 appears to be quite clumsy and seems to remain so in general [Dul00]. Exceptions are $\mathbb{Z}_2$ orbifolds in general and the $\mathbb{Z}_4$ orbifold in case $d = 4$ [EOTY89]. A straightforward but tedious calculation using appendix A shows:

Corollary 5.2.3
For a superconformal field theory $\mathcal{C} = \mathcal{T}(\Lambda, B)$ of arbitrary central charge $c = 3d/2$, the NS part of the partition function of $\mathcal{C}/\mathbb{Z}_2$ is

$$Z^{NS}_{\Gamma(\Lambda,B)}(\sigma, z) = \frac{1}{2}\left(Z_{\Gamma(\Lambda,B)}(\sigma) + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^d + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^d + \left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^d \right) \left|\frac{\vartheta_3(z)}{\eta}\right|^d. \quad (5.2.9)$$

If $\mathcal{C}$ is a superconformal field theory as in theorem 5.2.2 with $d = 4$ and $M = 4$, then the NS part of the partition function of $\mathcal{C}/\mathbb{Z}_4$ is

$$\begin{aligned} &Z^{NS}_{\Gamma(\Lambda,B)}(\sigma, z) \\ &= \tfrac{1}{2}\Bigg[\Bigg\{ \tfrac{1}{2}\left(Z_{\Gamma(\Lambda,B)}(\sigma) + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 + \left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4 \right) \left|\frac{\vartheta_3(z)}{\eta}\right|^4 \Bigg\} \\ &\qquad + \left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_4(z)}{\eta}\right|^4 + \left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4 \left|\frac{\vartheta_2(z)}{\eta}\right|^4 + \left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_1(z)}{\eta}\right|^4 \Bigg]. \end{aligned} \quad (5.2.10)$$

In particular, the $\mathbb{Z}_4$ orbifold partition function *looks like* the $\mathbb{Z}_2$ orbifold partition function of a theory whose NS partition function is the expression in curly brackets in (5.2.10). See section 7.3.4 for further comments on this coincidence.

5.2.2 Geometric interpretation for $\mathbb{Z}_M$ orbifolds

For the orbifolds of $N = (2, 2)$ superconformal toroidal theories with central charge $c = 3$ or $c = 6$ we can understand the appearance of twisted Ramond ground states much better. So far, we know that they are lowest weight states in representations of the super Virasoro algebra that describe a string winding around a fixed point of the $\mathbb{Z}_M$ action. Theorem 5.2.2 shows that the multiplicity every fixed point accounts with is the order of its stabilizer group minus 1. We will now give geometric arguments for this result.
Since the explanation in case $c = 6$ is quite common, we treat this first. The standard complex structure on $T^4 = T^2_{(1)} \times T^2_{(2)}$ is given by the choice of a complex coordinate z_k on each $T^2_{(k)}$. Then our $\mathbb{Z}_M$ action (5.2.2) does not destroy the complex structure, in other words $\mathbb{Z}_M$ acts as ALGEBRAIC AUTOMORPHISM on T^4 (see definition 7.3.1). Since there are no invariant onecycles on T^4 it follows that $\widehat{X} = T^4/\mathbb{Z}_M$ is the orbifold limit of a Calabi–Yau manifold with $b_1 = 0$, i.e. a $K3$ surface by the discussion in section 4.6. In particular, we can blow up the singularities resulting from the fixed points of $\mathbb{Z}_M$ on T^4 without destroying the Calabi–Yau condition; that is, we replace each singular point by a chain of exceptional divisors, which in the case of $\mathbb{Z}_m$-fixed points have as intersection matrix the Cartan matrix of A_{m-1}. In particular, the exceptional divisors themselves are rational curves, i.e. holomorphically embedded spheres with self intersection number -2. In terms of the homology of the resulting surface X these rational curves are elements of

$H_2(X,\mathbb{Z}) \cap H_{1,1}(X,\mathbb{C})$. To translate to cohomology we work with their Poincaré duals, which now are elements of $Pic(X)$ with length squared -2. One may check that for $M \in \{2,3,4,6\}$ this procedure changes the Hodge diamond by

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \longmapsto \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and indeed produces a $K3$ surface X [Wal88]. We also obtain a rational map $\pi : T^4 \to X$ of degree M that is defined outside the fixed points. To fix all necessary geometric data we additionally need to pick the class of a Kähler metric on X. To understand the geometric interpretation of $\mathcal{C}' = \mathcal{T}(\Lambda, B)/\mathbb{Z}_M$ we have to consider the ORBIFOLD LIMIT of our $K3$ surface, that is use the ORBIFOLD SINGULAR METRIC on X which is induced from the flat metric on T^4 and assigns volume zero to all the exceptional divisors. The corresponding Einstein metric is constructed by excising a sphere around each singular point of $T^4/\mathbb{Z}_M$ and gluing in an Eguchi Hanson sphere E_2 instead for $\mathbb{Z}_2$ fixed points, or a generalized version E_m with boundary $\partial E_m = \mathbb{S}^3/\mathbb{Z}_m$ at infinity and nonvanishing Betti numbers $b_0(E_m) = 1, b_2(E_m) = b_2^-(E_m) = m-1$, i.e. $\chi(E_m) = m$. The orbifold limit is the limit these Eguchi Hanson type spheres have shrunk to zero size in.
Summarizing, a $\mathbb{Z}_M$ orbifold limit X of $K3$ is obtained by replacing each $\mathbb{Z}_m$ fixed point on $T^4/\mathbb{Z}_M$ by a chain of $m-1$ rational spheres of volume zero. On the other hand, the stabilizer group $\mathbb{Z}_m$ of such a fixed point has order m, so there are $m-1$ twisted Ramond ground states associated to this fixed point. This leads to a natural interpretation of the multiplicity, namely winding a string around a $\mathbb{Z}_m$ fixed point leaves a choice of winding it around one of the $m-1$ rational spheres that replace the fixed point.
This interpretation is also natural in the context of chiral rings: Since by theorem 5.2.2 the orbifold conformal field theory $\mathcal{C}'$ is invariant under spectral flow (3.1.8), Ramond ground states with charges $(Q;\overline{Q})$ are in $1:1$ correspondence to (chiral,chiral) fields with charges shifted by $\frac{d}{4}$. On the other hand, as explained in section 3.1.1, it has been conjectured in [LVW89] and is strongly believed since then that for every $N=(2,2)$ superconformal field theory with nonlinear σ model interpretation on some Calabi–Yau manifold X the (chiral,chiral) ring is isomorphic to $H^{*,*}(X,\mathbb{C})$. Here, the (holomorphic,antiholomorphic) degree of a differential form is given by (left,right) charges of the corresponding (chiral,chiral) field.
Note that the Ramond ground states of the original toroidal theory which are $\mathbb{Z}_M$ invariant are in $1:1$ correspondence to the $\mathbb{Z}_M$ invariant forms on T^4. Hence the above counting already confirms that the number of generators of given bidegree agrees in each of the rings in case X is a $\mathbb{Z}_M$ orbifold limit of $K3$. In this case it is also not hard to see that the map described in [LVW89] indeed is a ring isomorphism.
The above reasoning cannot be directly applied in the case $d=2$. In the following, $\mathcal{C} = \mathcal{T}(\Lambda, B)$ with $\mathbb{Z}_M$ symmetric $T^2 = \mathbb{R}^2/\Lambda$ and parameters (τ, ρ) according to

(4.3.1). The orbifold $X = T^2/\mathbb{Z}_M$ is a Riemann sphere with metric singularities but smooth complex structure. The Calabi–Yau condition on a Kähler manifold X, $\dim_{\mathbb{C}} X = n$, is equivalent to the holonomy of X being contained in $SU(n)$. Now $SU(1) = \{\mathbb{1}\}$, so the only Calabi–Yau manifold in complex dimension one is a torus, and no blow up of the singularities is possible without destroying the Calabi–Yau condition. Nevertheless each $\mathbb{Z}_m$ fixed point accounts with multiplicity $m-1$ to the twisted sector of $\mathcal{C}' = \mathcal{C}/\mathbb{Z}_M$. A simple explanation can be given in terms of singularity theory.

We first introduce some notions from this theory, which can be learned from [Arn81, AGZV85]. In general, let $C = \{f(z_1, \ldots, z_n) = 0\} \subset \mathbb{C}^n$ denote an algebraic hypersurface with quasihomogeneous singularity in $z = 0$, i.e. f is a quasihomogeneous polynomial. This induces a natural $\mathbb{C}^*$ action on C. The MODALITY ν of the singularity with respect to this action is the least number such that a sufficiently small neighbourhood of 0 may be covered by a finite number of ν-parameter families of $\mathbb{C}^*$ orbits. Loosely speaking, it is the number of free complex parameters in f which do not change the type of singularity in $z = 0$. Quasihomogeneous hypersurface singularities of modality $0, 1, 2$ are entirely classified and called SIMPLE, UNIMODAL, and BIMODAL, respectively. The MULTIPLICITY or MILNOR NUMBER of the singularity is the index of the singularity $z = 0$ for the vector field $\mathrm{grad}(f)$, i.e. the dimension of the local ring $\mathbb{C}[z_1, \ldots, z_n]/\left(\frac{\partial f}{\partial z_i}\right)$ of C_0. It is the number of points into which $0 \in C$ will split under a sufficiently general (VERSAL) deformation.

Let us concentrate on the quotient singularity in a chosen $\mathbb{Z}_m$ fixed point α on $X = T^2/\mathbb{Z}_M$. This is the quasihomogeneous quotient singularity of

$$C^{(m)} = \{f_m(x, y) = 0\} \subset \mathbb{C}^2, \quad f_m(x, y) := y^2 - x^m. \tag{5.2.11}$$

It has modality 0 and is of A_{m-1} type in the (ADE–)classification of these singularities due to H. Cartan, D. Prill, E. Brieskorn, and V. Arnold [Car57, Pri67, Bri71, Arn72]. Its Milnor number is easily checked to be $\mu_\alpha = m-1$. The topology of the neighbourhood of a simple singularity $C \subset \mathbb{C}^2$ is uniquely determined by the link $K := \mathbb{S}^3_\varepsilon \cap C$, where $\varepsilon \in \mathbb{R}^+$ sufficiently small. In our case, $K = K_m$ is the torus knot $(2, m)$. Simplifying a bit, one can understand the topology of the singularity by replacing the singular $D^3_\varepsilon \cap C$ by a Riemann surface F with $\partial F = K$. Such a surface exists for every simple singularity and is uniquely defined. Namely, F is the fibre of the MILNOR FIBRATION

$$\mathbb{S}^3_\varepsilon - K \xrightarrow{\frac{f}{|f|}} \mathbb{S}^1,$$

and is called MILNOR FIBRE. For the singularities (5.2.11) of type A_{m-1} the Milnor fibre F_m is constructed as follows: Take two $2m$–gons and mark every second edge of each of them in cyclic order by numbers $1, \ldots, m$. Identify edges of the two polygons if they carry the same marking, but with orientation reversed. A simplified argument to show that the above indeed gives the Milnor fibre F_m is based on the observation that we have constructed the Riemann surface for

$\widetilde{f}_m^\lambda(x,y) = y^2 - x^m + \lambda$ in the limit $\lambda \to 0$. Namely, two sheets $\overline{\mathbb{C}}$ are glued along m branchcuts that end in m points $p_1, \dots, p_m \in \overline{\mathbb{C}}$, all of which approach $(x,y) = (0,0)$ in the limit $\lambda \to 0$. It is instructive to check that for $m = 3$ the trefoil knot K_3 is indeed the boundary of the Riemann surface F_3 constructed by the above prescription.
It is a deep result due to Milnor, that the topology of the Milnor fibre is just a bouquet of μ_α spheres $\mathbb{S}^1$. In other words, $\mu_\alpha = \text{rk } H_1(F_m, \mathbb{Z})$, and the noncontractable cycles in F_m can be visualized in the deformation $\widetilde{f}_m^\lambda(x,y) = y^2 - x^m + \lambda$ of C_m as encircling two branchpoints $p_1, p_j, j \in \{2, \dots, \mu_\alpha = m-1\}$ each. This means that for a string winding around the fixed point α there are $m-1$ topologically distinct possibilities to wind around different sets of branchpoints.
While the above gives a satisfactory geometric explanation for the multiplicity $m-1$ each fixed point accounts with in the orbifold conformal field theory, we would also like to understand the translation into the language of chiral rings (see section 3.1.1). By theorem 5.2.2 the (chiral,chiral) fields corresponding to twisted Ramond ground states have left and right handed charges $\frac{l}{M}$ in the γ^l twisted sector. An interpretation of these fields in terms of cohomology with rational degree $\frac{l}{M}$ on X seems natural. This would be in accord with the counting of Ramond ground states by the STRINGY EULER NUMBER for orbifolds, as introduced in [DHVW85, DHVW86]:

$$\chi_{st}(Y/G) = \frac{1}{|G|} \sum_{\substack{f,g \in G, \\ [f,g]=0}} \chi(Y^{f,g}), \quad Y^{f,g} := \{y \in Y \mid fy = gy = y\}.$$

Namely, one checks for $M \in \{2,3,4,6\}$ that

$$\chi_{st}(T^4/\mathbb{Z}_M) = 2 + \sum_j \mu_{\alpha_j} = \chi(\mathbb{S}^2) + \sum_j \mu_{\alpha_j},$$

where the sum runs over all fixed points on X and $\chi(\mathbb{S}^2)$ accounts for the two Ramond ground states with $Q = \overline{Q} = \pm 1$ of the original toroidal theory which are invariant under $\mathbb{Z}_M$. $\chi_{st}(T^4/\mathbb{Z}_M)$ also agrees with the Witten index of $\mathcal{C}/\mathbb{Z}_M$ (see section 3.1.2), since by theorem 5.2.2

$$\text{tr}_R(-1)^F = Z^{\Gamma(\Lambda,B),\widetilde{R}}_{\mathbb{Z}_M-orb}(\sigma, z=0) = \frac{1}{M} \sum_{\substack{l,m \in \{0,\dots,M-1\}, \\ (l,m)\neq(0,0)}} n_{l,m}$$

with $n_{l,m}$ given in theorem 5.2.1.
The more striking observation is that we can build the entire (chiral,chiral) ring of the orbifold conformal field theory from the local rings of singularities introduced above. In a different language this way we reobtain the standard Landau–Ginzburg description of chiral rings for $T^2/\mathbb{Z}_M, M \in \{3,4,6\}$ as discussed in [LLW89, VW89]. Let us review their analysis in a somewhat different approach to give the full picture.

The local ring of an A_{m-1} singularity (5.2.11) in a $\mathbb{Z}_m$ fixed point α_j of X is

$$\mathbb{C}[x,y]/\left(\tfrac{\partial f}{\partial x},\tfrac{\partial f}{\partial y}\right) \cong \mathbb{C}[\Phi_j]/\left(\Phi_j^{m-1}\sim 0\right) =: \mathcal{R}_j.$$

The generator Φ_j is now interpreted as the generator of the (chiral,chiral) ring of our orbifold conformal field theory of dimensions $h=\overline{h}=\frac{1}{2m}$ that corresponds to the twist field Σ_j on the fixed point α_j with minimal twist $\frac{1}{m}$. In the following, we will use the multiplication on the set of twist fields that is induced by the normal ordered product on the (c,c) ring. Then $\Phi_j \leftrightarrow \Sigma_j$ is consistent with the ring structure, since fusion gives $\Sigma_j^{m-1}\sim(\Sigma_j)^\dagger$, so the product of Σ_j^{m-1} with all other fields Σ_j^k is zero and therefore $\Sigma_j^{m-1}\sim 0$ in the ring of twist fields. By [Mar89, LLW89], $\mathcal{R}_j$ is isomorphic to the (c,c) ring of the $N=(2,2)$ minimal model $(m-2)$.

We now take the tensor product $\mathcal{R}_{\mathbb{Z}_M}$ of all local rings $\mathcal{R}_j$ over singular points of $X=T^2/\mathbb{Z}_M$:

$$\begin{aligned}
\mathcal{R}_{\mathbb{Z}_3} &= \mathrm{span}_\mathbb{C}\left\{\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\Phi_3^{\alpha_3} \mid \alpha_j\in\{0,1\}\right\},\\
\mathcal{R}_{\mathbb{Z}_4} &= \mathrm{span}_\mathbb{C}\left\{\Phi_1^{\alpha_1}\Phi_2^{\alpha_2} \mid \alpha_j\in\{0,1,2\}\right\},\\
\mathcal{R}_{\mathbb{Z}_6} &= \mathrm{span}_\mathbb{C}\left\{\Phi_1^{\alpha_1}\Phi_2^{\alpha_2} \mid \alpha_1\in\{0,\dots,4\},\alpha_2\in\{0,1\}\right\}.
\end{aligned} \tag{5.2.12}$$

Note that $\mathbb{Z}_2$ type singularities never account separately, since the corresponding twist field is obtained from others by fusion: $\Sigma_1\Sigma_2$ both in the $\mathbb{Z}_4$ and the $\mathbb{Z}_6$ case. Moreover, $\mathcal{R}_{\mathbb{Z}_M}$ has $\chi_{st}(T^2/\mathbb{Z}_M)=8,9,10$ elements for $M=3,4,6$, in accord with the desired interpretation as (c,c) ring of the $\mathbb{Z}_M$ orbifold conformal field theory. To keep with this interpretation, the relations between the Φ_j in $\mathcal{R}_{\mathbb{Z}_M}$ must be given by the normal ordered product in the corresponding (c,c) ring. This explicitly depends on the coefficients of the operator product expansion in the particular model, i.e. on the parameter ρ of the original toroidal theory, if we ensure $\mathbb{Z}_M$ symmetry of T^2 by fixing τ. The simplest case is the one where we can take the free product of the rings $\mathcal{R}_j$ such that the relations in $\mathcal{R}_{\mathbb{Z}_M}$ are just $\Phi_j^{m-1}\sim 0$ for a $\mathbb{Z}_m$ fixed point α_j. Anticipating theorem 5.6.4, we know that the fermionic tensor products $(2)\otimes(2), [(1)\otimes(1)\otimes(1)$ or $(1)\otimes(4)]$ of $N=(2,2)$ minimal models are the $\mathbb{Z}_4, [\mathbb{Z}_3$ or $\mathbb{Z}_6]$ orbifolds of the $N=(2,2)$ toroidal superconformal field theories with $c=3$ and parameters $\tau=\rho=i$ $[\tau=\rho=e^{2\pi i/3}]$, respectively. This directly shows that in these cases $\mathcal{R}_{\mathbb{Z}_M}$ is isomorphic to the (c,c) ring of the corresponding orbifold conformal field theory. In fact it is also not hard to check that for arbitrary parameters of ρ the point group selection rules [DFMS87] for orbifold conformal field theories give such severe restrictions on the fusion rules that the chiral ring structure can be determined. Namely, n point functions vanish unless the total monodromy of the inserted fields is trivial. For example, in the $\mathbb{Z}_3$ case, the only possible nonvanishing two point functions that contain Σ_1 are $\langle\Sigma_1(z)\Sigma_1^2(w)\rangle$ and $\langle\Sigma_1(z)\Sigma_2\Sigma_3(w)\rangle$. This implies the general relation $\Phi_1^2+a\Phi_2\Phi_3\sim 0$ in $\mathcal{R}_{\mathbb{Z}_3}$ for some parameter $a=a(\rho)$. All in all, by an appropriate rescaling of the Φ_j we find the following relations:

$$\begin{array}{ll} \mathcal{R}_{\mathbb{Z}_3}: & \Phi_1^2 + a_3\Phi_2\Phi_3 \sim 0,\ \Phi_2^2 + a_3\Phi_1\Phi_3 \sim 0,\ \Phi_3^2 + a_3\Phi_1\Phi_2 \sim 0; \\ \mathcal{R}_{\mathbb{Z}_4}: & \Phi_1^3 + a_4\Phi_1\Phi_2^2 \sim 0,\ \Phi_2^3 + a_4\Phi_1^2\Phi_2 \sim 0; \\ \mathcal{R}_{\mathbb{Z}_6}: & \Phi_1^5 + a_6\Phi_1\Phi_2^2 \sim 0,\ \Phi_2^2 + 2a_6\Phi_1^2\Phi_2 \sim 0. \end{array} \tag{5.2.13}$$

In fact, $\mathcal{R}_{\mathbb{Z}_M}$ with (5.2.12) and (5.2.13) is just the local ring of a singularity in $\mathbb{C}^3$:

$$\mathcal{R}_{\mathbb{Z}_M} = \mathbb{C}[\Phi_1, \Phi_2, \Phi_3]/\left(\tfrac{\partial W_M}{\partial \Phi_1}, \tfrac{\partial W_M}{\partial \Phi_2}, \tfrac{\partial W_M}{\partial \Phi_3}\right) \quad ,$$

$$\begin{array}{rlcl} a_3^3 \neq -1: & W_3(\Phi_1,\Phi_2,\Phi_3) & := & \Phi_1^3+\Phi_2^3+\Phi_3^3+3a_3\Phi_1\Phi_2\Phi_3 \\ a_4^2 \neq \ \ 1: & W_4(\Phi_1,\Phi_2,\Phi_3) & := & \Phi_1^4+\Phi_2^4+\Phi_3^2+2a_4\Phi_1^2\Phi_2^2 \\ 4a_6^3 \neq -1: & W_6(\Phi_1,\Phi_2,\Phi_3) & := & \Phi_1^6+\Phi_2^3+\Phi_3^2+3a_6\Phi_1^2\Phi_2^2. \end{array} \tag{5.2.14}$$

The above is the list of all unimodal parabolic singularities, which are labelled $\widehat{E}_6, \widehat{E}_7, \widehat{E}_8$ in the mathematics literature. Unimodality, i.e. one free parameter a_M, is in accord with the fact that the moduli spaces of $\mathbb{Z}_M$ orbifolds, $M \in \{3,4,6\}$, of $N=(2,2)$ superconformal field theories with central charge $c=3$ have complex dimension one and can be parametrized by $\rho \in \mathbb{H}/PSL(2,\mathbb{Z})$ (see sections 5.3 and (6.1.2)). Thus $a_M = a_M(\rho)$, and by the above $a_3(e^{2\pi i/3}) = a_4(i) = a_6(e^{2\pi i/3}) = 0$. Summarizing, we have reestablished the following theorem without recourse to the Landau–Ginzburg language, though this is quite fashionable in the context of chiral rings. We stress that the Landau–Ginzburg result is discussed in great detail for the $\mathbb{Z}_3$ case in [LLW89]. Application of the same technique to $\mathbb{Z}_4$ and $\mathbb{Z}_6$ is straightforward.

Theorem 5.2.4
The (chiral,chiral) ring of the $\mathbb{Z}_M$ orbifold of an $N=(2,2)$ toroidal superconformal field theory with $c=3$ for $M \in \{3,4,6\}$ is isomorphic to the local ring of the parabolic singularity $\widehat{E}_6, \widehat{E}_7, \widehat{E}_8$ given by $\{W_M(\Phi_1,\Phi_2,\Phi_3)=0\}$ in (5.2.14).

We remark that the precise relation $a_3 = a_3(\rho)$ has been determined in [LLW89]. Sayipjamal Dulat has counted the number of (chiral,chiral) states of given charge in the respective orbifolds in the formula of theorem 5.2.2 and checked agreement with the number of homogeneous polynomials of the same degree in the corresponding local rings [Dul00]. As we have seen above, this agreement actually follows directly from the geometric construction of orbifold conformal field theories.
As to the case of $\mathbb{Z}_2$ orbifolds we remark that no Landau–Ginzburg description is given in [LLW89]*. In fact they do not have a Landau–Ginzburg description, since the (chiral,antichiral) ring contains the $\mathbb{Z}_2$ invariant field $\psi_+^{(1)}\overline{\psi}_-^{(1)}$ of the original toroidal theory and thus is nontrivial. On the level of singularity theory, the reason is that the singularity of type A_1 in (5.2.11) is reducible: $f_2(x,y) = (y+x)(y-x)$, and the local ring is freely generated of rank one. Since there are two $\mathbb{Z}_2$ invariant (chiral,chiral) fields, $\mathbb{1}$ and $\psi_+^{(1)}\overline{\psi}_+^{(1)}$, in the original toroidal conformal field theory

*Accordingly, the corresponding statement about the (chiral,chiral) ring in [Dul00] is wrong.

and there are four $\mathbb{Z}_2$ fixed points, the (c,c) ring of a $\mathbb{Z}_2$ orbifold conformal field theory is isomorphic to the ring $\oplus_{i=1}^6 \mathbb{C}$, in accord with $\chi_{st}(T^2/\mathbb{Z}_2) = 6$.

5.3 Crystallographic orbifolds

In section 5.2 we explained that every geometric symmetry group G of the torus $T^d = \mathbb{R}^d/\Lambda$, i.e. an automorphism group of the lattice Λ, induces a symmetry of the toroidal conformal field theory $\mathcal{T}(\Lambda, B)$ if B satisfies (5.2.1). Then the orbifold construction yields a new theory $\mathcal{T}(\Lambda, B)/G$. In this section, we will construct all such "geometric" orbifolds of bosonic toroidal conformal field theories with $c = d = 2$. We will use the notation introduced in section 4.3 for twodimensional toroidal theories.

5.3.1 Crystallographic symmetry groups

We start by listing all possible geometric symmetry groups and their action on the corresponding toroidal conformal field theory. The discrete symmetries of twodimensional tori are classified. Namely, in two dimensions there are seventeen inequivalent crystallographic space groups [Pól24], i.e. discrete subgroups $G \subset O(2) \ltimes \mathbb{R}^2$ that leave invariant some lattice Λ' and therefore act on a torus $T^2 = \mathbb{R}^2/\Lambda$, where $\Lambda \subset \Lambda'$. Figure 5.3.1 shows all these symmetry groups by depicting the orbit of some symbol ▶ under G. A complete list of the corresponding groups is given in (5.3.9) after we have explained their action on T^2 and $\mathcal{T}(\Lambda, B)$.
Each lattice Λ' in figure 5.3.1 is formed by fixed combinations of the symbol ▶, which we call motive, in various orientations. Then $\Lambda \subset \Lambda'$ is given by those motives which have the same orientation. The space group G is a semi-direct product of a finite point group $P \subset O(2)$ and a "translationary" group $\triangle \subset O(2) \ltimes \mathbb{R}^2$ of elements which do not fix the origin. In figure 5.3.1 the group Δ is the minimal subgroup of G which acts transitively on motives. The finite group P is determined by inspection of the particular motives which comprise the orbit of the symbol ▶ under P each.
By the above, P is an automorphism group of the two–dimensional lattice Λ, and if $(S, \delta) \in \Delta$, then there is some $D \in \mathbb{Z}$ such that $D\delta \in \Lambda$. Therefore if $A \in P$ has order M then $M \in \{2, 3, 4, 6\}$. The values $M = 3$ or $M = 6$ require Λ to be a hexagonal lattice ($\tau = e^{2\pi i/3}$); $M = 4$ requires a square lattice ($\tau = i$). A lattice with these values of τ automatically has $\mathbb{Z}_M$ symmetry, and $\mathbb{Z}_M$ acts by (5.2.2) on the corresponding toroidal conformal field theory. Since rotations in two dimensions commute, (5.2.1) gives no restriction on the B–field.
The REFLECTION SYMMETRY R is an automorphism of lattices with $\tau_1 \in \{0, \frac{1}{2}\}$, where R acts on the coordinates of T^2 by

$$R = R_1 : (x^1, x^2) \mapsto (x^1, -x^2) \text{ or } R = R_2 : (x^1, x^2) \mapsto (-x^1, x^2). \qquad (5.3.1)$$

By inspection of the action on the respective fundamental cell one checks that an exchange of R_1 and R_2 is equivalent to a transformation of τ_2; with S, T as in

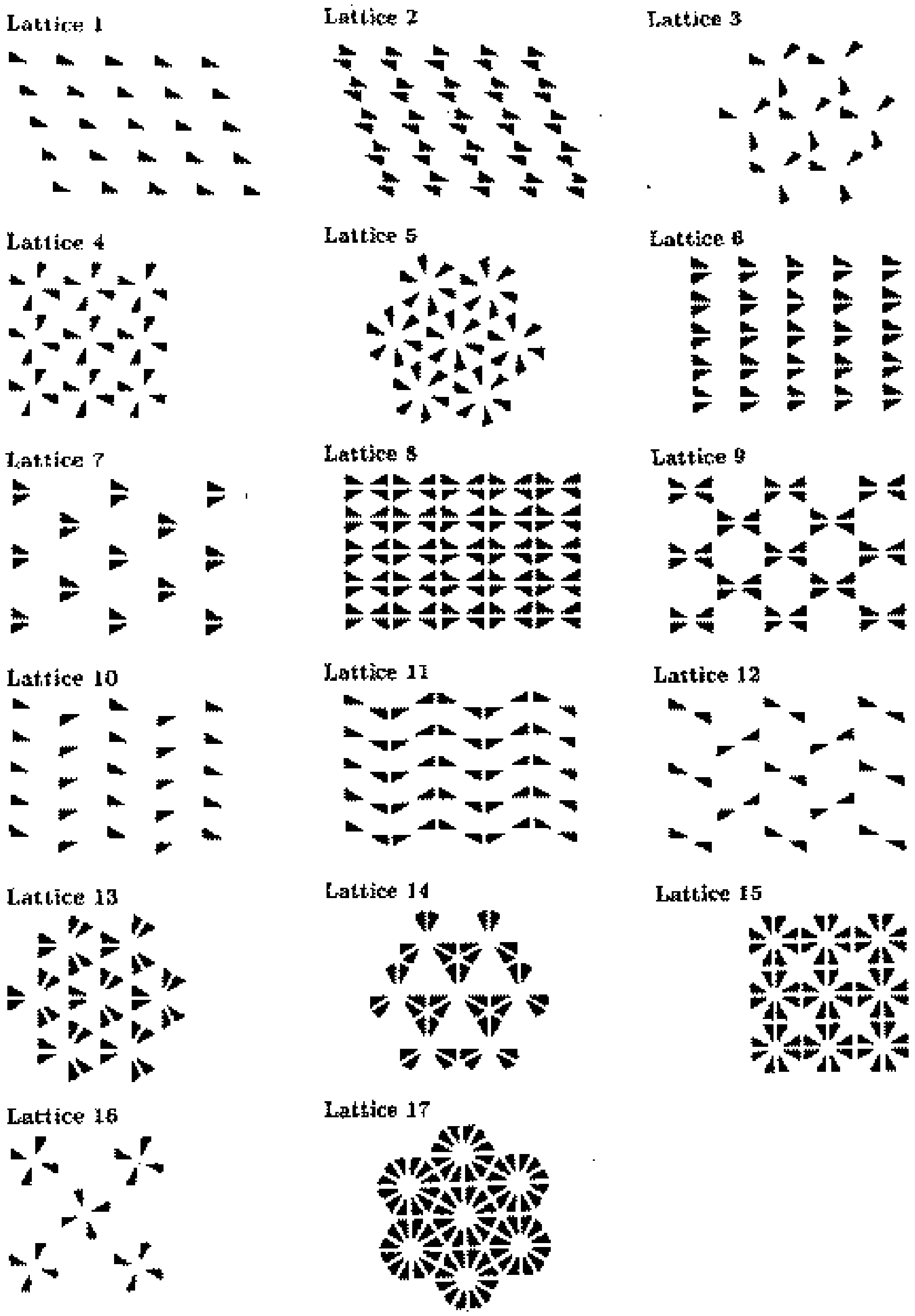

Figure 5.3.1: The seventeen inequivalent crystallographic space groups [Bue56].

(2.1.5) we define

$$\Theta\zeta := \begin{cases} S\zeta & \text{if } \zeta_1 = 0, \\ TST^2S\zeta & \text{if } \zeta_1 = \frac{1}{2}. \end{cases}$$

Then

$$R_1 \leftrightarrow R_2 \text{ is equivalent to } \tau \mapsto \Theta\tau, \text{ where } \begin{cases} \Theta(i\tau_2) &= \frac{i}{\tau_2}, \\ \Theta(\frac{1}{2} + i\tau_2) &= \frac{1}{2} + \frac{i}{4\tau_2}. \end{cases} \tag{5.3.2}$$

We can therefore restrict ourselves to the discussion of the symmetry R_1 in the following. To extend R_1 to the charge lattice (4.2.1), the B-field B must obey (5.2.1) which is true iff $\rho_1 \in \frac{1}{2}\mathbb{Z}$. Then by using (4.3.3) for the R_1 action $|m_1, m_2, n_1, n_2\rangle \mapsto \pm|m_1', m_2', n_1', n_2'\rangle$ we obtain

$$\begin{array}{rclrcl} m_1' &=& -m_1 \,, & n_1' &=& -n_1 + 2\tau_1 n_2 + 2\rho_1 m_2 + 4\tau_1\rho_1 m_1, \\ m_2' &=& m_2 + 2\tau_1 m_1 \,, & n_2' &=& n_2 + 2\rho_1 m_1, \end{array} \tag{5.3.3}$$

and the invariant vectors $(p_l; p_r)$ of the charge lattice correspond to $|0, m_2, n_1, n_2\rangle$,

$$p_{l,r} = \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} n_2\tau_2 \pm m_2\rho_2 \\ 0 \end{pmatrix}, \quad n_2, m_2 \in \mathbb{Z} \text{ such that } n_1 = n_2\tau_1 + m_2\rho_1 \in \mathbb{Z}. \tag{5.3.4}$$

Because (5.3.4) only depends on $\rho_1 \bmod \mathbb{Z}$ the same is true for the resulting orbifold theory and we can pick $\rho_1 \in \{0, \frac{1}{2}\}$. Note that in the case $\rho_1 = \frac{1}{2}$ the B-field of our theory is effectively shifted by an integer form if we apply R_1. This will be of some importance below.

In all cases except for $\tau_1 = \rho_1 = \frac{1}{2}$ we can fix the phases of the ground states such that R_1 acts by $|m_1, m_2, n_1, n_2\rangle \mapsto |m_1', m_2', n_1', n_2'\rangle$ with (5.3.3). If $\tau_1 = \rho_1 = \frac{1}{2}$, the charge lattice (4.3.3) of the toroidal theory is generated by the four vectors

$$v_{\delta,\epsilon} := \frac{1}{2\sqrt{2\tau_2\rho_2}} \left(\begin{pmatrix} \tau_2 + \delta\rho_2 \\ \epsilon\,(1/2 - 2\delta\tau_2\rho_2) \end{pmatrix}, \begin{pmatrix} \tau_2 - \delta\rho_2 \\ \epsilon\,(1/2 + 2\delta\tau_2\rho_2) \end{pmatrix} \right), \quad \delta, \epsilon \in \{\pm 1\}$$

which are pairwise interchanged by R_1 ($v_{\delta,1} \leftrightarrow v_{\delta,-1}$). The R_1 invariant part of the charge lattice is given by (5.3.4),

$$\begin{array}{rcl} p_{l,r} &=& \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} n_2\tau_2 \pm m_2\rho_2 \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{n}{r} \pm mr \\ 0 \end{pmatrix}, \\ n_2 &=& 2n, m_2 = 2m, n_1 = n + m \in \mathbb{Z},\ r = \sqrt{\rho_2/\tau_2}. \end{array} \tag{5.3.5}$$

Because $\langle v_{\delta,\epsilon}, v_{\delta,-\epsilon}\rangle = 1$, the vertex operators corresponding to generators of the invariant part of the charge lattice are obtained from operator product expansions

$$(V[v_{\delta,1}] + V[-v_{\delta,1}]) \times (V[v_{\delta,-1}] - V[-v_{\delta,-1}])\,.$$

Since this is a product between an R_1 even and an R_1 odd operator, the resulting vertex operators are R_1 odd. It follows that R_1 acts on ground states corresponding to invariant charge vectors (5.3.5) by $|m_1, m_2, n_1, n_2\rangle \mapsto (-1)^{n_2}|m_1, m_2, n_1, n_2\rangle$. The additional signs in the R_1 action for $\tau_1 = \rho_1 = \frac{1}{2}$ are due to the fact that the action of R_1 effectively shifts the B–field by an integer form, as was already mentioned above. In the discussion of the bicritical point (C15) in section 6.2 we will point out a very natural confirmation of this observation.

TRANSLATIONS T_δ by $\delta \in \Lambda$ are the basic symmetries of the torus $T^2 = \mathbb{R}^2/\Lambda$. The result of modding out any torus by a translation symmetry T_δ, $D\delta \in \Lambda$, $D \in \mathbb{N}$ minimal with this property, gives another torus with lattice generated by Λ and δ. To produce a surface different from the torus, and thereby non–toroidal conformal field theories, we must combine the translation with the reflection symmetry which we denote $T_R := RT_\delta$. More precisely, we will need this symmetry only in the case $\tau_1 = 0$ and $D = 2$, and we set

$$\delta_1 \;:=\; \sqrt{\tfrac{\rho_2}{\tau_2}}\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \qquad \delta_2 \;:=\; \sqrt{\tfrac{\rho_2}{\tau_2}}\begin{pmatrix} 0 \\ \tau_2/2 \end{pmatrix}, \qquad \delta' \;:=\; \sqrt{\tfrac{\rho_2}{\tau_2}}\begin{pmatrix} 1/2 \\ \tau_2/2 \end{pmatrix};$$

for $k \in \{1,2\}$:

$$T_{R_k} \;:=\; R_k e^{2\pi i p\cdot\frac{\delta_k}{\sqrt{2}}}, \qquad T'_{R_k} \;:=\; R_k e^{2\pi i p\cdot\frac{\delta'}{\sqrt{2}}}, \qquad \widehat{T}_{R_2} \;:=\; R_2 e^{2\pi i p\cdot\frac{\delta_1}{\sqrt{2}}}. \tag{5.3.6}$$

The groups of type $\mathbb{Z}_2$ generated by T_R or T'_R are denoted $\mathbb{Z}_2(T_R)$ or $\mathbb{Z}_2(T'_R)$, respectively, where either $R = R_1$ or $R = R_2$. To understand the action of the symmetry $T_R^{(\prime)} = RT_{\delta^{(\prime)}}$ on the space of states of a toroidal conformal field theory observe that $T_{\delta^{(\prime)}}$ only acts on the ground state sectors and leaves the oscillator modes invariant. On a state $|m_1, m_2, n_1, n_2\rangle$ corresponding to the charge vector $(p_l; p_r)(\mu, \lambda)$ the action of $T_{R_1}^{(\prime)}$ is given by the action (5.3.3) of R_1 combined with multiplication by $exp[2\pi i(p_l; p_r)(\mu, \lambda) \cdot \frac{1}{2}(p_l; p_r)(0, 2\delta^{(\prime)})] = (-1)^{\langle\mu, 2\delta^{(\prime)}\rangle}$, where we used (4.2.1). It is therefore a priori clear that as for the action of R we need to restrict the possible B–field values to $\rho_1 \in \{0, \frac{1}{2}\}$ for consistency of the action of $T_R^{(\prime)}$. In fact, $T_R^{(\prime)}$ actions are only needed in the case $\tau_1 = 0$. Using (4.3.3) one now checks that only for $\rho_1 = 0$ the order of $T_R^{(\prime)}$ is two, whereas for $\rho_1 = 1/2$ we find that $T_R^{(\prime)}$ generates a $\mathbb{Z}_4$ type group. The action of $g := (T_R^{(\prime)})^2$ is given by multiplication with ± 1 on the different sectors of the space of states. To mod out a toroidal theory A by this $\mathbb{Z}_4$ then is equivalent to performing a $\mathbb{Z}_2$ orbifold procedure on $A/\{1, g\}$. But $A/\{1, g\}$ is another toroidal theory, because both generic torus currents are invariant under g and give conserved currents in $A/\{1, g\}$ as well. The $T_R^{(\prime)}$ action with $\rho_1 = 1/2$ hence need not be considered separately. For $\rho_1 = 0$ by (5.3.2) we now have

$$T_{R_1}^{(\prime)} \leftrightarrow T_{R_2}^{(\prime)} \text{ is equivalent to } \tau\,(= i\tau_2) \mapsto \Theta\tau \left(= \frac{i}{\tau_2}\right). \tag{5.3.7}$$

Since by (5.3.6) $\delta^{(\prime)} = \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix} 1/2 \\ * \end{pmatrix}$, if $|m_1, m_2, n_1, n_2\rangle$ is R_1-invariant, then by (5.3.4) $m_1 = 0, n_1 = n_2\tau_1 + m_2\rho_1$, and $T_R^{(\prime)}$ acts by

$$T_R^{(\prime)}: \quad |m_1, m_2, n_1, n_2\rangle \mapsto (-1)^{n_2}|m_1, m_2, n_1, n_2\rangle. \tag{5.3.8}$$

We denote by $A(\vartheta) \in \mathbb{Z}_M$ the rotation by an angle of ϑ. Then $R_2 = A(\pi)R_1$, $T^{(\prime)}_{R_2} = A(\pi)T^{(\prime)}_{R_1}$, and we have the noncyclic crystallographic groups

$$\begin{array}{rcl rcl} D_2 &:=& \{1, A(\pi), R_1, R_2\}, & D_3(R) &:=& \mathbb{Z}_3 \cup R\mathbb{Z}_3, \\ D_4 &:=& \mathbb{Z}_4 \cup R_1\mathbb{Z}_4 = \mathbb{Z}_4 \cup R_2\mathbb{Z}_4, & D_6 &:=& \mathbb{Z}_6 \cup R\mathbb{Z}_6, \\ D_2(T_R) &:=& \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}, & D_2(T'_R) &:=& \{1, A(\pi), T'_{R_1}, T'_{R_2}\}, \\ D_4(T'_R) &:=& \mathbb{Z}_4 \cup T'_{R_1}\mathbb{Z}_4 = \mathbb{Z}_4 \cup T'_{R_2}\mathbb{Z}_4. & & & \end{array}$$

The symmetries that correspond to the lattices in figure 5.3.1 and the restrictions on the values of τ and ρ are

Lattice	1	2	3	4	5	6	7
Symmetries	$\{T_\delta, \delta \in \Lambda\}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_2(R)$	$\mathbb{Z}_2(R)$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$e^{2\pi i/3}$	i	$e^{2\pi i/3}$	$\tau_1 = 0$	$\tau_1 = \frac{1}{2}$
ρ	$\in \mathbb{H}$	$\in \mathbb{H}$	$\in \mathbb{H}$	$\in \mathbb{H}$	$\in \mathbb{H}$	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 \in \{0, \frac{1}{2}\}$

Lattice	8	9	10	11	12
Symmetries	D_2	D_2	$\mathbb{Z}_2(T_R)$	$D_2(T_R)$	$D_2(T'_R)$
τ	$\tau_1 = 0$	$\tau_1 = \frac{1}{2}$	$\tau_1 = 0$	$\tau_1 = 0$	$\tau_1 = 0$
ρ	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 = 0$	$\rho_1 = 0$	$\rho_1 = 0$

Lattice	13	14	15	16	17
Symmetries	$D_3(R_1)$	$D_3(R_2)$	D_4	$D_4(T'_R)$	D_6
τ	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$
ρ	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 \in \{0, \frac{1}{2}\}$	$\rho_1 = 0$	$\rho_1 \in \{0, \frac{1}{2}\}$

(5.3.9)

Note that all groups occurring in (5.3.9) are solvable. This is clear for the Abelian ones. For the dihedral groups D_n it follows from the fact that the subgroup $\mathbb{Z}_n$ of rotations in D_n is a normal subgroup with Abelian factor $D_n/\mathbb{Z}_n \cong \mathbb{Z}_2$. The finite reflection groups among the groups listed in (5.3.9) are $\mathbb{Z}_2(R), D_2, D_3(R), D_4$, and D_6. These are better known as Weyl groups of the semisimple Lie algebras $A_1, A_1 \oplus A_1, A_2, B_2$, and G_2, respectively.

As mentioned in sections 5.1 and 5.2, discrete torsion gives additional degrees of freedom to a G orbifold parametrized by $H^2(G, U(1))$. With (4.2.4) one checks

$$H^2(\mathbb{Z}_M, U(1)) = \{0\}, \quad H^2(D_M, U(1)) = \mathbb{Z}_M. \tag{5.3.10}$$

5.3.2 One loop partition functions of crystallographic orbifolds

Let us now give the construction for the sixteen different types of orbifold conformal field theories with $c = 2$ corresponding to compactification on T^2/G, G taken from

(5.3.9) (note that the first of them, corresponding to lattice 1, is the translation group $G \cong \Lambda$ which acts trivially on T^2). This has been achieved during the stay of Sayipjamal Dulat at Bonn University, who contributed the necessary Poisson resumations as well as consistency checks via computer calculation to our joint work [DW00]*. The partition function of the G-orbifold of a toroidal conformal field theory with parameters (τ, ρ) will be denoted $Z_{G-orb}(\tau, \rho)$.
Lattices 2-5 correspond to the standard $\mathbb{Z}_M$ orbifold discussed in section 5.2.1 with partition functions given in theorem 5.2.1. The calculation for the other lattices is simple if one uses the general orbifold prescription of section 5.1, employs (5.1.7) for the non Abelian groups, and computes box by box in (5.1.3) for the Abelian subgroups. ${}_{\mathbb{1}}\underset{\mathbb{1}}{\square}$ is just the partition function of the original toroidal theory, and for $g \neq \mathbb{1}$ boxes of type ${}_{g}\underset{\mathbb{1}}{\square}$ can be determined directly: The ground states $|m_1, m_2, n_1, n_2\rangle$ are pairwise orthogonal, so the only states that give a contribution are the ones that are built by an action of creation operators on ground states corresponding to vertex operators with g–invariant charge vectors. For the R and T_R action the latter are given in (5.3.4). If all other boxes are related to those of type ${}_{g}\underset{\mathbb{1}}{\square}$ by modular transformations (5.1.4), as is the case for lattices 6, 7, 10, 13, 14, one can directly determine the entire partition function.
Otherwise there are closed modular orbits in the twisted sector, and we have additional degrees of freedom due to discrete torsion. In all cases these are orbits of boxes of the type ${}_{g}\underset{A(\pi)}{\square}$, $g \in \{R_k, T^{(\prime)}_{R_k}, \widehat{T}_{R_k}\}$, and thus belong to D_2 type subgroups of the respective crystallographic group. Hence by (5.3.10) this leaves at most the choice of one sign for each of these orbits. For their calculation, recall from section 5.2 that the twisted sector $\mathcal{H}_{A(\pi)}$ of the ordinary $\mathbb{Z}_2$ orbifold by (5.2.3) corresponds to fields φ with half integer modes and $\varphi(z=0) = \alpha_0^j$, $j \in \{1,2,3,4\}$, a $\mathbb{Z}_2$ fixed point on T^2. Assume that k of the four corresponding $\mathbb{Z}_2$ twisted ground states are eigenstates of g. There eigenvalues must agree and be ± 1 in order for the $\mathbb{Z}_2$ action on the twisted sector to be well defined. Since by (5.2.4) the twisted ground states have dimensions $(h; \overline{h}) = (1/8; 1/8)$, we find

$$
\begin{aligned}
{}_{g}\underset{A(\pi)}{\square} &= \mathrm{tr}_{\mathcal{H}_{A(\pi)}} \left(g q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right) \\
&= \pm k \cdot (q\overline{q})^{-\frac{1}{12}} \frac{(q\overline{q})^{\frac{1}{8}}}{\prod_{n=1}^{\infty} (1 - q^{n-1/2})(1 - \overline{q}^{n-1/2})(1 + q^{n-1/2})(1 + \overline{q}^{n-1/2})} \\
&= \pm k \left| \frac{\eta^2}{\vartheta_3 \vartheta_4} \right| = \pm \frac{k}{2} \left| \frac{\vartheta_2}{\eta} \right| .
\end{aligned}
$$

All in all the modular orbit of ${}_{g}\underset{A(\pi)}{\square}$ is equal to $\pm 2k Z_{Ising}$, where Z_{Ising} denotes the partition function (3.3) of the Ising model, and $k \in \{0, 2, 4\}$, since g must map

*The interested reader can find a copy of the respective sections of [DW00] in Sayipjamal Dulat's PhD thesis.

twisted ground states onto twisted ground states. The correct factor k has to be determined separately in each case. For $g = T_R^{(l)}$ or $g = \widehat{T}_R$ no $\mathbb{Z}_2$ fixed points are g invariant, so clearly $k = 0$. For $g = R$ the result depends on the values of τ_1 and ρ_1. In case $\tau_1 = \rho_1 = 0$ the original toroidal theory decomposes into a tensor product of two $c = 1$ theories. The action of D_2 respects the product structure, hence

$$Z_{D_2^+-orb}(0, \tau_2, 0, \rho_2) = Z^{c=1}_{orb}(\sqrt{\tau_2\rho_2})\, Z^{c=1}_{orb}(\sqrt{\rho_2/\tau_2}), \tag{5.3.11}$$

where $Z^{c=1}_{orb}(r)$ is the $\mathbb{Z}_2$ orbifold partition function of a single boson compactified on a circle of radius r,

$$Z^{c=1}_{orb}(r) = \frac{1}{2}\left(Z^{c=1}(r) + 2\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right| \right), \tag{5.3.12}$$

see [EGRS87, Sal87, Yan87]. $Z^{c=1}(r)$ is a special case of (4.1.7),

$$Z^{c=1}(r) := \frac{1}{|\eta|^2} \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2}. \tag{5.3.13}$$

One now checks from (5.3.11) that in this case $k = 4$, in agreement with the geometric observation that all the four $\mathbb{Z}_2$ fixed points on T^2 are invariant under the R actions. For $\tau_1 = 1/2, \rho_1 = 0$ one can argue that only two of the four fixed points are invariant, thus $k = 2$. If $\rho_1 = 1/2$, this geometric argument breaks down since, as noted in our general discussion in section 5.3.1, in this case the symmetries R_1, R_2 effectively shift the B–field by an integer form. The correct factor for $\tau_1 = 0, \rho_1 = 1/2$ is $k = 2$, as well. This follows from the construction of the D_4 orbifold conformal field theory (lattice 15), where one sees that the D_2 orbifold at $\tau_1 = 0, \rho_1 = 1/2$ must always contain an even number of fields with dimensions $h = \overline{h} = 1/16$. For $\tau_1 = \rho_1 = 1/2$ we find $k = 0$. This follows from the fact that $1\underset{R_1}{\square}$ by the result for the R–orbifold generically does not get any contributions from fields with dimensions $h = \overline{h} = 1/16$. Hence $A(\pi)\underset{R_1}{\square} = \pm\frac{k}{2}\left|\frac{\vartheta_3}{\eta}\right|$ cannot give such contributions either. We stress that we have been discussing a perhaps counterintuitive effect of "turning on the B–field": The action of R_1, R_2 on twisted ground states depends severely on the value of ρ_1. In particular, they must not be interpreted from a purely geometric point of view.

With the above, all D_2 type orbifolds can be performed (lattices 8, 9, 11, 12). The dihedral groups D_4 and D_6 have D_2 type subgroups, where discrete torsion occurs as well. Concerning lattice 15, the maximal Abelian subgroups of D_4 are $\mathbb{Z}_4$, $D_2 = \{1, A(\pi), R_1, R_2\}$, and $D_2' = \{1, A(\pi), A(\pi/2)R_1, A(\pi/2)R_2\}$. The two order four groups D_2 and D_2' give different contributions to the partition function, since these groups are not conjugate in D_4. The fundamental cells of lattice 15 we have to pick in order to interpret them as reflections along the edges of the cell have different shape. For D_2 it is a unit square giving a contribution $Z_{D_2-orb}(\tau = i, \rho)$, whereas for D_2' it is a rhombus giving a contribution $Z_{D_2-orb}(\tau = \frac{1}{2} + \frac{i}{2}, \rho)$. As to discrete torsion, both subgroups must contribute with the same factor* since $H^2(D_4, U(1)) \cong \mathbb{Z}_4$ by (5.3.10) does not contain $\mathbb{Z}_2 \times \mathbb{Z}_2$. Lattice 16 is treated analogously. In the case of lattice 17, the maximal Abelian subgroups of D_6 are $\mathbb{Z}_6$, and three groups of type D_2, namely $\{1, A(\pi), R_1, R_2\}$, $\{1, A(\pi), A(\pi/3)R_1, A(\pi/3)R_2\}$, $\{1, A(\pi), A(2\pi/3)R_1, A(2\pi/3)R_2\}$. These order four groups give identical contributions to the partition function since they are conjugate in D_6. This also means that in order for the action of D_6 on the twisted sector to be well defined, discrete torsion must be the same for all the three of them.

We will now list the results for all one–loop partition functions of crystallographic orbifolds that are not $\mathbb{Z}_M$ orbifolds and thus were given in theorem 5.2.1; for more details see [DW00]. In general, if $\rho_1 = \tau_1$, we set $r := \sqrt{\rho_2/\tau_2}, r' := \sqrt{\rho_2\tau_2}$, and if $\rho_1 \neq \tau_1$, we have $r := \sqrt{2\rho_2/\tau_2}, r' := \sqrt{2\rho_2\tau_2}$. By (5.3.2), an exchange of R_1 and R_2 is equivalent to exchanging r and r'. Moreover, we set $h^\pm_{mn} := \frac{1}{4}(\frac{n}{r} \pm mr)^2$.

$$
\begin{aligned}
Z_{R_1-orb}(0,\tau_2,0,\rho_2) &= Z^{c=1}(r) Z^{c=1}_{orb}(r'),\\
Z_{R_1-orb}(0,\tau_2,\frac{1}{2},\rho_2) &= \frac{1}{2}\Bigg(Z + \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n} q^{2h^+_{mn}} \overline{q}^{2h^-_{mn}}\\
&\quad + \left|\frac{\vartheta_3\vartheta_2}{2\eta^4}\right| \sum_{m,n} q^{\frac{1}{2}h^+_{mn}} \overline{q}^{\frac{1}{2}h^-_{mn}}\\
&\quad + \left|\frac{\vartheta_4\vartheta_2}{2\eta^4}\right| \sum_{m,n} (-1)^{mn} q^{\frac{1}{2}h^+_{mn}} \overline{q}^{\frac{1}{2}h^-_{mn}} \Bigg),\\
Z_{R_1-orb}(\frac{1}{2},\tau_2,0,\rho_2) &= Z_{R_1-orb}(0,\tau_2,\frac{1}{2},\rho_2)\\
Z_{R_1-orb}(\frac{1}{2},\tau_2,\frac{1}{2},\rho_2) &= \frac{1}{2}\Bigg(Z + \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \Bigg(\sum_{m,n\in\mathbb{Z}} - \sum_{m,n\in\mathbb{Z}+1/2} \Bigg) q^{4h^+_{mn}} \overline{q}^{4h^-_{mn}}\\
&\quad + \left|\frac{\vartheta_3\vartheta_2}{2\eta^4}\right| \sum_{m,n\in\mathbb{Z}, m+n\equiv 1(2)} q^{\frac{1}{4}h^+_{mn}} \overline{q}^{\frac{1}{4}h^-_{mn}}\\
&\quad + \left|\frac{\vartheta_4\vartheta_2}{2\eta^4}\right| \sum_{m,n\in\mathbb{Z}, m+n\equiv 1(2)} i^{nm} q^{\frac{1}{4}h^+_{mn}} \overline{q}^{\frac{1}{4}h^-_{mn}} \Bigg),\\
Z_{D_2^{\pm}-orb}(\tau_1,\tau_2,\rho_1,\rho_2) &= \frac{1}{2}(Z_{\mathbb{Z}_2-orb} + Z_{R_1-orb} + Z_{R_2-orb} \pm k Z_{Ising} - Z),\\
&\quad k = 4(1-\tau_1-\rho_1), \quad \tau_1,\rho_1 \in \{0,\frac{1}{2}\},\\
Z_{T_{R_1}-orb}(0,\tau_2,0,\rho_2) &= \frac{1}{2}\Bigg(Z + \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n} (-1)^n q^{h^+_{mn}} \overline{q}^{h^-_{mn}}\\
&\quad + \left|\frac{\vartheta_3\vartheta_2}{\eta^4}\right| \sum_{n\in\mathbb{Z}, m\in\mathbb{Z}+1/2} q^{h^+_{mn}} \overline{q}^{h^-_{mn}}
\end{aligned}
$$

*We remark that we have unfortunately made this observation only after the publication of [DW00], where accordingly the corresponding statements are wrong.

$$
\begin{aligned}
&+\left|\frac{\vartheta_4\vartheta_2}{\eta^4}\right|\sum_{n\in\mathbb{Z},m\in\mathbb{Z}+1/2}(-1)^n q^{h^+_{mn}}\overline{q}^{h^-_{mn}}\Bigg),\\
Z_{D_2(T_R)-orb}(0,\tau_2,0,\rho_2) &= \frac{1}{2}(Z_{\mathbb{Z}_2-orb}+Z_{R_2-orb}+Z_{T_{R_1}-orb}-Z),\\
Z_{D_2(T'_R)-orb}(0,\tau_2,0,\rho_2) &= \frac{1}{2}(Z_{\mathbb{Z}_2-orb}+Z_{T_{R_1}-orb}+Z_{T_{R_2}-orb}-Z),\\
Z_{D_3(R_1)-orb}(1/2,\sqrt{3}/2,\rho_1,\rho_2) &= \frac{1}{2}\left(Z_{\mathbb{Z}_3-orb}+2Z_{R_1-orb}-Z\right),\\
Z_{D_3(R_2)-orb}(1/2,\sqrt{3}/2,\rho_1,\rho_2) &= \frac{1}{2}\left(Z_{\mathbb{Z}_3-orb}+2Z_{R_2-orb}-Z\right),\\
&\qquad \rho_1\in\{0,\frac{1}{2}\},\\
Z_{D_4^\pm-orb}(0,1,0,\rho_2) &= \frac{1}{2}\Big(Z_{\mathbb{Z}_4-orb}(0,1,0,\rho_2)+Z_{D_2^\pm-orb}(0,1,0,\rho_2)\\
&\quad +Z_{D_2^\pm-orb}(1/2,1/2,0,\rho_2)-Z_{\mathbb{Z}_2-orb}(0,1,0,\rho_2)\Big),\\
Z_{D_4^\pm-orb}(0,1,1/2,\rho_2) &= \frac{1}{2}\Big(Z_{\mathbb{Z}_4-orb}(0,1,1/2,\rho_2)+Z_{D_2^\pm-orb}(0,1,1/2,\rho_2)\\
&\quad +Z_{D_2-orb}(1/2,1/2,1/2,\rho_2)-Z_{\mathbb{Z}_2-orb}(0,1,1/2,\rho_2)\big),\\
Z_{D_4(T'_R)^\pm-orb}(0,1,0,\rho_2) &= \frac{1}{2}\left(Z_{\mathbb{Z}_4-orb}(0,1,0,\rho_2)+Z_{D_2(T'_R)-orb}(0,1,0,\rho_2)\right.\\
&\quad +Z_{D_2^\pm-orb}(1/2,1/2,0,\rho_2)-Z_{\mathbb{Z}_2-orb}(0,1,0,\rho_2)\Big),\\
Z_{D_6^{(\pm)}-orb}(1/2,\sqrt{3}/2,\rho_1,\rho_2) &= \frac{1}{12}(6Z_{\mathbb{Z}_6-orb}+3(4Z_{D_2^{(\pm)}-orb}-2Z_{\mathbb{Z}_2-orb}))\\
&= \frac{1}{2}(Z_{\mathbb{Z}_6-orb}+2Z_{D_2^{(\pm)}-orb}-Z_{\mathbb{Z}_2-orb}),\\
&\qquad \rho_1\in\{0,\frac{1}{2}\}.
\end{aligned}
\tag{5.3.14}
$$

Here, Z denotes the original toroidal partition function (4.3.4), the partition functions $Z_{\mathbb{Z}_M-orb}$ of the $\mathbb{Z}_M$ orbifolds were given in theorem 5.2.1, and Z_{Ising} is the partition function (3.3) of the Ising model.

5.4 Orbifolds involving the spacetime fermion number operator

The orbifolds discussed in this section are all constructed by the general procedure described in section 5.1 and are not motivated geometrically. Namely, they involve the SPACETIME FERMION NUMBER OPERATOR $(-1)^{F_S}$, which is defined to act trivially on the Neveu–Schwarz sector of a superconformal field theory and by multiplication with -1 on the Ramond sector. We will see, however, that we can always translate into settings where either geometric interpretations are at hand or everything can be reduced to the Ising model.

In the case of an $N=(2,2)$ superconformal theory which is invariant under spectral flow, the operators of spectral flow are Ramond fields and so are projected out if we perform an orbifold involving $(-1)^{F_S}$. In other words, spacetime supersymmetry is broken and since the entire $N=(2,2)$ superconformal algebra is invariant under $(-1)^{F_S}$ we are able to construct examples of superconformal field theories which are *not invariant* under spectral flow. Though for all theories constructed below the charges $(Q;\overline{Q})$ obey $Q-\overline{Q}\in\mathbb{Z}$, the condition of $Q-\overline{Q}$ being even for bosonic fields is violated. In particular, the worldsheet fermion number operator $(-1)^F$ may not be identified with $e^{\pi i(J_0-\overline{J}_0)}$.

In [DGH88] all orbifolds of $N=(1,1)$ toroidal superconformal field theories with $c=\frac{3}{2}$ have been constructed that apart from geometric symmetries involve the spacetime fermion number operator. It is easy to generalize the analysis to arbitrary dimensions, and we will use the notations introduced in [DGH88] to do so. The $\mathbb{Z}_2\times\mathbb{Z}_2$ gradings of the resulting orbifold conformal field theories, which are not given in [DGH88], can also be determined in general, since the $\widetilde{R}$ part of the partition function must transform covariantly under modular transformations, as was remarked as a comment to (3.1.9).

By definition, $(-1)^{F_S}$ leaves invariant spacetime fermions with twisted boundary conditions in space direction of the worldsheet torus $\Xi(\sigma)$. Since modular transformations permute the boundary conditions of the fermions, the theory obtained by modding out $(-1)^{F_S}$ from a superconformal field theory $\mathcal{C}$ will have the same space $\mathcal{H}=\mathcal{H}^{NS}\oplus\mathcal{H}^R$ of states as the original, but in $\mathcal{H}^R$ the sectors $\mathcal{H}^R_b$ and $\mathcal{H}^R_f$ are interchanged. On the level of partition functions, only $Z_{\widetilde{R}}$ changes its sign. Since by our general approach to superconformal field theories in chapter 3 the sectors $\mathcal{H}^R_b$ and $\mathcal{H}^R_f$ are on equal footing, this renaming of bosonic into fermionic Ramond fields can always be done consistently. If $\mathcal{C}=\mathcal{T}(\Lambda,B)$ is a toroidal superconformal field theory, as we will assume for the rest of this section, we even have $\mathcal{H}^R_b\cong\mathcal{H}^R_f$ (theorem 4.1.4) as a remnant of the order–disorder duality of the critical Ising model. In the $N=(2,2)$ case where $c=3d/2, d/2\in\mathbb{N}$, the Ramond fields in the orbifold $\mathcal{C}/(-1)^{F_S}$ will carry different charges from the ones in $\mathcal{C}$, however, but in this case we can interpret the orbifold geometrically. Namely, the Dirac fermions in $\mathcal{C}_f$ (notations as in definition 4.1.3) can be bosonized, and the operator $(-1)^{F_S}$ acts as a shift on the charge lattice of the resulting toroidal theory. Since the shift is left–right symmetric, level matching conditions are satisfied automatically. The Ising and the geometric interpretation of the fermion number operator ensures that all orbifolds discussed below can indeed be explicitly constructed.

Let us now combine $(-1)^{F_S}$ with other symmetries of toroidal superconformal field theories. Firstly, consider a shift symmetry T_Δ with $\Delta=(p_l;p_r)(0,\delta), D\delta\in\Lambda$ for some $D\in\mathbb{N}$ as discussed in (5.1.5). Set $S_\delta:=(-1)^{F_S}T_\Delta$, then since S_δ^2 is a pure shift symmetry and generates a normal subgroup of $\mathrm{span}\{S_\delta\}$, we can perform the S_δ orbifold stepwise and assume $2\delta\in\Lambda$ from now on. The corresponding orbifold conformal field theory is called SUPER–AFFINE ORBIFOLD with partition function $Z^\Gamma_{s-a}, \Gamma:=\Gamma(\Lambda,B)$. We use the notations of (5.1.5) and remark that modular transformations (2.1.5) act by

$$S : \left\{ \begin{array}{lcl} Z_{\Gamma_0^+} & \mapsto & \frac{1}{2}\left(Z_{\Gamma_0^+} + Z_{\Gamma_0^-} + Z_{\Gamma_\Delta^+} + Z_{\Gamma_\Delta^-}\right) \\ Z_{\Gamma_0^-} & \mapsto & \frac{1}{2}\left(Z_{\Gamma_0^+} + Z_{\Gamma_0^-} - Z_{\Gamma_\Delta^+} - Z_{\Gamma_\Delta^-}\right) \\ Z_{\Gamma_\Delta^+} & \mapsto & \frac{1}{2}\left(Z_{\Gamma_0^+} - Z_{\Gamma_0^-} + Z_{\Gamma_\Delta^+} - Z_{\Gamma_\Delta^-}\right) \\ Z_{\Gamma_\Delta^-} & \mapsto & \frac{1}{2}\left(Z_{\Gamma_0^+} - Z_{\Gamma_0^-} - Z_{\Gamma_\Delta^+} + Z_{\Gamma_\Delta^-}\right) \end{array} \right. , \quad T : \left\{ \begin{array}{lcl} Z_{\Gamma_0^\pm} & \mapsto & Z_{\Gamma_0^\pm}, \\ Z_{\Gamma_\Delta^\pm} & \mapsto & \pm Z_{\Gamma_\Delta^\pm}. \end{array} \right.$$

Then it is easy to determine the entire partition function from

$$\underset{\mathbb{1}}{{}_{S_\delta}\square} = \left(Z_{\Gamma_0^+} - Z_{\Gamma_0^-}\right) \frac{1}{2} \left(- \left|\frac{\vartheta_1(z)}{\eta}\right|^d - \left|\frac{\vartheta_2(z)}{\eta}\right|^d + \left|\frac{\vartheta_3(z)}{\eta}\right|^d + \left|\frac{\vartheta_4(z)}{\eta}\right|^d \right)$$

by modular transformations (as usual, set $z = 0$ for odd d). Moreover, since $Z^{\widetilde{R}}_{s-a}$ must transform covariantly under modular transformations,

$$\begin{array}{lcl} Z^{NS}_{s-a} & = & \frac{1}{2}\left(Z_{\Gamma_0^+} + Z_{\Gamma_\Delta^-}\right)\left|\frac{\vartheta_3(z)}{\eta}\right|^d, \quad Z^{\widetilde{NS}}_{s-a} \;=\; \frac{1}{2}\left(Z_{\Gamma_0^+} - Z_{\Gamma_\Delta^-}\right)\left|\frac{\vartheta_4(z)}{\eta}\right|^d, \\ Z^{R}_{s-a} & = & \frac{1}{2}\left(Z_{\Gamma_0^-} + Z_{\Gamma_\Delta^+}\right)\left|\frac{\vartheta_2(z)}{\eta}\right|^d, \quad Z^{\widetilde{R}}_{s-a} \;=\; \frac{1}{2}\left(Z_{\Gamma_0^-} - Z_{\Gamma_\Delta^+}\right)\left|\frac{\vartheta_1(z)}{\eta}\right|^d. \end{array}$$

If we combine $(-1)^{F_S}$ with the generator of an ordinary $\mathbb{Z}_M$ orbifold (5.2.2), by the same reasoning as above we can restrict ourselves to the case $M = 2$. The resulting $\mathbb{Z}_2$ type orbifold by $S_R := (-1)^{F_S} \cdot (-\mathbb{1})$ is called ORBIFOLD PRIME with partition function $Z^{\Gamma}_{orb'}$. By analogous calculations as in the superaffine case, we find

$$\begin{array}{lcl} Z^{NS}_{orb'} & = & \frac{1}{2}\left(Z_\Gamma \left|\frac{\vartheta_3(z)}{\eta}\right|^d + \left|\frac{\vartheta_3\vartheta_4\vartheta_4(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_3\vartheta_2(z)}{\eta^3}\right|^d - \left|\frac{\vartheta_2\vartheta_4\vartheta_1(z)}{\eta^3}\right|^d\right), \\ Z^{\widetilde{NS}}_{orb'} & = & \frac{1}{2}\left(Z_\Gamma \left|\frac{\vartheta_4(z)}{\eta}\right|^d + \left|\frac{\vartheta_3\vartheta_4\vartheta_3(z)}{\eta^3}\right|^d - \left|\frac{\vartheta_2\vartheta_3\vartheta_1(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_4\vartheta_2(z)}{\eta^3}\right|^d\right), \\ Z^{R}_{orb'} & = & \frac{1}{2}\left(Z_\Gamma \left|\frac{\vartheta_2(z)}{\eta}\right|^d - \left|\frac{\vartheta_3\vartheta_4\vartheta_1(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_3\vartheta_3(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_4\vartheta_4(z)}{\eta^3}\right|^d\right), \\ Z^{\widetilde{R}}_{orb'} & = & -\frac{1}{2}\left(Z_\Gamma \left|\frac{\vartheta_1(z)}{\eta}\right|^d + \left|\frac{\vartheta_3\vartheta_4\vartheta_2(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_3\vartheta_4(z)}{\eta^3}\right|^d + \left|\frac{\vartheta_2\vartheta_4\vartheta_3(z)}{\eta^3}\right|^d\right). \end{array}$$

Note that in general this is not just the $(-1)^{F_S}$ orbifold of the $\mathbb{Z}_2$ orbifold, but comparison with theorem 5.2.2 shows

$$Z^{\Gamma}_{orb'}(\sigma, z = 0) = Z^{\Gamma}_{\mathbb{Z}_2-orb}(\sigma, z = 0) - 3 \cdot 2^{d-1}. \tag{5.4.1}$$

Finally, let us discuss possible combinations of symmetries S_δ, S_R, and generators of $\mathbb{Z}_M$, $M \in \{2, 3, 4, 6\}$. Since the group generated by S_R and S_δ contains a purely geometric normal subgroup span$\{S_\delta \cdot S_R\}$ of index 2, the corresponding orbifold can be obtained stepwise by a geometric symmetry and by $(-1)^{F_S}$. The group generated by S_R and a generator of the standard $\mathbb{Z}_M$ action (5.2.2) will neither produce anything new, since for even M we get $(-1)^{F_S}$ applied to the ordinary $\mathbb{Z}_M$ orbifold, and for odd M it is generated by a single symmetry of type S_R which was discussed above. The only nontrivial combination is a $\mathbb{Z}_M \times \mathbb{Z}_2$ type group containing the ordinary $\mathbb{Z}_M$ and the additional generator S_δ. This type of orbifold is called SUPER-M-ORBIFOLD, and the partition function is

$$Z^\Gamma_{s-M-orb} = \tfrac{1}{2M}\left(2Z^\Gamma_{s-a} + MZ^{NS}_{\mathbb{Z}_M-orb} + MZ^{\widetilde{NS}}_{\mathbb{Z}_M-orb} + MZ^{R}_{\mathbb{Z}_M-orb} - 2Z_\Gamma\right).$$

5.5 The generalized GSO projection

It is known from Calabi–Yau compactification, and we have also seen in section 3.1.5, that those $N=(2,2)$ superconformal field theories are particularly simple which have all integer charges on the left and right handed side. To be consistent with theorem 3.1.4 we have to relax this condition to hold only in the NS sector and require $Q \pm \frac{c}{6}, \overline{Q} \pm \frac{c}{6} \in \mathbb{Z}$ in the R sector. In particular, we must have $c = 3d/2, d/2 \in \mathbb{N}$, which is assumed throughout this section. Analogously to the proof of theorem 3.1.4 we can argue that the above condition on the charges is equivalent to the theory being invariant under all combinations $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}, U_{\pm\frac{1}{2}}\overline{U}_{\mp\frac{1}{2}}$ of spectral flows, i.e. these operators being realized as fields in the theory. The aim is to associate a theory where this is the case to every superconformal field theory by an orbifold procedure. Unfortunately we will have to make additional assumptions on our theories in order for this idea to work.

Definition 5.5.1
Let $\mathcal{C}$ denote an $N=(2,2)$ superconformal field theory with central charge $c = 3d/2, d/2 \in \mathbb{N}$, such that all left and right charges are multiples of a fixed fraction $\frac{1}{M}$, $M \in \mathbb{N}$. Then by GENERALIZED GSO PROJECTION* *we mean the orbifold procedure performed with the group G_{GSO} which is generated by*

$$\zeta_M := \begin{cases} e^{2\pi i J_0} & \text{on } \mathcal{H}^{NS} \\ e^{\pi i \frac{c}{3}} e^{2\pi i J_0} & \text{on } \mathcal{H}^{R} \end{cases}, \qquad \kappa_M := \begin{cases} e^{\pi i (J_0 - \overline{J}_0)} & \text{on } \mathcal{H}_b \\ -e^{\pi i (J_0 - \overline{J}_0)} & \text{on } \mathcal{H}_f \end{cases}.$$

We claim that under appropriate assumptions on $\mathcal{C}$, the generalized GSO projection can be used to achieve our above aim:

Theorem 5.5.2
Suppose $\mathcal{C}$ is an $N=(2,2)$ superconformal field theory with central charge $c = 3d/2, d/2 \in \mathbb{N}$, such that the set of charges $Q, \overline{Q}$ occurring in $\mathcal{C}$ does not have an accumulation point. Then by the generalized GSO projection we can construct an $N=(2,2)$ superconformal field theory $\widetilde{\mathcal{C}} = \mathcal{C}/G_{GSO}$ which is invariant under all combinations of spectral flow.

* "GSO" refers to Gliozzi, Scherk and Olive [GSO77] who proposed a similar projection to arrive at tachyon free consistent string theories.

Proof:
Since by assumption the set of charges $Q, \overline{Q}$ occurring in $\mathcal{C}$ does not have an accumulation point and on the other hand forms a lattice in $\mathbb{R}$, we see that there exists an integer $M \in \mathbb{N}$ such that all charges are multiples of $\frac{1}{M}$. Thus the group G_{GSO} of definition 5.5.1 is a well defined finite group, actually a subgroup of $\mathbb{Z}_{2M} \times \mathbb{Z}_{2M}$. The orbifold by G_{GSO} is well defined, since G_{GSO} can be generated by the left–right (anti–)symmetric $\zeta_M \kappa_M^{-1}, \kappa_M$, and thus level matching conditions are automatically satisfied. By construction, $\mathcal{C}/G_{GSO}$ is invariant under G_{GSO}, which directly translates into the conditions on the charges that we know are equivalent to invariance under all combinations of spectral flow. □

At the end of section 5.1 we have remarked that orbifolds by solvable groups can always be rescinded by an orbifold of the same type. Hence we can use theorem 5.5.2 for a first result towards a classification of $N = (2,2)$ superconformal field theories:

Theorem 5.5.3
Let $\mathcal{C}$ denote an $N = (2,2)$ superconformal field theory with central charge $c = 3d/2, d/2 \in \mathbb{N}$ such that the set of charges $Q, \overline{Q}$ occurring in $\mathcal{C}$ does not have an accumulation point. Then $\mathcal{C}$ can be obtained as orbifold conformal field theory $\mathcal{C} = \widetilde{\mathcal{C}}/G$ from a theory $\widetilde{\mathcal{C}}$ which is invariant under all combinations of spectral flow. The group G is a product of cyclic groups.
If $d = 2$, then $\widetilde{\mathcal{C}}$ is a toroidal superconformal field theory.

Proof:
Only the assertion on the case $d = 2$ remains to be shown. Since $\widetilde{\mathcal{C}}$ is an $N = (2,2)$ superconformal field theory with $c = 3$ that is invariant under all combinations of spectral flow, in particular the operators $U_1 = e^{iH}, \overline{U}_1 = e^{i\overline{H}}$ of (3.1.7) are realized as fields in the theory. They have quantum numbers $(h, Q; \overline{h}, \overline{Q}) = (\frac{1}{2}, 1; 0, 0)$ and $(h, Q; \overline{h}, \overline{Q}) = (0, 0; \frac{1}{2}, 1)$, respectively. Thus $\psi^1 = U_1$ and $\overline{\psi}^1 = \overline{U}_1$ are free Dirac fermions, and their superpartners $j^1, \overline{j}^1$ give $U(1)$ currents. Since $c = 3$, by definition 4.1.4 this suffices to show that $\widetilde{\mathcal{C}}$ is a toroidal superconformal field theory. □

Unfortunately, it is not clear how strong the condition on the charges in theorem 5.5.3 really is. Theorem 5.5.3 is obviously important, since all theories that satisfy the conditions asserted for $\widetilde{\mathcal{C}}$ are believed to have a nonlinear sigma model interpretation in terms of string compactification. On the other hand it is not very handy, since the GSO projection is explicitly not motivated by geometry, so for its inverse we cannot expect anything better. Nevertheless, the situation we find for the known examples gives reason to hope:

Theorem 5.5.4
Suppose that $\mathcal{C}$ is an $N = (2,2)$ superconformal field theory with central charge $c = 3d/2, d/2 \in \mathbb{N}$ odd, obtained as orbifold $\mathcal{C} = \widetilde{\mathcal{C}}/G$ from a toroidal superconformal field theory $\widetilde{\mathcal{C}} = \mathcal{T}(\Lambda, B)$. Moreover assume that G is a product of cyclic groups $\mathbb{Z}_M$, acting either asstandard $\mathbb{Z}_M$ (theorem 5.2.2) or superaffine, orbifold–

prime or super–M– orbifold as described in section 5.4. Then the generalized GSO projection reconstructs the toroidal theory $\widetilde{\mathcal{C}}$ from $\mathcal{C}$.

Proof:
It suffices to show that the generalized GSO projection inverts the standard $\mathbb{Z}_M$, the orbifold prime and the superaffine orbifold separately. Our assumptions on $\mathcal{C}$ are such that the generalized GSO projection is well defined in any case.
Since by theorem 5.2.2 the $\mathbb{Z}_M$ orbifold conformal field theories are invariant under spectral flow, κ_M of definition 5.5.1 acts trivially on $\widetilde{\mathcal{C}}/\mathbb{Z}_M$. Let γ denote a generator of $\mathbb{Z}_M$, then to show theorem 5.2.2 we argued that for a single Dirac fermion $\psi_\pm^{(k)}$ twisted by γ^m the Ramond ground states correspond to Neveu–Schwarz states with left and right handed charge $(-1)^k \frac{m}{M}$. Since $\frac{c}{3} = \frac{d}{2}$ was assumed to be odd, the total charge of a Ramond ground state in $\mathcal{H}_{\gamma^m}$ is $Q = \overline{Q} = \frac{m}{M} - \frac{c}{6}$. Hence $G_{GSO} \cong \mathbb{Z}_M$ acts by multiplication with $e^{\frac{2\pi i m}{M}}$ on $\mathcal{H}_{\gamma^m}$ and thus indeed inverts the standard $\mathbb{Z}_M$ orbifold.
For the orbifold prime and the superaffine orbifold it follows from our results of section 5.4 that only (half) integer charges occur in the respective sectors, but the condition of $Q - \overline{Q} \in 2\mathbb{Z}$ exactly for bosonic states is violated. Thus only κ_1 of definition 5.5.1 acts nontrivially on these orbifold conformal field theories, $G_{GSO} \cong \mathbb{Z}_2$. It is straightforward to check that κ_1 acts by multiplication with -1 exactly in the twisted sectors of the orbifold prime and superaffine orbifold and so inverts these orbifolds, as asserted. $\Box$

The following observation is reassuring: In section 5.3 we have constructed various orbifolds of toroidal conformal field theories with central charge $c = 2$ by non Abelian groups. The action of the corresponding crystallographic group can be extended to toroidal superconformal field theories with central charge $c = 3$ such that the fermionic fields are treated equally as their bosonic superpartners. But now one checks that none of the non Abelian groups leaves invariant the generators $G^\pm, J$ of the superconformal algebra (3.1.1), basically since the reflection symmetry (5.3.1) does not. This is in accordance with theorem 5.5.3.

5.6 Gepner models revisited

In theorem 3.1.16 we have seen that Gepner models exactly satisfy the conditions on charges that make theories as simple as required in section 5.5. It is therefore natural to ask whether Gepner models in turn can be constructed as orbifolds of other $N = (2, 2)$ superconformal field theories. Such a construction indeed is the more common one [Gep87, Gep88, GVW89], as we will see in theorem 5.6.1 below. Some knowledge on the discrete symmetries of $N = (2, 2)$ minimal models and orbifolds of their tensor products will facilitate the discussion.
The minimal model (k) inherits a $\mathbb{Z}_{k+2}$ symmetry from the parafermionic subtheory whose generator acts by

$$\Phi^l_{m,s;\overline{m},\overline{s}} \longmapsto e^{\frac{2\pi i}{2(k+2)}(m+\overline{m})}\, \Phi^l_{m,s;\overline{m},\overline{s}}. \tag{5.6.1}$$

Hence the tensor product $(k_1) \otimes \cdots \otimes (k_r)$ possesses an Abelian symmetry group $\prod_{j=1}^r \mathbb{Z}_{k_j+2}$, whose elements are denoted $a = [a_1, \ldots, a_r]$. To study the corresponding orbifolds of $(k_1) \otimes \cdots \otimes (k_r)$ (see also [GP90]) it proves convenient to introduce a scalar product on $\prod_{j=1}^r \mathbb{Z}_{k_j+2}$ by

$$a, b \in \prod_{j=1}^r \mathbb{Z}_{k_j+2}: \quad a \bullet b := \sum_{j=1}^r \frac{a_j b_j}{k_j + 2}.$$

Then the level matching conditions for an orbifold by $H := \mathrm{span}(a) \subset \prod_{j=1}^r \mathbb{Z}_{k_j+2}$ read $Aa \bullet a \in \mathbb{Z}$, if $H \cong \mathbb{Z}_A$ [GP90]. Using the fusion rules (3.1.18) as well as (3.1.17) one checks that a (Ramond) Neveu–Schwarz field $\otimes_{j=1}^r \Phi^{l_j}_{m_j,s_j;\overline{m}_j,\overline{s}_j}$ is invariant under a iff it is (semi–)local to $V^a := \otimes_{j=1}^r \Phi^0_{a_j,a_j;-\overline{a}_j,-\overline{a}_j}$. To be able to directly include the twisted sectors into the discussion, here we have assumed arbitrary left–right coupling. Namely, since it is local to all fields in the theory, by property 8 for our conformal field theories (section 2.1) $(k_1) \otimes \cdots \otimes (k_r)/H$ automatically contains the field V^a. Note that this is a SIMPLE CURRENT in the sense of [SY89b, SY89a, SY90], i.e. fusion with V^a always produces only one conformal family. In fact, it is easy to see that up to projection onto H–invariant states the twisted sector $\mathcal{H}_{a^m}$ of $(k_1) \otimes \cdots \otimes (k_r)/H$ is obtained from the original Hilbert space $\mathcal{H}$ by application of $(V^a)^m$. But this means that orbifoldizing by H is equivalent to extending the holomorphic W–algebra of $(k_1) \otimes \cdots \otimes (k_r)$ by the simple current

$$U^a := \bigotimes_{j=1}^r \Phi^0_{2a_j,2a_j;0,0}. \tag{5.6.2}$$

We remark that $\Phi^0_{b,b;0,0} = (\Phi^0_{1,1;0,0})^b$ in fact are the only holomorphic simple currents of (k) and since for any factor $\Phi^l_{m,s;\overline{m},\overline{s}}$ of a field in a consistent theory we must assume $s - \overline{s} \equiv 0 \bmod 2$ for consistency of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading, we have to assume b to be even. This leaves us with the U^a as the only candidates an extension of the W–algebra by simple currents may work with. We are now ready to show

Theorem 5.6.1
The Gepner model $(k_1) \cdots (k_r)$ of definition 3.1.15 is obtained by GSO projection (definition 5.5.1) from $(k_1) \otimes \cdots \otimes (k_r)$.

Proof:
We set $\mathcal{C} := (k_1) \otimes \cdots \otimes (k_r)/G_{GSO}$ and must show $\mathcal{C} = (k_1) \cdots (k_r)$. To keep sign errors out of the game, we restrict to the bosonic Neveu–Schwarz sector, which is legitimate since both $\mathcal{C}$ and the Gepner model are invariant under spectral flow. By (3.1.17) we find $M = \mathrm{lcm}\,\{k_i + 2, i = 1, \ldots, r\}$ in the definition 5.5.1 of G_{GSO}. Then κ_M acts trivially, since this is so on each minimal model factor. On the other hand, since G_{GSO} projects onto integer charges in the Neveu–Schwarz sector, ζ_M acts by

$$\zeta_M : \bigotimes_{j=1}^r \Phi^{l_j}_{m_j,s_j;m_j,\overline{s}_j} \longmapsto \left(\prod_{j=1}^r e^{\frac{2\pi i}{k_j+2} m_j} \right) \bigotimes_{j=1}^r \Phi^{l_j}_{m_j,s_j;m_j,\overline{s}_j}.$$

Since all fields in $(k_1) \otimes \cdots \otimes (k_r)$ have left–right symmetric quantum numbers $m_j = \overline{m}_j$, the GSO projection therefore is just an orbifold by $H = \text{span}(\beta) \subset \prod_{j=1}^r \mathbb{Z}_{k_j+2}$ with $\beta = [1, \ldots, 1]$ as discussed above. Hence $\mathcal{C}$ is obtained from $(k_1) \otimes \cdots \otimes (k_r)$ by enhancement of the holomorphic W–algebra with the operator of twofold lefthanded spectral flow $U^\beta = U_1$, which is just our construction of the Gepner model in definition 3.1.15. □

It is now clear that also the Gepner model inherits a residual Abelian symmetry group from its minimal model factors: Namely, for $(k_1) \cdots (k_r)$ we find $\mathbb{Z}_2 \times \mathcal{G}_{ab}$, where $\mathbb{Z}_2$ denotes charge conjugation and $\mathcal{G}_{ab} = (\prod_{j=1}^r \mathbb{Z}_{k_j+2})/\mathbb{Z}_M$, $M = \text{lcm}\,\{k_i + 2, i = 1, \ldots, r\}$. Here, $\mathbb{Z}_M$ acts by

$$\prod_{j=1}^r \mathbb{Z}_{k_j+2} \longrightarrow \prod_{j=1}^r \mathbb{Z}_{k_j+2}, \quad [a_1, \ldots, a_r] \longmapsto [a_1 + 1, \ldots, a_r + 1]$$

(see also [GP90]). Note that only elements of the subgroup

$$\mathcal{G}_{ab}^{alg} := \left\{ [a_1, \ldots, a_r] \in \mathcal{G}_{ab} \;\middle|\; \sum_{j=1}^r \frac{a_j}{k_j + 2} \in \mathbb{Z} \right\} \subset \mathcal{G}_{ab} \tag{5.6.3}$$

leave invariant the operators of spectral flow $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$, i.e. commute with spacetime supersymmetry. Elements of $\mathcal{G}_{ab} - \mathcal{G}_{ab}^{alg}$ describe R-symmetries [Gep87]. We directly deduce the following

Theorem 5.6.2
Consider a Gepner type model $\widetilde{\mathcal{C}} = (\widetilde{k_1}) \cdots (\widetilde{k_r})$ as in definition 3.1.17, where $\mathcal{W} \supset \mathcal{W}_{Gepner}$ is obtained by extending with simple currents $U^a, U^b, \ldots$ as in (5.6.2), and $H := \text{span}(a, b, \ldots) \subset \mathcal{G}_{ab}^{alg}$. Then $\widetilde{\mathcal{C}}$ is obtained as orbifold by H from the Gepner model $(k_1) \cdots (k_r)$.

The discussion now is fairly close to a remarkable duality on the moduli space of superconformal field theories, known as MIRROR SYMMETRY:

Theorem 5.6.3 [GP90]
Let $\mathcal{C}$ denote a Gepner model and $H := \mathcal{G}_{ab}^{alg}$. Then the theory obtained by modding out H is isomorphic to $\mathcal{C}$. More generally, assume $H \subset \mathcal{G}_{ab}^{alg}$ such that $\mathcal{C}/H$ is well defined. Then one can define a dual group

$$H^* := \left\{ b \in \mathcal{G}_{ab}^{alg} \mid a \bullet b \in \mathbb{Z} \; \forall \, a \in H \right\},$$

and $\mathcal{C}/H$ and $\mathcal{C}/H^*$ are isomorphic $N = (2,2)$ superconformal field theories, so–called MIRROR PARTNERS. *Their spectrum differs by the sign chosen for the right handed charges.*

Since by the above we obtain Gepner models by performing the GSO projection on tensor products of minimal models, it is a natural question to ask what kind of orbifold on the Gepner model will reproduce the tensor product of minimal models. At least in case of central charge $c = 3$ the answer is simple:

Theorem 5.6.4
The fermionic tensor product $(2)\otimes(2)$ $[(1)\otimes(1)\otimes(1)$ *or* $(1)\otimes(4)]$ *of* $N=(2,2)$ *minimal models has a geometric interpretation as* $\mathbb{Z}_4$ $[\mathbb{Z}_3$ *or* $\mathbb{Z}_6]$ *orbifold of the* $N=(2,2)$ *toroidal superconformal field theory with* $c=3$ *and parameters* $\tau=\rho=i$ $[\tau=\rho=e^{2\pi i/3}]$, *respectively.*

Proof:
First note that the Gepner models $(2)^2$ $[(1)^3$ or $(1)(4)]$ possess geometric interpretations, since they are just the toroidal superconformal conformal field theories with $c=3$ and parameters $\tau=\rho=i$ $[\tau=\rho=e^{2\pi i/3}]$. This is generally regarded as a well known fact, and it is easy to check that the partition functions indeed agree. Moreover, the current algebra of each of these models contains $u(1)^3$, and so they are toroidal conformal field theories by definition 4.1.1. To see that they are indeed the asserted toroidal superconformal field theories in the sense of definition 4.1.3 one has to show that the charge lattices with respect to the $u(1)^3$ current algebras agree. This is explicitly done for the case $(2)^2$ in theorem 7.3.24. For $(1)^3, (1)(4)$ it already suffices to note that both of them have left and right $u(1)\oplus su(3)$ current algebras and thereby are uniquely determined within the moduli space of toroidal superconformal field theories with $c=3$ (see theorem 6.1.1).
We now need to show that the inverse of the GSO projection on the Gepner models induces the standard $\mathbb{Z}_4$ $[\mathbb{Z}_3$ or $\mathbb{Z}_6]$ action (5.2.2) on the respective toroidal theory. It clearly is a $\mathbb{Z}_M$ type orbifold, $M=4$ [3 or 6], generated by

$$\widetilde{\zeta}_M : \bigotimes_{j=1}^{r} \Phi^{l_j}_{m_j,s_j;\overline{m}_j,\overline{s}_j} \longmapsto e^{2\pi i \frac{\overline{m}_1-m_1}{2M}} \bigotimes_{j=1}^{r} \Phi^{l_j}_{m_j,s_j;\overline{m}_j,\overline{s}_j}.$$

On $(2)^2$, $\widetilde{\zeta}_4$ leaves two of the seven $(1,0)$ fields invariant, whereas the eigenvalues $\pm i$ occur twice and -1 occurs once. The two $SU(2)$ current algebras are interchanged. The same is true for the standard $\mathbb{Z}_4$ action on the $SU(2)^2$ torus. By the explicit field identifications of theorem 7.3.24 one finds that the corresponding actions indeed are $su(2)^2\oplus u(1)$ conjugate. A similar analysis holds in the other two cases. Again, it is also easy to show that the respective partition functions agree. □

Theorem 5.6.4 matches nicely with theorem 5.5.4 but in fact is a stronger result.

Chapter 6

The moduli space of unitary conformal field theories with central charge $c = 2$

This chapter is devoted to the study of the moduli space $\mathcal{M}^2$ of unitary conformal field theories with central charge $c = 2$. Most of the results have been published in [DW00]. We use the notations for toroidal theories in two dimensions and crystallographic symmetries that were introduced in sections 4.3 and 5.3.

In section 6.1 we give a classification of all nonisolated nonexceptional orbifold components of the moduli space. The global structure of $\mathcal{M}^2$ is investigated in section 6.2, where all intersection points and lines of nonisolated nonexceptional orbifold components are determined. In section 6.3 we discuss theories obtained as tensor products of known models with central charge $c < 2$. We relate our results to those on $c = 3/2$ superconformal field theories [DGH88] and are able to interpret all the orbifolds discussed there in terms of crystallographic orbifolds. We also correct the statements on multicritical points on the moduli space of $N = (1, 1)$ superconformal field theories with $c = 3/2$ made in [DGH88]. Section 6.4 ends the chapter with a summary on the picture we have obtained so far.

6.1 Classification of orbifolds with $c = 2$

In this section we will argue why our constructions of crystallographic orbifolds of section 5.3 already suffice to determine all nonisolated nonexceptional orbifold components of the moduli space $\mathcal{M}^2$. In section 6.1.1 we discuss all symmetries of two dimensional conformal field theories we will need. In section 6.1.2 we show that they indeed suffice, and give the parameter spaces of the corresponding components of the moduli space. The proof of theorem 6.1.3 below is based on ideas of Werner Nahm's.

6.1.1 Discrete and continuous symmetries of toroidal conformal field theories

In order to determine all orbifold components of the moduli space we need to find all possible symmetries of toroidal conformal field theories $\mathcal{C} \in \mathcal{M}_2^{Narain}$. The simplest ones are those which are induced by geometric symmetries of the torus $T^2 = \mathbb{R}^2/\Lambda$ for a given geometric interpretation $\mathcal{C} = \mathcal{T}(\Lambda, B)$. All of these symmetries have been discussed in section 5.3.

One can expect to find discrete quantum symmetries without geometric interpretation on fixed points of the additional generators U, V, UV of the modular group $PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$ of $\mathcal{M}_2^{Narain}$ in (4.3.6). But it turns out that neither of them gives new orbifold components of the moduli space. Firstly, target space orientation change V induces a trivial action on toroidal conformal field theories. The fixed lines of U and UV in the Teichmüller space $\mathbb{H} \times \mathbb{H}$ of $\mathcal{M}_2^{Narain}$ have parameter spaces

$$\mathcal{E}_U := \left\{(\tau_1, t, \tau_1, t) \mid t \in \mathbb{R}^+\right\}, \quad \mathcal{E}_{UV} := \left\{(\tau_1, t, -\tau_1, t) \mid t \in \mathbb{R}^+\right\}. \tag{6.1.1}$$

Then, for mirror symmetry $g = U$ we read off an induced action $n_2 \leftrightarrow m_2$ on the charge lattice (4.3.3). Moreover, all theories in $\mathcal{E}_U$ have a righthanded $SU(2) \times U(1)$ symmetry, two of whose commuting generators are invariant under this action. Since one checks that both generic Abelian lefthanded $U(1)$ currents are invariant under the action of U as well, we find that the theory we produce by modding out U contains at least two left– and two righthanded Abelian currents and thus is a torus theory again. U therefore is $SU(2)$ conjugate to a shift on the charge lattice, which acts by multiplication with $i^{n_2 - m_2}$ on states created from the Hilbert space ground state $|m_1, m_2, n_1, n_2\rangle$. It is now a straightforward calculation to check that this shift orbifold reproduces the original theory. The case $g = UV$ is treated analogously, since $\mathcal{E}_{UV}$ is obtained from $\mathcal{E}_U$ by a parity change $(\tau, \rho) \mapsto (\tau, -\overline{\rho})$.

Classically, tori cannot have continuous symmetries beyond translations. However, at certain parameters τ, ρ it is well known that quantum effects can lead to enhanced gauge symmetry groups. This seems to be equivalent to the occurrence of additional $(1,0)$ fields that generate an affine Kac–Moody algebra in our theory, though a proof for this observation is lacking. It is not clear, when a Noether method can be applied to associate conserved currents to a global symmetry of a conformal field theory, unless the theory is described by functional integrals.

In the case of toroidal conformal field theories, however, it is not hard to determine all points in $\mathcal{M}_2^{Narain}$ where additional $(1,0)$ fields occur left–right symmetrically and generate an enhanced continuous quantum symmetry. We remark that our idea of proof is fairly similar to methods used by B. Rostand [Ros90, Ros91]. Assume that a toroidal conformal field theory with charge lattice Γ has left–right symmetrically enhanced symmetry. By $\{(\pm p_i; 0), (0; \pm p_i'), i \in \{1, \dots, d\}\} \subset \Gamma$ we denote the charge vectors corresponding to the additional vertex operators of dimensions $(1;0)$ and $(0;1)$, respectively. In particular, $|p_i|^2 = |p_i'|^2 = 2$, and since these vertex operators are pairwise local, for $i \neq j$ we may assume $p_i \cdot p_j = p_i' \cdot p_j' \in \{0, 1\}$. Then the $\mathbb{R}$–span of $\{(p_i; p_i'), i \in \{1, \dots, d\}\} \subset \Gamma$ is totally isotropic with respect to the scalar product (4.1.4). By the discussion in section 4.2 this means that we may choose a geometric interpretation (Λ, B) of our toroidal theory such that $p_i = p_i'$ for all $i \in \{1, \ldots, d\}$. Moreover, by the above restrictions on the scalar products between the p_i, these vectors generate the root lattice of a simply laced Lie group. Since the rank of this group can be at most two, the only possible groups are $A_2 = SU(3), A_1^2 = SU(2)^2$ or $A_1 = SU(2)$. If we now write the charge vectors $(p_i; 0)$ and $(0; -p_i)$ in the form (4.2.1), we find

$$\forall\, i \in \{1, \ldots, d\}: \quad \pm p_i = \frac{1}{\sqrt{2}} \left(\mu_i^\pm - B\lambda_i \pm \lambda_i \right), \lambda_i \in \Lambda, \mu_i^\pm \in \Lambda^*.$$

In particular, $2\lambda_i, 2B\lambda_i \in \Lambda^*$ for all $i \in \{1, \ldots, d\}$. These conditions are sufficient to determine all theories in $\mathcal{M}_2^{Narain}$ with left–right symmetrically enhanced symmetry:

Theorem 6.1.1
There are two theories with maximally, that is rank two, enhanced symmetry, namely the $SU(2)^2$ torus theory at $\tau = \rho = i$ and the $SU(3)$ torus theory at $\tau = \rho = e^{2\pi i/3}$ with gauge symmetries $SU(2)_l^2 \times SU(2)_r^2, SU(3)_l \times SU(3)_r$, respectively. Tori with $\tau = \rho \notin \{i, e^{2\pi i/3}\}$ and $\tau_1 \in \{0, \frac{1}{2}\}$ exhibit an enhanced $SU(2)_l \times SU(2)_r$ symmetry.

We stress that (6.1.1) contains examples where enhanced symmetry occurs nonsymmetrically: For generic parameters $\tau = \rho$ (i.e. $\tau_1 \notin \{0, \frac{1}{2}\}$), we have a right handed $SU(2)_1$, whereas for $\tau = -\overline{\rho}, \tau_1 \notin \{0, \frac{1}{2}\}$, we have a left handed one.

6.1.2 Irreducible nonexceptional nonisolated components of the moduli space

Suppose that a nonisolated component of $\mathcal{M}^2$ with Teichmüller space $\mathcal{E} \subset \mathbb{H} \times \mathbb{H}$ is obtained by modding out a common symmetry group G of all toroidal theories with parameters in $\mathcal{E}$. Assume further that $\mathcal{E}$ is a maximal connected subset of $\mathbb{H} \times \mathbb{H}$ corresponding to theories with symmetry G. In particular, G acts as group of isometries on $\mathcal{E}$, and the $(1,1)$ fields which describe deformations within $\mathcal{E}$ are invariant under G. Thus $\mathcal{E}$ is totally geodesic. Since by the discussion of (4.3.6) the metric on the Teichmüller space $\mathbb{H}\times\mathbb{H}$ is just the product of hyperbolic metrics on each of the factors $\mathbb{H}$, geodesics are well known: The projection on each of the $\mathbb{H}$–factors is a half circle with center on the real axis, a half line parallel to the imaginary axis of $\mathbb{H}$, or constant.
If $\mathcal{E}$ contains a large volume theory, then the action of G has a geometric interpretation. Namely, recall from (4.2.7) that a large volume theory in $\mathcal{E}$ has a unique preferred geometric interpretation (Λ, B) with large $\rho_2 = \det(G_{\mu\nu})$ in terms of a nonlinear σ model. Then the action of G on that toroidal theory will not change this preferred geometric interpretation. Since $\widetilde{\Gamma}$ as defined in (4.2.7) has the property $\mathrm{span}_{\mathbb{Z}}\widetilde{\Gamma} = \{\frac{1}{\sqrt{2}}(\mu; \mu) \mid \mu \in \Lambda^*\}$, the action of G is given by a geometric symmetry on the corresponding torus $T^2 = \mathbb{R}^2/\Lambda$.

Let us assume that G maps the set $\{j_k\bar{\jmath}_l \mid k,l \in \{1,2\}\}$ of generic $(1,1)$ fields of theories in $\mathcal{M}_2^{Narain}$ into itself. By construction of toroidal conformal field theories, this means that G induces an action on the entire Teichmüller space $\mathbb{H}\times\mathbb{H}$ of $\mathcal{M}_2^{Narain}$, which identifies isomorphic theories and fixes $\mathcal{E}$. This action will be denoted $\mathcal{G}$ in the following. By construction (4.3.6) of the moduli space $\mathcal{M}_2^{Narain}$ of toroidal theories, we must have $\mathcal{G} \subset PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$. Note that in general $\mathcal{G}$ will be different from G, since an action of G on vertex operators $V[p]$ by multiplication with phases will be invisible in its induced action on $\mathcal{M}_2^{Narain}$.
If $\mathcal{E}$ contains a geodesic with the property that its projection on one of the factors of $\mathbb{H}$ in the Teichmüller space is constant, then by (4.3.3) one checks that $\mathcal{E}$ contains a large volume limit and thus G acts geometrically by the above. Otherwise, note that $\mathcal{E}$ is the fixed point set of a subgroup $\mathcal{G} \subset PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$, and that by what was said in section 6.1.1 the action of V need not be discussed. Thus $\mathcal{E}$ must be one of the spaces $\mathcal{E}_U, \mathcal{E}_{UV}$ of (6.1.1), or a Möbius transform thereof. Since $\mathcal{E}$ is maximal, we may assume $G = \{1, g\}$, where $g \in \{U, UV\}$ and $\mathcal{E} = \mathcal{E}_g$. But then in section 6.1.1 we argued that in neither of these cases we find new components of $\mathcal{M}^2$ by modding out such a nongeometric symmetry.
To summarize what we have shown up to now we use the following

Definition 6.1.2
Let $\mathcal{E} \subset \mathbb{H}\times\mathbb{H}$ denote the Teichmüller space of a nonisolated irreducible orbifold component $\widehat{\mathcal{E}}$ of $\mathcal{M}^2$, such that $\widehat{\mathcal{E}}$ is obtained by orbifolding with the group G. If the set $\{j_k\bar{\jmath}_l \mid k,l \in \{1,2\}\}$ of generic $(1,1)$ fields of theories with parameters in $\mathcal{E}$ is not mapped onto itself by the action of G, we call the action as well as the corresponding orbifold component of $\mathcal{M}^2$ EXCEPTIONAL.

Above we have shown

Theorem 6.1.3
Each nonexceptional nonisolated component $\widehat{\mathcal{E}}$ of $\mathcal{M}^2$ which is obtained by an orbifold procedure form a subspace $\mathcal{M}_2^{Narain}$ is obtained by a geometric orbifold. By the discussion of section 5.3 it thus is obtained by a crystallographic orbifold.

In fact, since the Teichmüller space $\mathcal{E}$ of an exceptional component is totally geodesic, to give an estimate of how many exceptional components one may find it suffices to determine all geodesics in $\mathbb{H}\times\mathbb{H}$ that parametrize theories which generically possess more than four $(1,1)$ fields. By explicit calculation using (4.3.3) one checks that all such geodesics have the form $f(t) = (\tau_1, t, \pm\tau_1, t) \in \mathbb{H}\times\mathbb{H}, t \in \mathbb{R}^+$, or are Möbius transforms thereof. In other words, without loss of generality $\mathcal{E} = \mathcal{E}_U$ or $\mathcal{E} = \mathcal{E}_{UV}$ as defined in (6.1.1). Thus in all exceptional cases the toroidal conformal field theories with parameters in $\mathcal{E}$ possess an additional $SU(2)_l$ or $SU(2)_r$ symmetry, and the exceptional action is given by a binary tetrahedral, octahedral or icosahedral subgroup T, O, I of $SU(2)$ (see [Gin88b]), possibly in combination with some other symmetry. For instance, if $\tau_1 = 0$ the toroidal theories in $\mathcal{E}_U = \mathcal{E}_{UV}$ decompose into tensor products of $c = 1$ circle theories at radii $r = 1, r' = t$, respectively. Then the possible actions of T, O, I on the first factor theory are clear from the results on conformal field theories with central charge $c = 1$ [Gin88b].

Discrete subgroups of, for example, $SU(3)$ cannot lead to the construction of non-isolated components of the moduli space, since by the discussion at the end of section 6.1.1 we know that an enhanced $SU(3)$ symmetry only occurs at the isolated point $\tau = \rho = e^{2\pi i/3}$ of $\mathcal{M}_2^{Narain}$. In general, exceptional components of $\mathcal{M}^2$ are an interesting issue to be studied separately, which exceeds the scope of the present work.

We rather concentrate on the nonexceptional components of $\mathcal{M}^2$ in the following. Note that equivalent toroidal theories need not always be mapped onto equivalent orbifold theories if we mod out a symmetry group G, since the action of G in some cases does depend on the particular choice of coordinates on T^2. In other words, $\widehat{\mathcal{E}}$ is obtained from $\mathcal{E}$ by modding out a subgroup of $\{A \in PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2 \mid A\mathcal{E} = \mathcal{E}\}$ which needs to be determined for every group G separately.

Recall on the other hand from the end of section 5.1 that every theory that was constructed as orbifold by a solvable group G possesses a symmetry which one can mod out to regain the original theory. The list of crystallographic symmetries (5.3.9) shows that indeed only orbifolds by solvable groups are of relevance to us. Thus no information distinguishing two theories may be lost under our orbifold procedures. In other words, if we mod out two distinct toroidal theories by the same symmetry, then the resulting theories must be distinct as well.

Let us now determine the parameter spaces of all nonexceptional nonisolated orbifold components of $\mathcal{M}^2$. By theorem 6.1.3 all of them are crystallographic orbifolds by some group G in our list (5.3.9), and the corresponding component is denoted $\mathcal{M}_{G-orb}$. From (5.3.9) one also reads the Teichmüller spaces, for an illustration and the numbering of the respective lattices see figure 5.3.1.

If $G = \mathbb{Z}_M$ (lattices 2–5), we find that G commutes with Möbius transformations, and in case $M = 2$ also with the entire $PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$ of (4.3.6). Thus for the families of $\mathbb{Z}_M$ orbifold conformal field theories with $c = 2$ we get the following irreducible components of $\mathcal{M}^2$:

$$\begin{array}{rrcl} & \mathcal{M}_{\mathbb{Z}_2-orb} & \cong & \mathcal{M}_2^{Narain} \cong \mathbb{H} \times \mathbb{H}/PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2, \\ \text{for } M \in \{3,4,6\}: & \mathcal{M}_{\mathbb{Z}_M-orb} & = & \left\{(\tau,\rho) \mid \tau = e^{2\pi i/M}, \rho \in \mathbb{H}/PSL(2,\mathbb{Z})\right\} \\ & & \cong & \mathbb{H}/PSL(2,\mathbb{Z}). \end{array} \tag{6.1.2}$$

For the lattices 6 to 17, by (5.3.9) we get irreducible components $\mathcal{M}_{G-orb}^{(\tau_1,\rho_1)}$ of the moduli space $\mathcal{M}^2$ with $\tau_1, \rho_1 \in \{0, \frac{1}{2}\}$. In some cases discrete torsion gives additional degrees of freedom, increasing the number of irreducible components to $\mathcal{M}_{G^\pm-orb}^{(\tau_1,\rho_1)}$. The Teichmüller space of each such irreducible component is $(\mathbb{R}^+)^k$, where $k = 1$ if τ_2 must be fixed for the particular lattice, too, and $k = 2$ otherwise. To find the correct parameter spaces, we must determine the subgroup $\mathcal{P}$ of $PSL(2,\mathbb{Z})^2 \rtimes \mathbb{Z}_2^2$ in (4.3.6) which maps the respective Teichmüller space $(\mathbb{R}^+)^k$ onto itself. Then we must discuss which elements of $\mathcal{P}$ map equivalent orbifold theories onto each other.

Restrict $\mathcal{P}$ to one of the factors $\mathbb{R}^+ \subset \mathbb{H}$ of the Teichmüller space $(\mathbb{R}^+)^k$, specified by $\zeta_1 = 0$ or $\zeta_1 = \frac{1}{2}$. We claim that

$$\mathcal{P} \cap PSL(2, \mathbb{Z}) = \{1, \Theta\},$$

where Θ was given in (5.3.2). By definition, Θ acts on $I^0 := \{\zeta \in \mathbb{H} \mid \zeta_1 = 0\}$ by $\zeta_2 \mapsto \frac{1}{\zeta_2}$ and on $I^+ := \{\zeta \in \mathbb{H} \mid \zeta_1 = \frac{1}{2}\}$ by $\zeta_2 \mapsto \frac{1}{4\zeta_2}$. Now $I^0 = J^0 \cup \Theta J^0$, where $J^0 := \{\zeta \in I^0 \mid \zeta_2 \geq 1\}$. Because J^0 does not contain any two points identified by Möbius transformations, the assertion follows for the case $\zeta_1 = 0$. For $\zeta_1 = \frac{1}{2}$ observe that $I^+ = (J^+ \cup TSTJ^1) \cup \Theta(J^+ \cup TSTJ^1)$, where $J^+ := \{\zeta \in I^+ \mid \zeta_2 \geq \frac{\sqrt{3}}{2}\}$ and $J^1 := \{\zeta \in \mathbb{H} \mid \|\zeta\| = 1, \zeta_1 \in [-\frac{1}{2}, 0]\}$. Because no two points in $J^+ \cup J^1$ are related by Möbius transformations, the assertion follows. For the respective factor of the Teichmüller space under discussion, Θ will be called T–DUALITY.

By our convention to fix $\tau_1, \rho_1 \in \{0, \frac{1}{2}\}$ it is clear that target space orientation change $V : (\tau, \rho) \mapsto (-\overline{\tau}, -\overline{\rho})$ in (4.3.5) can only be contained in $\mathcal{P}$ if $\tau_1 = \rho_1 \in \{0, \frac{1}{2}\}$, in which case it acts trivially. Mirror symmetry $U : (\tau, \rho) \mapsto (\rho, \tau)$ is contained iff $\tau_1 = \rho_1$ and τ_2 is not fixed. Inspection of the charge lattice (4.2.1) and the action (5.3.3) of R_1 shows that mirror symmetry commutes with R_1, R_2 on toroidal conformal field theories. But a priori it is not clear whether it indeed commutes with the action of each of the symmetry groups corresponding to lattices 6 to 17. Therefore, a case by case study is necessary to decide which of Θ, U map a G orbifold onto an equivalent one and thus determine all the parameter spaces $\mathcal{M}^{(\bullet)}_{G^\bullet-orb}$. We will also see that not all of the lattices yield different components of the moduli space $\mathcal{M}^2$.

By our general discussion of crystallographic symmetry groups in section 5.3.1, to find the correct parameter space for the irreducible components of $\mathcal{M}^2$ obtained by $\mathbb{Z}_2(R)$ orbifolding, the Teichmüller spaces are constructed by considering $R = R_1$ only. T–duality applied to τ alone, which by (5.3.2) is equivalent to $R_1 \leftrightarrow R_2$, i.e. $r \leftrightarrow r'$ in (5.3.14), does not generically map onto an isomorphic theory. Application of T–duality to both τ and ρ simultaneously, which will be denoted by

$$\mathcal{S} : \ (\tau, \rho) \mapsto \left(-\frac{1}{\tau}, -\frac{1}{\rho} \right)$$

and called SIMULTANEOUS T–DUALITY in the following, amounts to $r \mapsto \frac{1}{r}$, $r' \mapsto \frac{1}{r'}$, mapping the $\mathbb{Z}_2(R)$ orbifold to an isomorphic theory (see (5.3.14)). Mirror symmetry $\tau \leftrightarrow \rho$ acts by $r \mapsto \frac{1}{r}$, $r' \mapsto r'$, which it is invariant under, too. In particular, lattice 6 ($\tau_1 = 0$) with $\rho_1 = \frac{1}{2}$ and lattice 7 ($\tau_1 = \frac{1}{2}$) with $\rho_1 = 0$ correspond to families of isomorphic orbifold conformal field theories.

Summarizing, we have constructed the following three irreducible components of the moduli space:

$$\begin{aligned} \mathcal{M}^{(0,0)}_{\mathbb{Z}_2(R)-orb} &\cong \left(\mathbb{R}^+\right)^2 / \{U, \mathcal{S}\}, \qquad \mathcal{M}^{(\frac{1}{2},\frac{1}{2})}_{\mathbb{Z}_2(R)-orb} \cong \left(\mathbb{R}^+\right)^2 / \{U, \mathcal{S}\}, \\ \mathcal{M}^{(0,\frac{1}{2})}_{\mathbb{Z}_2(R)-orb} &= \mathcal{M}^{(\frac{1}{2},0)}_{\mathbb{Z}_2(R)-orb} \cong \left(\mathbb{R}^+\right)^2 / \mathcal{S}. \end{aligned}$$

In the other cases an analogous discussion leads to the determination of the correct parameter spaces which we list in (6.1.3) below. As to the case of $D_3(R)$ (lattices

13 and 14) it is worth mentioning that by (5.3.14) we know that Z_{R_2-orb} is obtained from Z_{R_1-orb} by application of T–duality (5.3.2) on τ. Using mirror symmetry we see that we can equally apply T–duality to ρ and find

$$\begin{aligned} Z_{D_3(R_2)-orb}(1/2,\sqrt{3}/2,0,\rho_2) &= Z_{D_3(R_1)-orb}(1/2,\sqrt{3}/2,0,1/\rho_2), \\ Z_{D_3(R_2)-orb}(1/2,\sqrt{3}/2,1/2,\rho_2) &= Z_{D_3(R_1)-orb}(1/2,\sqrt{3}/2,1/2,1/4\rho_2), \end{aligned}$$

The above actually is the equation for T–duality on $\mathcal{M}^{(\rho_1)}_{D_3(R_1)-orb}$. In particular, the $D_3(R_2)$ orbifold procedure does not yield a new component of the moduli space $\mathcal{M}^2$ but only reproduces $\mathcal{M}^{(\rho_1)}_{D_3(R_1)-orb}$, $\rho_1 \in \{0, \frac{1}{2}\}$. All in all we find

$$\begin{array}{rclrcl} \mathcal{M}^{(0,0)}_{D_2^\pm-orb} &\cong& \left(\mathbb{R}^+/\Theta\right)^2/U, & \mathcal{M}^{(0,\frac{1}{2})}_{D_2^\pm-orb} &\cong& \mathcal{M}^{(\frac{1}{2},0)}_{D_2^\pm-orb} \cong \left(\mathbb{R}^+/\Theta\right)^2, \\ &&& \mathcal{M}^{(\frac{1}{2},\frac{1}{2})}_{D_2-orb} &\cong& \left(\mathbb{R}^+/\Theta\right)^2/U, \\ \mathcal{M}_{\mathbb{Z}_2(T_R)-orb} &\cong& \left(\mathbb{R}^+\right)^2/U, &&& \\ \mathcal{M}_{D_2(T_R)-orb} &\cong& \left(\mathbb{R}^+\right)^2/U\mathcal{S}, & \mathcal{M}_{D_2(T'_R)-orb} &\cong& \left(\mathbb{R}^+/\Theta\right)\times\mathbb{R}^+, \\ \mathcal{M}^{(\rho_1)}_{D_3(R_1)-orb} &\cong& \mathbb{R}^+, & \mathcal{M}^{(\rho_1)}_{D_4^\pm-orb} &\cong& \mathbb{R}^+/\Theta, \quad \rho_1\in\{0,\frac{1}{2}\}, \\ \mathcal{M}_{D_4(T'_R)^\pm-orb} &\cong& \mathbb{R}^+, &&& \\ \mathcal{M}^{(0)}_{D_6^\pm-orb} &\cong& \mathbb{R}^+/\Theta, & \mathcal{M}^{(1/2)}_{D_6-orb} &\cong& \mathbb{R}^+/\Theta. \end{array} \tag{6.1.3}$$

As a first consistency check we remark that if our description of nonisolated nonexceptional components of $\mathcal{M}^2$ is complete, it must be possible to find all nonisolated components known so far. In particular, we should consider tensor products of known models. The simplest case is the product of two models with central charge $c = 1$. The possible factor theories then are $\mathcal{C}^{c=1}(r), \mathcal{C}^{c=1}_{orb}(r), \mathcal{C}^{c=1}_T, \mathcal{C}^{c=1}_O$, and $\mathcal{C}^{c=1}_I$, corresponding to compactification on a circle with radius r, its $\mathbb{Z}_2$ orbifold, or one of the three isolated components of the $c = 1$ moduli space, respectively. Models containing one of the latter three factor theories are exceptional but of course easily constructed, as was mentioned above. Moreover,

$$\begin{aligned} \mathcal{C}^{c=1}(r)\otimes\mathcal{C}^{c=1}(r') &= \mathcal{C}_T(0,\tfrac{r'}{r},0,rr'), \\ \mathcal{C}^{c=1}(r)\otimes\mathcal{C}^{c=1}_{orb}(r') &= \mathcal{C}_{R_1-orb}(0,\tfrac{r'}{r},0,rr'), \\ \mathcal{C}^{c=1}_{orb}(r)\otimes\mathcal{C}^{c=1}_{orb}(r') &= \mathcal{C}_{D_2^+-orb}(0,\tfrac{r'}{r},0,rr'). \end{aligned}$$

Using the results of [DGH88], nonisolated components of the moduli space can also be obtained by tensoring $N = (1,1)$ superconformal field theories $\mathcal{C}^{c=3/2}_\bullet(r)$ with central charge $c = 3/2$ with the unique unitary conformal field theory at $c = 1/2$. In section 6.3 we will discuss how the resulting models $\mathcal{C}^M_\bullet(r)$ can be found within the components of $\mathcal{M}^2$ we have determined above.

6.2 Multicritical points and lines

We now determine all intersections of the 26 nonexceptional components $\mathcal{M}^{(\bullet)}_{G^{(\bullet)}-orb}$ of the moduli space that we constructed in section 6.1. We find that all but three of them can be connected directly or indirectly to the moduli space $\mathcal{M}_2^{Narain}$ of toroidal theories, and $\mathcal{M}^2$ exhibits a complicated structure with various loops.
The procedure closely follows the proof for the isomorphy of the $c = 1$ circle theory at radius $r = 2$ to the orbifold theory at radius $r = 1$ (see, e.g., [DHVW85, Gin88a]). The main idea is to exploit the enhanced $SU(2)$ symmetry of the circle theory at radius $r = 1$. Namely, $SU(2)$ relates two generically different $\mathbb{Z}_2$ actions in this theory by conjugation. Thus the resulting orbifold theories are isomorphic. One of them is the circle theory at doubled radius $r = 2$, the other is the ordinary $\mathbb{Z}_2$ orbifold theory at radius $r = 1$.
Using results of B. Rostand's we can show that the generalization of the above procedure to $c = 2$ will suffice to find all intersections of our 27 nonexceptional nonisolated components of $\mathcal{M}^2$. Namely, in [Ros90, Ros91] it is shown that every multicritical point on the moduli space $\mathcal{M}_2^{Narain}$ of toroidal theories is an orbifold of another toroidal theory with enhanced symmetry. By our discussion in section 6.1, we may restrict ourselves to the study of left–right symmetric orbifolds. In particular, to find all intersections of $\mathcal{M}_2^{Narain}$ with one of the 26 nonexceptional orbifold components it suffices to determine all toroidal theories with enhanced left and right symmetry (which in the following are simply called theories with enhanced symmetry) and mod out all symmetries which are conjugate to some shift on the charge lattice. As anticipated in [DVV87] each of the toroidal multicritical points generates a series of further multicritical points or lines, since we can mod out further symmetries. But even better, this procedure will lead to the determination of all intersection points: By the discussion in sections 6.1 and 5.3, all the 26 nonexceptional components of $\mathcal{M}^2$ are obtained by modding out solvable groups from toroidal theories. This means that we can always regain the original toroidal theory by performing another orbifold procedure. In particular, any intersection point between nonexceptional nonisolated components of $\mathcal{M}^2$ corresponds to a multicritical point on $\mathcal{M}_2^{Narain}$.
One can simplify things by stepwise modding out [DGH88]: If a symmetry group G contains a normal subgroup H, then the G orbifold conformal field theory $\mathcal{C}/G$ of a theory $\mathcal{C}$ is isomorphic to the G/H orbifold conformal field theory of $\mathcal{C}/H$. Moreover, the G/H action on $\mathcal{C}/H$ translates to an action on any other theory $\mathcal{C}'$ which was identified with $\mathcal{C}/H$. For $H' \subset G/H, G' \cong H \times H'$ this leads to possibly new identifications $\mathcal{C}/G' \cong \mathcal{C}'/H'$ which need not correspond to conjugate actions on the original $\mathcal{C}$. In $\mathcal{C}/H \cong \mathcal{C}'$ we may have gotten rid of all states which the G' action has no consistent conjugate on.
In theorem 6.1.1 we have determined all points of enhanced symmetry in $\mathcal{M}_2^{Narain}$. Namely, only tori with enhanced $SU(2)_l \times SU(2)_r$ or $SU(3)_l \times SU(3)_r$ symmetry are of relevance. In section 6.2.1 we discuss all the multicritical points and lines obtainable by modding out conjugate $\mathbb{Z}_2$ symmetries of tori with enhanced $SU(2)$ symmetry. In sections 6.2.2–6.2.6 we determine all multicritical points and lines

obtainable from those identifications we found in 6.2.1 by modding out further symmetries. Afterwards (section 6.2.7) we follow the same procedure for the $SU(3)$ torus theory at $\tau = \rho = e^{2\pi i/3}$. The slightly technical discussion results in a list of all multicritical points and lines in nonexceptional nonisolated components of $\mathcal{M}^2$. We remark that about half of the discrete identifications below have been conjectured by Sayipjamal Dulat on the basis of numerical calculations of partition functions. She has confirmed all the others numerically as well. Some of these identifications also follow directly from the literature [DHVW85, Gin88a, KS88]. We will denote the $G^{(\bullet)}$ orbifold theory of the toroidal theory $\mathcal{C}_T(\tau_1, \tau_2, \rho_1, \rho_2)$ with parameters (τ, ρ) by $\mathcal{C}_{G^{(\bullet)}-orb}(\tau_1, \tau_2, \rho_1, \rho_2)$ in the following.

6.2.1 Multicritical lines on the torus moduli space $\mathcal{M}_2^{Narain}$: Conjugate $\mathbb{Z}_2$ actions

To compare all $\mathbb{Z}_2$ symmetries of the $SU(2)^2$ torus theory at $\tau = \rho = i$ we discuss their action on the $(1,0)$ fields. As in definition 4.1.1, the conserved currents of the generic toroidal theory are called j^μ. The additional vertex operators of dimensions $(1,0)$ are denoted $j^\mu_\pm, \mu \in \{1,2\}$, such that each triple $j^\mu, j^\mu_\pm$ generates an $SU(2)_1$ Kac–Moody algebra. Each of these $SU(2)_1$ Kac–Moody algebras belongs to one of the $c = 1$ factors of the torus theory. Let us list all $\mathbb{Z}_2$ symmetries with two positive and four negative eigenvalues on the set of $(1,0)$ fields. By $\widetilde{\mathbb{Z}_2(R)}$ we denote the $\mathbb{Z}_2(R)$ symmetry applied to the torus theory with fundamental cell such that $\tau = \rho = 1/2 + i/2$ (remember the phases on Hilbert space ground states that were discussed for lattice 9 around (5.3.5)):

$$\begin{array}{lll}
\mathbb{Z}_2 \text{ rotational group}: & j^\mu \mapsto -j^\mu, & j^\mu_\pm \leftrightarrow j^\mu_\mp, \\
\text{shift orbifold by } \delta' = \frac{1}{2}\binom{1}{1}: & j^\mu \mapsto j^\mu, & j^\mu_\pm \mapsto -j^\mu_\pm, \\
\widetilde{\mathbb{Z}_2(R_1)}: & j^1 \mapsto j^1, j^2 \mapsto -j^2, & j^1_\pm \mapsto -j^1_\pm, j^2_+ \leftrightarrow j^2_-, \\
\mathbb{Z}_2(T_{R_1}): & j^1 \mapsto j^1, j^2 \mapsto -j^2, & j^1_\pm \mapsto -j^1_\pm, j^2_+ \leftrightarrow j^2_-.
\end{array}$$

None of the above symmetries mixes currents from different $c = 1$ factors of the torus theory or j^μ with $j^\mu_\pm$ currents. Moreover, their eigenvalue spectrum is identical on each $c = 1$ factor, so we may use the corresponding $c = 1$ result to show that the four $\mathbb{Z}_2$ orbifolds by the above listed symmetries give isomorphic theories when applied to the $SU(2)^2$ theory. This generates a quadrucritical point. The shift orbifold by the half lattice vector δ', as explained around (5.1.5), results in a torus theory with additional generator δ' of the lattice and half volume and B–field ($\mathcal{C}_T(0,1,0,2) = \mathcal{C}_T(0,1,0,1/2)$ by T–duality):

$$\begin{aligned}
\mathcal{C}_T(0,1,0,2) &= \mathcal{C}_{T_R-orb}(0,1,0,1) \\
&= \mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1) = \mathcal{C}_{R-orb}(1/2,1/2,1/2,1/2).
\end{aligned} \tag{Q1}$$

The equality $\mathcal{C}_T(0,1,0,2) = \mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$ has already been proven in [KS88], both on the level of partition function and operator algebra.

The above quadrucritical point turns out to actually be the intersection of four bicritical lines. First consider the family of torus theories at parameters $\tau = \rho = it, t \in \mathbb{R}^+$ which decompose into tensor products of two $c = 1$ circle theories at radii $r = 1$ and $r' = t$, respectively. For all values of t the first factor possesses an $SU(2)$ symmetry. Since the actions of T_{R_2} and the shift by $\delta' = \frac{1}{2}\binom{1}{t}$ only differ on this first factor, where they are generally conjugate by the $SU(2)$ symmetry, we find ($\mathcal{C}_T(1/2, t/2, 0, t/2) = \mathcal{C}_T(0, t/2, 1/2, t/2)$ by mirror symmetry)

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_T(0, t/2, 1/2, t/2) = \mathcal{C}_{T_{R_2}-orb}(0, t, 0, t), \tag{L1}$$

and analogously

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{T_{R_1}-orb}(0, t, 0, t) = \mathcal{C}_{\mathbb{Z}_2-orb}(0, t, 0, t). \tag{L2}$$

Next consider the family of toroidal theories at parameters $\tau = \rho = 1/2 + it, t \in \mathbb{R}^+$. We also have a generic $SU(2) \times U(1)$ symmetry for this family. Inspection of the charge lattice shows that as before we have conjugate $\mathbb{Z}_2$ symmetries now giving bicritical lines

$$\begin{aligned} \forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{\mathbb{Z}_2-orb}(1/2, t, 1/2, t) &= \mathcal{C}_{R_1-orb}(1/2, t, 1/2, t), && \text{(L3)} \\ \mathcal{C}_{R_2-orb}(1/2, t, 1/2, t) &= \mathcal{C}_T(0, 2t, 1/4, t/2). && \text{(L4)} \end{aligned}$$

There are two more $\mathbb{Z}_2$ symmetries which are conjugate on the entire family of toroidal theories with parameters $\tau = \rho = it, t \in \mathbb{R}^+$, by $SU(2)$ symmetry on the first factor. They have four positive and two negative eigenvalues on $(1, 0)$ fields:

$$\begin{array}{llll} \mathbb{Z}_2(R_2) : & j^1 \mapsto -j^1, j^2 \mapsto j^2, & j^1_+ \leftrightarrow j^1_-, & j^2_\pm \mapsto j^2_\pm, \\ \text{shift orbifold by } \delta_1 = \frac{1}{2}\binom{1}{0} : & j^\mu \mapsto j^\mu, & j^1_\pm \mapsto -j^1_\pm, & j^2_\pm \mapsto j^2_\pm. \end{array}$$

In particular,

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{R_2-orb}(0, t, 0, t) = \mathcal{C}_T(0, t/2, 0, 2t). \tag{L5}$$

We remark that $\mathbb{Z}_2(R)$ applied to the theory with fundamental cell such that $\tau = 1/2 + i/2, \rho = i$ has three positive and three negative eigenvalues on the set of $(1, 0)$ fields. Hence it is not conjugate to any other crystallographic symmetry of $\mathcal{C}_T(0, 1, 0, 1)$.

6.2.2 Series of multicritical lines and points obtainable from (L1) and (L5)

We are now going to mod out further symmetries on both sides of the equalities obtained above. The main problem is to find the correct translation for the action of a symmetry from one model to the other. The simplest case is (L5) from which we mod out R_1 on both sides. Because all the symmetries used so far respect the factorization of $\mathcal{C}_T(0, t, 0, t)$ into a tensor product of two circle theories and commute, we directly get

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{D_2^+-orb}(0, t, 0, t) = \mathcal{C}_{R_1-orb}(0, t/2, 0, 2t). \tag{L6}$$

Note that by mirror symmetry and T–duality (5.3.2) we have $\mathcal{C}_{R_1-orb}(0,2,0,1/2) = \mathcal{C}_{R_2-orb}(0,2,0,2)$, hence the above multicritical line and the one found in (L5) intersect in a tricritical point:

$$\mathcal{C}_{D_2^+-orb}(0,1,0,1) = \mathcal{C}_{R_1-orb}(0,1/2,0,2) = \mathcal{C}_T(0,1,0,4). \tag{T1}$$

We now systematically mod out all symmetries of the torus theory $\mathcal{C}_T(0,t/2,0,2t)$ in (L5). The procedure is similar in all cases, namely, the charge lattices of the underlying toroidal theories on both sides of an identification must be determined, as well as twisted ground states, if present. After having performed a state by state identification, symmetries can be translated from one side to the other. This way the details which we partly omit in the proofs below can easily be filled.
As to (L5), by (5.3.9) the actions we can generically mod out on the torus theory $\mathcal{C}_T(0,t/2,0,2t)$ are $\mathbb{Z}_2, \mathbb{Z}_2(R), \mathbb{Z}_2(T_R), D_2^\pm, D_2(T_R)$ and $D_2(T_R')$. At $t=2$ one has additional $\widetilde{Z_2(R)}$ and $\mathbb{Z}_4$ actions which give no new identifications, though.
Modding out by $\mathbb{Z}_2(R_1)$ gives the bicritical line (L6) as discussed above. The reflection R_2 on the torus side acts as a shift by $\delta_1 = \frac{1}{2}\binom{1}{0}$ on the underlying torus theory of $\mathcal{C}_{R_2}(0,t,0,t)$ leading to a trivial identity. The symmetry T_{R_1} applied to the torus side differs in its action from R_1 by additional signs on those vertex operators (of lowest dimension) in $\mathcal{C}_T(0,t/2,0,2t)$ which correspond to twisted ground states in $\mathcal{C}_{R_2}(0,t,0,t)$. Therefore, comparison with (L6) shows

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{D_2^--orb}(0,t,0,t) = \mathcal{C}_{T_{R_1}-orb}(0,t/2,0,2t). \tag{L7}$$

Modding out by T_{R_2} instead of T_{R_1} again gives a trivial identity, since T_{R_2} acts on the underlying torus of $\mathcal{C}_{R_2-orb}(0,t,0,t)$ by the shift T_{δ_1}. Note that a comparison of (L7) with (L6) gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion. Since $\mathcal{C}_{T_{R_1}-orb}(0,1/2,0,2) = \mathcal{C}_{T_{R_2}-orb}(0,2,0,2)$ by T–duality (see (5.3.14)), the multicritical lines (L7) and (L1) intersect in a tricritical point:

$$\mathcal{C}_{D_2^--orb}(0,1,0,1) = \mathcal{C}_{T_{R_2}-orb}(0,2,0,2) = \mathcal{C}_T(0,1,1/2,1). \tag{T2}$$

Next, we mod out the ordinary $\mathbb{Z}_2$ action on (L5). The multicritical line (L5) can also be written as $\mathcal{C}_{\widehat{T}_{R_2}-orb}(0,t,0,t) = \mathcal{C}_T(0,t/2,0,2t)$. Recall that $\mathcal{C}_T(0,t,0,t)$ as well as $\mathcal{C}_T(0,2t,0,t/2)$ are tensor products of circle theories at radii $r=1, r'=t$ and $r=2, r'=t$, respectively. Now consider the residual action of $D_2(T_R)$ of the original torus theory $\mathcal{C}_T(0,t,0,t)$ on the orbifoldized theory $\mathcal{C}_{\widehat{T}_{R_2}-orb}(0,t,0,t)$ and note that it acts as ordinary $\mathbb{Z}_2$ on the invariant sector. The twisted ground states of the first circle factor are interchanged, so all in all we get an ordinary $\mathbb{Z}_2$ action on $\mathcal{C}_T(0,t/2,0,2t)$. This yields

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{D_2(T_R)-orb}(0,t,0,t) = \mathcal{C}_{\mathbb{Z}_2-orb}(0,t/2,0,2t). \tag{L8}$$

By analogous arguments one finds that modding out (L1) by $\mathbb{Z}_2$ on the torus side yields

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2) = \mathcal{C}_{D_2(T_R')-orb}(0,t,0,t). \tag{L9}$$

As mentioned above, R_2 applied to $\mathcal{C}_T(0,t/2,1/2,t/2)$ acts as shift T_{δ_1} on the underlying torus theory of $\mathcal{C}_{R_2-orb}(0,t,0,t)$. Applying this to the bicritical line (L8), if R_2 acts with positive sign on the $\mathbb{Z}_2$ twisted ground states of the right hand side we obtain a trivial identity. On the other hand, if we use negative discrete torsion on the right hand side we find

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{D_2(T_R)-orb}(0,t/2,0,2t) = \mathcal{C}_{D_2^- -orb}(0,t/2,0,2t). \tag{L10}$$

Note that the bicritical lines (L7) and (L10) intersect in a tricritical point which can be interpreted as the result of modding out (T1) by T_{R_1}:

$$\mathcal{C}_{T_{R_1}-orb}(0,1,0,4) = \mathcal{C}_{D_2^- -orb}(0,2,0,2) = \mathcal{C}_{D_2(T_R)-orb}(0,1/2,0,2). \tag{6.2.1}$$

To mod out (L5) by $D_2(T_R')$ on the torus side amounts to modding out (L7) by T_{R_2}' which acts as shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{t}$, on the underlying torus of $\mathcal{C}_{D_2^- -orb}(0,t,0,t)$. Thus

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{D_2^- -orb}(1/2,t/2,0,t/2) = \mathcal{C}_{D_2(T_R')-orb}(0,t/2,0,2t). \tag{L11}$$

Note that because of T–duality $\mathcal{C}_{D_2(T_R')-orb}(0,2,0,2) = \mathcal{C}_{D_2(T_R')-orb}(0,1/2,0,2)$ as can be seen from (5.3.14), so (L11) intersects (L9) in a tricritical point which can be understood as the result of modding out (T2) by $\mathbb{Z}_2$:

$$\mathcal{C}_{D_2^- -orb}(1/2,1/2,0,1/2) = \mathcal{C}_{D_2(T_R')-orb}(0,1/2,0,2) = \mathcal{C}_{\mathbb{Z}_2}(0,1,1/2,1). \tag{T3}$$

We now turn to a systematic discussion of intersection lines and points obtained from (L1). From $\mathcal{C}_T(0,t/2,1/2,t/2)$ we can generically mod out $\mathbb{Z}_2, \mathbb{Z}_2(R)$, and $D_2^\pm$. The additional symmetries for $t = 1$ and $t = 2$ produce nothing new.
Modding out by the ordinary $\mathbb{Z}_2$ action on the torus side gives the bicritical line (L9), as was mentioned above. We claim that the result of modding out a $\mathbb{Z}_2(R_1)$ action leads to the bicritical line

$$\forall\, t \in \mathbb{R}^+ : \quad \mathcal{C}_{R_1-orb}(0,t/2,1/2,t/2) = \mathcal{C}_{D_2(T_R)-orb}(0,1/t,0,t). \tag{L12}$$

Actually, the slightly surprising parameters on the right hand side are due to an apparent asymmetry in the definition of $D_2(T_R) = \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}$. If we use $\widehat{D_2(T_R)} = \{1, A(\pi), T_{R_2}, \widehat{T}_{R_1}\}$ instead, then by T–duality the parameters on the right hand side of (L12) are $(0,t,0,t)$. Our claim thus amounts to the fact that R_1 as applied to $\mathcal{C}_T(0,t/2,1/2,t/2)$ induces an ordinary $\mathbb{Z}_2$ action (or equivalently $\widehat{T}_{R_1}$) on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$. For the $(1,0)$ fields this is easy to check: R_1 leaves one of the Abelian currents of the torus theory invariant and multiplies the other by -1. So do $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$, where the T_{R_2} invariant generic Abelian current of the underlying torus theory is multiplied by -1, and the T_{R_2} invariant combination of vertex operators remains invariant. To give a full proof for (L12), note that the charge lattice of $\mathcal{C}_T(1/2,t/2,0,t/2)$ by (4.3.3) is generated by vectors

$$(p;\overline{p}) \in \left\{ \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\-\frac{1}{t}\end{pmatrix};\begin{pmatrix}1\\-\frac{1}{t}\end{pmatrix}\right), \quad \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix};\begin{pmatrix}-1\\0\end{pmatrix}\right), \right.$$
$$\left. \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\\frac{2}{t}\end{pmatrix};\begin{pmatrix}0\\\frac{2}{t}\end{pmatrix}\right), \quad \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}\frac{1}{2}\\\frac{t}{2}\end{pmatrix};\begin{pmatrix}-\frac{1}{2}\\-\frac{t}{2}\end{pmatrix}\right) \right\}.$$

The four vertex operators of dimension $\frac{1}{4}(1+\frac{1}{t^2})$ given by

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi^1+\delta\varphi^2/t)}e^{\frac{i\epsilon}{\sqrt{2}}(\overline{\varphi}^1+\delta\overline{\varphi}^2/t)},\ \epsilon,\delta\in\{\pm1\}$$

correspond to the following T_{R_2} invariant vertex operators of $\mathcal{C}_T(0,t,0,t)$:

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi_\prime^1+\varphi_\prime^2/t)}e^{\frac{i\epsilon}{\sqrt{2}}(\delta\overline{\varphi}_\prime^1+\overline{\varphi}_\prime^2/t)}-e^{\frac{i\epsilon}{\sqrt{2}}(-\varphi_\prime^1+\varphi_\prime^2/t)}e^{\frac{i\epsilon}{\sqrt{2}}(-\delta\overline{\varphi}_\prime^1+\overline{\varphi}_\prime^2/t)},\quad \epsilon,\delta\in\{\pm1\}$$

(see (4.3.3) to determine the charge lattice of $\mathcal{C}_T(0,t,0,t)$; $\varphi_\prime^\mu$ denote the bosonic fields in this torus theory to distinguish them from φ^μ on $\mathcal{C}_T(0,t/2,1/2,t/2)$). Both R_1 on $\mathcal{C}_T(1/2,t/2,0,t/2)$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$ pairwise interchange these vertex operators. The four vertex operators of dimension $\frac{1}{16}(1+t^2)$ given by

$$e^{\frac{i}{2\sqrt{2}}(\epsilon\varphi^1+\delta t\varphi^2)}e^{-\frac{i}{2\sqrt{2}}(\epsilon\overline{\varphi}^1+\delta t\overline{\varphi}^2)},\ \epsilon,\delta\in\{\pm1\},$$

correspond to the twisted ground states on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$, both being pairwise interchanged by R_1 on $\mathcal{C}_T(1/2,t/2,0,t/2)$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$ as well. This proves (L12). Modding out R_2 instead of R_1 gives the same result, up to T–duality. Note that the point (6.2.1) actually lies on (L12), hence we have found another quadrucritical point:

$$\begin{aligned}\mathcal{C}_{T_R-orb}(0,1,0,4) &= \mathcal{C}_{D_2^--orb}(0,2,0,2)\\ &= \mathcal{C}_{D_2(T_R)-orb}(0,1/2,0,2)=\mathcal{C}_{R-orb}(0,1,1/2,1).\end{aligned}\tag{Q2}$$

Moreover, (L12) intersects the bicritical lines (L2) and (L8), so there is another quadrucritical point:

$$\begin{aligned}\mathcal{C}_{R-orb}(1/2,1/2,0,2) &= \mathcal{C}_{D_2(T_R)-orb}(0,1,0,1)\\ &= \mathcal{C}_{\mathbb{Z}_2-orb}(0,2,0,2)=\mathcal{C}_{T_{R_1}-orb}(0,2,0,2).\end{aligned}\tag{Q3}$$

We proceed with the above reasoning to see that the $\mathbb{Z}_2$ action on the toroidal $\mathcal{C}_T(0,t/2,1/2,t/2)$ translates to a T'_{R_1} action on the theory $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)=\mathcal{C}_{T'_{R_2}-orb}(0,t,0,t)$ (this is the proof of (L9)). Therefore, in order to determine the action induced by D_2^+ on $\mathcal{C}_T(0,t/2,1/2,t/2)$, we note that on $\mathcal{C}_{T_{R_2}-orb}(0,t,0,t)$ the additional symmetry to mod out compared to (L12) on the underlying torus theory $\mathcal{C}_T(0,t,0,t)$ is the combination $T'_{R_1}\widehat{T}_{R_1}$, i.e. a shift by $\delta_1=\frac{1}{2}\binom{1}{0}$. Moreover, the $\mathbb{Z}_2$ twisted ground states in $\mathcal{C}_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2)$ are given by vertex operators which are $\widehat{T}_{R_1}$ invariant, and therefore

$$\forall\, t\in\mathbb{R}^+:\quad \mathcal{C}_{D_2^+-orb}(0,t/2,1/2,t/2)=\mathcal{C}_{D_2(T_R)-orb}(0,2/t,0,2t).\tag{L13}$$

This can also be seen by applying R_1 to $\mathcal{C}_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2)$ in (L9). Modding out the D_2^- action on the torus side analogously gives (L11), again. Note that the bicritical line (L13) intersects (L8) and (L10), so we have found two more tricritical points:

$$\mathcal{C}_{D_2^+-orb}(0,1/2,1/2,1/2)=\mathcal{C}_{D_2(T_R)-orb}(0,2,0,2)=\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,4),\tag{T4}$$

$$\mathcal{C}_{D_2^+-orb}(0,1,1/2,1)=\mathcal{C}_{D_2(T_R)-orb}(0,1,0,4)=\mathcal{C}_{D_2^--orb}(0,1,0,4).\tag{T5}$$

6.2.3 Series of multicritical lines and points obtainable from (L2)-(L4)

To gain further identifications from (L2) we can only mod out further symmetries of the underlying torus theory $\mathcal{C}_T(0,t,0,t)$. If we add generators of order four we only get trivial identities. An action of $\mathbb{Z}_2(R)$ type basically acts as a shift on the $\mathcal{C}_{T_{R_1}}(0,t,0,t)$ theory, so we arrive at the bicritical lines (L6) and (L7) again. All other symmetries give trivial identities.

Next we consider (L3). The symmetries we can generically mod out are $\mathbb{Z}_2, \mathbb{Z}_2(R)$ and $\mathbb{Z}_2(T_R)$, all giving trivial identities. For $t=\sqrt{3}/2$ we can mod out additional symmetries containing a $\mathbb{Z}_3$ action, but this does not produce anything new. For the special value $t=1/2$, where we have $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)=\mathcal{C}_{R_1-orb}(1/2,1/2,1/2,1/2)$ all but the modding out of T'_{R_1} give trivial identities as well. The symmetry T'_{R_1} multiplies both $\mathbb{Z}_2$ invariant $(1,0)$ fields in $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$ by -1, and the generators of the invariant part of the $\mathcal{C}_T(0,1,0,1)$ charge lattice are pairwise interchanged. The same is true for the $\mathbb{Z}_2$ twisted ground states. We claim that this translates to an R_2 action on $\mathcal{C}_{R_1-orb}(1/2,1/2,1/2,1/2)$. Namely, as a result of the discussion for lattice 7 around (5.3.5) we found that on $\mathcal{C}_T(1/2,1/2,1/2,1/2)$ the action of D_2 leaves invariant none of the combinations of vertex operators of dimensions $(1,0)$. The respective $(1/8,1/8)$ and $(1/2,1/2)$ fields are also pairwise interchanged in $\mathcal{C}_{R_1-orb}(1/2,1/2,1/2,1/2)$, thus

$$\mathcal{C}_{D_2(T'_R)-orb}(0,1,0,1)=\mathcal{C}_{D_2-orb}(1/2,1/2,1/2,1/2).$$

By (L9) and $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1/2,1/2,1/2)=\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)$ we see that we have actually found a tricritical point on a bicritical line:

$$\mathcal{C}_{D_2-orb}(1/2,1/2,1/2,1/2)=\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)=\mathcal{C}_{D_2(T'_R)-orb}(0,1,0,1). \tag{T6}$$

We remark that the above can be seen more directly by showing that in the notation of section 6.2.1 the groups $\widetilde{\mathbb{Z}_2(R_1)}\times\widetilde{\mathbb{Z}_2(R_2)}, \mathbb{Z}_2\times\mathbb{Z}_2(T_{\delta'})$ and $D_2(T'_R)$ are conjugate symmetry groups of type D_2 of the $SU(2)^2$ torus theory.

In the discussion of lattice 15 in section 5.3.2 we have found that $D_4^\pm$ acting on $\mathcal{C}_T(0,1,1/2,1/2)$ has a subgroup $D'_2\subset D_4^\pm$ which effectively acts on the toroidal $\mathcal{C}_T(1/2,1/2,1/2,1/2)=\mathcal{C}_T(0,1,0,1)$. By the above this action is conjugate to the one of $D_2(T'_R)$ on $\mathcal{C}_T(0,1,0,1)$, where $D_2(T'_R)\subset D_4^\pm(T'_R)$ generically exactly gives the distinction between $D_4^\pm(T'_R)$ and $D_4^\pm$. This means

$$\mathcal{C}_{D_4^+-orb}(0,1,1/2,1/2)=\mathcal{C}_{D_4(T'_R)^+-orb}(0,1,0,1), \tag{6.2.2}$$

$$\mathcal{C}_{D_4^--orb}(0,1,1/2,1/2)=\mathcal{C}_{D_4(T'_R)^--orb}(0,1,0,1). \tag{C1}$$

Let us now turn to the discussion of (L4). Generically, we can only mod out a $\mathbb{Z}_2$ action on $\mathcal{C}_T(0,2t,1/4,t/2)$. This leads to another bicritical line:

$$\forall\, t\in\mathbb{R}^+ :\quad \mathcal{C}_{D_2-orb}(1/2,t,1/2,t)=\mathcal{C}_{\mathbb{Z}_2-orb}(0,2t,1/4,t/2), \tag{L14}$$

as follows directly from (L3) and (L4). Note that (L14) intersects the bicritical line (L9) in (T6).
We can mod out additional symmetries of (L4) at special values of t, namely if $\rho = 1/4 + it/2$ is equivalent to ρ' with $\rho'_1 \in \{0, 1/2\}$ by Möbius transformations. This is true for $t \in \{1/2, \sqrt{3}/2, \sqrt{7}/2, \sqrt{5/12}, \sqrt{3/20}, \sqrt{1/28}\}$, but only for $t = \sqrt{3}/2$ we produce a new identification by our methods. Here, (L4) gives $\mathcal{C}_{R_2-orb}(1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2) = \mathcal{C}_T(0, \sqrt{3}, 0, \sqrt{3})$, and the torus theory decomposes into a tensor product of two $c = 1$ circle theories at radii $r = 1$ and $r' = \sqrt{3}$, respectively. The latter only contains one $(1, 0)$ field which is identified with the vertex operator $e^{i\sqrt{2/3}\varphi_1} e^{i\sqrt{2/3}\overline{\varphi}_1} + e^{-i\sqrt{2/3}\varphi_1} e^{-i\sqrt{2/3}\overline{\varphi}_1}$ in the $\mathcal{C}_{R_2-orb}(1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2)$ model. The $SU(2)$ generators of the first circle factor are identified with the two other R_2 invariant vertex operators and the Abelian current j^2 of $\mathcal{C}_T(1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2)$. The only symmetry we can mod out to find a new identification is T_{R_1}. Then by definition, of the $(1, 0)$ fields on the torus side only one is invariant, namely the Abelian current of the first factor theory. The same is true for the R_1 action on $\mathcal{C}_{R_2-orb}(1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2)$, where only one combination of vertex operators is invariant. Actually, the actions match entirely, showing

$$\mathcal{C}_{D_2-orb}(1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2) = \mathcal{C}_{T_{R_1}-orb}(0, \sqrt{3}, 0, \sqrt{3}). \qquad \text{(C2)}$$

6.2.4 Series of multicritical points obtainable from (Q1)

The identifications in section 6.2.1 we have not yet used by our discussions of the bicritical lines (L1)-(L5) are $\mathcal{C}_T(0, 1, 0, 2) = \mathcal{C}_{\mathbb{Z}_2-orb}(0, 1, 0, 1)$ and $\mathcal{C}_{T_{R_1}-orb}(0, 1, 0, 1)$ $= \mathcal{C}_{R-orb}(1/2, 1/2, 1/2, 1/2)$, taken from (Q1). In the latter case we can mod out additional symmetries on the underlying tori, but this produces no new identifications. Namely, the ordinary $\mathbb{Z}_2$ action applied to the left hand side gives the identification $\mathcal{C}_{D_2(T_R)-orb}(0, 1, 0, 1) = \mathcal{C}_{R_1-orb}(0, 1/2, 1/2, 1/2)$ on (L12), and $\mathbb{Z}_2$ applied to the right hand side gives $\mathcal{C}_{D_2(T'_R)-orb}(0, 1, 0, 1) = \mathcal{C}_{D_2-orb}(1/2, 1/2, 1/2, 1/2)$, see (T6). In fact, by the discussion at the beginning of the section we know that it suffices to mod out further symmetries of identities that contain toroidal theories.
We are now going to mod out further symmetries on both sides of the equality $\mathcal{C}_{\mathbb{Z}_2-orb}(0, 1, 0, 1) = \mathcal{C}_T(0, 1, 0, 2)$. We mostly use the description in terms of the toroidal theory $\mathcal{C}_T(0, 1, 0, 2)$, which by (4.3.3) has charge vectors

$$p_{\substack{l\\r}} = \frac{1}{2}\left\{\begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \pm 2\begin{pmatrix} m_2 \\ m_1 \end{pmatrix}\right\}, \quad m_i, n_i \in \mathbb{Z}. \qquad (6.2.3)$$

On the $\mathcal{C}_{\mathbb{Z}_2-orb}(0, 1, 0, 1)$ side, the torus currents J^1, J^2 of $\mathcal{C}_T(0, 1, 0, 2)$ are $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h; \overline{h}) = (1; 0)$ in the two $c = 1$ factors of $\mathcal{C}_T(0, 1, 0, 1)$. The states $|0, 0, \pm1, 0\rangle$, $|0, 0, 0, \pm1\rangle$ in $\mathcal{C}_T(0, 1, 0, 2)$ by (6.2.3) correspond to the $(1/8, 1/8)$ fields of the theory and therefore are identified with the four twisted ground states of the $\mathbb{Z}_2$ orbifold $\mathcal{C}_{\mathbb{Z}_2-orb}(0, 1, 0, 1)$. Further generators of the Hilbert space of $\mathcal{C}_T(0, 1, 0, 2)$ are vertex operators corresponding to $|\pm1, 0, 0, 0\rangle$, $|0, \pm1, 0, 0\rangle$ which are identified with the $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h;\overline{h}) = (1/2;1/2)$ of the $\mathcal{C}_T(0,1,0,1)$ side. These do not live in one of the separate factor theories.

The $\mathbb{Z}_2$ action on $\mathcal{C}_T(0,1,0,2)$ induces a $\widetilde{\mathbb{Z}_2(R)}$ action on the underlying torus of $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$, and we arrive at $\mathcal{C}_{D_2-orb}(1/2,1/2,1/2,1/2) = \mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)$ reproducing part of (T6). The R_1 action on $\mathcal{C}_T(0,1,0,2)$ translates to the orbifold theory $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$ in the following way: Among the $(1,0)$ fields in $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$ only the combination in the first factor of $\mathcal{C}_T(0,1,0,1)$ is invariant; two of the twisted ground states of the $\mathbb{Z}_2$ orbifold are exchanged, whereas two of them are fixed. Among the $(1/2,1/2)$ fields, again two are fixed and two are exchanged; this is just the R_1 action on $\mathcal{C}_{\mathbb{Z}_2-orb}(1/2,1/2,0,1)$, hence

$$\mathcal{C}_{D_2^+-orb}(1/2,1/2,0,1) = \mathcal{C}_{R-orb}(0,1,0,2). \tag{C3}$$

If we combine the $\mathbb{Z}_2$ and $\mathbb{Z}_2(R)$ actions on $\mathcal{C}_T(0,1,0,2)$, the $\mathbb{Z}_2$ now will act as a shift on the underlying torus of $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$. It is easier to understand the resulting identification by considering the $\mathbb{Z}_2$ orbifold theory $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)$. T'_{R_1} acts on $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)$ by pairwise interchanging the $\mathbb{Z}_2$ twisted ground states and multiplying the $\mathbb{Z}_2$ invariant vertex operators of dimensions $(1/8,1/8)$ in $\mathcal{C}_T(0,1,0,2)$ by -1. On the other hand, R_1 with negative discrete torsion will multiply the two T'_{R_1} invariant twisted ground state combinations by -1 but leave invariant the two $\mathbb{Z}_2$ invariant $(1/8,1/8)$ fields of $\mathcal{C}_T(0,1,0,2)$. These $\mathbb{Z}_2$ actions are conjugate, since the action on the invariant $\mathbb{Z}_2$ twisted ground state combinations of $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1) = \mathcal{C}_T(0,1,0,2)$ is merely exchanged with that on two combinations of twisted ground states of $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,2)$. This again is possible because of the $c=1$ identification between the circle theory at radius $r=1$ and the orbifold theory at radius $r=2$. In summary,

$$\mathcal{C}_{D_2(T'_R)-orb}(0,1,0,2) = \mathcal{C}_{D_2^--orb}(0,1,0,2). \tag{C4}$$

The T_{R_1} action on $\mathcal{C}_T(0,1,0,2)$ differs from the R_1 action by a sign in the action on the $(1/8,1/8)$ fields, i.e. the twisted ground states of the $\mathbb{Z}_2$ orbifold on the $\mathcal{C}_{\mathbb{Z}_2-orb}(1/2,1/2,0,1)$ side. Therefore by comparison with (C3)

$$\mathcal{C}_{D_2^--orb}(1/2,1/2,0,1) = \mathcal{C}_{T_R-orb}(0,1,0,2). \tag{C5}$$

Comparison of (C3) with (C5) also gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion.

If we mod out $\widetilde{\mathbb{Z}_2(R)}$ and the corresponding D_2 type symmetries on $\mathcal{C}_T(0,1,0,2)$, i.e. consider $\mathbb{Z}_2(R)$ on $\mathcal{C}_T(1/2,1/2,0,2)$, we only reproduce identities we have found already above: $\mathcal{C}_{D_2(T_R)-orb}(0,1,0,1) = \mathcal{C}_{R-orb}(1/2,1/2,0,2)$ on (L12), as well as $\mathcal{C}_{D_2(T_R^{(\prime)})-orb}(0,2,0,2) = \mathcal{C}_{D_2^\pm-orb}(1/2,1/2,0,2)$ on (L13) and (L11), respectively.

Next we discuss the action of T_{R_1} on $\mathcal{C}_T(0,1,0,1/2)$ instead of $\mathcal{C}_T(0,1,0,2)$. In (6.2.3) this exchanges the roles of m_i and n_i, such that compared to the action of R_1 on $\mathcal{C}_T(0,1,0,2)$ we now have additional signs on $(1/2,1/2)$ fields. In particular, only one combination of $(1/2,1/2)$ fields is invariant, as well as three of the twisted ground state combinations in $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1)$. We claim that this is the residual action of an ordinary $\mathbb{Z}_4$ rotation on $\mathcal{C}_T(0,1,0,1)$. It acts by interchanging the two circle factors of $\mathcal{C}_T(0,1,0,1)$, but the generators of the Hilbert space of the second factor are multiplied with an additional sign. Indeed, this is exactly the T_{R_1} action on a torus whose lattice has an additional generator $(1/2,1/2)$ compared to $\mathbb{Z}^2$ for $\mathcal{C}_T(0,1,0,1)$, i.e. on $\mathcal{C}_T(0,1,0,1/2)$. Hence,

$$\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,1) = \mathcal{C}_{T_R-orb}(0,1,0,1/2). \tag{C6}$$

Using (C6) we can further mod out T_{R_2} on the underlying torus theory of the above $\mathcal{C}_{T_{R_1}-orb}(0,1,0,1/2)$. This translates to a $\widetilde{\mathbb{Z}_2(R_2)}$ action on the underlying torus theory of $\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,1)$, so

$$\mathcal{C}_{D_4^+-orb}(0,1,1/2,1/2) = \mathcal{C}_{D_2(T_R)-orb}(0,1,0,1/2).$$

By (6.2.2) we see that we have actually found a tricritical point:

$$\mathcal{C}_{D_4^+-orb}(0,1,1/2,1/2) = \mathcal{C}_{D_2(T_R)-orb}(0,1,0,1/2) = \mathcal{C}_{D_4(T_R')^+-orb}(0,1,0,1). \tag{T7}$$

We now rewrite (C6) as $\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,1) = \mathcal{C}_{T_{R_1}'-orb}(0,1,0,1/2)$ and mod out by T_{R_2}' on the underlying torus of the right hand side. Analogously to the T_{R_2} action on $\mathcal{C}_{T_{R_1}-orb}(0,t/2,0,2t)$ in (L7), which induced a shift on the underlying torus theory of $\mathcal{C}_{D_2^--orb}(0,t,0,t)$, in (C6) we get a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,1)$. Then we obtain

$$\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,2) = \mathcal{C}_{D_2(T_R')-orb}(0,1,0,1/2). \tag{C7}$$

Back to the identification $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,1) = \mathcal{C}_T(0,1,0,1/2)$ in (Q1) we now mod out groups containing $\mathbb{Z}_4$ on the torus side. With the ordinary $\mathbb{Z}_4$ action we reproduce the above bicritical point (C7), but in combination with $D_2(T_R')$, the $\mathbb{Z}_4$ generator acts as a shift on the underlying torus theory of $\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,2)$ in (C7):

$$\begin{aligned} \mathcal{C}_{D_4(T_R')^+-orb}(0,1,0,1/2) &= \mathcal{C}_{\mathbb{Z}_4-orb}(0,1,0,4), && \text{(C8)} \\ \mathcal{C}_{D_4(T_R')^--orb}(0,1,0,1/2) &= \mathcal{C}_{\mathbb{Z}_4-orb}(0,1,1/2,1). && \text{(C9)} \end{aligned}$$

The latter identification is more easily understood when we mod out symmetries on the tricritical point (T2), as we will do in section 6.2.6.
The effect of D_4 type actions is most easily understood from the fact that by (C4) the action of $D_2(T_R') \subset D_4(T_R')^\pm$ on $\mathcal{C}_T(0,1,0,2)$ is conjugate to that of $D_2^- \subset D_4^-$. Therefore,

$$\mathcal{C}_{D_4(T_R')^--orb}(0,1,0,2) = \mathcal{C}_{D_4^--orb}(0,1,0,2). \tag{6.2.4}$$

6.2.5 Series of multicritical points obtainable from (T1)

From the multicritical points and lines determined so far we can find further multicritical points by modding out further symmetries. By the systematic procedure we followed above, this can only give something new, if we use an identification obtained as intersection of bicritical lines. Moreover, because by the discussion at the beginning of the section it suffices to use identifications containing a toroidal theory, only (T1) and (T2) are left to be discussed in this and the following section. For the point (T1) only the identification $\mathcal{C}_T(0,1,0,4) = \mathcal{C}_{D_2^+}(0,1,0,1)$ has not been used yet. By modding out $\mathbb{Z}_2$ we yield (T4) from (T1), in particular we find $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,0,4) = \mathcal{C}_{D_2^+-orb}(0,1/2,1/2,1/2)$. Modding out a $\mathbb{Z}_2(R)$ action yields $\mathcal{C}_{R-orb}(0,1,0,4) = \mathcal{C}_{D_2^+-orb}(0,2,0,2)$ on (L6). Note that this shows that $\mathbb{Z}_2$ and R on $\mathcal{C}_T(0,1,0,4)$ both induce shifts on the underlying torus theory of $\mathcal{C}_{D_2^+}(0,1,0,1)$, namely $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$, and $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, respectively. The combined action gives a trivial identity for D_2^+, and $\mathcal{C}_{D_2^--orb}(0,1,0,4) = \mathcal{C}_{D_2^+-orb}(0,1,1/2,1)$ in (T5). Modding out $\mathbb{Z}_2(T_R), D_2(T_R)$ and $D_2(T'_R)$ reproduces the points at $t = 2$ in (L7), (L10), and (L11), respectively. Modding out $\mathbb{Z}_4$ reproduces (C8). To determine the result of modding out D_4 actions, note that by the above the action of R induces a shift T_{δ_1} on the underlying torus theory of $\mathcal{C}_{D_2^+}(0,1,0,1)$, so from (C8) we obtain

$$\mathcal{C}_{D_4^--orb}(0,1,0,4) = \mathcal{C}_{D_4(T'_R)^+-orb}(0,1,0,4). \tag{6.2.5}$$

All the other choices of discrete torsion give trivial identities. Modding out by $D_4(T'_R)^\pm$ gives the same or a trivial identity again.

Next we mod out $\widetilde{\mathbb{Z}_2(R)}$, i.e. $\mathbb{Z}_2(R)$ on $\mathcal{C}_T(1/2,1/2,0,4)$. This interchanges the two circle factors of the original $\mathcal{C}_T(0,1,0,1)$ in $\mathcal{C}_{D_2^+-orb}(0,1,0,1)$ above and thus is equivalent to adding a $\mathbb{Z}_4$ generator to D_2. Therefore,

$$\mathcal{C}_{R-orb}(1/2,1/2,0,4) = \mathcal{C}_{D_4^+-orb}(0,1,0,1). \tag{C10}$$

To mod out the corresponding D_2 actions we again use the above observation that $\mathbb{Z}_2$ on $\mathcal{C}_T(1/2,1/2,0,4)$ acts as T_{δ_1} on the torus theory underlying $\mathcal{C}_{D_2^+-orb}(0,1,0,1)$ to find

$$\mathcal{C}_{D_2^+-orb}(1/2,1/2,0,4) = \mathcal{C}_{D_4^+-orb}(0,1,0,2), \tag{C11}$$

and

$$\mathcal{C}_{D_2^--orb}(1/2,1/2,0,4) = \mathcal{C}_{D_4(T'_R)^+-orb}(0,1,0,2). \tag{C12}$$

6.2.6 Series of multicritical points obtainable from (T2)

We now discuss additional identifications that can be obtained from (T2). The only identity not used up to now is $\mathcal{C}_T(0,1,1/2,1) = \mathcal{C}_{D_2^--orb}(0,1,0,1)$. If we mod out a $\mathbb{Z}_2$ action from the torus theory, (T2) is transformed into (T3), in particular we yield $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,1/2,1) = \mathcal{C}_{D_2^--orb}(1/2,1/2,0,1/2)$. The $\mathbb{Z}_2$ action thus induces a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $\mathcal{C}_{D_2^--orb}(0,1,0,1)$. The R action on $\mathcal{C}_T(0,1,1/2,1)$ induces a shift as well, now by $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, yielding

$\mathcal{C}_{R-orb}(0,1,1/2,1) = \mathcal{C}_{D_2^- -orb}(0,2,0,2)$ in (Q2). The combined R and $\mathbb{Z}_2$ actions thus yield a trivial identity for D_2^- and $\mathcal{C}_{D_2^+ -orb}(0,1,1/2,1) = \mathcal{C}_{D_2(T_R)-orb}(0,1,0,4)$ on (L13). Modding out $\mathbb{Z}_4$ is equivalent to modding out another $\mathbb{Z}_2$ action on $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,1/2,1) = \mathcal{C}_{D_2^- -orb}(1/2,1/2,0,1/2)$ which interchanges the circle factors of the underlying geometric torus (i.e. $\mathbb{Z}_2$ invariant vertex operators with $h = \overline{h}$). The action matches a $\widetilde{D_4}$ action on $\mathcal{C}_{D_2^- -orb}(1/2,1/2,0,1/2)$, where the additional D_2^- invariant vertex operators as compared to $\mathcal{C}_{D_2^- -orb}(1/2,1/2,0,1)$ correspond to the $\mathbb{Z}_2$ twisted ground states of $\mathcal{C}_{\mathbb{Z}_2-orb}(0,1,1/2,1)$. We thus obtain $\mathcal{C}_{\mathbb{Z}_4-orb}(0,1,1/2,1) = \mathcal{C}_{D_4(T_R')^- -orb}(0,1,0,1/2)$ reproducing (C9). Since by the above we know that R_1 on $\mathcal{C}_T(0,1,1/2,1)$ induces a T_{δ_1} shift on the underlying torus theory of $\mathcal{C}_{D_2^- -orb}(0,1,0,1)$, it also follows that

$$\mathcal{C}_{D_4^- -orb}(0,1,1/2,1) = \mathcal{C}_{D_4(T_R')^- -orb}(0,1,0,4). \tag{C13}$$

Flipping the sign of discrete torsion on both sides of the above equivalence we find

$$\mathcal{C}_{D_4^+ -orb}(0,1,1/2,1) = \mathcal{C}_{D_4(T_R')^+ -orb}(0,1,0,4),$$

which together with (6.2.5) yields a tricritical point:

$$\mathcal{C}_{D_4^+ -orb}(0,1,1/2,1) = \mathcal{C}_{D_4(T_R')^+ -orb}(0,1,0,4) = \mathcal{C}_{D_4^- -orb}(0,1,0,4). \tag{T8}$$

We now mod out $\widetilde{\mathbb{Z}_2(R)}$ on $\mathcal{C}_T(0,1,1/2,1)$, i.e. $\mathbb{Z}_2(R)$ on $\mathcal{C}_T(1/2,1/2,1/2,1)$. Similarly to (C10) we find

$$\mathcal{C}_{R-orb}(1/2,1/2,1/2,1) = \mathcal{C}_{D_4^- -orb}(0,1,0,1). \tag{C14}$$

Because by the above, $\mathbb{Z}_2$ on $\mathcal{C}_T(1/2,1/2,1/2,1)$ induces a shift $T_{\delta'}$ on the underlying torus theory of $\mathcal{C}_{D_4^- -orb}(0,1,0,1)$ in (C14), we find

$$\mathcal{C}_{D_2-orb}(1/2,1/2,1/2,1) = \mathcal{C}_{D_4(T_R')^- -orb}(0,1,0,2).$$

Together with (6.2.4) this gives another tricritical point:

$$\mathcal{C}_{D_2-orb}(1/2,1/2,1/2,1) = \mathcal{C}_{D_4(T_R')^- -orb}(0,1,0,2) = \mathcal{C}_{D_4^- -orb}(0,1,0,2). \tag{T9}$$

6.2.7 Multicritical points obtained from conjugate $\mathbb{Z}_3, D_3$, $\mathbb{Z}_6$ and D_6 type actions

We start by comparing all $\mathbb{Z}_3$ type symmetries of the $SU(3)$ torus theory at parameters $\tau = \rho = \omega$, $\omega := e^{2\pi i/3}$. The generically conserved currents of the torus theory we call j^1, j^2, and k^1, k^2, k^3 together with $l^\mu = (k^\mu)^\dagger, \mu \in \{1,2,3\}$, denote the additional vertex operators with dimensions $(h;\overline{h}) = (1;0)$. The fields j^μ, k^μ, l^μ generate an $SU(3)_1$ Kac–Moody algebra, and $\{k^\mu\}$, $\{l^\mu\}$ form closed orbits under the ordinary $\mathbb{Z}_3$ action. In passing we remark that among all possible $\mathbb{Z}_2$ symmetries of $\mathcal{C}_T(\omega,\omega)$, those conjugate only reproduce (L3).

Among the $\mathbb{Z}_3$ actions on one hand we have the ordinary rotational $\mathbb{Z}_3$ which leaves two fields $k^1+k^2+k^3$ and $l^1+l^2+l^3$ invariant, three fields $j_+ = j^1+ij^2, k^1+\omega k^2+\omega^2 k^3, l^1+\omega l^2+\omega^2 l^3$ have eigenvalue ω. On the other hand, the shift orbifold by $\delta = \frac{1}{2}(\lambda_1 - \lambda_2)$ exhibits the same spectrum, where the λ_i as usual denote a basis of the lattice associated to the parameters $\tau = \rho = \omega$. Here, j^1, j^2 are invariant, and k^1, k^2, k^3 have eigenvalue ω. We particularly see that the two $\mathbb{Z}_3$ actions are conjugate, thus modding out $\mathcal{C}_T(\omega,\omega)$ by these two symmetries gives isomorphic theories. The shift orbifold again produces a torus theory with same parameter $\tau = \omega$, but ρ reduced by a factor of three; in the following we use $\alpha := 1/2+i3\sqrt{3}/2$ which is related to $\omega/3$ by the Möbius transformation T^2S and state

$$\mathcal{C}_{\mathbb{Z}_3-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) = \mathcal{C}_T(1/2,\sqrt{3}/2,1/2,3\sqrt{3}/2). \qquad \text{(C15)}$$

We will now mod out additional symmetries on both sides of the above equality. Only those of order two give new identifications. Note that both R_2 and the ordinary $\mathbb{Z}_2$ on $\mathcal{C}_T(\omega,\omega)$ interchange the two $\mathbb{Z}_3$–invariant $(1,0)$ fields $k^1+k^2+k^3$ and $l^1+l^2+l^3$. Thus $R_2, -\mathbb{1}$ must act as R_1, R_2 on the torus theory $\mathcal{C}_T(\omega,\alpha)$. Study the action on the charge lattice to check that the order above is indeed correct. This means that the R_1 action on $\mathcal{C}_T(\omega,\omega)$ must induce the ordinary $\mathbb{Z}_2$ action on $\mathcal{C}_T(\omega,\alpha)$. In particular, the fields $k^1+k^2+k^3$ and $l^1+l^2+l^3$ are multiplied by -1 under R_1. Here we can confirm our result of the discussion of lattice 7 around (5.3.5): The signs obtained there occur in a completely natural way in the present example.
All in all for the $\mathbb{Z}_2$ actions on $\mathcal{C}_T(\omega,\omega)$ compared to $\mathcal{C}_T(\omega,\alpha)$ we have found $(R_1,R_2,-\mathbb{1}) \mapsto (R_2,-\mathbb{1},R_1)$ and therefore directly obtain the following bicritical points:

$$\begin{aligned}
\mathcal{C}_{D_3(R_1)-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) &= \mathcal{C}_{\mathbb{Z}_2-orb}(1/2,\sqrt{3}/2,1/2,3\sqrt{3}/2), && \text{(C16)}\\
\mathcal{C}_{D_3(R_2)-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) &= \mathcal{C}_{R_1-orb}(1/2,\sqrt{3}/2,1/2,3\sqrt{3}/2), && \text{(C17)}\\
\mathcal{C}_{\mathbb{Z}_6-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) &= \mathcal{C}_{R_2-orb}(1/2,\sqrt{3}/2,1/2,3\sqrt{3}/2), && \text{(C18)}\\
\mathcal{C}_{D_6-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) &= \mathcal{C}_{D_2-orb}(1/2,\sqrt{3}/2,1/2,3\sqrt{3}/2). && \text{(C19)}
\end{aligned}$$

6.3 Dixon, Ginsparg and Harvey's results on $c = 3/2$ revisited

We can construct conformal field theories $\mathcal{C}^B_\bullet$ with central charge $c = 2$ by tensorizing the bosonic subtheories $\mathcal{C}^{c=3/2}_\bullet$ of $N = (1,1)$ superconformal field theories with the bosonic subtheory of the Ising model (3.3). By [DGH88], the moduli space of $N = (1,1)$ superconformal field theories with $c = 3/2$ contains five connected lines. The CIRCLE LINE $\mathcal{C}^{c=3/2}_{circ}(r)$ is just the moduli space of "toroidal" superconformal field theories of definition 4.1.3 in dimension $d = 1$, such that $\mathcal{C}_f$ is just the Ising model (3.3). The tensor product of two Ising models is the critical Ashkin–Teller model and has a bosonic description as $\mathbb{Z}_2$ orbifold of the $c = 1$ circle theory at radius $r' = \sqrt{2}$ [Zam87b]. By the discussion of $\mathbb{Z}_2(R)$ orbifolds in section 5.3 we therefore directly obtain

$$\mathcal{C}^B_{circ}(\sqrt{2}r) = \mathcal{C}^{c=1}_{orb}(\sqrt{2}) \otimes \mathcal{C}^{c=1}(\sqrt{2}r) = \mathcal{C}_{R_2-orb}(0, r, 0, 2r).$$

The other four lines in the $c = 3/2$ moduli space are obtained as orbifold models of $\mathcal{C}^{c=3/2}_{circ}(r)$. The ordinary $\mathbb{Z}_2$ orbifold (see theorem 5.2.2) gives the so–called ORBIFOLD LINE $\mathcal{C}^{c=3/2}_{orb}(r)$. For the fermions the orbifold procedure effectively only exchanges boundary conditions, which we here forget about since we only consider the bosonic subtheories. Therefore, we can regard $\mathbb{Z}_2$ as only acting on the second circle factor of $\mathcal{C}^B_{circ}(\sqrt{2}r) = \mathcal{C}_{R_2-orb}(0, r, 0, 2r)$. This amounts to modding out an R_1 action, i.e.

$$\mathcal{C}^B_{orb}(\sqrt{2}r) = \mathcal{C}_{D_2^+-orb}(0, r, 0, 2r).$$

Note that by the results of section 6.2 and in agreement with [DGH88] the only intersection point of the above lines is situated on (L6):

$$\mathcal{C}^B_{circ}(2) = \mathcal{C}^B_{orb}(1). \tag{6.3.1}$$

The remaining three lines are the superaffine, super–2–orbifold (or simply SUPER–ORBIFOLD), and the orbifold–prime lines in dimension $d = 1$ as introduced in section 5.4. To determine $\mathcal{C}^B_{s-a}(\sqrt{2}r)$, we trivially continue the action of $S_\delta = (-1)^{F_S} T_\Delta$ to $\mathcal{C}^B_{circ}(\sqrt{2}r)$. Then S_δ remains to act as ordinary shift orbifold on the second factor theory in $\mathcal{C}^B_{circ}(\sqrt{2}r)$, the $c = 1$ circle theory at radius $\sqrt{2}r$. On the first factor, we have the action of $(-1)^{F_S}$ on one of the Majorana fermions. We use the bosonic description as $\mathbb{Z}_2$ orbifold of the $c = 1$ circle theory at radius $\sqrt{2}$. Here, the Ramond sector is built on those Hilbert space ground states with odd label of the momentum mode. Thus on the underlying $c = 1$ circle theory, $(-1)^{F_S}$ acts as shift orbifold as well. This means that $\mathcal{C}^B_{s-a}(\sqrt{2}r)$ can be obtained as shift orbifold by $T_{\delta'}$, $\delta' = \frac{1}{\sqrt{2}}\binom{1}{r}$, on the underlying torus theory $\mathcal{C}_T(0, r, 0, 2r)$ of $\mathcal{C}^B_{circ}(\sqrt{2}r)$:

$$\mathcal{C}^B_{s-a}(\sqrt{2}r) = \mathcal{C}_{R_2-orb}(1/2, r/2, 0, r).$$

Recall that the superorbifold line $\mathcal{C}^{c=3/2}_{s-orb}(r)$ is a D_2 type orbifold of $\mathcal{C}^{c=3/2}_{circ}(r)$ by the group generated by the ordinary $\mathbb{Z}_2$ action and S_δ. Since by the above $\mathbb{Z}_2$ and S_δ act as reflection R_1 and shift $T_{\delta'}$ on the underlying torus theory $\mathcal{C}_T(0, r, 0, 2r)$ of $\mathcal{C}^B_{circ}(\sqrt{2}r)$, respectively, we find

$$\mathcal{C}^B_{s-orb}(\sqrt{2}r) = \mathcal{C}_{D_2^+-orb}(1/2, r/2, 0, r).$$

By the results of section 6.2 we see that only the superorbifold line intersects one of the other three lines discussed so far, namely in (C3):

$$\mathcal{C}^B_{s-orb}(\sqrt{2}) = \mathcal{C}^B_{circ}(\sqrt{2}). \tag{6.3.2}$$

This agrees with the results of [DGH88].

Recall that the orbifold-prime line $\mathcal{C}^{c=3/2}_{orb'}(r)$ is obtained by modding out $S_R := (-1)^{F_S} \cdot (-1\!\!1)$ from $\mathcal{C}^{c=3/2}_{circ}(r)$. Since the generator $-1\!\!1$ of the ordinary $\mathbb{Z}_2$ action on $\mathcal{C}^B_{circ}(\sqrt{2}r)$ acts as reflection R_1, and $(-1)^{F_S}$ induces the shift orbifold on the underlying $c = 1$ circle theory at radius $\sqrt{2}$ of the first factor in $\mathcal{C}^B_{circ}(\sqrt{2})$, S_R acts as T_{R_1} on the underlying torus theory $\mathcal{C}_T(0, r, 0, 2r)$ of $\mathcal{C}^B_{circ}(\sqrt{2}r)$. Therefore,

$$\mathcal{C}^B_{orb'}(\sqrt{2}r) = \mathcal{C}_{D_2(T_R)-orb}(0, r, 0, 2r).$$

Concerning intersections of the orbifold–prime line with the other lines discussed above, again we are in exact agreement with the results of [DGH88]: We find multicritical points on (L13) and (L12), namely

$$\begin{aligned} \mathcal{C}^B_{orb'}(2) &= \mathcal{C}^B_{s-orb}(2), && (6.3.3) \\ \mathcal{C}^B_{orb'}(1) &= \mathcal{C}^B_{s-a}(2). && (6.3.4) \end{aligned}$$

It is a straightforward calculation to check (5.4.1) for our $c = 2$ models, i.e.

$$Z_{D_2(T_R)-orb}(0, r, 0, 2r) = Z_{D_2^+-orb}(0, r, 0, 2r) - 3Z_{Ising}$$

from (5.3.14) and (3.3).

The above in particular gives a geometric interpretation in terms of crystallographic orbifolds to all the nonisolated orbifolds discussed in [DGH88].

Up to now we have carefully restricted the discussion to bosonic subtheories of the $N = (1, 1)$ superconformal field theories. The reason is that three of the four intersection points of lines given in [DGH88] do not hold on the level of superconformal field theories. In the notation of section 3, only the bosonic sectors $\mathcal{H}_b$ agree in these points, not the fermionic ones $\mathcal{H}_f$. Let us discuss the alleged intersection points case by case, which for convenience are labelled by the formula of the corresponding $c = 2$ theory above, that expresses the correct observation that the bosonic parts of the $c = 3/2$ factor theories coincide.

Concerning (6.3.1), the $\mathbb{Z}_2$ orbifold of the $N = (1, 1)$ "toroidal" superconformal field theory at $r = 1$ does not contain any fields of dimensions $(h; \overline{h}) = (\frac{1}{2}; 0)$, since the Majorana fermion of the toroidal theory is not $\mathbb{Z}_2$ invariant and no holomorphic vertex operators with $h = \frac{1}{2}$ exist. On the other hand, the $N = (1, 1)$ "toroidal" superconformal field theory at $r = 2$ does possess a free left handed Majorana fermion with dimensions $(h; \overline{h}) = (\frac{1}{2}; 0)$ in the fermionic sector $\mathcal{H}_f$ which therefore cannot agree with the fermionic sector on the orbifold line.

The intersection point (6.3.2) of circle and super–orbifold lines is the only one that Dixon, Ginsparg and Harvey argue for on the level of operator algebras in [DGH88]. Their argument proves that this point is a true intersection point of the two lines, also on the level of superconformal field theories.

To understand the bosonic intersection point (6.3.3) we make use of the results on partition functions of the respective theories in section 5.4. Though the orbifold prime and super–orbifold theories at $r = 2$ have the same vacuum character in each sector, counting of the $(h; \overline{h}) = (\frac{1}{16}; \frac{1}{16})$ fields shows that the theories do not agree.

The orbifold prime theory possesses one bosonic Ramond and one bosonic Neveu–Schwarz as well as two fermionic Ramond fields with these dimensions, whereas the fermionic Ramond ground states are missing in the super–orbifold theory.
For the intersection point (6.3.4) of orbifold prime and super–affine orbifold lines again a counting of the $(h;\overline{h}) = (\frac{1}{2};0)$ fields suffices. The super-affine orbifold at $r=2$ possesses a fermionic Neveu–Schwarz field with these dimensions which is missing in the orbifold prime theory at $r=1$.
Despite the above corrections to the statements of [DGH88], one can consider fermionic tensor products $\mathcal{C}^F_\bullet$ of $N=(1,1)$ superconformal field theories with the Ising model and search for the resulting components of $\mathcal{M}^2$ in our description. We obtain

$$\begin{array}{rclrcl} \mathcal{C}^F_{circ}(\sqrt{2}r) &=& \mathcal{C}_T(0,r,0,2r), & \mathcal{C}^F_{orb}(\sqrt{2}r) &=& \mathcal{C}_{\mathbb{Z}_2-orb}(0,r,0,2r), \\ \mathcal{C}^F_{s-a}(\sqrt{2}r) &=& \mathcal{C}_T(\frac{1}{2},\frac{r}{2},0,r), & \mathcal{C}^F_{s-orb}(\sqrt{2}r) &=& \mathcal{C}_{\mathbb{Z}_2-orb}(\frac{1}{2},\frac{r}{2},0,r), \\ \mathcal{C}^F_{orb'}(\sqrt{2}r) &=& \mathcal{C}_{\mathbb{Z}_2-orb}(0,r,0,2r). &&& \end{array} \quad (6.3.5)$$

Since above we argued that the intersection point (6.3.2) is a true intersection point also on the level of superconformal field theories, we should expect a corresponding intersection point for $\mathcal{C}^F_{circ}(r)$ and $\mathcal{C}^F_{s-orb}(r)$. Indeed, from (6.3.5) we read

$$\mathcal{C}^F_{s-orb}(\sqrt{2}) = \mathcal{C}_{\mathbb{Z}_2-orb}(\tfrac{1}{2},\tfrac{1}{2},0,1) = \mathcal{C}_T(0,1,0,2) = \mathcal{C}^F_{circ}(\sqrt{2})$$

by $PSL(2,\mathbb{Z})^2$ invariance of $\mathcal{M}_{\mathbb{Z}_2-orb}$ and (Q1). On the other hand, $\mathcal{C}^F_{circ}(2) \neq \mathcal{C}^F_{orb}(1)$, $\mathcal{C}^F_{orb'}(2) \neq \mathcal{C}^F_{s-orb}(2), \mathcal{C}^F_{orb'}(1) \neq \mathcal{C}^F_{s-a}(2)$, in full agreement with our above observation that these are no intersection points on the level of $N=(1,1)$ superconformal field theories with $c=3/2$.

6.4 Summary: A glimpse on the structure of $\mathcal{M}^2$

We can now give a complete description of those nonisolated parts of the moduli space $\mathcal{M}^2$ of unitary conformal field theories with $c=2$ that can be constructed by an orbifold procedure from toroidal theories and are nonexceptional. The exceptional cases are those related to the binary tetrahedral, octahedral and icosahedral subgroups of $SU(2)$, see definition 6.1.2. All the nonexceptional cases are obtained as orbifolds with geometric interpretation by crystallographic symmetries (theorem 6.1.3). Apart from the moduli space $\mathcal{M}_2^{Narain}$ of toroidal theories we find 26 components of $\mathcal{M}^2$, which exhibit a complicated graph like structure. There are fourteen bicritical lines and 31 multicritical points, among them three quadrucritical and nine tricritical points. We have proven multicriticality on the level of the operator algebra for all these lines and points. Our analysis of multicritical points shows that all but three of the irreducible crystallographic components of the moduli space are directly or indirectly connected to the moduli space of toroidal theories. The remaining three components are $\mathcal{M}^{(0)}_{D_6^\pm-orb}, \mathcal{M}^{(0)}_{D_3(R)-orb}$.
Our results are consistent with those on $c=3/2$ superconformal field theories [DGH88], as long as their bosonic subtheories are concerned only: We have determined the tensor products of the bosonic subtheories of the five continuous lines of

$c = 3/2$ superconformal field theories discussed in [DGH88] with that of an Ising model in terms of our description of $\mathcal{M}^2$. All multicritical points in the $c = 3/2$ moduli space are reidentified by our results on $\mathcal{M}^2$. In particular, this gives geometric interpretations to all nonisolated orbifolds discussed in [DGH88] in terms of crystallographic orbifolds.
We also find the bosonic parts of the fermionic tensor products of the five lines with the Ising model in our picture. Contrary to the statements in [DGH88] these lines have only one intersection point on the level of superconformal field theories. The latter observation is also in full agreement with our picture of $\mathcal{M}^2$.
A discussion of the exceptional components of $\mathcal{M}^2$ is not carried out in this work. By our results, these would yield the only possible examples of asymmetric orbifold conformal field theories [NSV87] with $c = 2$ and therefore should be studied separately. Neither do we touch the determination of isolated components of the moduli space, which is expected to be even more involved. Apart from that, our results do not give a complete classification of unitary conformal field theories with central charge $c = 2$, since we are lacking a theorem which would tell us that all nonisolated components of the moduli space may be obtained by some orbifold procedure from a subspace of the toroidal component. It would also be interesting to determine those theories in $\mathcal{M}^2$ which admit supersymmetry.

Chapter 7

The moduli space of superconformal field theories with central charge $c = 6$

In this chapter, we study the moduli space $\mathcal{M}$ of $N = (4,4)$ superconformal field theories with central charge $c = 6$. Section 7.1 is devoted to its global description in an emended version as compared to the literature. In section 7.2 we study partition functions on the moduli space, we in particular construct "topological" or rather "a generic part of" partition functions of theories in $\mathcal{M}$. In section 7.3 we discuss the component of $\mathcal{M}$ which consists of theories that are associated to $K3$. Some geometric features of $K3$ surfaces, in particular algebraic automorphisms, the determination of generic Picard numbers, and the description of the integer cohomology for $\mathbb{Z}_M$ orbifold limits of $K3$ surfaces are given in section 7.3.1. Section 7.3.2 deals with $\mathbb{Z}_2$ orbifold conformal field theories in $\mathcal{M}$. The results enable us to discuss Nahm and Fourier–Mukai transforms from a purely conformal field theoretic point of view in section 7.3.3, such that we can prove T–duality and justify our global description of $\mathcal{M}$ without leaving this framework. The embedding of the other $\mathbb{Z}_M$ orbifold conformal field theories within $\mathcal{M}^{K3}$, i.e. $M \in \{3,4,6\}$, is the object of section 7.3.4. This in particular allows us to show in section 7.3.5 that the $\mathbb{Z}_4$ orbifold of the nonlinear σ model on the torus with lattice $\Lambda = \mathbb{Z}^4$ has a geometric interpretation on the Fermat quartic hypersurface. Two further models on the quartic are determined, too. In section 7.3.6 we find the locations of the Gepner model $(2)^4$ and of some of its orbifolds within $\mathcal{M}$ by proving isomorphisms to nonlinear σ models. In particular, by identifying $(2)^4$ with the above $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\mathbb{Z}^4, 0)$ we are able to show that this Gepner model has a geometric interpretation with Fermat quartic target space. We find a meeting point of the moduli spaces of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold conformal field theories different from the one conjectured in [EOTY89]. In section 7.4 we close with a panoramic view of $\mathcal{M}$ that summarizes the results of the present chapter.

7.1 The moduli space: General discussion

By $\mathcal{M}$ we denote the moduli space of $N = (4,4)$ superconformal field theories with central charge $c = 6$, where the superconformal algebra is obtained as extension of an $N = (2,2)$ superconformal algebra by an $su(2)_1$ Kac–Moody algebra as introduced in section 3.2. The general features of $\mathcal{M}$ discussed in the present section have been extracted from the literature [Sei88, Cec91, AM94, Asp97] and emended in joint work with Werner Nahm that is accepted for publication in [NW].
For any $N = (4,4)$ superconformal theory the space of states $\mathcal{H}$ contains four–dimensional vector spaces Q_l and Q_r of real left and right supercharges. Since we consider left and right central charge $c = 6$ and restrict to the Ademollo et al algebra (3.2.1), (3.2.2) at $k = 1$, $\mathcal{H}$ carries the action of an $su(2) \oplus su(2)$ current algebra of level 1. The (3+3)-dimensional Lie group generated by the corresponding charges will be denoted $SU(2)_l^{susy} \times SU(2)_r^{susy}$ and its $\{(\mathbb{1}, \mathbb{1}), (-\mathbb{1}, -\mathbb{1})\}$ quotient by $SO(4)^{susy}$. The commutant of $SU(2)_l^{susy}$ in $SO(Q_l)$ will be called $SU(2)_l$. One can identify $SU(2)_l^{susy}$ with $SU(2)_l$ by selecting one vector in Q_l. Namely, the stabilizer subgroup of $SO(Q_l)$ for this vector is of type $SO(3)$ with surjective projections to the two $SU(2)$ groups modulo their centers. This allows an identification of the images. Such an identification seems to be implicit in many discussions in the literature, but will not be used in this section.
We assume the existence of a quartet of spectral flow fields $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$, $U_{\pm\frac{1}{2}}\overline{U}_{\mp\frac{1}{2}}$ as in (3.1.7), i.e. $(h, Q; \overline{h}, \overline{Q}) = (\frac{1}{4}, \varepsilon_1; \frac{1}{4}, \varepsilon_2), \varepsilon_i \in \{\pm 1\}$, for $c = 6$. By the discussion in section 5.5 this is equivalent to the assumption that only integer charges $(Q; \overline{Q})$ occur in our theories. By what was said in section 3.2, instead of using $N = (4,4)$ supersymmetry it suffices to start with $N = (2,2)$ and this quartet.
Our assumptions are natural in the context of superstring compactification. There, unbroken extended spacetime supersymmetry is obtained from $N = (2,2)$ worldsheet supersymmetry with spectral flow operators [Sen86, Sen87]. Thus our superconformal theories may be used as a background for $N = 4$ supergravity in six dimensions. Here, however, we concentrate on the internal conformal field theory. External degrees of freedom are not taken into account.
Let us give a brief summary on what is known about the moduli space $\mathcal{M}$ so far. The spaces of states of the conformal field theories form a vector bundle over $\mathcal{M}$ with local gradings by finite dimensional subbundles. They can be decomposed into irreducible representations of the left and right $N = 4$ supersymmetries. The irreducible representations are determined by their lowest weight values of (h, Q). As we have seen in section 3.2.2, these representations can be deformed continuously with respect to the value of h, except for those of non-zero Witten index, the massless ones. Let us enumerate the representations which are massless with respect to both the left and the right handed side. Apart from the vacuum we already mentioned the spectral flow operators with $(h, Q; \overline{h}, \overline{Q}) = (\frac{1}{4}, \varepsilon_1; \frac{1}{4}, \varepsilon_2), \varepsilon_i \in \{\pm 1\}$. They form a vector multiplet under $SO(4)^{susy}$. Since the vacuum is unique, there is exactly one multiplet of such fields. On the other hand, the dimension of the vector space of real $(\frac{1}{4}, 0; \frac{1}{4}, 0)$ fields is not fixed a priori. We shall denote it by $4 + \delta$ and show $\delta \in \{0, 16\}$ in theorem 7.1.1 below. With a slight abuse of notation,

the orthogonal group of this vector space will be called $O(4+\delta)$. These are all the possibilities of massless representations in the Ramond sector. The corresponding ground state fields describe the entire cohomology of Landau-Ginzburg or σ model descriptions of our theories [LVW89].

Theorem 7.1.1
The moduli space $\mathcal{M}$ of $N=(2,2)$ superconformal field theories with $c=6$ and four operators of spectral flow $U_{\pm\frac{1}{2}}\overline{U}_{\pm\frac{1}{2}}$, $U_{\pm\frac{1}{2}}\overline{U}_{\mp\frac{1}{2}}$ realized as fields of the theory decomposes into two components $\mathcal{M}^{tori}$ and $\mathcal{M}^{K3}$. The elliptic genus of theories in $\mathcal{M}^{tori}$ vanishes, and that of theories in $\mathcal{M}^{K3}$ agrees with the geometric elliptic genus of a $K3$ surface given in theorem 3.1.10.
We call one of our conformal field theories ASSOCIATED TO TORUS OR $K3$, *depending on the elliptic genus. For the theories associated to the torus one has $\delta=0$, and for those associated to $K3$ one has $\delta=16$.*

Proof:
By our assumptions on theories contained in $\mathcal{M}$ we can use theorem 3.1.7 to characterize components $\mathcal{M}^\bullet$ of $\mathcal{M}$ by the conformal field theoretic elliptic genus $\mathcal{E}^\bullet$ of theories in $\mathcal{M}^\bullet$. By theorem 3.1.12 we know $\mathcal{E}^\bullet=\frac{A}{2}\mathcal{E}_{X=K3}, A\in\mathbb{Z}$. Thus the leading order terms of the elliptic genus are

$$\mathcal{E}^\bullet(\sigma,z)=\frac{A}{2}\,\mathrm{tr}_{L_0=\overline{L}_0=\frac{c}{24}}\,e^{\pi i\overline{J}_0}y^{J_0}+\cdots=\frac{A}{2}\left(2y+20+2y^{-1}\right)+\cdots \tag{7.1.1}$$

Consider first a component $\mathcal{M}^\bullet$ of $\mathcal{M}$ where there exist fields $\overline{\psi}_\pm^{(1)}$ with quantum numbers $(h,Q;\overline{h},\overline{Q})=(0,0;\frac{1}{2},\pm1)$ in the Neveu–Schwarz sector. Since they are complex conjugate, from the existence of a real structure on the space of states of our theory by property 1 of section 2.1 it is clear that if either of the two exists, so does the other. Since there is only one irreducible representation of the Virasoro algebra with these quantum numbers [FQS84], $\overline{\psi}_\pm^{(1)}$ is a Dirac fermion, and $\overline{J}_1:=\frac{1}{2}\overline{\psi}_+^{(1)}\overline{\psi}_-^{(1)}$ is a right moving $U(1)$ current. We now decompose the right moving $U(1)$ current $\overline{J}$ of a given theory in $\mathcal{M}^\bullet$ into $\overline{J}=\overline{J}_1+\overline{J}_2$ by the GKO construction and bosonize $\overline{J}_k=i\partial\overline{H}_k$. Then $\overline{\psi}_\pm^{(1)}=\,:e^{\pm i\overline{H}_1}:$. By our assumptions on theories in $\mathcal{M}$ we can find Neveu–Schwarz fields $\overline{J}_\pm$ which together with $\overline{J}$ generate an $su(2)_r$ Kac–Moody algebra. Since $\overline{J}_\pm$ must be local to both $\overline{\psi}_\pm^{(1)}$, we find $\overline{J}_\pm=\,:e^{\pm i(\overline{H}_1+\overline{H}_2)}:$. OPE with $\overline{\psi}_\pm^{(1)}$ then shows that $\overline{\psi}_\pm^{(2)}:=\,:e^{\pm i\overline{H}_2}:$ is a second Dirac fermion which is realized as Neveu–Schwarz field of our theory. It also follows that no further right moving Dirac fermions can be contained in the theory. Thus by application of the spectral flow $U_{-\frac{1}{2}}\overline{U}_{-\frac{1}{2}}$ as given by (3.1.8) we can determine all Ramond ground states of the theory with $\overline{Q}=-1$. Namely, two of them with $Q=0$ correspond to the two Dirac fermions, and one with $Q=\pm1$ each corresponds to the operators $U_{\pm\frac{1}{2}}\overline{U}_{-\frac{1}{2}}$ of spectral flow. The coefficient in front of y^{-1} in (7.1.1) vanishes, so $A=0$, and $\mathcal{M}^\bullet=\mathcal{M}^{tori}$.
In any other component $\mathcal{M}^\bullet$ of $\mathcal{M}$ we have no fields with $(h,Q;\overline{h},\overline{Q})=(0,0;\frac{1}{2},\pm1)$. Again, by spectral flow all Ramond ground states of the theory with $\overline{Q}=-1$ are

determined, and now the coefficient in front of y^{-1} in (7.1.1) is 2. Thus $A = 2$ and $\mathcal{M}^\bullet = \mathcal{M}^{K3}$ as asserted above.
The respective values of δ are read off from (7.1.1). □

In theorem 7.1.1 we have only used the conformal field theoretic elliptic genus $\mathcal{E}$ which a priori need not be left–right symmetric by definition 3.1.6. The RIGHT HANDED ELLIPTIC GENUS

$$\overline{\mathcal{E}}(\overline{\sigma}, \overline{z}) := \mathrm{tr}_{\mathcal{H}^R}(-1)^F \overline{y}^{\overline{J}_0} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}$$

has analogous properties and will be used in the following

Theorem 7.1.2
For all theories in $\mathcal{M}$ the left and right handed elliptic genera have the same power series expressions. In particular, theories in $\mathcal{M}^{tori}$ are toroidal superconformal field theories in the sense of definition 4.1.3, i.e. $\mathcal{M}^{tori} = \mathcal{M}_4^{tori}$ as in (4.2.6).

Proof:
Since theorems 3.1.7 and 3.1.12 have direct analogs for $\overline{\mathcal{E}}$, theorem 7.1.1 can also be directly translated to the right handed elliptic genus. To prove the assertion, it therefore suffices to exclude the case $\mathcal{E} = \mathcal{E}_{X=K3}, \overline{\mathcal{E}} = 0$. Let $h^{p,q}$ denote the generic number of Ramond–Ramond ground states of the theories in a given component of $\mathcal{M}$ with charge $(Q; \overline{Q}) = (p-1; q-1)$. Then from our assumption on the existence of a quartet of operators of spectral flow we have $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$. If $\mathcal{E} = \mathcal{E}_{X=K3}$, it follows that $h^{1,0} = h^{1,2} = 0$, and from $\overline{\mathcal{E}} = 0$ we read $h^{0,1} = h^{2,1} = 2$. But then $\mathcal{E} = \mathcal{E}_{X=K3}$ implies $h^{1,1} = 24$, whereas $\overline{\mathcal{E}} = 0$ implies $h^{1,1} = 0$, so $\mathcal{E} = \mathcal{E}_{X=K3}, \overline{\mathcal{E}} = 0$ cannot hold simultaneously.
In particular we see that the existence of one right handed Dirac fermion already implies the existence of two left and two right handed Dirac fermions. The superpartners of the corresponding Majorana fermions give four left and four right moving Abelian $U(1)$ currents. Hence the respective theory indeed is a toroidal superconformal field theory in the sense of definition 4.1.3. □

To understand the local structure of the moduli space $\mathcal{M}$ we must determine the tangent space $\mathcal{H}_1$ in a given point of $\mathcal{M}$, i.e. describe the deformation moduli of a given theory. As discussed in section 2.2, this space consists of real fields of dimensions $h = \overline{h} = 1$ in the space of states $\mathcal{H}$ over the chosen point. The Zamolodchikov metric on $\mathcal{H}_1$ establishes on $\mathcal{M}$ the structure of a Riemannian manifold, with holonomy group contained in $O(\mathcal{H}_1)$. To preserve the supersymmetry algebra, $\mathcal{H}_1$ by conjecture 3.1.1 must consist of $SO(4)^{susy}$ invariant fields in the image of $\mathcal{F}_{1/2}$ under $(Q_l)_{1/2} \otimes (Q_r)_{1/2}$, where the latter subscripts denote Fourier components. The vector space $\mathcal{F}_{1/2}$ is spanned by the fields with $(h, Q; \overline{h}, \overline{Q}) = (\frac{1}{2}, \varepsilon_1; \frac{1}{2}, \varepsilon_2), \varepsilon_i \in \{\pm 1\}$, and is obtained from the $(\frac{1}{4}, 0; \frac{1}{4}, 0)$ Ramond fields by spectral flow. Thus it gives an irreducible $4(4+\delta)$–dimensional representation of $su(2)_l^{susy} \oplus su(2)_r^{susy} \oplus o(4+\delta)$. Accordingly, $\mathcal{F}_{1/2} \oplus \mathcal{H}_1$ yields a well–known representation of the $osp(2,2)$ superalgebra spanned by $(Q_l)_{\pm 1/2}$, $su(2)_l^{susy}$, and the Virasoro operator L_0. In particular, $\mathcal{H}_1$ should be $4(4+\delta)$–dimensional and form an irreducible representation of

$su(2)_l \oplus su(2)_r \oplus o(4+\delta)$. We shall assume that all elements of $\mathcal{H}_1$ really give integrable deformations, as has been shown to all orders in perturbation theory [Dix87]. Note, however, that there is no complete proof yet.
The holonomy group of $\mathcal{M}$ projects to an $O(4+\delta)$ action on the uncharged massless Ramond representations and to an $SO(4)$ action on $Q_l \otimes Q_r$. Thus its Lie algebra is contained in $su(2)_l \oplus su(2)_r \oplus o(4+\delta)$. The two Lie algebras are equal for $\mathcal{M}^{tori}$ and one expects the same for $\delta = 16$. In section 7.3.2 we shall find an isometry from $\mathcal{M}^{tori}$ to a subvariety of $\mathcal{M}^{K3}$, such that the holonomy Lie algebra of the latter space is at least $su(2)_l \oplus su(2)_r \oplus so(4)$. Moreover, this isometry shows that $\mathcal{M}^{K3}$ is not compact. Since one has the inclusion

$$su(2) \oplus su(2) \oplus o(4+\delta) \cong sp(1) \oplus sp(1) \oplus o(4+\delta) \hookrightarrow sp(1) \oplus sp(4+\delta),$$

the moduli space of $N = (4,4)$ superconformal field theories with $c = 6$ associated to torus or $K3$ is a quaternionic Kähler manifold of real dimension $4(4+\delta)$. To determine its local structure, recall that we are looking for a noncompact space. By Berger's classification of quaternionic Kähler manifolds [Ber55] it can only be reducible or quaternionic symmetric [Sim62, Th. 9]. Because non–Ricci flat quaternionic Kähler manifolds are (even locally) de Rham irreducible [Wol65], this means that it can only be Ricci flat or quaternionic symmetric. The former is excluded because geodesic submanifolds on which all holomorphic sectional curvatures are negative and bounded away from zero have been found [PS90, CFG89, Cec90]. Hence the moduli space must locally be the Wolf space

$$\begin{aligned} \mathcal{T}^{4,4+\delta} &= O^+(4,4+\delta;\mathbb{R})/SO(4)\times O(4+\delta) \\ &\cong SO^+(4,4+\delta;\mathbb{R})/SO(4)\times SO(4+\delta) \\ &\cong O(4,4+\delta;\mathbb{R})/O(4)\times O(4+\delta), \end{aligned} \tag{7.1.2}$$

i.e. one component of the Grassmannian of oriented spacelike four–planes $x \subset \mathbb{R}^{4,4+\delta}$ [Cec91], reproducing Narain's and Seiberg's previous results [CENT85, Nar86, Sei88]. The Zamolodchikov metric on $\mathcal{T}^{4,4+\delta}$ is the group invariant one. In case $\delta = 0$ this indeed is the Teichmüller space of $\mathcal{M}_4^{tori}$, as was stated in (4.2.2).
Generic examples for our conformal theories are nonlinear σ models which by the discussion of the elliptic genus and theorem 7.1.1 must have the oriented four–torus or the $K3$ surface as target space X. In the $K3$ case, the existence of these quantum field theories has not been proven yet, but their conformal dimensions and operator product coefficients have a well defined perturbation theory in terms of inverse powers of the volume. We tacitly make the assumption that a rigorous treatment is possible and warn the reader that many of our statements depend on this assumption.
Around (4.2.3) we argued that the parameter space of nonlinear σ models has the form $\{$*Ricci flat metrics*$\} \times \{B-$*fields*$\}$. To understand the parameter spaces for the two components of $\mathcal{M}$ corresponding to nonlinear σ models on a four–torus or a $K3$ surface X, note that metric and orientation define a Hodge star operator on X which on $\Lambda^2 X$ has eigenvalues ± 1. The corresponding eigenspaces $\Lambda^\pm X$ have

dimension 3 and $3 + \delta$, where as above $\delta = 0$ for the torus and $\delta = 16$ for the $K3$ case. One finds that Λ^+X has vanishing curvature [Hit74]. Hence one can choose three parallel sections $\omega_1, \omega_2, \omega_3 \in H^2(X, \mathbb{R})$ which span a positive definite three plane $\Sigma \subset H^2(X, \mathbb{R})$ with respect to the intersection product, which agrees with the cup product (on homology: use Poincaré duality). This means that the choice of an Einstein metric on X induces the choice of a positive definite three plane $\Sigma \subset H^2(X, \mathbb{R})$. The orientation on X induces an orientation on Σ. One can show that Ricci flat metrics are locally uniquely specified by Σ, apart from a scale factor given by the volume. Since the Hodge star operator in the middle dimension does not change under a rescaling of the metric, the volume V must be specified separately.

We consider the vector space $H^2(X, \mathbb{R})$ together with the intersection product, such that $H^2(X, \mathbb{R}) \cong \mathbb{R}^{3,3+\delta}$. In other words, positive definite subspaces have at most dimension three, negative definite ones at most dimension $3 + \delta$. On $K3$ this choice of sign determines a canonical orientation. As explained at the end of section 4.4, when one wants to study $\mathcal{M}^{tori}$ by itself, the choice of a torus orientation is superfluous. One of our main interests, however, is the study of torus orbifolds. For a canonical blow–up of the resulting singularities one needs an orientation. The effect of a change of orientation on the torus will be considered in section 7.3.2.

It follows that $\mathcal{T}^{3,3+\delta} \times \mathbb{R}^+$ is the Teichmüller space of Einstein metrics on X. Explicitly, we have

$$\mathcal{T}^{3,3+\delta} = O^+(H^2(X, \mathbb{R}))/SO(3) \times O(3 + \delta). \qquad (7.1.3)$$

The $SO(3)$ group in the denominator is to be interpreted as $SO(\Sigma_0)$ for some positive definite reference three–plane in $H^2(X, \mathbb{R})$, while $O(3+\delta)$ is the corresponding group for the orthogonal complement of Σ_0. Equivalently, $\mathcal{T}^{3,3+\delta}$ could have been written as $SO^+(H^2(X, \mathbb{R}))/SO(3) \times SO(3 + \delta)$. We choose the description (7.1.3) for later convenience in the construction of the entire moduli space.

Since also B–field degrees of freedom have to be taken into account, the Teichmüller spaces for parameter spaces of nonlinear σ models in our cases are

$$\mathcal{T}^{3,3+\delta} \times \mathbb{R}^+ \times H^2(X, \mathbb{R}). \qquad (7.1.4)$$

Their elements will be denoted by (Σ, V, B). The Zamolodchikov metric gives a warped product structure to this space.

In the context of σ models it often is useful to choose a complex structure on X. When such a structure is given, the real and imaginary parts of any generator of $H^{2,0}(X, \mathbb{C})$ span an oriented twoplane $\Omega \subset \Sigma$. Conversely, any such subspace Ω defines a complex structure. This means that the choice of an Einstein metric is nothing but the choice of an $\mathbb{S}^2$ of complex structures on X, in other words a HYPERKÄHLER STRUCTURE. In terms of cohomology, Ω specifies $H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$. The orthogonal complement of Ω in $H^2(X, \mathbb{R})$ yields $H^{1,1}(X, \mathbb{R})$. Any vector $\omega \in H^{1,1}(X, \mathbb{R})$ of positive norm yields a Kähler class compatible with the complex structure and the hyperkähler structure Σ spanned by Ω and ω.

By a result of Kodaira's, X is algebraic, if $NS(X)$ (definition 4.6.1) contains an element ρ of positive length squared [Kod64], where by the discussion after definition 4.6.2 in case X is $K3$ or a torus we have $NS(X) \cong H^{1,1}(X) \cap H^2(X,\mathbb{Z})$. Given a hyperkähler structure Σ we can always find $\Omega \subset \Sigma$ such that X becomes an algebraic surface. It suffices to choose ω as the projection of ρ on Σ and Ω as the corresponding orthogonal complement. The projection is non-vanishing, since the orthogonal complement of Σ in $H^2(X,\mathbb{R})$ is negative definite. Varying ρ one obtains a countable infinity of algebraic structures on X. Thus the occasionally encountered interpretation of moduli of conformal field theories as corresponding to nonalgebraic deformations of $K3$ surfaces does not make sense. This was already pointed out in [Cec91] by different arguments.
Worldsheet parity transformations $(\xi_0,\xi_1) \mapsto (-\xi_0,\xi_1)$ change the sign of the cycles, or equivalently the sign of B, which yields an automorphism of the parameter space. Target space parity for $B=0$ yields a specific worldsheet parity transformation and thus an identification of $su(2)_l$ with $su(2)_r$. The corresponding diagonal Lie algebra $su(2)_{l+r}$ generates an $SO(3)$ subgroup of $SO(4)$. Under the action of this subgroup a four–plane $x \in \mathcal{T}^{4,20}$ decomposes into a line and its orthogonal three–plane $\Sigma \subset x$. The $\mathbb{S}^2 \times \mathbb{S}^2$ bundle over $\mathcal{M}$ now has a diagonal $\mathbb{S}^2$ subbundle. Each point in the fibre corresponds to the choice of an $SO(2)$ subgroup of $SO(3)$ or a subalgebra $u(1)_{l+r}$ of $su(2)_{l+r}$. Geometrically this yields a complex structure in the target space. Thus the $\mathbb{S}^2$ bundle over the $B=0$ subspace of $\mathcal{M}$ is the bundle of complex structures over the moduli space of Ricci flat metrics on the target space.
For higher dimensional Calabi-Yau spaces the σ model description works only for large volume due to instanton corrections. In our case, however, the metric on the moduli space does not receive corrections [NS95]. Therefore the Teichmüller space (7.1.4) of σ models on X should be a covering of a component of $\mathcal{M}$, thus isomorphic to the Teichmüller space $\mathcal{T}^{4,4+\delta}$ obtained in (7.1.2). Indeed, for $\delta = 16$ a natural isomorphism

$$\mathcal{T}^{4,4+\delta} \cong \mathcal{T}^{3,3+\delta} \times \mathbb{R}^+ \times H^2(X,\mathbb{R}) \tag{7.1.5}$$

was given in [AM94, Asp97], with a correction and clarification by [RW98, Dij99]. The same construction actually works for $\delta = 0$, too, see (4.4.3). It uses the identification

$$\mathcal{T}^{4,4+\delta} = O^+(H^{even}(X,\mathbb{R}))/SO(4) \times O(4+\delta),$$

where $SO(4)$ is to be interpreted as $SO(x_0)$ for some positive definite reference four–plane in $H^{even}(X,\mathbb{R})$, while $O(4+\delta)$ is the corresponding group for the orthogonal complement of x_0. In other words, the elements of $\mathcal{T}^{4,4+\delta}$ are interpreted as positive definite oriented four–planes $x \subset H^{even}(X,\mathbb{R})$ by $H^{even}(X,\mathbb{R}) \cong \mathbb{R}^{4,4+\delta}$. Note that all the cohomology of $K3$ is even, whereas $H^{odd}(X,\mathbb{R}) \cong \mathbb{R}^{4,4}$ when X is a four–torus.
From the preceding discussion, x can be interpreted as the $SO(4)^{susy}$ invariant part of the tensor product of $Q_l \otimes Q_r$ with the four-dimensional space of charged Ramond ground states. Note that the action of $so(4) = su(2)_l \oplus su(2)_r$ discussed above generates orthogonal transformations of the four–plane $x \in \mathcal{T}^{4,4+\delta}$ that

corresponds to the theory under inspection, whereas $o(4+\delta)$ acts on its orthogonal complement.
We have repeatedly used the splitting $so(4) = su(2)_l \oplus su(2)_r$. Consider the antisymmetric product $\Lambda^2 x$ of the above four–plane x. We choose the orientation of x such that $su(2)_l$ fixes the anti–selfdual part $(\Lambda^2 x)^-$ of $\Lambda^2 x$ with respect to the group invariant metric on $O^+(4,4+\delta;\mathbb{R})$. When the theory has a parity operation which interchanges Q_l and Q_r, this induces a change of orientation of x. The choice of an $N=(2,2)$ subalgebra within the $N=(4,4)$ superconformal algebra corresponds to the selection of a Cartan torus $u(1)_l \oplus u(1)_r$ of $su(2)_l \oplus su(2)_r$. This induces the choice of an oriented twoplane in $\widetilde{\Omega} \subset x$. The rotations of x in this twoplane are generated by $u(1)_{l+r}$, those perpendicular to the plane by $u(1)_{l-r}$. Thus the moduli space of $N=(2,2)$ superconformal field theories with central charge $c=6$ is given by a Grassmann bundle over $\mathcal{M}$, with fibre $SO(4)/(SO(2)_{l+r} \times SO(2)_{l-r}) \cong \mathbb{S}^2 \times \mathbb{S}^2$. Given an image (Σ, V, B) of x under the isomorphism (7.1.5), the twoplane $\widetilde{\Omega} \subset x$ actually is the lift of a twoplane $\Omega \subset \Sigma$ that corresponds to the choice of a complex structure on X. Though we refer to the choice of such a twoplane $\widetilde{\Omega}$ as fixing a complex structure we see that, more precisely, the twoplane $\widetilde{\Omega}$ specifies a complex structure in *every* such image (Σ, V, B).
To explicitly realize the isomorphism (7.1.5) one also needs the positive generators υ of $H^4(X,\mathbb{Z})$ and υ^0 of $H^0(X,\mathbb{Z})$, which are Poincaré dual to points and to the whole oriented cycle X, respectively. They are null vectors in $H^{even}(X,\mathbb{R})$ and satisfy $\langle \upsilon, \upsilon^0 \rangle = 1$. Thus over $\mathbb{Z}$ they span an even, unimodular lattice isomorphic to the standard hyperbolic lattice U with bilinear form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now consider a triple (Σ, V, B) in the right hand side of (7.1.5). Define

$$\begin{aligned} &\xi : \Sigma \to H^{even}(X,\mathbb{R}), \quad \xi(\sigma) := \sigma - \langle B, \sigma \rangle \upsilon, \\ &x := \mathrm{span}_{\mathbb{R}} \left(\xi(\Sigma),\ \xi_4 := \upsilon^0 + B + \left(V - \tfrac{\|B\|^2}{2} \right) \upsilon \right). \end{aligned} \tag{7.1.6}$$

Then $\widetilde{\Sigma} = \xi(\Sigma)$ is a positive definite oriented three–plane in $H^{even}(X,\mathbb{R})$, and the vector ξ_4 is orthogonal to $\widetilde{\Sigma}$. Since $\|\xi_4\|^2 = 2V$, it has positive square. Together, $\widetilde{\Sigma}$ and ξ_4 span an oriented four–plane $x \subset H^{even}(X,\mathbb{R})$. Obviously, the map $(\Sigma, V, B) \mapsto x$ is invertible, once υ and υ^0 are given.
Note that the rôle of triality for toroidal conformal theories (see section 4.4) is now visible upon comparison of the geometric interpretations (7.1.6) and (4.2.1). The analogy between choices of nullplanes Y, Y^0 as described in section 4.2 and null vectors υ, υ^0 above is apparent. Indeed, part of the triality manifests itself in a one to one correspondence between maximal isotropic subspaces $Y \subset \mathbb{R}^{4,4}$ and null Weyl spinors υ such that $Y = \{y \in \mathbb{R}^{d,d} \mid c(y)(\upsilon) = 0\}$ where c denotes Clifford multiplication on the spinor bundle [BT88]. One can regard this as further justification for interpreting υ as volume form which generates $H^4(T,\mathbb{Z})$ in our

geometric interpretation. Note also that in both cases different choices of Y^0, v^0 correspond to B-field shifts by integral forms (see below).
To describe the projection from Teichmüller space to $\mathcal{M}$ we need to consider the lattices $H^2(X,\mathbb{Z})$ and $H^{even}(X,\mathbb{Z})$. They are even, unimodular, and have signature $(p, p+\delta)$ with $p=3$ and $p=4$, respectively. Such lattices are isometric to $\Gamma^{p,p+\delta} = U^p \oplus (E_8(-1))^{\delta/8}$. Here each summand is a free $\mathbb{Z}$ module, E_8 has as bilinear form the Cartan matrix of E_8, and for any lattice Γ we denote by $\Gamma(n)$ the same $\mathbb{Z}$ module Γ with quadratic form scaled by n.
We now consider the projection from Teichmüller space to $\mathcal{M}$. First we have to identify all points in $\mathcal{T}^{3,3+\delta}$ which yield the same Ricci flat metric. This means that we have to quotient the Teichmüller space (7.1.3) by the so–called CLASSICAL SYMMETRIES. The projection is given by

$$O^+(H^2(X,\mathbb{Z}))\backslash\mathcal{T}^{3,3+\delta} \tag{7.1.7}$$

[KT87]. The interpretation of the quotient space (7.1.7) as moduli space of Einstein metrics of volume 1 on X is straightforward in the torus case. For $X = K3$ one has to include orbifold limits [KT87] (see section 5.2.2), and as was shown by Anderson [And92] one can define an EXTRINSIC L^2-METRIC on the space $\mathbb{E}$ of regular Einstein metrics of volume 1 on $K3$ such that the completion of $\mathbb{E}$ is contained in the set of regular and orbifold singular Einstein metrics. The σ models corresponding to orbifold limits are not expected to exist for all values of B [Wit95]. To simplify the discussion we include such CONIFOLD POINTS in $\mathcal{M}$. On $\mathcal{T}^{4,4+\delta}$ the group of classical symmetries lifts by (7.1.6) to the subgroup of $O^+(H^{even}(X,\mathbb{Z}))$ which fixes both lattice vectors v and v^0.
Next we consider the shifts of B by elements $\lambda \in H^2(X,\mathbb{Z})$, which neither change the physical content. One easily calculates that this also yields a left action on $\mathcal{T}^{4,4+\delta}$ by a lattice automorphism in $O^+(H^{even}(X,\mathbb{Z}))$, generated by $w \mapsto w - \langle w,\lambda\rangle v$ for $\langle w,v\rangle = 0$ and $v^0 \mapsto v^0 + \lambda - \frac{\|\lambda\|^2}{2}v$. These transformations fix v and shift v^0 to arbitrary null vectors dual to v. Thus the choice of v^0 is physically irrelevant.
Below, we shall argue that the projection from Teichmüller space to $\mathcal{M}$ is given by

$$\mathcal{T}^{4,4+\delta} \longrightarrow O^+(H^{even}(X,\mathbb{Z}))\backslash\mathcal{T}^{4,4+\delta}. \tag{7.1.8}$$

The group $O^+(H^{even}(X,\mathbb{Z}))$ acts transitively on pairs of primitive lattice vectors of equal length [LP81, Nik80b]. Thus (7.1.8) would imply that different choices of v, v^0 are equivalent. Anticipating this result in general, we call the choice of an arbitrary primitive null vector $v \in H^{even}(X,\mathbb{Z})$ a GEOMETRIC INTERPRETATION of a positive oriented four–plane $x \subset H^{even}(X,\mathbb{Z})$. Such a choice yields a family of σ models with physically equivalent data (Σ, V, B). A conformal field theory has various different geometric interpretations, and the choice of v is comparable to a choice of a chart of $\mathcal{M}$.
Aspinwall and Morrison also identify theories which are related by the worldsheet parity transformation [AM94]. We regard the latter as a symmetry of $\mathcal{M}$. It is given by change of orientation of the four–plane x or equivalently by a conjugation

of $O^+(H^{even}(X,\mathbb{R}))$ with an element of $O(H^{even}(X,\mathbb{R})) - O^+(H^{even}(X,\mathbb{R}))$ which transforms the lattice $H^{even}(X,\mathbb{Z})$ and the reference four–plane x_0 into themselves. To stay in the classical context, one may choose an element which fixes υ and υ^0. More canonically, parity corresponds to $(\upsilon,\upsilon^0) \mapsto (-\upsilon,-\upsilon^0)$. The latter induces $\xi_4 \mapsto -\xi_4$ and $(\Sigma, V, B) \mapsto (\Sigma, V, -B)$.
Let us consider the general pattern of identifications. When two points in Teichmüller space are identified, the same is true for their tangent spaces. Higher derivatives can be treated by perturbation theory in terms of tensor products of the tangent spaces $\mathcal{H}_1$. Assuming the convergence of the perturbation expansion in conformal field theory, any such isomorphism can be transported to all points of $\mathcal{T}^{4,4+\delta}$. Therefore σ model isomorphisms are given by the action of a group $\mathcal{G}^{(\delta)}$ on this space. In the previous considerations we have found a subgroup of $\mathcal{G}^{(\delta)}$.
In theorem 7.3.17 we shall prove that the interchange of υ and υ^0, which is the Fourier-Mukai transform [RW98], also belongs to $\mathcal{G}^{(\delta)}$. When $B = 0$, this yields the map $(\Sigma, V, 0) \mapsto (\Sigma, V^{-1}, 0)$. In the torus case, it is known as T-duality and it seems natural to extend this name to $X = K3$. We will not use the name mirror symmetry for this transformation.
It is obvious that classical symmetries, integral B-field shifts, and T-duality generate all of $O^+(H^{even}(X,\mathbb{Z}))$. Thus $\mathcal{G}^{(\delta)}$ contains all of this group. As argued in [AM94, Asp97], it cannot be larger, since otherwise the quotient of $\mathcal{T}^{4,4+\delta}$ by $\mathcal{G}^{(\delta)}$ plus the parity automorphism would not be Hausdorff [All66]. For a proof of the Hausdorff property of $\mathcal{M}$ one will need some features of the superconformal field theories, which should be easy to verify once they are somewhat better understood along the lines that were drawn in sections 2.1 and 3. First, one has to check that all fields are generated by the iterated operator products of a finite dimensional subspace of basic fields. Next one has to show that the operator product coefficients are determined in terms of a finite number of basic coefficients, and that the latter are constrained by algebraic equations only. This would show that $\mathcal{M}$ is an algebraic space. In particular, every point has a neighborhood which contains no isomorphic point. All of these features are true in the known examples of conformal field theories with finite effective central charge, in particular for the unitary ones. They certainly should be true in our case.

7.2 The topological part of partition functions

Let us now study generic features of theories contained in the moduli space $\mathcal{M}$. In particular, one can ask whether certain fields exist generically, as is the case, e.g., for the quartet of spectral flow fields by assumption. As a first step of such an analysis we ask for a generic part of the partition function of all theories in a given component of $\mathcal{M}$. The idea is similar to the determination of the "topological part Z_{top} of the partition function" of Eguchi, Ooguri, Taormina and Yang [EOTY89, (3.10)]. The analysis in [EOTY89] is not carried out explicitly, though, and we suggest an improvement to the result presented there.
We use the ansatz of [Tao90] for the partition function of any theory $\mathcal{C}$ in $\mathcal{M}$.

Namely, by our assumptions $\mathcal{C}$ is invariant under spectral flow, such that it is sufficient to study the partition function in one of the sectors $\mathcal{S} \in \{NS, \widetilde{NS}, R, \widetilde{R}\}$ and use the flows (3.1.9). Since $\mathcal{C}$ possesses $N = (4,4)$ superconformal symmetry, we can split the partition function into combinations of the $N = (4,4)$ characters $ch^{\mathcal{S}}_{0,0}, ch^{\mathcal{S}}_{1/2,1/2}, ch^{\mathcal{S}}_{h,0}$ introduced in section 3.2.2. By theorems 7.1.1 and 7.1.2 we have either no left–right coupling between $ch^{\mathcal{S}}_{0,0}$ and $ch^{\mathcal{S}}_{1/2,1/2}$, if $\mathcal{C} \in \mathcal{M}^{K3}$, or otherwise $\varepsilon = 2$ left and right handed couplings each, if $\mathcal{C} \in \mathcal{M}^{tori}$. With

$$\widetilde{ch}^{\mathcal{S}} := \lim_{h\to 0} ch^{\mathcal{S}}_{h,0}$$

we then find

$$\begin{aligned} Z_{\mathcal{S}} \;=\; & ch^{\mathcal{S}}_{0,0}\left(ch^{\mathcal{S}}_{0,0}\right)^* \;+\; h^{1,1}\, ch^{\mathcal{S}}_{1/2,1/2}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* \\ & + \;\varepsilon\left[ch^{\mathcal{S}}_{0,0}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* \;+\; ch^{\mathcal{S}}_{1/2,1/2}\left(ch^{\mathcal{S}}_{0,0}\right)^*\right] \\ & + \;F\,\widetilde{ch}^{\mathcal{S}}\left(ch^{\mathcal{S}}_{0,0}\right)^* \;+\; ch^{\mathcal{S}}_{0,0}\left(F'\,\widetilde{ch}^{\mathcal{S}}\right)^* \\ & + \;G\,\widetilde{ch}^{\mathcal{S}}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* + ch^{\mathcal{S}}_{1/2,1/2}\left(G'\,\widetilde{ch}^{\mathcal{S}}\right)^* + H\,\widetilde{ch}^{\mathcal{S}}\left(\widetilde{ch}^{\mathcal{S}}\right)^*. \end{aligned} \tag{7.2.1}$$

If $\mathcal{C} \in \mathcal{M}^{tori}$, we have $\varepsilon = 2, h^{1,1} = 4$, whereas for $\mathcal{C} \in \mathcal{M}^{K3}$, we have $\varepsilon = 0, h^{1,1} = 20$. Moreover, $F = F(\sigma, z)$ and analogously for F', G, G', and $H = H(\sigma, z; \overline{\sigma}, \overline{z})$. Note that F, G, H may only have nonnegative integer coefficients in $q, \overline{q}$.
By theorem 7.1.1 we explicitly know the elliptic genus $\mathcal{E}$ of $\mathcal{C}$, and since $\mathcal{E}$ can be obtained from the partition function by (3.1.11) we can deduce restrictions on F, G and F', G'. Namely,

Theorem 7.2.1
Consider a conformal field theory $\mathcal{C}$ in $\mathcal{M}$ with partition function (7.2.1). Define

$$e := G - 2F, \quad e' := G' - 2F'.$$

Then

$$e = e' = \begin{cases} 0 & \text{on } \mathcal{M}^{tori} \\ e_{K3}(\sigma) := 2 + 2q^{\frac{1}{8}}\eta\left(\dfrac{\vartheta_2^4 - \vartheta_4^4}{\eta^4} - 12h_3\right) & \text{on } \mathcal{M}^{K3}. \end{cases}$$

Proof:
We use (3.1.11) to determine the elliptic genus $\mathcal{E}$ from (7.2.1). By inserting the Witten indices (3.2.7) we find

$$\mathcal{E}(\sigma, z) = Z_{\widetilde{R}}(\sigma, z; \overline{\sigma}, \overline{z} = 0) = (\varepsilon - 2)\, ch^{\widetilde{R}}_{0,0} + (h^{1,1} - 2\varepsilon)\, ch^{\widetilde{R}}_{1/2,1/2} + e\, \widetilde{ch}^{\widetilde{R}}.$$

Since $\overline{\mathcal{E}}(\overline{\sigma}, \overline{z}) = \mathcal{E}(\overline{\sigma}, \overline{z})$ by theorem 7.1.2 we immediately deduce $e = e'$. If $\mathcal{C} \in \mathcal{M}^{tori}$, $\mathcal{E} = 0$, so with $\varepsilon = 2, h^{1,1} = 4$ we get $e = 0$. In case $\mathcal{C} \in \mathcal{M}^{K3}$ we insert the formula for $\mathcal{E}$ given in theorem 3.1.10 as well as the explicit forms of $ch^{\widetilde{R}}_{0,0}, ch^{\widetilde{R}}_{1/2,1/2}, \widetilde{ch}^{\widetilde{R}}$ (see (3.2.5), (3.2.10)) to show the assertion of the theorem. □

Theorem 7.2.1 allows us to rewrite (7.2.1) in a more suggestive way:

$$\begin{aligned}
Z_{\mathcal{S}} \;=\;& ch^{\mathcal{S}}_{0,0}\left(ch^{\mathcal{S}}_{0,0}\right)^* \;+\; h^{1,1}\, ch^{\mathcal{S}}_{1/2,1/2}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* \\
&+\; \varepsilon\left[ch^{\mathcal{S}}_{0,0}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* \;+\; ch^{\mathcal{S}}_{1/2,1/2}\left(ch^{\mathcal{S}}_{0,0}\right)^*\right] \\
&+\; \left[e\,\widetilde{ch}^{\mathcal{S}}\left(ch^{\mathcal{S}}_{1/2,1/2}\right)^* \;+\; ch^{\mathcal{S}}_{1/2,1/2}\left(e\,\widetilde{ch}^{\mathcal{S}}\right)^*\right] \\
&+\; F\,\widetilde{ch}^{\mathcal{S}}\left(ch^{\mathcal{S}}_{0,0}+2ch^{\mathcal{S}}_{1/2,1/2}\right)^* + \left(ch^{\mathcal{S}}_{0,0}+2ch^{\mathcal{S}}_{1/2,1/2}\right)\left(F'\,\widetilde{ch}^{\mathcal{S}}\right)^* \\
&+\; H\,\widetilde{ch}^{\mathcal{S}}\left(\widetilde{ch}^{\mathcal{S}}\right)^* .
\end{aligned} \tag{7.2.2}$$

Note that the vacuum character of $\mathcal{C}$ is given by

$$\chi_{vac} = \tfrac{1}{2}\left(ch^{NS}_{0,0} + ch^{\widetilde{NS}}_{0,0}\right) + F\;\cdot\tfrac{1}{2}\left(\widetilde{ch}^{NS} + \widetilde{ch}^{\widetilde{NS}}\right),$$

i.e. F counts the primary holomorphic fields of $\mathcal{C}$ that are not contained in the $N=4$ vacuum representation with character $\frac{1}{2}(ch^{NS}_{0,0} + ch^{\widetilde{NS}}_{0,0})$. By our discussion in section 3.2, nongeneric holomorphic fields may occur if a right handed massive representation hits the unitarity bound and splits into three massless ones,

$$\left(\widetilde{ch}^{\mathcal{S}}\right)^* = \lim_{h\to 0}\left(ch^{\mathcal{S}}_{h,0}\right)^* = \left(ch^{\mathcal{S}}_{0,0} + 2ch^{\mathcal{S}}_{1/2,1/2}\right)^*,$$

as in (3.2.6). Hence (7.2.2) shows that all fields counted by e are generic, even if all holomorphic fields counted by F are nongeneric. The latter is clearly not the case on $\mathcal{M}^{tori}$, since there we always have seven additional $(1,0)$ fields by the very definition 4.1.3 of a toroidal superconformal field theory. In the $K3$ case we could easily give lower bounds on the number of generic holomorphic fields if the function e_{K3} had negative coefficients. Alone, numerical calculations up to order q^{70}, where the coefficients of e_{K3} appear to be growing rapidly, give reason to

Conjecture 7.2.2
The function $e_{K3}(\sigma)$ of theorem 7.2.1 has only integer exponents and nonnegative coefficients in q.

Proof of the first assertion:
By (3.2.12), the function

$$\widetilde{h}_3 := q^{\frac{1}{8}}\eta\left(\tfrac{1}{8}\left(\frac{\vartheta_2}{\eta}\right)^4 - h_3\right)$$

has only integer exponents in q. With (A2.2) one now checks

$$e_{K3}(\sigma) = 24\widetilde{h}_3 + 2 - \frac{q^{\frac{1}{8}}}{\eta^3}\left(\vartheta_3^4 + \vartheta_4^4\right),$$

from which the first assertion of the conjecture follows. □

The second assertion of conjecture 7.2.2 is left as a challenge to the mathematical reader. Note that if this conjecture is true, Zamolodchikov's method for the determination of generic holomorphic fields as discussed at the end of section 2.2 must fail on $\mathcal{M}^{K3}$. On the other hand, we can then write down a generic part of partition functions on the two components of $\mathcal{M}$. Namely, for $\mathcal{M}^{tori}$ we use (4.1.7) and (4.1.11), whereas for $\mathcal{M}^{K3}$ we set $F=0, F'=0, H=0$ in (7.2.2), insert the theta function expressions of section 3.2 for the $N=4$ characters, and perform persevering theta function training. Since it is based on conjecture 7.2.2 we formulate

Conclusion 7.2.3
Generic partition functions on $\mathcal{M}^{tori}$ and on $\mathcal{M}^{K3}$ are given by

$$\begin{aligned} Z_{gen}^{tori} &= \frac{1}{|\eta|^8}\cdot\frac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(z)}{\eta}\right|^4, \\ Z_{gen}^{K3} &= \frac{1}{2}\left\{\left|\frac{\vartheta_2\vartheta_4}{\eta^2}\right|^4+\left|\frac{\vartheta_2\vartheta_3}{\eta^2}\right|^4+\left|\frac{\vartheta_3\vartheta_4}{\eta^2}\right|^4-\left|\frac{\vartheta_2}{\eta}\right|^8-\left|\frac{\vartheta_4}{\eta}\right|^8\right. \\ &\qquad\left.-48\left|h_3\right|^2+8\,\mathrm{Re}\,\overline{h}_3\left(\frac{\vartheta_2^4-\vartheta_4^4}{\eta^4}\right)+2\left|\frac{q^{-\frac{1}{8}}}{\eta}\right|^2\right\}\cdot\frac{1}{2}\sum_{i=1}^{4}\left|\frac{\vartheta_i(z)}{\eta}\right|^4, \end{aligned}$$

respectively.

We remark that in contrast to Z_{top} as suggested in [EOTY89, (3.11)] for $K3$, our Z_{gen}^{K3} above has only positive coefficients, if conjecture 7.2.2 is true. None of the generic partition functions is modular invariant. From (7.2.2), since e_{K3} has only integer exponents in q by the part of conjecture 7.2.2 we could prove, we see that all generic primary fields counted by e_{K3} are fermionic. They are nonholomorphic, since

$$e_{K3}(\sigma)=90q+462q^2+1540q^3+\cdots.$$

From the properties of the elliptic genus it should be possible to interpret all these fields in terms of deformations of parameters of the theory. The leading order term, for example, corresponds to the 90 deformations of any of our theories that preserve conformal invariance but break the left handed supersymmetry.

7.3 The moduli space of theories associated to $K3$

For the rest of this work, we will concentrate on the moduli space $\mathcal{M}^{K3}$ of conformal field theories associated to $K3$, namely

$$\mathcal{M}^{K3}=O^+(H^{even}(X,\mathbb{Z}))\backslash\mathcal{T}^{4,20} \tag{7.3.1}$$

by (7.1.8), where X always denotes a $K3$ surface in the following. For other presentations see [AM94, RW98, Dij99].

In the decomposition (7.1.5) we determine the product metric such that it becomes an isometry. In particular, it faithfully relates moduli of the conformal field theory to deformations of geometric objects. Recall that the structure of the tangent space $\mathcal{H}_1$ of $\mathcal{M}^{K3}$ in a given superconformal field theory is best understood by examining the $(\frac{1}{2}, \frac{1}{2})$-fields in $\mathcal{F}_{1/2}$. In our case we have related it to the $su(2)_l^{susy} \oplus su(2)_r^{susy}$ invariant subspace of the tensor product $Q_l \otimes Q_r \otimes \mathcal{H}_{1/4}^{(4)} \otimes \mathcal{H}_{1/4}^{(0)}$, where $\mathcal{H}_{1/4}^{(4)}$ denotes the charged and $\mathcal{H}_{1/4}^{(0)}$ the uncharged Ramond ground states. The invariant subspace of $Q_l \otimes Q_r \otimes \mathcal{H}_{1/4}^{(4)}$ yields a four–plane with an orthogonal group generated by $su(2)_l \oplus su(2)_r$. When a frame in $Q_l \otimes Q_r$ is chosen, the latter tensor product factor can be omitted. The description of $\mathcal{M}$ implies that $\mathcal{H}_{1/4}^{(4)} \oplus \mathcal{H}_{1/4}^{(0)}$ has a natural non-degenerate indefinite metric and remains invariant under deformations, but it has not been understood from a pure conformal field theoretic point of view how its signature comes about. In terms of the four–plane $x \in \mathcal{T}^{4,20}$ giving the location of our theory in moduli space, specific vectors in the tangent space $T_x\mathcal{T}^{4,20}$ are described by infinitesimal deformations of one generator $\xi \in x$ in direction $x^\perp$ leaving $\xi^\perp \cap x$ invariant.

To formulate this in terms of a geometric interpretation (Σ, V, B) specified by (7.1.6), pick a basis $\eta_1, \dots, \eta_{19}$ of $\Sigma^\perp \subset H^2(X, \mathbb{R}) \cong \mathbb{R}^{3,19}$. Then $x^\perp$ is spanned by $\{\eta_i - \langle \eta_i, B \rangle\, \upsilon; i = 1, \dots, 19\}$ and $\eta_{20} := \upsilon^0 + B - (\frac{\|B\|^2}{2} + V)\upsilon$. The following is most easily achieved by making use of R. Dijkgraaf's description of the quaternionic structure of $\mathcal{M}^{K3}$ [Dij99]. In each of the $SO(4)$ fibres of $\mathcal{H}_1$ over $\eta_i - \langle \eta_i, B \rangle\, \upsilon$, $i = 1, \dots, 19$, we find a three dimensional subspace deforming generators of Σ by η_i, as well as the deformation of B in direction of η_i. The fibre over η_{20} contains B-field deformations in direction of Σ and the deformation of volume. All in all, a $3 \cdot 19 = 57$ dimensional subspace of $\mathcal{H}_1 = T_x\mathcal{M}^{K3}$ is mapped onto deformations of Σ by $(1,1)$-forms $\eta \in \Sigma^\perp \cap H^2(X, \mathbb{R}) \subset H^{1,1}(X, \mathbb{R})$, no matter what complex structure we pick in Σ. The 23 dimensional complement of this subspace is given by $19 + 3$ deformations of the B-field by forms $\eta \in H^2(X, \mathbb{R})$ and the volume deformation.

One of the most valuable tools for understanding the structure of the moduli space is the study of symmetries. Such an analysis will be initiated in the next section, where we also review some of the mathematical background.

7.3.1 Making use of $K3$ geometry: Symmetries and lattices

In order to take advantage of the mathematicians' insight on $K3$ geometry we first study how to translate symmetries of a given superconformal field theory to its geometric interpretations. Those symmetries which commute with our $su(2)_l \oplus su(2)_r$ action leave the four–plane x invariant and are called ALGEBRAIC SYMMETRIES. When the $N = (4,4)$ supersymmetric theories are constructed in terms of $(2,2)$ supersymmetric theories one has a natural framing. In this context, algebraic symmetries are those which leave the entire vector space $Q_l \otimes Q_r$ of supercharges invariant. More generally, any Abelian symmetry group of our theory projects to a $u(1)_l \oplus u(1)_r$ subgroup of $su(2)_l \oplus su(2)_r$ and fixes the corresponding $N = (2,2)$

subalgebra. When corresponding supercharges are fixed, the Abelian symmetry group acts diagonally on the charge generators $J^\pm, \overline{J}^\pm$ of $su(2)_l^{susy} \oplus su(2)_r^{susy}$. The algebraic subgroup of this symmetry group is the one which fixes these charges.
If the primitive null vector υ specifying our geometric interpretation (Σ, V, B) is invariant upon the induced action of an algebraic symmetry we call the latter a CLASSICAL SYMMETRY of the geometric interpretation (Σ, V, B). Because a classical symmetry α^* fixes x by definition we get an induced automorphism of $H^2(X,\mathbb{R})$ which leaves $\Sigma \subset H^2(X,\mathbb{R})$ and $B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})$ invariant. Moreover, because ξ_4 in (7.1.6) is invariant as well, $\eta_{20} = \upsilon^0 + B - (\frac{\|B\|^2}{2} + V)\upsilon$ is fixed. Thus α^* acts trivially on moduli of volume and B-field deformation in direction of Σ. Because α^* acts as automorphism on $H^{1,1}(X,\mathbb{R}) = \Omega^\perp \cap H^2(X,\mathbb{R})$ for any choice of complex structure $\Omega \subset \Sigma$ on X leaving the onedimensional $H^{1,1}(X,\mathbb{R}) \cap \Sigma$ invariant, all in all, $x \mapsto (\Sigma, V, B)$ maps the action of α^* to an automorphism of $H^2(X,\mathbb{R})$ which on $H^{1,1}(X,\mathbb{R})$ has exactly the same spectrum as α^* on $(\frac{1}{2},\frac{1}{2})$-fields with charge, say, $Q = \overline{Q} = 1$.
We thus find that we are naturally led to a discussion of algebraic automorphisms of $K3$ surfaces:

Algebraic automorphisms and discrete symmetries of Gepner models

Definition 7.3.1 [Nik80a]
Consider an automorphism $\alpha \in Aut(X)$ of finite order on a $K3$ surface X, whose induced action α^* on $H^2(X,\mathbb{C})$ is trivial on $H^{2,0}(X)$. Then α is called an ALGEBRAIC AUTOMORPHISM.

This notion of course only makes sense after a choice of complex structure. Below (7.1.5) we have seen that in conformal field theory language such a choice arises from selecting an $N = (2,2)$ subalgebra of the $N = (4,4)$ superconformal algebra and so fixing generators $J, J^\pm, \overline{J}, \overline{J}^\pm$ of $su(2)_l \oplus su(2)_r$. Still, because in the present context the metric always is invariant under α^* as well, i.e. $\Sigma \subset H^2(X,\mathbb{R})^{\alpha^*}$, we see that for α^* with integral action on $H^2(X,\mathbb{C})$ which is induced by an automorphism $\alpha \in Aut(X)$ of finite order, α is an algebraic automorphism, independently of the choice of complex structure $\Omega \subset \Sigma$. On the other hand, given an algebraic automorphism α of X which induces an automorphism of $H^2(X,\mathbb{R})$ that leaves the B-field invariant, α induces a symmetry of our conformal field theory which leaves $J, J^\pm, \overline{J}, \overline{J}^\pm$ invariant. This gives a precise notion of how to continue such an algebraic automorphism to the conformal field theory level.
Algebraic automorphisms are mathematically well understood thanks to the work of Nikulin [Nik80a] for the Abelian and Mukai [Muk88] for the general case. The first to explicitly take advantage of their special properties in the context of conformal field theory was P.S. Aspinwall [Asp95]. From [Nik80a, Th. 4.3,4.7,4.15] one can deduce the following consequence of the global Torelli theorem:

Theorem 7.3.2
Let g denote an automorphism of $H^2(X,\mathbb{C})$ of finite order which maps forms corresponding to effective divisors of self intersection number -2 in $Pic(X)$ to forms

corresponding to effective divisors. Then g is induced by an algebraic automorphism of X iff $(H^2(X,\mathbb{Z})^g)^\perp \cap H^2(X,\mathbb{Z}) \subset Pic(X)$ is negative definite with respect to the intersection form and does not contain elements of length squared -2.

If for a geometric interpretation (Σ, V, B) of $x \in O^+(H^{even}(X,\mathbb{Z}))\backslash\mathcal{T}^{4,20}$ we have classical symmetries which act effectively on what we read off as $H^2(X,\mathbb{C})$ but are not induced by an algebraic automorphism of the $K3$ surface X by theorem 7.3.2, then our interpretation of x as giving a superconformal field theory breaks down. Such points should be conifold points of the moduli space $\mathcal{M}^{K3}$, characterized by too high an amount of symmetry. One can regard Nikulin's theorem 7.3.2 as harbinger of Witten's result that in points of enhanced symmetry on the moduli space of type IIA string theories compactified on $K3$ the conformal field theory description breaks down [Wit95].
By abuse of notation we will often renounce to distinguish between an algebraic automorphism on $K3$ and its induced action on cohomology.
From Mukai's work [Muk88, Th. 1.4] one may learn that the induced action of any algebraic automorphism group G on the total rational cohomology $H^*(X,\mathbb{Q})$ is a MATHIEU REPRESENTATION of G over $\mathbb{Q}$, i.e. a representation with character

$$\chi(g) = \mu(\mathrm{ord}(g)), \text{ where for } n \in \mathbb{N}: \; \mu(n) := \frac{24}{n \prod\limits_{\substack{p \text{ prime},\\ p|n}} (1+\frac{1}{p})}. \tag{7.3.2}$$

This imposes such severe restrictions on G that all possible finite algebraic automorphism groups can be classified [Nik80a, Muk88]. It also follows that

$$\dim_\mathbb{Q} H^*(X,\mathbb{Q})^G = \mu(G) := \frac{1}{|G|}\sum_{g\in G} \mu(\mathrm{ord}(g)) \tag{7.3.3}$$

[Muk88, Prop. 3.4]. Since G acts algebraically, we have $\dim_\mathbb{Q} H^*(X,\mathbb{Q})^G = \dim_\mathbb{R} H^*(X,\mathbb{R})^G = \dim_\mathbb{C} H^*(X,\mathbb{C})^G$. By definition of algebraic automorphisms $H^*(X,\mathbb{C})^G \supset H^0(X,\mathbb{C}) \oplus H^{2,0}(X,\mathbb{C}) \oplus H^{0,2}(X,\mathbb{C}) \oplus H^{2,2}(X,\mathbb{C})$, so

$$\mu(G) - 4 = \dim_\mathbb{R} H^{1,1}(X,\mathbb{R})^G. \tag{7.3.4}$$

Moreover, from theorem 7.3.2 we know that $\left(H^2(X,\mathbb{R})^G\right)^\perp \subset H^{1,1}(X,\mathbb{R})$ is negative definite, and because $H^{1,1}(X,\mathbb{R})$ has signature $(1,19)$, we may conclude that it contains an invariant element with positive length squared. Thus $\mu(G) \geq 5$ for every algebraic automorphism group G [Muk88, Th. 1.4]. Moreover [Muk88, Cor. 3.5, Prop. 3.6],

$$G \neq \{\mathbb{1}\} \Longrightarrow \mu(G) \leq 16. \tag{7.3.5}$$

Finally let us consider the special case of an algebraic automorphism α of order 4, which will be useful in due course. By n_k we denote the multiplicity of the eigenvalue i^k of the induced action α^* on $H^{1,1}(X,\mathbb{C})$. Because by (7.3.2) and (7.3.3) $\mu(\mathbb{Z}_4) = 10$ and $\mu(\mathbb{Z}_2) = 16$, using (7.3.4) we find $n_0 = 10 - 4 = 6, n_2 = 16 - 4 - n_0 = 6$. The automorphism α^* acts on the lattice $H^2(X,\mathbb{Z})$, so it must have integer trace. On the other hand $20 = \dim_{\mathbb{C}} H^{1,1}(X,\mathbb{C}) = n_0 + n_1 + n_2 + n_3$, hence

$$n_0 = n_2 = 6, \quad n_1 = n_3 = 4. \tag{7.3.6}$$

It now is natural to investigate algebraic symmetries of superconformal field theories in $\mathcal{M}^{K3}$. A program to find a stratification of the moduli space could even be formulated as follows: Determine all subspaces of theories having a geometric interpretation (Σ, V, B) with given algebraic automorphism group G. Relations between such subspaces may be described by the modding out of algebraic automorphisms. Any infinitesimal deformation of Σ by an element of $H^{1,1}(X,\mathbb{R})^G$ will preserve the symmetries in G, as well as volume deformations and B-field deformations by elements in $H^2(X,\mathbb{R})^G$. The subspace of theories with given classical symmetry group G in a geometric interpretation therefore can maximally have real dimension $3(\mu(G)-5)+1+\mu(G)-2 = 4(\mu(G)-4)$ in accord with (7.3.4). In particular, for the minimal value $\mu(G)=5$, the only deformations preserving the entire symmetry are deformations of volume and those of the B-field by elements of Σ. Of course, the above program is far from utterly realizable, even in the pure geometric context, but it might serve as a useful line of thought.
Let us discuss algebraic symmetries of Gepner models (sections 3.1.3 and 5.6) associated to $K3$. Assume we can locate our Gepner model $\prod_{j=1}^r (k_j)$ within $\mathcal{M}^{K3}$, that is we explicitly know the corresponding four–plane $x \subset H^{even}(X,\mathbb{R})$ as described in section 7.1. Furthermore assume that by picking a primitive null vector $v \in H^{even}(X,\mathbb{Z})$ we have chosen a specific geometric interpretation (Σ, V, B). By construction, a Gepner model comes with a fixed choice of the $N=(2,2)$ subalgebra corresponding to a specific twoplane $\Omega \subset \Sigma$. We stress that this is true for any geometric interpretation of $\prod_{j=1}^r (k_j)$: The choice of the $N=(2,2)$ subalgebra does *not* fix a complex structure *a priori*, it fixes a choice of complex structure *in every geometric interpretation* of our model, as was explained in section 7.1. Still, we now assume our $K3$ surface X to be equipped with complex structure and Kähler metric. By the above discussion we know that any symmetry of the Gepner model which leaves the $su(2)_l^{susy} \oplus su(2)_r^{susy}$ currents $J, J^\pm, \overline{J}, \overline{J}^\pm$ and the vector v invariant may act as an algebraic automorphism on X. Since the operators of twofold spectral flow together with the $U(1)$ currents $J, \overline{J}$ of $\prod_{j=1}^r (k_j)$ are the generators of $su(2)_l^{susy} \oplus su(2)_r^{susy}$ we may identify $J^\pm = \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes r}$ and $\overline{J}^\pm = \left(\Phi^0_{0,0;\mp 2,2}\right)^{\otimes r}$. Now recall the discussion of discrete Abelian symmetry groups $\mathcal{G}_{ab}$ of Gepner models in section 5.6. In particular,

$$\mathcal{G}_{ab}^{alg} = \left\{ [a_1,\dots,a_r] \in \prod_{j=1}^r \mathbb{Z}_{k_j+2}/\mathbb{Z}_M \;\middle|\; \sum_{j=1}^r \frac{a_j}{k_j+2} \in \mathbb{Z} \right\} \subset \mathcal{G}_{ab}$$

($M = \mathrm{lcm}\,(k_j+2, j=1,\dots,r)$) as defined in (5.6.3) is the stabilizer group of $J^\pm, \overline{J}^\pm$, indeed, the subgroup of algebraic symmetries of $\prod_{j=1}^r (k_j)$ in $\mathcal{G}_{ab}$. We conclude that elements of $\mathcal{G}_{ab}^{alg}$ can act as algebraic automorphisms on X fixing the B-field $B \in H^2(X,\mathbb{R})$, and vice versa. More explicitly by what was said at the beginning of this section, the action of such a Gepner symmetry on the $\left(\frac{1}{2},\frac{1}{2}\right)$-fields with charges, say, $Q = \overline{Q} = 1$ should be identified with the induced action of an algebraic automorphism of X on $H^{1,1}(X,\mathbb{R})$. With reference to its possible geometric interpretation we call $\mathcal{G}_{ab}^{alg}$ the ABELIAN ALGEBRAIC SYMMETRY GROUP OF THE GEPNER MODEL. It is not hard to determine $\mathcal{G}_{ab}^{alg}$ for all Gepner models associated to $K3$:

Theorem 7.3.3
The Abelian algebraic symmetry groups $\mathcal{G}_{alg}^{ab}$ of the Gepner models associated to $K3$ are given by

$\prod_{j=1}^r(k_j)$	$(1)^6$	$(1)^4(4)$	$(2)^4$	$(1)(2)^2(4)$	$(1)^2(4)^2$	$(1)^2(2)(10)$
$\mathcal{G}_{ab}^{alg}$	$\mathbb{Z}_3^4$	$\mathbb{Z}_3^3$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2$	$\mathbb{Z}_3^2$	$\mathbb{Z}_3$

$\prod_{j=1}^r(k_j)$	$(4)^3$	$(3)^2(8)$	$(2)(6)^2$	$(2)(4)(10)$	$(2)(3)(18)$
$\mathcal{G}_{ab}^{alg}$	$\mathbb{Z}_6$	$\mathbb{Z}_5$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\{\mathbb{1}\}$

$\prod_{j=1}^r(k_j)$	$(1)(10)^2$	$(1)(8)(13)$	$(1)(7)(16)$	$(1)(6)(22)$	$(1)(5)(40)$
$\mathcal{G}_{ab}^{alg}$	$\mathbb{Z}_3$	$\{\mathbb{1}\}$	$\mathbb{Z}_3$	$\{\mathbb{1}\}$	$\{\mathbb{1}\}$.

In section 7.3.6 we will investigate where in the moduli space of superconformal field theories associated to $K3$ to locate the Gepner model $(2)^4$ and some of its orbifolds by elements of $\mathcal{G}_{ab}^{alg} \cong (\mathbb{Z}_4)^2$. From the above discussion it is clear that given a definite geometric interpretation for $(2)^4$ the geometric interpretation of its orbifold models is obtained by modding out the corresponding algebraic automorphisms.
Apart from symmetries in $\mathbb{Z}_2 \times \mathcal{G}_{ab}$ our Gepner model will possess permutation symmetries involving identical factor theories. Their discussion is a bit more subtle, because as noted in [FKS92] a permutation of fermionic fields will involve additional signs (3.1.19). This in particular applies to $J^\pm = \left(\Phi^0_{\mp 2,2;0,0}\right)^{\otimes r}$, meaning that odd permutations can only act algebraically when accompanied by a phase symmetry

$$[a_1,\dots,a_r] \in \mathcal{G}_{ab}: \quad \sum_{j=1}^r \frac{a_j}{k_j+2} \in \mathbb{Z} + \tfrac{1}{2}. \tag{7.3.7}$$

Let us discuss this phenomenon in detail for the example of prime interest to us, namely the Gepner model $(2)^4$. Here $\mathcal{G}_{ab}^{alg} \cong (\mathbb{Z}_4)^2$ by theorem 7.3.3, and the entire algebraic symmetry group is generally believed to be $\mathcal{G}^{alg} \cong (\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ [Asp95]. Moreover, based on Landau-Ginzburg computations and comparison of symmetries [GVW89, GP90, FKSS90, Asp95] it is generally believed that $(2)^4$ has a geometric interpretation (Σ_Q, V_Q, B_Q) given by the Fermat quartic

$$\begin{aligned} \psi\in\mathbb{C}:\ \mathcal{Q}_\psi &= \Big\{(x_0:x_1:x_2:x_3)\in\mathbb{CP}^3 \ \Big| \\ &\qquad f_\psi(x_0:x_1:x_2:x_3) := \sum_{i=0}^3 x_i^4 - 4\psi\prod_{i=0}^3 x_i = 0\Big\} \end{aligned} \tag{7.3.8}$$

at $\psi = 0$. Indeed, $\mathcal{Q} = \mathcal{Q}_0$ is a $K3$ surface with algebraic automorphism group $(\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ [Muk88], and arguments in favour of the viewpoint that it yields a geometric interpretation of $(2)^4$ will arise from the following discussion. It is proved in corollary 7.3.28.
To give the action of the two generators $[1,3,0,0]$ and $[1,0,3,0]$ of $\mathcal{G}_{ab}^{alg} \cong (\mathbb{Z}_4)^2$ on the $\left(\frac{1}{2},\frac{1}{2}\right)$-fields with charges $Q = \overline{Q} = 1$ we use the shorthand notation

$$\begin{aligned} E &:= (\Phi^1_{1,0;-3,2})^{\otimes 4}, \\ F(n_1,n_2,n_3,n_4) &:= \Phi^{n_1}_{n_1,0;n_1,0} \otimes \Phi^{n_2}_{n_2,0;n_2,0} \otimes \Phi^{n_3}_{n_3,0;n_3,0} \otimes \Phi^{n_4}_{n_4,0;n_4,0} \end{aligned} \tag{7.3.9}$$

($n_i \in \mathbb{N}$) and find

$[1,3,0,0] \rightarrow$ $\downarrow [1,0,3,0]$	1	-1	i	$-i$
1	$F(1,1,1,1),\ E$	$F(0,2,0,2)$, $F(2,0,2,0)$	$F(1,0,1,2)$	$F(1,2,1,0)$
-1	$F(2,2,0,0)$, $F(0,0,2,2)$	$F(2,0,0,2)$, $F(0,2,2,0)$	$F(2,1,0,1)$	$F(0,1,2,1)$
i	$F(1,1,0,2)$	$F(2,0,1,1)$	$F(2,1,1,0)$	$F(1,2,0,1)$
$-i$	$F(1,1,2,0)$	$F(0,2,1,1)$	$F(1,0,2,1)$	$F(0,1,1,2)$

(7.3.10)

Note first that by (7.3.3) we have $\mu(\mathbb{Z}_4 \times \mathbb{Z}_4) = 6$, in accordance with (7.3.4) and $2 = 6 - 4$ invariant fields in the above table. One moreover easily checks that the spectrum of every element $g \in \mathcal{G}_{ab}^{alg}$ of order four agrees with the one computed in (7.3.6) for algebraic automorphisms of order four on $K3$ surfaces. This is a strong and highly non-trivial evidence for the fact that one possible geometric interpretation of $(2)^4$ is given by a $K3$ surface whose algebraic automorphism group contains $(\mathbb{Z}_4)^2$.
As stated above, further discussion is due concerning the action of $\mathcal{S}_4$ because transpositions of fermionic modes introduce sign flips (3.1.19). In particular, odd elements of $\mathcal{S}_4$ do not leave $J^\pm$ invariant. To have an algebraic action of the entire group $\mathcal{S}_4$ we must therefore accompany $\sigma \in \mathcal{S}_4$ by a phase symmetry $a_\sigma = [a_1(\sigma), a_2(\sigma), a_3(\sigma), a_4(\sigma)] \in \mathcal{G}_{ab}$ which for odd σ satisfies (7.3.7). Thus a transposition $(\alpha,\omega) \in \mathcal{S}_4$ must be represented by $\rho((\alpha,\omega)) = (\alpha,\omega) \circ a_{(\alpha,\omega)} = a_{(\alpha,\omega)} \circ (\alpha,\omega)$ in order to have $\rho((\alpha,\omega))^2 = \mathbb{1}$. With any such choice of ρ on generators (α_j,ω_j) of $\mathcal{S}_4$ one may then check explicitly that ρ defines an algebraic action of $\mathcal{S}_4$, i.e. its spectrum on the $\left(\frac{1}{2},\frac{1}{2}\right)$-fields coincides with the spectrum of the algebraic automorphism group $\mathcal{S}_4$. Namely, any element of order two (or three, four) in $\mathcal{S}_4$ leaves $\mu(\mathbb{Z}_2) - 4 = 12$ (or $\mu(\mathbb{Z}_3) - 4 = 8, \mu(\mathbb{Z}_4) - 4 = 6$) states invariant, and elements of order four have the spectrum given in (7.3.6). Note in particular that by (7.3.7) with any consistent choice of $\sigma \mapsto a_\sigma$ the group $\mathcal{S}_4$ acts by $\sigma \mapsto \mathrm{sign}(\sigma)$ on $F(1,1,1,1)$ and trivially on E. This leaves $E = (\Phi^1_{1,0;-3,2})^{\otimes 4}$ as the unique

invariant state upon the action of $(\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ in accordance with $\mu((\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4) = 5$ and (7.3.4).
Summarizing, we have shown that the action of the entire algebraic symmetry group $\mathcal{G}^{alg} = (\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$ of $(2)^4$ as described above exhibits a spectrum consistent with its interpretation as group of algebraic automorphisms of a $K3$ surface, e.g. the Fermat quartic with geometric interpretation $(\Sigma_\mathcal{Q}, V_\mathcal{Q}, B_\mathcal{Q})$. Remember that $\mu\left(\mathcal{G}^{alg}\right) = 5$ is the minimal possible value of μ by the above discussion. Thus the only four invariant $(\frac{1}{2}, \frac{1}{2})$-fields $(\Phi^1_{\pm 1,0;\mp 3,2})^{\otimes 4}$, $(\Phi^1_{\pm 1,0;\mp 1,0})^{\otimes 4}$ are those corresponding to moduli of volume deformations and of B-field deformations in direction of $\Sigma_\mathcal{Q}$.

On Picard numbers and Fermat quartic hypersurfaces

V. Nikulin has found and explained to us an elegant method to compute lower bounds on the Picard number (see definition 4.6.1) of a $K3$ surface*. Since the idea is based on exploiting algebraic automorphisms of the respective manifold and seems not to be known in general, we take the opportunity and digress to explain it and apply it to the family of Fermat hypersurfaces in $\mathbb{CP}^3$.
Consider the induced action of an algebraic automorphism group $G \subset Aut(X)$ on $H^2(X, \mathbb{Z})$ for a $K3$ surface X. By $T_G := H^2(X, \mathbb{Z})^G \subset H^2(X, \mathbb{Z})$ we denote the G invariant part of this lattice, and we set $S_G := T_G^\perp \cap H^2(X, \mathbb{Z})$. By [Nik80a, Lemma 4.2], S_G is contained in the Picard lattice $Pic(X) = NS(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ of X. In the proof of this lemma, V. Nikulin uses the following construction: Let Y denote the minimal resolution of X/G, then we obtain a birational map $\pi : X \to Y$ of degree $|G|$ which is defined outside the singular points. Loosely speaking, each exceptional divisor of the blow up on Y corresponds to an element of $S_G \subset Pic(X)$, such that we get a lower bound on the Picard number $\rho(X) = \mathrm{rk}\ Pic(X)$. More precisely, let $P \subset H^2(Y, \mathbb{Z})$ denote the lattice generated by the exceptional divisors obtained from the blow up. Set $K := P^\perp \cap H^2(Y, \mathbb{Z})$. Then π induces a map $\pi^* : K \hookrightarrow H^2(X, \mathbb{Z})$ which obeys

$$\forall\, x, y \in K: \quad \pi^*(x) \cdot \pi^*(y) = |G|(x \cdot y).$$

Hence $\pi^* K \subset T_G$ is a nondegenerate sublattice, and by construction $\mathrm{rk}\,(K) = \mathrm{rk}\,(T_G) = \mu(G) - 2$, where we used (7.3.3) and (7.3.4). Since $\mathrm{rk}\,(H^2(X, \mathbb{Z})) = 22$, we find

$$\rho(X) = \mathrm{rk}\ Pic(X) \geq \mathrm{rk}\,(S_G) = 24 - \mu(G),$$

a lower bound on the Picard number of X. If X is algebraic, $Pic(X)$ contains an element of positive norm. The above procedure only counts contributions to the negative definite part of $Pic(X)$, thus for an algebraic $K3$ surface X

$$\rho(X) \geq 25 - \mu(G). \tag{7.3.11}$$

We will now apply Nikulin's method to the Fermat family (7.3.8):

*We thank F. Laytimi for her truly enjoyable story on this subject [Nah].

Theorem 7.3.4
The Fermat family $\mathcal{Q}_\psi$ given in (7.3.8) has generic Picard number 19; the Fermat quartic hypersurface $\mathcal{Q}_0$ has Picard number 20.

Proof:
The Abelian group of algebraic automorphisms of $\mathbb{CP}^3$ which leave $\mathcal{Q}_\psi$ invariant is

$$G := \left\{(n_0, n_1, n_2, n_3) \in (\mathbb{Z}_4)^4 \,|\sum_{i=0}^{3} n_i \equiv 0(4)\right\},$$

where $(n_0, n_1, n_2, n_3) \in (\mathbb{Z}_4)^4$ acts by

$$(n_0, n_1, n_2, n_3).(x_0 : x_1 : x_2 : x_3) = (i^{n_0}x_0 : i^{n_1}x_1 : i^{n_2}x_2 : i^{n_3}x_3).$$

It is also easy to check that the holomorphic twoform $\Omega_\psi^{2,0}$ of $\mathcal{Q}_\psi$ is invariant under G, since for homogeneous coordinates z_1, z_2, z_3 and $\widetilde{f}_\psi$ the defining polynomial for $\mathcal{Q}_\psi$ with respect to these coordinates,

$$\Omega^{2,0} = \frac{dz_1 \wedge dz_2}{\partial \widetilde{f}_\psi / \partial z_3}.$$

Since in the above notation $(1,1,1,1) \in G$ generates the $\mathbb{C}^*$ action that acts trivially on $\mathbb{CP}^3$, we find $G \cong \mathbb{Z}_4 \times \mathbb{Z}_4$. Because each of the $\mathcal{Q}_\psi$ is algebraic by construction, $\mu(G) = 6$ now leaves us with $\rho(\mathcal{Q}_\psi) \geq 19$ by (7.3.11). By theorem 4.6.6 singular $K3$ surfaces are discrete in the moduli space of complex structures on $K3$, hence we cannot have generic Picard number 20 on the Fermat family, proving the first assertion of the theorem. Indeed, one checks that the entire algebraic symmetry group of $\mathcal{Q}_\psi$ is $G_\bullet^{alg} = \mathbb{Z}_4^2 \rtimes \mathcal{A}_4$, where $\mathcal{A}_4 \subset \mathcal{S}_4$ is the group of even permutations of the four coordinates. By (7.3.3) we have $\mu(G_\bullet^{alg}) = 6$.
On the other hand, at $\psi = 0$ one finds an algebraic symmetry group $G_0^{alg} = \mathbb{Z}_4^2 \rtimes \mathcal{S}_4$ with $\mu(G_0^{alg}) = 5$, so $\mathcal{Q}_0$ must have maximal Picard number 20 by (7.3.11). □

From theorem 4.6.6 it follows that the complex structure of the singular Fermat quartic $\mathcal{Q} = \mathcal{Q}_0$ is uniquely determined by the quadratic form on its transcendental lattice, which by [Ino76] is

$$Q_{\mathcal{Q}} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}. \tag{7.3.12}$$

Another special point on the family $\mathcal{Q}_\psi$ is the one at $\psi = 1$ (or any other power of i): Here, the complex structure attains singularities at 16 points $x = (i^{n_0} : i^{n_1} : i^{n_2} : i^{n_3})$, $\sum_{i=0}^{3} n_i \equiv 0(4)$. All of them are NODES, i.e. ordinary double points. To check this for $x = (1:1:1:1)$, one rewrites the defining polynomial f_1 of $\mathcal{Q}_1$ with respect to coordinates $Z_0 := z_0, Z_i := z_i - z_0, i = 1, 2, 3$, to obtain a polynomial F. Now

$$\begin{aligned} F(Z_0, Z_1, Z_2, Z_3) &= 2(Z_0)^2 A(Z_1, Z_2, Z_3) + \cdots, \\ A(Z_1, Z_2, Z_3) &= (Z_1, Z_2, Z_3) \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \end{aligned}$$

where the dots refer to lower order terms with respect to Z_0. It suffices to check that A is a nondegenerate quadratic form in order to show that $(1 : 1 : 1 : 1)$ is a node. Then the other singular points are nodes as well by symmetry.

$K3$ lattices

In section 7.1 we observed that the integer cohomology of the complex surface X is of major importance to describe the moduli space $\mathcal{M}$. Here, we will concentrate on the geometric part of the moduli space $\mathcal{M}^{K3}$, i.e. the space (7.1.7) of Einstein metrics on a $K3$ surface X. An Einstein metric is specified by the relative position of the positive definite three–plane $\Sigma \subset H^2(X, \mathbb{R})$ to a fixed reference lattice $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$. In many cases the structure of the lattice $\Sigma^\perp \cap H^2(X, \mathbb{Z})$ gives insight in the particular geometry, especially for the orbifold limits of $K3$ (see section 5.2.2).
$\mathbb{Z}_2$ orbifold limits of $K3$ are known to mathematicians as KUMMER SURFACES, denoted by $\mathcal{K}(\Lambda)$ if obtained by the $\mathbb{Z}_2$ orbifold procedure from the four–torus $T = \mathbb{R}^4/\Lambda$. Generators of the lattice Λ are denoted by $\lambda_1, \ldots, \lambda_4$. From (4.4.3) we obtain an associated three–plane $\Sigma_T \subset H^2(T, \mathbb{R})$, i.e. an Einstein metric on T, and we want to describe how the Teichmüller space $\mathcal{T}^{3,3}$ of Einstein metrics of volume 1 on the torus is mapped into the corresponding space $\mathcal{T}^{3,19}$ for $K3$. In our notation $H^2(T, \mathbb{Z})$ is generated by $\mu_j \wedge \mu_k, j, k \in \{1, \ldots, 4\}$, where $(\mu_1, \ldots, \mu_4)$ is the basis dual to $(\lambda_1, \ldots, \lambda_4)$. Note that in order to simplify the following argument we regard $\Sigma_T \subset H^2(T, \mathbb{Z})$ as giving the position of the lattice $H^2(T, \mathbb{Z}) = \mathrm{span}_\mathbb{Z}(\mu_j \wedge \mu_k)$ relative to a fixed three–plane $\mathrm{span}_\mathbb{R}(e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2, e_1 \wedge e_4 + e_2 \wedge e_3)$ with respect to the standard basis $(e_1, \ldots, e_4)$ of $\mathbb{R}^4$.
To make contact with the theory of Kummer surfaces we pick a complex structure $\Omega_T \subset \Sigma_T$. The $\mathbb{Z}_2$ action on T has 16 fixed points $\frac{1}{2}\sum_{k=1}^4 \varepsilon_k \lambda_k, \varepsilon \in \mathbb{F}_2^4$. We can therefore choose indices in $\mathbb{F}_2^4$ to label the fixed points**. Note that this is not only a labelling, but the torus geometry indeed induces a natural affine $\mathbb{F}_2^4$-structure on the set I of fixed points [Nik75, Cor. 5]. The twoforms that correspond to the 16 exceptional divisors obtained from blowing up the fixed points are denoted by $\{E_i \mid i \in I\}$. They are elements of $Pic(X) = NS(X)$ no matter what complex structure Ω_T we choose, because we are working in the orbifold limit, i.e. $\forall\, i \in I: \ E_i \perp \Sigma$. Let $\Pi \subset Pic(X)$ denote the primitive sublattice of the Picard lattice containing $\{E_i \mid i \in I\}$. It is called KUMMER LATTICE and by [Nik75, Th. 3]:

Theorem 7.3.5
The Kummer lattice Π is spanned by the exceptional divisors $\{E_i \mid i \in I\}$ and $\{\frac{1}{2}\sum_{i\in H} E_i \mid H \subset I$ is a hyperplane$\}$. On the other hand, a $K3$ surface X is a Kummer surface iff $Pic(X)$ contains a primitive sublattice isomorphic to Π.

The next step is to understand how the lattice Π is embedded in $H^2(X, \mathbb{Z})$. We will review the construction given in [Nik75]. Let $\pi : T \to X$ be the birational degree two map from the torus to the Kummer surface. By Poincaré duality one gets maps π_* from the homology and cohomology groups of T to those of X, and π^* in the

**$\mathbb{F}_2$ denotes the unique finite field with two elements.

other direction. In particular, this gives the natural embedding $\pi_* : H^2(T,\mathbb{Z})(2) \hookrightarrow H^2(X,\mathbb{Z})$ [Nik75, Remark 2] (here $\Gamma(2)$ denotes Γ with quadratic form scaled by 2). The image lattice will be called K_2. We prefer to work with metric isomorphisms and therefore denote the image in K_2 of an element $a \in H^2(T,\mathbb{Z})$ by $\sqrt{2}a$. In particular, we write $\sqrt{2}\mu_j \wedge \mu_k, j,k \in \{1,\dots,4\}$, for the generators of K_2.
We remark that by the results in [Nik75], X is a Kummer surface iff there is a primitive embedding $\Gamma^{3,3}(2) \hookrightarrow Pic(X)$. In particular, a singular $K3$ surface X (see definition 4.6.3) is Kummer iff the quadratic form on its transcendental lattice is $Q_X = Q_T(2)$, where Q_T is an even quadratic form as well. Now (7.3.12) shows that the Fermat quartic $\mathcal{Q}_0$ of (7.3.8) is a Kummer surface. Indeed, from (4.6.2) we find that $\widetilde{T/\mathbb{Z}_2}$ with $T = \mathbb{R}^4/D_4^T$ has the same complex structure as $\mathcal{Q}_0$.
The lattice $H^2(X,\mathbb{Z})$ obeys $K_2 \oplus \Pi \subset H^2(X,\mathbb{Z}) \subset K_2^* \oplus \Pi^*$. One finds $K_2^*/K_2 \cong (\mathbb{Z}_2)^6 \cong \Pi^*/\Pi$, where Π^*/Π is generated by $\{\frac{1}{2}\sum_{i\in P} E_i \mid P \subset I \text{ is a plane}\}$. The isomorphism $\gamma : K_2^*/K_2 \longrightarrow \Pi^*/\Pi$ is most easily understood in terms of homology by assigning the image in X of a twocycle through four fixed points in a plane $P \subset I$ to $\frac{1}{2}\sum_{i\in P} E_i$. For example, $\gamma(\frac{1}{\sqrt{2}}\mu_j \wedge \mu_k) = \frac{1}{2}\sum_{i\in P_{jk}} E_i$, $P_{jk} = \mathrm{span}_{\mathbb{F}_2}(f_j, f_k) \subset \mathbb{F}_2^4$, $f_j \in \mathbb{F}_2^4$ the jth standard basis vector. Note that P_{jk} may be exchanged by any of its translates $l + P_{jk}, l \in \mathbb{F}_2^4$. The discriminant forms of K_2^*/K_2 and Π^*/Π, i.e. the induced $\mathbb{Q}/2\mathbb{Z}$ valued quadratic forms, agree up to a sign. One now uses

Lemma 7.3.6 [Nik80a, §1]
Let $L \subset \Gamma$ be a primitive sublattice of an even, selfdual lattice Γ. Let L' denote its orthogonal complement and assume $L \cap L' = \{0\}$. Then $L^*/L \cong (L')^*/L'$ with identical quadratic forms, up to a sign, where the isomorphism is denoted γ. If conversely L, L' are nondegenerate even lattices with discriminant forms that agree up to a sign,

$$\Gamma \cong \left\{ (l,l') \in L^* \oplus (L')^* \mid \gamma(\bar{l}) = \bar{l'} \right\}.$$

Hence in our setting

$$H^2(X,\mathbb{Z}) \cong \{(\kappa,\pi) \in K_2^* \oplus \Pi^* \mid \gamma(\overline{\kappa}) = \overline{\pi}\}, \qquad (7.3.13)$$

$\overline{\kappa}, \overline{\pi}$ denoting the images of κ, π under projection to K_2^*/K_2, Π^*/Π. Let us give a geometric explanation for Nikulin's method that we found together with Werner Nahm: Loosely speaking, the Poincaré dual of a representative κ of $\overline{\kappa} \in K_2^*/K_2$ can be interpreted as the π_* image of a torus cycle which contains $\mathbb{Z}_2$ fixed points. It is not a cycle on X, since it has boundaries where the exceptional divisors were glued in instead of the fixed points by the blow up procedure. Since the discriminant forms of K_2^*/K_2 and Π^*/Π agree up to a sign, there is a representative π of the Poincaré dual of $\overline{\pi} = \gamma(\overline{\kappa})$ with the same boundary as that of κ but orientation reversed. In other words, we can glue a part of a rational sphere corresponding to π into the boundary of the Poincaré dual of κ such that the intersection numbers of κ and π with the $E_i^{(\bullet)}$ agree up to a sign, to obtain a cocycle $\kappa + \pi \in H^2(X,\mathbb{Z})$. This can be seen explicitly in a local coordinate chart around a fixed point. Let us formulate it directly for general $\mathbb{Z}_M$ orbifolds, $M \in \{2,3,4,6\}$. For example, near

$(z_1, z_2) = (0, 0) \in T$ introduce $\mathbb{Z}_M$ invariant coordinates $x = z_1^M, y = z_2^M, z = z_1 z_2$. The (first) blow up of the fixed point $(x, y, z) = (0, 0, 0)$ on X in this chart is given by

$$U = \left\{(x, y, z; s_1, s_2, s_3) \in \mathbb{C}^3 \times \mathbb{CP}^2 \mid (x, y, z) \sim (s_1, s_2, s_3),\ s_1 s_2 = s_3^M\right\}.$$

Near $(0, 0, 0; 1, 0, 0)$ we may use coordinates (x, s_3) to write

$$U \ni \quad (x, y, z; s_1, s_2, s_3) = (x, x s_3^M, x s_3; 1, s_3^M, s_3).$$

The twocycle $\{z_2 = \varepsilon = const\}$ on T maps onto $\{y = \varepsilon^M\}$ and is described by the equation $x s_3^M = \varepsilon^M$. It is Poincaré dual to $\sqrt{M}\mu_1 \wedge \mu_2$ and has intersection number zero with the exceptional divisor Poincaré dual to $E_{(0,0,0,0)}$ (or $E^{(+)}_{(0,0)}, E^{(+)}_{(0,0,0,0)}, E^{(1)}_0$ for $M = 3, 4, 6$, respectively, see the notations introduced below). For $\varepsilon \to 0$ our cycle decomposes into $M + 1$ new cycles, $M\{s_3 = 0\} \cup \{x = 0\}$, where $\{x = 0\}$ is Poincaré dual to $E_{(0,0,0,0)}$ with self intersection number -2. Since $M\{s_3 = 0\} \cup \{x = 0\}$ must remain orthogonal to $\{x = 0\}$, we find $\{s_3 = 0\} = \{(x, 0, 0; 1, 0, 0)\}$ Poincaré dual to $\frac{1}{\sqrt{M}}\mu_1 \wedge \mu_2 - \frac{1}{M} E_{(0,0,0,0)} +$ contributions from other blow ups.
The isomorphism (7.3.13) provides a natural primitive embedding $K_2 \perp \Pi \hookrightarrow H^2(X, \mathbb{Z})$, which is unique up to isomorphism [Nik75, Lemma 7]. Hence $H^2(X, \mathbb{Z}) \cong \Gamma^{3,19}$ is generated by

$$M_2 := \left\{\tfrac{1}{\sqrt{2}}\mu_j \wedge \mu_k - \tfrac{1}{2}\sum_{i \in P_{jk}} E_{i+l}, l \in I\right\} \text{ and } \mathcal{E}_2 := \operatorname{span}_{\mathbb{Z}}(E_i, i \in I), \qquad (7.3.14)$$

and $\Gamma^{3,3}(2) \cong H^2(T, \mathbb{Z})(2) \hookrightarrow H^2(X, \mathbb{Z}) \cong \Gamma^{3,19}$ is naturally embedded. The three–plane $\Sigma \subset H^2(X, \mathbb{R})$ which describes the location of the singular Kummer surface within the moduli space (7.1.7) of Einstein metrics of volume 1 on $K3$ is given by $\Sigma = \pi_* \Sigma_T$. In particular $\Sigma \subset H^2(X, \mathbb{R}) \cong H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ is obtained directly by regarding $\Sigma_T \subset H^2(T, \mathbb{R}) \cong H^2(T, \mathbb{Z}) \otimes \mathbb{R} \hookrightarrow H^2(X, \mathbb{Z}) \otimes \mathbb{R}$ as three–plane in $H^2(X, \mathbb{R})$.
Nikulin's construction can be generalized to the other $\mathbb{Z}_M$ orbifold limits of $K3$, $M \in \{3, 4, 6\}$, if we can determine the analogs $P_M \in \{\,\text{Ш}, \text{ШІ}, \text{VI}\,\}$ of the Kummer lattice Π for these cases. Vice versa, we know that the orbifold limits exist and can use lemma 7.3.6 for the primitive sublattice $K_M := \pi_*(H^2(T, \mathbb{Z})^{\mathbb{Z}_M}) \subset H^2(X, \mathbb{Z})$ and its complement P_M. A representative κ of an element of K_M^*/K_M again can be interpreted as image of a torus cycle that contains $\mathbb{Z}_M$ fixed points. Hence its image under the gluing isomorphism γ in P_M^*/P_M that allows to construct a cycle on X from κ is determined by the intersection number of κ with the respective exceptional divisors. This procedure allows to find the analogs of (7.3.14) in all cases, once K_M is known.

Lemma 7.3.7
Let X denote a $\mathbb{Z}_3$ orbifold limit of $K3$, i.e. $X = \widetilde{T/\mathbb{Z}_3}$, $T = T^2_{(1)} \times T^2_{(2)}$ as described in section 5.2.2, with birational map $\pi : T \longrightarrow X$ of degree 3. Then $K_3 = \pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3} \subset H^2(X, \mathbb{Z})$ has intersection form

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus U(3).$$

*Let $\Pi\!\Pi\!\Pi$ denote the primitive sublattice of $H^2(X,\mathbb{Z})$ that contains all exceptional divisors $\mathcal{E}_3 := \{E_t^{(\pm)}, t \in \mathbb{F}_3^2\}$ of the blow up, where $t \in \mathbb{F}_3^2$ labels $\mathbb{Z}_3$ fixed points*** on T, each $\mathbb{F}_3$ referring to one factor $T^2_{(k)}$. With $E_t := E_t^{(+)} - E_t^{(-)}$, $\Pi\!\Pi\!\Pi$ is generated by $\mathcal{E}_3$ and*

$$\tfrac{1}{3}\left(\sum_{t\in L_1} E_t - \sum_{s\in L_2} E_s\right), \qquad L_1, L_2 \subset \mathbb{F}_3^2 \text{ parallel lines.}$$

The lattice generated by $\mathcal{E}_3$ and the set M_3 which consists of

$$\begin{aligned}
&\tfrac{1}{\sqrt{3}}\mu_1\wedge\mu_2 - \tfrac{1}{3}\left(E_{(0,i)} + E_{(1,i)} + E_{(2,i)}\right),\\
&\tfrac{1}{\sqrt{3}}\mu_3\wedge\mu_4 + \tfrac{1}{3}\left(E_{(i,0)} + E_{(i,1)} + E_{(i,2)}\right), \quad i \in \mathbb{F}_3;\\
&\tfrac{1}{\sqrt{3}}(\mu_1-\mu_3)\wedge(\mu_2 - \tfrac{1}{2}\mu_1 + \mu_4 - \tfrac{1}{2}\mu_3) + \tfrac{1}{3}\left(E_{(0,0)} + E_{(1,1)} + E_{(2,2)}\right);\\
&\tfrac{1}{\sqrt{3}}(\mu_1+\mu_3)\wedge(\mu_2 - \tfrac{1}{2}\mu_1 - \mu_4 + \tfrac{1}{2}\mu_3) + \tfrac{1}{3}\left(E_{(0,0)} + E_{(1,2)} + E_{(2,1)}\right)
\end{aligned}$$

is isomorphic to $\Gamma^{3,19}$. In particular, $\Pi\!\Pi\!\Pi^*/\Pi\!\Pi\!\Pi \cong (\mathbb{Z}_3)^3$ with generators

$$\tfrac{1}{3}\left(E_{(0,0)} + E_{(1,0)} + E_{(2,0)}\right),\ \tfrac{1}{3}\left(E_{(0,0)} + E_{(0,1)} + E_{(0,2)}\right),\ \tfrac{1}{3}\left(E_{(0,0)} + E_{(1,1)} + E_{(2,2)}\right).$$

This gives a natural embedding $K_3 \perp \Pi\!\Pi\!\Pi \hookrightarrow H^2(X,\mathbb{Z})$ and in particular an embedding $\pi_*(H^2(T,\mathbb{Z}))^{\mathbb{Z}_3} \hookrightarrow H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$. Given a Kähler–Einstein metric in $\mathcal{T}^{3,3}$ defined by $\Sigma_T \subset H^2(T,\mathbb{R})^{\mathbb{Z}_3}$, its image Σ under the $\mathbb{Z}_3$ orbifold procedure is read off from $\Sigma_T \subset H^2(T,\mathbb{R})^{\mathbb{Z}_3} \cong H^2(T,\mathbb{Z})^{\mathbb{Z}_3} \otimes \mathbb{R} \hookrightarrow H^2(X,\mathbb{Z}) \otimes \mathbb{R} \cong H^2(X,\mathbb{R})$.

Proof:
With respect to complex coordinates (z_1, z_2) on T as induced by the decomposition $T = T^2_{(1)} \times T^2_{(2)}$ the cycle $\{z_2 = \text{const.}\}$ corresponds to the twoform $\mu_1 \wedge \mu_2$. It can contain three $\mathbb{Z}_3$ fixed points with labels $(0,i),(1,i),(2,i) \in \mathbb{F}_3^2$, and as explained before (7.3.14) we have to find the Poincaré dual of $\{s_3 = 0\}$ in $\{y = \varepsilon^3\} \stackrel{\varepsilon\to 0}{\longrightarrow} 3\{s_3 = 0\} \cup \{x = 0\}$, where $\{x = 0\}$ is the Poincaré dual of $E^{(+)}_{(0,0)}$. Since $-\frac{1}{3}(4E^{(+)}_{(0,0)} + 2E^{(-)}_{(0,0)})$ has the correct intersection numbers, we find

$$\tfrac{1}{\sqrt{3}}\mu_1\wedge\mu_2 - \tfrac{1}{3}\left(E_{(0,i)} + E_{(1,i)} + E_{(2,i)}\right) \in H^2(X,\mathbb{Z}).$$

The other two types of elements of M_3 are determined analogously and correspond to cycles $\{z_1 = \text{const.}\}$ and $\{z_1 = \overline{z}_2\}$, respectively. Since $i \in \mathbb{F}_3$ arbitrary for these generators of $H^2(X,\mathbb{Z})$, all the additional generators of $\Pi\!\Pi\!\Pi$ listed in the theorem are indeed contained in $H^2(X,\mathbb{Z})$. Moreover, $\Pi\!\Pi\!\Pi^*/\Pi\!\Pi\!\Pi$ is generated by the asserted elements, hence $|\Pi\!\Pi\!\Pi^*/\Pi\!\Pi\!\Pi| \geq 3^3$. On the other hand, if T is the $SU(3)$ torus, then X is singular and by [SI77, Lemma 5.1] the quadratic form on the transcendental lattice is $Q_X = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Thus K_3 is contained in a lattice with intersection form as asserted above, and $|K^*/K| \leq 3^3$. Now $|K^*/K| = |\Pi\!\Pi\!\Pi^*/\Pi\!\Pi\!\Pi|$ by lemma

***$\mathbb{F}_3$ denotes the unique finite field with three elements.

7.3.6 proves the theorem, since no additional generators can occur in either of the lattices discussed above. $\square$

The other two cases are treated analogously, where we use $Q_X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for $X = \widetilde{T_{SU(2)^4}/\mathbb{Z}_4}$ [SI77, Lemma 5.2], and $Q_X = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ for $\widetilde{T_{SU(3)^2}/\mathbb{Z}_6}$. The latter can be understood since similarly to theorem 7.3.13, this orbifold limit can be obtained from a $\mathbb{Z}_3$ orbifold by another $\mathbb{Z}_2$ orbifold construction but must have different quadratic form from the $\mathbb{Z}_3$ orbifold limit by theorem 4.6.6.
As to notations, for the twoforms corresponding to the exceptional divisors of the $\mathbb{Z}_4$ orbifold we adopt the labelling of fixed points by $I \cong \mathbb{F}_2^4$ as used in the $\mathbb{Z}_2$ orbifold case. Here, we have six $\mathbb{Z}_2$ fixed points labelled by $i \in I^{(2)} := \{(j_1, j_2, 1, 0), (1, 0, j_3, j_4) \mid j_k \in \mathbb{F}_2\}$. The four true $\mathbb{Z}_4$ fixed points are labelled by $i \in I^{(4)} := \{(i, i, j, j) \mid i, j \in \mathbb{F}_2\}$. The corresponding twoforms are denoted by E_i for $i \in I^{(2)}$, and for each $\mathbb{Z}_4$ fixed point $i \in I^{(4)}$ we have three exceptional divisors Poincaré dual to $E_i^{(\pm)}, E_i^{(0)}$ such that $\langle E_i^{(\pm)}, E_i^{(0)}\rangle = 1, \langle E_i^{(+)}, E_i^{(-)}\rangle = 0$. For ease of notation we also use the combination $E_i := 3E_i^{(+)} + 2E_i^{(0)} + E_i^{(-)}$ if $i \in I^{(4)}$. We adopt the notation $P_{jk} = \mathrm{span}_{\mathbb{F}_2}(f_j, f_k)$ used above. Remember to count $\mathbb{Z}_2$ fixed points only once, e.g. $P_{12} = \{(0,0,0,0), (1,0,0,0), (1,1,0,0)\}$. We then have

Lemma 7.3.8
The lattice generated by the set M_4 which consists of

$$\tfrac{1}{2}\mu_1 \wedge \mu_2 - \tfrac{1}{2}E_{(0,1,0,0)+\varepsilon(0,0,1,1)} - \tfrac{1}{4}\sum_{i \in P_{12}\cap I^{(4)}} E_{i+\varepsilon(0,0,1,1)}, \quad \varepsilon \in \{0,1\};$$
$$\tfrac{1}{2}\mu_3 \wedge \mu_4 + \tfrac{1}{2}E_{(0,0,0,1)+\varepsilon(1,1,0,0)} + \tfrac{1}{4}\sum_{i \in P_{34}\cap I^{(4)}} E_{i+\varepsilon(1,1,0,0)}, \quad \varepsilon \in \{0,1\};$$
$$\tfrac{1}{2}\left(\mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2\right) - \tfrac{1}{2}\sum_{i \in P_{13}} E_{i+j}, \quad j \in I^{(4)};$$
$$\tfrac{1}{2}\left(\mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3\right) - \tfrac{1}{2}\sum_{i \in P_{14}} E_{i+j}, \quad j \in I^{(4)},$$

and by $\mathcal{E}_4 := \{E_i^{(\pm)}, E_i^{(0)}, i \in I^{(4)}; E_i, i \in I^{(2)}\}$ is isomorphic to $\Gamma^{3,19}$. In particular, $\Pi\!\Pi$ is generated by $\mathcal{E}_4$ and

$$\tfrac{1}{4}\left(E_{(0,0,0,0)} + E_{(1,1,1,1)} - E_{(0,0,1,1)} - E_{(1,1,0,0)}\right) + \tfrac{1}{2}\left(E_{(0,1,0,1)} + E_{(0,1,1,0)}\right),$$

$$\tfrac{1}{2}\left(E_{(0,0,0,0)} + E_{(0,0,1,1)} + E_{(0,1,0,0)} + E_{(0,1,1,1)} + E_{(0,1,0,1)} + E_{(0,1,1,0)}\right),$$

$$\tfrac{1}{2}\left(E_{(1,1,0,0)} + E_{(0,0,1,1)} + E_{(0,0,0,1)} + E_{(0,1,0,0)} + E_{(1,1,0,1)} + E_{(0,1,1,1)}\right).$$

This gives a natural embedding $K_4 \perp \Pi\!\Pi \hookrightarrow H^2(X,\mathbb{Z})$ and in particular an embedding $\pi_*(H^2(T,\mathbb{Z}))^{\mathbb{Z}_4} \hookrightarrow H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$. Given a Kähler–Einstein metric in $\mathcal{T}^{3,3}$ defined by $\Sigma_T \subset H^2(T,\mathbb{R})^{\mathbb{Z}_4}$, its image Σ under the $\mathbb{Z}_4$ orbifold procedure is read off from $\Sigma_T \subset H^2(T,\mathbb{R})^{\mathbb{Z}_4} \cong H^2(T,\mathbb{Z})^{\mathbb{Z}_4} \otimes \mathbb{R} \hookrightarrow H^2(X,\mathbb{Z}) \otimes \mathbb{R} \cong H^2(X,\mathbb{R})$.

A $\mathbb{Z}_6$ orbifold limit has one $\mathbb{Z}_6$ type fixed point 0, where the twoforms corresponding to the exceptional divisors are denoted $E_0^{(k)}, k \in \{0, \dots, 5\}$, such that $\langle E_0^{(k)}, E_0^{(k+1)} \rangle = 1, \|E_0^{(k)}\|^2 = -2$, and all other intersection numbers are zero. We set $E_0 := E_0^{(1)} + 2E_0^{(2)} + \dots + 5E_0^{(5)}$, and for the four $\mathbb{Z}_3$ and five $\mathbb{Z}_2$ type fixed points adopt the labelling from the $\mathbb{Z}_3$ and $\mathbb{Z}_2$ orbifold cases. Then we find

Lemma 7.3.9
The lattice generated by the set M_6 which consists of

$$\begin{aligned}
&\tfrac{1}{\sqrt{6}}\mu_1 \wedge \mu_2 - \tfrac{1}{6}E_0 - \tfrac{1}{3}E_{(1,0)} - \tfrac{1}{2}E_{(1,1,0,0)},\ \tfrac{1}{\sqrt{6}}\mu_3 \wedge \mu_4 + \tfrac{1}{6}E_0 + \tfrac{1}{3}E_{(0,1)} + \tfrac{1}{2}E_{(0,0,1,1)},\\
&\tfrac{2}{\sqrt{6}}\mu_1 \wedge \mu_2 - \tfrac{1}{3}\left(E_{(0,1)} + E_{(1,1)} + E_{(1,-1)}\right), \tfrac{2}{\sqrt{6}}\mu_3 \wedge \mu_4 + \tfrac{1}{3}\left(E_{(1,0)} + E_{(1,1)} + E_{(1,-1)}\right),\\
&\tfrac{3}{\sqrt{6}}\mu_1 \wedge \mu_2 - \tfrac{1}{2}\left(E_{(0,0,1,1)} + E_{(0,1,1,1)} + E_{(1,0,1,1)} + E_{(1,0,1,0)}\right),\\
&\tfrac{3}{\sqrt{6}}\mu_3 \wedge \mu_4 + \tfrac{1}{2}\left(E_{(1,1,0,0)} + E_{(1,1,1,0)} + E_{(1,1,0,1)} + E_{(1,0,1,0)}\right),\\
&\tfrac{1}{\sqrt{6}}(\mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3 + \mu_3 \wedge \mu_1 + \mu_1 \wedge \mu_2 - \mu_3 \wedge \mu_4)\\
&\qquad\qquad - \tfrac{1}{6}E_0 - \tfrac{1}{3}E_{(1,1)} - \tfrac{1}{2}\left(E_{(0,1,0,1)} + E_{(1,0,1,0)} + E_{(1,1,1,1)}\right),\\
&\tfrac{1}{\sqrt{6}}(\mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3 + \mu_3 \wedge \mu_1 - \mu_1 \wedge \mu_2 + \mu_3 \wedge \mu_4)\\
&\qquad\qquad + \tfrac{1}{6}E_0 + \tfrac{1}{3}E_{(1,-1)} + \tfrac{1}{2}\left(E_{(0,1,0,1)} + E_{(1,0,1,0)} + E_{(1,1,1,1)}\right),
\end{aligned}$$

and by $\mathcal{E}_6 := \{E_0^{(k)}, k \in \{0, \dots, 5\}; E_t^{(\pm)}, t \in \mathbb{F}_3^2; E_i, i \in I^{(2)}\}$ is isomorphic to $\Gamma^{3,19}$. In particular, $\mathbb{\Pi}$ is generated by $\mathcal{E}_6$ and

$$\begin{aligned}
&\tfrac{1}{6}E_0 + \tfrac{1}{3}\left(E_{(1,-1)} + E_{(1,0)} + E_{(0,1)} + E_{(1,1)}\right)\\
&\qquad\qquad + \tfrac{1}{2}\left(E_{(0,0,1,1)} + E_{(1,1,0,0)} + E_{(1,0,0,1)} + E_{(0,1,1,0)} + E_{(0,1,0,1)}\right).
\end{aligned}$$

This gives a natural embedding $K_6 \perp \mathbb{\Pi} \hookrightarrow H^2(X, \mathbb{Z})$ and in particular an embedding $\pi_*(H^2(T, \mathbb{Z}))^{\mathbb{Z}_6} \hookrightarrow H^2(X, \mathbb{Z}) \cong \Gamma^{3,19}$. Given a Kähler–Einstein metric in $\mathcal{T}^{3,3}$ defined by $\Sigma_T \subset H^2(T, \mathbb{R})^{\mathbb{Z}_6}$, its image Σ under the $\mathbb{Z}_6$ orbifold procedure is read off from the embedding $\Sigma_T \subset H^2(T, \mathbb{R})^{\mathbb{Z}_6} \cong (H^2(T, \mathbb{Z}))^{\mathbb{Z}_6} \otimes \mathbb{R} \hookrightarrow H^2(X, \mathbb{Z}) \otimes \mathbb{R} \cong H^2(X, \mathbb{R})$.

As noted above, the study of symmetries is an important tool in order to understand the moduli space $\mathcal{M}$ of superconformal field theories with central charge $c = 6$. It will prove particularly useful to understand algebraic automorphisms of $\mathbb{Z}_M$ orbifold conformal field theories. Here, we investigate algebraic automorphisms of Kummer surfaces which fix the orbifold singular metric. Such an automorphism induces an automorphism of the Kummer lattice Π because by $K_2 \cong H^2(T, \mathbb{Z})(2)$ and (7.3.13) all the lattice vectors of length squared -2 in $\Sigma^\perp$ belong to Π, and $\Pi \otimes \mathbb{R}$ by theorem 7.3.5 is spanned by the lattice vectors $E_i, i \in I$, of length squared -2. Vice versa,

Lemma 7.3.10
The action of an algebraic automorphism α which fixes the orbifold singular metric on a Kummer surface X is uniquely determined by its action on the set $\{E_i \mid i \in I\}$ of forms corresponding to exceptional divisors, i.e. by an affine transformation $A_\alpha \in \mathrm{Aff}(I)$.

Proof:
Let α^* denote the induced automorphism on the Kummer lattice Π. By theorem 7.3.5 and (7.3.13) the intersection form on Π is negative definite and the $\pm E_i, i \in I$, are the only lattice vectors of length squared -2. Therefore, α^* is uniquely determined by $\alpha^*(E_i) = \varepsilon_i(\alpha) e_{A_\alpha(i)}$ for $i \in I$, where $\varepsilon_i(\alpha) \in \{\pm 1\}$ and $A_\alpha \in \mathrm{Aff}(I)$. Actually, $\varepsilon_i(\alpha) = \varepsilon_i(A_\alpha)$, because $A_\alpha(i) = \mathbb{1} \Longrightarrow \varepsilon_i(\alpha) = 1$ for otherwise $E_i \in (H^2(X,\mathbb{Z})^{\alpha^*})^\perp$ with length squared -2 contradicting theorem 7.3.2. Assume $A_\alpha = A_{\alpha'}$ for another algebraic automorphism α' fixing the metric. Then $g := (\alpha^{-1} \circ \alpha')^*$ acts trivially on Π, and because Σ is fixed by g as well, for the group G generated by $\alpha^{-1} \circ \alpha'$ we find $\mu(G) \geq 2 + 3 + 16 = 21$. Now (7.3.5) shows that G is trivial and proves $\alpha = \alpha'$. $\square$

Note that every singular $K3$ surface possesses an infinite algebraic automorphism group by [SI77, Th. 5]. By abuse of language in the following we will frequently use the induced action of an algebraic automorphism on Π or in $\mathrm{Aff}(I)$ as a shorthand for the entire action.

Theorem 7.3.11
For every Kummer surface X the group of algebraic automorphisms fixing the orbifold singular metric contains $\mathbb{F}_2^4 \subset \mathrm{Aff}(I)$, which acts by translations on I.

Proof:
Any translation $t_i \in \mathrm{Aff}(I)$ by $i \in I$ acts trivially on Π^*/Π. Thus t_i can be continued trivially to $H^2(X,\mathbb{Z})$ by (7.3.13). One now easily checks that the resulting automorphism of $H^2(X,\mathbb{C})$ satisfies the criteria of theorem 7.3.2. $\square$

Next we will determine the group of algebraic automorphisms for the Kummer surface associated to a torus with enhanced symmetry:

Theorem 7.3.12
The group of algebraic automorphisms fixing the orbifold singular metric of $X = \mathcal{K}(\Lambda), \Lambda \sim \mathbb{Z}^4$, is $\mathcal{G}^+_{Kummer} = \mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4$. Here, $\mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4 \subset GL(\mathbb{F}_2^4) \ltimes \mathbb{F}_2^4 = \mathrm{Aff}(I)$ is equipped with the standard semidirect product.
For $\widetilde{X} = \mathcal{K}(\widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1,2$, the group of algebraic automorphisms fixing the orbifold singular metric generically is $\widetilde{\mathcal{G}}^+_{Kummer} = \mathbb{Z}_2 \ltimes \mathbb{F}_2^4$.

Proof:
To demonstrate $\mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4 \subset \mathcal{G}^+_{Kummer}$ we will show that certain algebraic automorphisms on the underlying torus $T = \mathbb{R}^4/\Lambda$ can be pushed to X and generate an additional group of automorphisms $\mathbb{Z}_2^2 \subset GL(\mathbb{F}_2^4)$ on Π. Namely, in terms of standard coordinates $(x_1, \ldots, x_4)$ on T, we are looking for automorphisms which leave

the forms

$$dx_1 \wedge dx_3 + dx_4 \wedge dx_2, \quad dx_1 \wedge dx_4 + dx_2 \wedge dx_3, \quad dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \quad (7.3.15)$$

invariant. This is true for

$$\begin{aligned} r_{12}: &\quad (x_1,x_2,x_3,x_4) \mapsto (-x_2,x_1,x_4,-x_3), \\ r_{13}: &\quad (x_1,x_2,x_3,x_4) \mapsto (-x_3,-x_4,x_1,x_2), \\ r_{14} = r_{12} \circ r_{13}: &\quad (x_1,x_2,x_3,x_4) \mapsto (x_4,-x_3,x_2,-x_1). \end{aligned} \quad (7.3.16)$$

The induced action on Π is described by permutations $A_{kl} \in \mathrm{Aff}(I)$ of the $\mathbb{F}_2^4$-coordinates, namely $r_{12} \mathrel{\widehat{=}} A_{12} = (12)(34), r_{13} \mathrel{\widehat{=}} A_{13} = (13)(24)$. To visualize this action we introduce the following helpful pictures first used by H. Inose [Ino76]: The vertical line labelled by $j \in \mathbb{F}_2^2$ symbolizes the image of the twocycle $\{x \in T \mid$

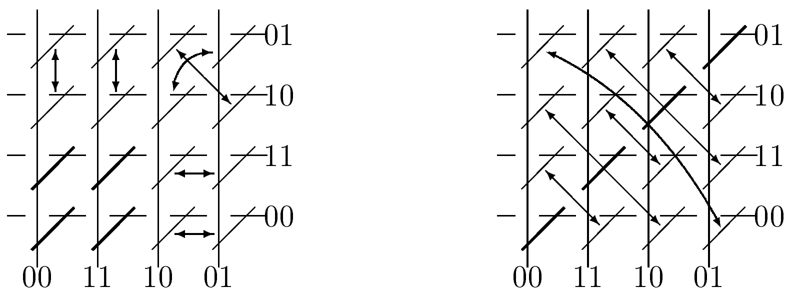

Figure 7.3.1: Action of the algebraic automorphisms r_{12} (left) and r_{13} (right) on Π.

$(x_1,x_2) = \frac{1}{2}j\}$ in X, and analogously for the horizontal line labelled by $j' \in \mathbb{F}_2^2$ we have $\{x \in T \mid (x_3,x_4) = \frac{1}{2}j'\}$. Then the diagonal lines from cycle j to cycle j' symbolize the exceptional divisor obtained from blowing up the fixed point labelled $(j,j') \in I$. Fat diagonal lines mark those exceptional divisors which are fixed by the respective automorphism.

One may now easily check that the automorphisms (7.3.16), viewed as automorphisms on $H^2(X,\mathbb{C})$, satisfy the criteria of theorem 7.3.2 and thus indeed are induced by algebraic automorphisms of X.

To see that $\mathcal{G}^+_{Kummer}$ does not contain any further elements, by lemma 7.3.10 it will suffice to show that no other element of $Aut(\Pi)$ can be continued to $H^2(X,\mathbb{Z})$ consistently such that it satisfies the criteria of theorem 7.3.2. Because all the translations of I are already contained in $\mathcal{G}^+_{Kummer}$ we can restrict our investigation to those elements $A \in GL(\mathbb{F}_2^4) \subset \mathrm{Aff}(I)$ which can be continued to $H^2(X,\mathbb{Z})$ preserving the symplectic forms on $\mathbb{F}_2^4$ that correspond to (7.3.15). After some calculation one finds that A must commute with all the transformations listed in (7.3.16). This means that A acts on I by $A'_{kl}(i) = A_{kl}(i) + |i|(1,1,1,1), |i| = \sum_k i_k \in \mathbb{F}_2$. But if any such $A'_{kl} \in \mathcal{G}^+_{Kummer}$, then also $A' \in \mathcal{G}^+_{Kummer}$, where $A'(i) = i + |i|(1,1,1,1)$. A' leaves invariant a sublattice of Π of rank 12. But then, because of (7.3.5) and from (7.3.4) A' cannot be induced by an algebraic automorphism fixing the orbifold singular metric of X.

The result for $\widetilde{\mathcal{G}}^+_{Kummer}$ follows from the above proof. Namely, if (x_1, x_2) are standard coordinates on $\Lambda_1 \otimes \mathbb{R}$ and (x_3, x_4) on $\Lambda_2 \otimes \mathbb{R}$, then among the automorphisms (7.3.16) only r_{12} is generically defined on $\widetilde{\Lambda}$. □

It is not hard to translate our results on algebraic automorphisms of Kummer surfaces to the $\mathbb{Z}_4$ orbifold limits of $K3$. To have a better understanding of their location within the moduli space and their geometric properties we now show that $\mathbb{Z}_4$ orbifolds can be constructed by another orbifold procedure from Kummer surfaces with enhanced symmetries:

Theorem 7.3.13
Let $\widetilde{\Lambda}$ denote a lattice generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$. Consider the $K3$ surface X obtained from the Kummer surface $\mathcal{K}(\widetilde{\Lambda})$ by modding out the algebraic automorphism $r_{12} \in \widetilde{\mathcal{G}}^+_{Kummer}$, blowing up the singularities and using the induced orbifold singular metric. Then X is the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\widetilde{\Lambda}$.

Proof:
By construction (7.3.16), r_{12} is induced by the automorphism $(x_1, x_2, x_3, x_4) \mapsto (-x_2, x_1, x_4, -x_3)$ with respect to standard coordinates on T. In terms of complex coordinates as induced by the decomposition $T = T^2_{(1)} \times T^2_{(2)}$ this is just the action $\rho : (z_1, z_2) \mapsto (iz_1, -iz_2)$, and because $\mathcal{K}(\Lambda) = \widetilde{T/\rho^2}$, the assertion follows. □

Remark:
Study figure 7.3.1 to see how the structure $A_1^6 \oplus A_3^4$ of the exceptional divisors in the $\mathbb{Z}_4$ orbifold comes about: Twelve of the fixed points in $\mathcal{K}(\widetilde{\Lambda})$ are identified pairwise to yield six $\mathbb{Z}_2$ fixed points in the $\mathbb{Z}_4$ orbifold, that is A_1^6. The four points labelled $i \in \{(0,0,0,0), (1,1,0,0), (0,0,1,1), (1,1,1,1)\}$ are true $\mathbb{Z}_4$ fixed points. The induced action of r_{12} on the corresponding exceptional divisor $\mathbb{CP}^1 \cong \mathbb{S}^2$ is just a 180° rotation about the north-south axis, and north and south poles are fixed points. Blow up the resulting singularities in $\mathcal{K}(\Lambda)/r_{12}$ to see how an A_3 arises from the A_1 over each true $\mathbb{Z}_4$ fixed point.
Concerning the algebraic automorphism group of $\mathbb{Z}_4$ orbifolds we find

Theorem 7.3.14
Let X denote the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\Lambda$. Then the group $\mathcal{G}$ of algebraic automorphisms fixing the orbifold singular metric of X consists of all the residual symmetries induced by algebraic automorphisms of $\mathcal{K}(\Lambda)$ which commute with r_{12}. Thus, generically $\mathcal{G} \cong \mathbb{F}_2^2$ is generated by the induced actions of t_{1100} and t_{0011}. If $\Lambda \sim \mathbb{Z}^4$, $\mathcal{G} \cong D_4$ is generated by the induced actions of t_{1100} and r_{13}.

Proof:
From theorem 7.3.13 it is clear that the residual symmetries induced by algebraic automorphisms of $\mathcal{K}(\Lambda)$ which commute with r_{12} induce algebraic automorphisms on X. Any further algebraic automorphism α^* would have a lift to $\mathcal{K}(\Lambda)$, so without loss of generality we can assume that α^* acts only by permutation on the ROOTS (lattice vectors of length squared -2) that correspond to each $\mathbb{Z}_4$ fixed point $i \in I^{(4)}$. There are twelve such roots, $E_i^{(\pm)}, E_i^{(0)}, E_i^{(\pm)} + E_i^{(0)}, E_i^{(+)} + E_i^{(0)} + E_i^{(-)}$,

and there negatives, where the previous ones are effective since the set of effective classes on a Kähler $K3$ surface contains the semigroup generated by the nodal classes [BPdV84, Prop.37]. Because α^* preserves intersection numbers, one checks that apart from the identity no consistent action α of this type exists. Now the assertion of the theorem follows from theorems 7.3.11 and 7.3.12. □

7.3.2 $\mathbb{Z}_2$ Orbifolds within the moduli space

The present section contains joint work with Werner Nahm that has been accepted for publication in [NW]. Some comments on $\mathbb{Z}_2$ orbifold conformal field theories as described in section 5.2.2 are due, before we can show where they are located within the moduli space $\mathcal{M}^{K3}$. We denote the $\mathbb{Z}_2$ orbifold obtained from the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ by $\mathcal{K}(\Lambda, B_T)$. If the theory on the torus has an enhanced symmetry G we sometimes simply write $G/\mathbb{Z}_2$, e.g. $SU(2)_1^4/\mathbb{Z}_2$ for $\mathcal{K}(\mathbb{Z}^4, 0)$.
In the nonlinear σ model on the torus $T = \mathbb{R}^4/\Lambda$ as described in section 4.2 the current j^k generates translations in direction of coordinate x_k. This induces a natural correspondence between tangent vectors of T and fields of the nonlinear σ model which is compatible with the $so(4)$ action on the tangent spaces of T and the moduli space, respectively. After selection of an appropriate framing of $Q_l \otimes Q_r$ to identify $su(2)_{l,r}^{susy}$ with $su(2)_{l,r}$ as described in section 7.1 the ψ^k are the superpartners of the j^k. Hence the choice of complex coordinates $z_1 := \frac{1}{\sqrt{2}}(x_1 + ix_2), z_2 := \frac{1}{\sqrt{2}}(x_3 + ix_4)$ corresponds to the definition of $\psi_\pm^{(k)}, j_\pm^{(k)}$ in (4.1.10). By (3.2.3) and (3.2.4) the holomorphic W-algebra of our theory has an $su(2)_1^2$-subalgebra generated by

$$\begin{array}{rclrclrcl} J &:=& \psi_+^{(1)}\psi_-^{(1)} + \psi_+^{(2)}\psi_-^{(2)}, & J^+ &:=& \psi_+^{(1)}\psi_+^{(2)}, & J^- &:=& \psi_-^{(2)}\psi_-^{(1)}; \\ A &:=& \psi_+^{(1)}\psi_-^{(1)} - \psi_+^{(2)}\psi_-^{(2)}, & A^+ &:=& \psi_+^{(1)}\psi_-^{(2)}, & A^- &:=& \psi_+^{(2)}\psi_-^{(1)}. \end{array} \tag{7.3.17}$$

Its geometric counterpart on the torus is the Clifford algebra generated by the twoforms $dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2, dz_1 \wedge dz_2, d\overline{z}_1 \wedge d\overline{z}_2; dz_1 \wedge d\overline{z}_1 - dz_2 \wedge d\overline{z}_2, dz_1 \wedge d\overline{z}_2, dz_2 \wedge d\overline{z}_1$ upon Clifford multiplication.
The nonlinear σ model on the Kummer surface $\mathcal{K}(\Lambda)$ is the $\mathbb{Z}_2$ orbifold of the above, where $\mathbb{Z}_2$ acts by $j^k \mapsto -j^k, \psi^k \mapsto -\psi^k$, $k = 1, \dots, 4$. Note that the entire $su(2)_1^2$-algebra (7.3.17) is invariant under this action, thus any nonlinear σ model on a Kummer surface possesses an $su(2)_1^2$-current algebra.
This orbifold model has an $N = (16, 16)$ supersymmetry. We are interested in deformations which conserve $N = (4, 4)$ subalgebras. By conjecture 3.1.1 the latter are given by chiral and antichiral $(\frac{1}{2}, \frac{1}{2})$–fields. Generically, the Neveu-Schwarz sector contains 144 fields with dimensions $(h; \overline{h}) = (\frac{1}{2}; \frac{1}{2})$. Their quantum numbers under $(J, A; \overline{J}, \overline{A})$ are $(\varepsilon_1, \varepsilon_2; \varepsilon_3, \varepsilon_4), \varepsilon_i \in \{\pm 1\}$ (16 fields), $(\varepsilon_1, 0; \varepsilon_3, 0)$ (64 fields), and $(0, \varepsilon_2; 0, \varepsilon_4)$ (64 fields). The 80 fields which are charged under $(J; \overline{J})$ yield the $N = (4, 4)$ supersymmetric deformations which conserve the superalgebra containing the J currents. The 80 fields which are charged under $(A; \overline{A})$ yield deformations conserving a different $N = (4, 4)$ superalgebra. The latter corresponds to the opposite torus orientation.

By construction orbifold conformal field theories have a preferred geometric interpretation in the sense of (7.1.6). We will now investigate this geometric interpretation for $\mathbb{Z}_2$ orbifolds and generalize Nikulin's technique of embedding $\mathcal{T}^{3,3}$ into $\mathcal{T}^{3,19}$ to the quantum level. We have to lift π_* to an embedding $\widehat{\pi}_* : H^{even}(T,\mathbb{Z})(2) \hookrightarrow H^{even}(X,\mathbb{Z})$. The image will be denoted by $\widehat{K}_2$. Apart from $\mu_j \wedge \mu_k$ the lattice $H^{even}(T,\mathbb{Z})$ has generators υ, υ^0 as defined in (7.1.6). Note that $\widehat{K}_2$ cannot be embedded as primitive sublattice in $\Gamma^{4,20}$ such that $\widehat{K}_2 \perp \Pi$ because $\widehat{K}_2^*/\widehat{K}_2 \cong (\mathbb{Z}_2)^8 \ncong (\mathbb{Z}_2)^6 \cong \Pi^*/\Pi$ would contradict lemma 7.3.6. This means that the B-field of the orbifold theory must have components in the Picard lattice. The torus model is given by a four–plane $x_T \subset H^{even}(T,\mathbb{R})$, the corresponding orbifold model by its image $x = \widehat{\pi}_* x_T$ in $H^{even}(X,\mathbb{Z}) \otimes \mathbb{R}$. To arrive at a complete description, we must find the embedding of $H^{even}(X,\mathbb{Z})$ in $\widehat{K}_2 \otimes \mathbb{R} + H^2(X,\mathbb{R})$. Since scalar products with elements of $\widehat{K}_2$ must be integral and $\sqrt{2}\upsilon^0 \in \widehat{K}_2$, every $a \in \Pi$ must have a lift $\frac{1}{\sqrt{2}}\upsilon + a$ or $0 + a$ in $H^{even}(X,\mathbb{Z})$. Those elements for which the lift has the form $0+a$ must form an $O^+(H^{even}(T,\mathbb{Z}))$ invariant sublattice of Π. One may easily check that this sublattice cannot contain the exceptional divisors $E_i, i \in I$. Moreover, as unimodular lattice $H^{even}(X,\mathbb{Z})$ has to contain an element of the form $\frac{1}{\sqrt{2}}\upsilon^0 + a$ with $a \in \Pi^*$. One finds that $H^{even}(X,\mathbb{Z})$ must contain the set of elements

$$\widehat{M}_2 := M_2 \cup \left\{ \tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I} E_i;\ -\tfrac{1}{\sqrt{2}}\upsilon + E_i, i \in I \right\}, \tag{7.3.18}$$

where M_2 was defined in (7.3.14). In analogy to Nikulin's description (7.3.13) and (7.3.14) of $H^2(X,\mathbb{Z}) \cong \Gamma^{3,19}$ we now find

Lemma 7.3.15
The lattice Γ spanned by $\widehat{M}_2$ and $\{\pi \in \Pi \mid \forall m \in \widehat{M} : \langle \pi, m\rangle \in \mathbb{Z}\}$ is isomorphic to $\Gamma^{4,20}$.

Proof:
Define

$$\widehat{\upsilon} := \sqrt{2}\upsilon, \quad \widehat{\upsilon}^0 := \tfrac{1}{\sqrt{2}}\upsilon^0 - \tfrac{1}{4}\sum_{i\in I} E_i + \sqrt{2}\upsilon, \quad \widehat{E_i} := -\tfrac{1}{\sqrt{2}}\upsilon + E_i. \tag{7.3.19}$$

Then Γ is generated by $\widehat{\upsilon}, \widehat{\upsilon}^0$ and the lattice

$$\widehat{\Gamma} := \mathrm{span}_{\mathbb{Z}}\left(\tfrac{1}{\sqrt{2}}\mu_j \wedge \mu_k + \tfrac{1}{2}\sum_{i\in P_{jk}} \widehat{E}_{i+l}, l \in I;\ \widehat{E_i}, i \in I \right).$$

Because $\langle \widehat{E_i}, \widehat{E_j}\rangle = -2\delta_{ij}$ and upon comparison to (7.3.14) it is now easy to see that $\widehat{\Gamma} \cong \Gamma^{3,19}$. Moreover, $\widehat{\upsilon}, \widehat{\upsilon}^0 \perp \widehat{\Gamma}$ and $\mathrm{span}_{\mathbb{Z}}(\widehat{\upsilon}, \widehat{\upsilon}^0) \cong U$ completes the proof. $\square$

In particular, lemma 7.3.15 gives a natural embedding $\Gamma^{4,4}(2) \cong H^{even}(T,\mathbb{Z})(2) \hookrightarrow H^{even}(X,\mathbb{Z}) \cong \Gamma^{4,20}$. As in the case of embedding the Teichmüller spaces $\mathcal{T}^{3,3} \hookrightarrow \mathcal{T}^{3,19}$ this enables us to locate the image under $\mathbb{Z}_2$ orbifold of a conformal field theory corresponding to a four–plane $x \subset H^{even}(T,\mathbb{R}) \cong \Gamma^{4,4}\otimes\mathbb{R}$ within $\mathcal{M}^{K3}$ by regarding x as four–plane in $H^{even}(X,\mathbb{R}) \cong \Gamma^{4,20}\otimes\mathbb{R}$. Note that in this geometric interpretation $\widehat{\upsilon}, \widehat{\upsilon}^0$ are the generators of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$.

Theorem 7.3.16
Let (Σ_T, V_T, B_T) denote a geometric interpretation of the nonlinear sigma model $\mathcal{T}(\Lambda, B_T)$ as given by (4.4.3). Then the corresponding orbifold conformal field theory $\mathcal{K}(\Lambda, B_T)$ associated to the Kummer surface $X=\mathcal{K}(\Lambda)$ has geometric interpretation (Σ, V, B) where $\Sigma \in \mathcal{T}^{3,19}$ as described after theorem 7.3.5, $V=\frac{V_T}{2}$ and $B=\frac{1}{\sqrt{2}}B_T+\frac{1}{2}B^{(2)}_{\mathbb{Z}}$, $B^{(2)}_{\mathbb{Z}}=\frac{1}{2}\sum_{i\in I}\widehat{E_i}\in H^{even}(X,\mathbb{Z})$ with $\widehat{E_i}\in H^{even}(X,\mathbb{Z})$ of length squared -2 given in (7.3.19).
In particular, the $\mathbb{Z}_2$ orbifold procedure induces an embedding $\mathcal{M}^{tori}\hookrightarrow\mathcal{M}^{K3}$ as quaternionic submanifold.

Proof:
Pick a basis $\sigma_i, i\in\{1,2,3\}$, of Σ_T. Then by (7.1.6) the nonlinear σ model $\mathcal{T}(\Lambda, B_T)$ is given by the four–plane x with generators $\xi_i=\sigma_i-\langle\sigma_i,B_T\rangle\upsilon, i\in\{1,2,3\}$, and $\xi_4=\upsilon^0+B_T+\left(V_T-\frac{\|B_T\|^2}{2}\right)\upsilon$. By the embedding $\Gamma^{4,4}\otimes\mathbb{R}\cong H^{even}(T,\mathbb{R})\hookrightarrow H^{even}(X,\mathbb{R})\cong\Gamma^{4,20}\otimes\mathbb{R}$ given in lemma 7.3.15 it is now a simple task to reexpress the generators of x using the generators $\widehat{\upsilon},\widehat{\upsilon}^0$ of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$:

$$\begin{aligned}
\sqrt{2}\,(\sigma_i-\langle\sigma_i,B_T\rangle\upsilon) &= \sqrt{2}\sigma_i-\langle\sqrt{2}\sigma_i,\tfrac{1}{\sqrt{2}}B_T\rangle\widehat{\upsilon}\\
\tfrac{1}{\sqrt{2}}\left(\upsilon^0+B_T+\left(V_T-\tfrac{\|B_T\|^2}{2}\right)\upsilon\right) &= \widehat{\upsilon}^0+\tfrac{1}{\sqrt{2}}B_T+\tfrac{1}{2}B^{(2)}_{\mathbb{Z}}\\
&\quad+\left(\tfrac{V_T}{2}-\tfrac{1}{2}\left\|\tfrac{1}{\sqrt{2}}B_T+\tfrac{1}{2}B^{(2)}_{\mathbb{Z}}\right\|^2\right)\widehat{\upsilon}.
\end{aligned}$$

Comparison with (7.1.6) gives the assertion of the theorem. □

Theorem 7.3.16 makes precise how the statement that orbifold conformal field theories tend to give value $B=\frac{1}{2}$ to the B-field in direction of exceptional divisors [Asp97, §4] is to be understood: For $B_T=0$, integration of B over any of the exceptional divisors that are Poincaré dual to an E_i gives $-\frac{1}{2}$. Note that $x^\perp\cap\Gamma^{4,20}$ does not contain vectors of length squared -2, namely $E_i\in x^\perp$, $\|E_i\|^2=-2$, but $E_i\notin H^{even}(X,\mathbb{Z})$. In the context of compactifications of the type IIA string on $K3$ this proves that $\mathbb{Z}_2$ orbifold conformal field theories do not have enhanced gauge symmetry. A similar statement was made in [Asp95] and widely spread in the literature, but we were unable to follow the argument up to our result of theorem 7.3.16.
Since in section 7.3.4 we will discuss how to locate the other $\mathbb{Z}_M$ orbifold models with $M\in\{3,4,6\}$ in $\mathcal{M}^{K3}$, it will prove useful to give some more comments on the above construction. Firstly, to embed $\widehat{K}_2=\widehat{\pi}_*H^{even}(T,\mathbb{Z})$ we had to fix images of υ,υ^0 in $H^{even}(X,\mathbb{Z})$, namely primitive null vectors $\sqrt{2}\upsilon,\sqrt{2}\upsilon^0\in H^{even}(X,\mathbb{Z})$ with $\langle\sqrt{2}\upsilon,\sqrt{2}\upsilon^0\rangle=2$. In terms of homology, the Poincaré dual of $\sqrt{2}\upsilon$ is the class of the image of a *generic* point on T under $\pi: T\to X$: recall that π is only defined

away from the $\mathbb{Z}_2$ fixed points on T. To understand the Poincaré dual of $\sqrt{2}\upsilon^0$, consider the following commutative diagram for $M = 2$ [Nik75, (9)]:

$$\begin{tikzcd}
\widetilde{T} \arrow[r, "\sigma_1"] \arrow[d, "\mathrm{f}", "\mathrm{M:1}"'] & T \arrow[d, "\mathrm{g}"] \arrow[dl, "\pi"'] \\
X \arrow[r, "\sigma"] & X_{sing}
\end{tikzcd} \tag{7.3.20}$$

Here, g is the factorization by $\mathbb{Z}_2$, and $\widetilde{T}$ is the two sheeted covering of X ramified over the divisor $\sum_{i\in I} e_i$, where e_i denotes the Poincaré dual of E_i. Then $C_i = f^{-1}(e_i)$ is a nonsingular rational curve with self intersection number -1 and can be contracted by the Castelnuovo–Grauert criterion [Šaf65]. σ_1 denotes this contraction. Note that $f^*(e_i) = 2C_i$, and the fact that $\sum_i e_i$ is the branch locus of f proves $\frac{1}{2}\sum_i E_i \in \Pi$, as stated in theorem 7.3.5. Now $\sqrt{2}\upsilon^0$ is the Poincaré dual of $f_*[\widetilde{T}]$. This is consistent with $\langle\sqrt{2}\upsilon, \sqrt{2}\upsilon^0\rangle = 2$ since f is $2:1$, and so is π outside the fixed points. Hence $\sqrt{2}\upsilon^0 - \frac{1}{2}\sum_i E_i$ is the Poincaré dual of $f_*([\widetilde{T}] - \sum_i C_i)$. Since f is $2:1$ and unbranched* on $([\widetilde{T}] - \sum_i C_i)$, we find $\frac{1}{2}(\sqrt{2}\upsilon^0 - \frac{1}{2}\sum_i E_i) \in H^{even}(X, \mathbb{Z})$, in agreement with (7.3.18). Note that this form has noninteger intersection products with the E_i, showing $E_i \notin H^2(X, \mathbb{Z})$ as above. We now find lemma 7.3.15 by setting $\check{B}_2 := \frac{1}{4}\sum_i E_i$ and

$$\widehat{\Pi} := \left\{\pi \in \Pi \mid \langle \check{B}_2, \pi\rangle \in \mathbb{Z}\right\}. \tag{7.3.21}$$

Indeed, then $\widehat{\Pi}^*/\widehat{\Pi} \cong (\mathbb{Z}_2)^8 \cong \widehat{K_2}^*/\widehat{K_2}$ with quadratic forms of opposite sign, so Nikulin's method given in lemma 7.3.6 works to reproduce the description for $H^2(X, \mathbb{Z})$ of lemma 7.3.15.

Recall the comment to theorem 7.3.2, where we noted that if some four–plane $x \in O^+(H^{even}(X, \mathbb{Z}))\backslash\mathcal{T}^{4,20}$ has an apparent geometric interpretation which allows too many symmetries, something must have gone wrong. This observation can be used to give another proof of the fact that the B–field cannot be zero for the orbifold limit geometric interpretation of $\mathbb{Z}_2$ orbifold conformal field theories. Let υ, υ^0 denote the null vectors that correspond to a geometric interpretation of $\mathcal{K}(\mathbb{Z}^4, 0)$ as nonlinear σ model on $\mathcal{K}(\mathbb{Z}^4)$. Assume that $B = 0$. Then $\xi_4 = \upsilon^0 + \frac{1}{2}\upsilon$ in (7.1.6). One now checks that every element of $O^+(H^{even}(X, \mathbb{Z}))$ that fixes ξ_4 automatically fixes υ, υ^0 as well. In other words, $\mathcal{K}(\mathbb{Z}^4, 0)$ does not possess nonclassical symmetries. All classical symmetries have been determined in theorem 7.3.12 to $\widetilde{\mathcal{G}}^+_{Kummer} = \mathbb{Z}_2 \ltimes \mathbb{F}_2^4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$. On the other hand, in theorem 7.3.25 below we will show that $\mathcal{K}(\mathbb{Z}^4, 0)$ can be described as Gepner type model $(\widehat{2})^4$. Below (7.3.39) we find its algebraic symmetry group $\widehat{\mathcal{G}}^{alg} \cong (\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes D_4 \neq \widetilde{\mathcal{G}}^+_{Kummer}$, a contradiction, unless nonclassical symmetries exist. Indeed, explicit translation of the symmetries from one model into the other shows that the Gepner symmetry $[1, 0, 3, 0]$ (see (5.6.3) for the notation) cannot be realized as classical symmetry on $\mathcal{K}(\mathbb{Z}^4, 0)$.

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7.3.3 T–duality, Fourier–Mukai, and Nahm transform

By theorem 7.3.16 any automorphism on the Teichmüller space $\mathcal{T}^{4,4}$ of $\mathcal{M}^{tori}$ is conjugate to an automorphism on the Teichmüller space $\mathcal{T}^{4,20}$ of $\mathcal{M}^{K3}$. In particular, nonlinear σ models on tori related by T–duality must give isomorphic theories on $K3$ under $\mathbb{Z}_2$ orbifoldizing. To show this explicitly and discuss the duality transformation on $\mathcal{M}^{K3}$ obtained this way is the object of the present section. Finally we will point out the relation between Fourier–Mukai and Nahm transform and also comment on its meaning in the context of "brane physics".

For simplicity first assume that our σ model on the torus $T = \mathbb{R}^4/\Lambda$ has vanishing B-field, where we have chosen a geometric interpretation $(\Sigma_T, V_T, 0)$. Then T–duality acts by $(\Sigma_T, V_T, 0) \mapsto (\Sigma_T, 1/V_T, 0)$ and in terms of $O^+(H^{even}(T,\mathbb{Z}))$ is given by the exchange of υ and υ^0. By theorem 7.3.16 the corresponding $\mathbb{Z}_2$ orbifold theories have geometric interpretations $(\Sigma, V_T/2, B)$ and $(\Sigma, 1/2V_T, B)$, respectively, where Σ is obtained as image of the embedding $\Sigma_T \subset H^2(T,\mathbb{R}) \hookrightarrow H^2(X,\mathbb{R})$ and $B = \frac{1}{2}B^{(2)}_{\mathbb{Z}} = \frac{1}{4}\sum_{i\in I}\widehat{E_i}$. We will now construct an automorphism Θ of the lattice $H^{even}(X,\mathbb{Z})$ which fixes the four–plane x corresponding to the model with geometric interpretation $(\Sigma, V_T/2, B)$ and acts by $V_T/2 \mapsto 1/2V_T$. In other words, we will explicitly construct the duality transformation induced by torus T–duality on $\mathcal{M}^{K3}$. Our transformation Θ below was already given in [RW98] but not with complete proof. Within the context of boundary conformal field theories, in [BER99] it was shown that Θ induces an isomorphism on the corresponding conformal field theories. The relation to the Fourier–Mukai transform which we will show in theorem 7.3.17 has not been clarified up to now and was obtained in joint work with Werner Nahm [NW].

By (7.1.6) the four–plane $x \subset H^{even}(X,\mathbb{Z})$ is spanned by $\widetilde{\Sigma} = \xi(\Sigma)$ and the vector $\xi_4 = \widehat{\upsilon}^0 + B + (\frac{V_T}{2}+1)\widehat{\upsilon}$ (notations as in theorem 7.3.16). Because by the above Θ fixes x and $\widetilde{\Sigma}$ pointwise, the unit vector $\xi_4/\sqrt{V_T} \in \widetilde{\Sigma}^\perp \cap x$ must be invariant, too, i.e. invariant under the transformation $V_T \mapsto 1/V_T$. Hence

$$\tfrac{1}{\sqrt{V_T}}\widehat{\upsilon}^0 + \tfrac{1}{\sqrt{V_T}}B + \left(\tfrac{1}{2}\sqrt{V_T} + \tfrac{1}{\sqrt{V_T}}\right)\widehat{\upsilon} = \sqrt{V_T}\widetilde{\upsilon}^0 + \sqrt{V_T}\widetilde{B} + \left(\tfrac{1}{2\sqrt{V_T}} + \sqrt{V_T}\right)\widetilde{\upsilon}$$

for any value of V_T. We set $\widetilde{\upsilon} := \Theta(\widehat{\upsilon}), \widetilde{\upsilon}^0 := \Theta(\widehat{\upsilon}^0)$ etc. and deduce

$$\widetilde{\upsilon}^0 + \widetilde{B} + \widetilde{\upsilon} = \tfrac{1}{2}\widehat{\upsilon}, \quad \widehat{\upsilon}^0 + B + \widehat{\upsilon} = \tfrac{1}{2}\widetilde{\upsilon}. \tag{7.3.22}$$

The first equation together with $\langle \widetilde{B}, \widetilde{\upsilon}\rangle = \langle \widetilde{B}, \widetilde{\upsilon}^0\rangle = 0, \|\widetilde{B}\|^2 = -2$ implies $\langle \widetilde{B}, \widehat{\upsilon}\rangle = -4$ and justifies the ansatz

$$\widetilde{B} = -4\widehat{\upsilon}^0 - \sum_{i\in I}\alpha_i\widehat{E_i} + a\widehat{\upsilon} \implies \sum_{i\in I}(\alpha_i - 1)^2 = 1, \quad \sum_{i\in I}\alpha_i = 8 - 2a.$$

The only solutions satisfying $\sum_{i\in I}\alpha_i\widehat{E_i} \in H^{even}(X,\mathbb{Z})$, which must be true by (7.3.22), are $\alpha_{i_0} \in \{0,2\}$ for some $i_0 \in I$ and $\alpha_i = 1$ for $i \neq i_0$, correspondingly $a \in \{-\frac{7}{2}, -\frac{9}{2}\}$. We conclude that if the automorphism Θ exists, then it is already uniquely determined up to the choice of a and of one point $i_0 \in I$. The two possible choices of a turn out to be related by the B-field shift $\widetilde{B} \mapsto \widetilde{B} - 2\widetilde{B} = -\widetilde{B}$ and yield equivalent results. In the following we pick $a = -\frac{7}{2}$ and find

$$\widetilde{\upsilon} = 2(\widehat{\upsilon} + \widehat{\upsilon}^0) + \tfrac{1}{2}\sum_{i\in I}\widehat{E_i}, \quad \widetilde{\upsilon}^0 = 2(\widehat{\upsilon} + \widehat{\upsilon}^0) + \tfrac{1}{2}\sum_{i\in I}\widehat{E_i} - \widehat{E}_{i_0}. \tag{7.3.23}$$

One easily checks that $\widetilde{U} := \mathrm{span}_{\mathbb{Z}}(\widetilde{\upsilon}, \widetilde{\upsilon}^0) \cong U$. By $\widetilde{\Pi}$ we denote the orthogonal complement of $\widetilde{U}$ in $\mathrm{span}_{\mathbb{Z}}(\widehat{\upsilon}, \widehat{\upsilon}^0) \perp \Pi \cong U \perp \Pi$, where Π is the Kummer lattice of X as introduced in theorem 7.3.5. Note that in I there are 15 hyperplanes $H_i, i \in I_0 = I - \{i_0\}$, which do not contain i_0. The label $i \in I_0$ is understood as the vector dual to the hyperplane H_i. Since the choice of i_0 can be seen as the choice of an origin in the affine space $\mathbb{F}_2^4$, the latter can be regarded as a vector space, and we have a unique natural isomorphism $(\mathbb{F}_2^4)^* \cong \mathbb{F}_2^4$. One now checks that $\widetilde{\Pi}$ is spanned by $\widetilde{E_i}, i \in I$, with

$$\widetilde{E}_{i_0} \;:=\; \widehat{\upsilon} - \widehat{\upsilon}^0, \quad \widetilde{E_i} \;:=\; -\tfrac{1}{2}\sum_{j\in H_i}\widehat{E_j} - \widehat{\upsilon} - \widehat{\upsilon}^0 \qquad (i \neq i_0) \tag{7.3.24}$$

as well as $\frac{1}{2}\sum_{i\in H}\widetilde{E_i}$ for any hyperplane $H \subset I$. The signs of the $\widetilde{E_i}$ have been chosen such that $\widetilde{B} = \frac{1}{2}\widetilde{B}^{(2)}_{\mathbb{Z}} = \frac{1}{4}\sum_{i\in I}\widetilde{E_i}$.
Since $\langle\widetilde{E_i}, \widetilde{E_j}\rangle = -2\delta_{ij}$, one has $\widetilde{\Pi} \cong \Pi$. Hence $\Theta(\widehat{E_i}) = \widetilde{E_i}$ is a continuation of (7.3.23) to an automorphism of lattices $U \perp \Pi \cong \widetilde{U} \perp \widetilde{\Pi}$, and we find $\Theta^2 = \mathbb{1}$. Note that the action of Θ can be viewed as a duality transformation exchanging vectors $i \in I$ with hyperplanes $H_i, i \in I$. Twoplanes $P \subset I$ are exchanged with their duals P^* which shows that Θ can be continued to a map on the entire lattice $H^*(X, \mathbb{Z})$ consistently with (7.3.13). The induced action on $K_2 = \pi_* H^2(T, \mathbb{Z})$ leaves Σ invariant. We also see that the above procedure is easily generalized to arbitrary nonlinear σ models $\mathcal{T}(\Lambda, B_T)$.
Let S denote the classical symmetry which changes the sign of $\widehat{E}_{i_0}$ and leaves the other lattice generators $\widehat{E_i}$, $i \neq i_0$, $\widehat{\upsilon}$, $\widehat{\upsilon}^0$, $\mu_j \wedge \mu_k$ invariant. By (7.3.23) and (7.3.24) one has $\Theta S = T_{FM}\Theta$, where T_{FM} is the Fourier–Mukai transform which exchanges $\widehat{\upsilon}$ with $\widehat{\upsilon}^0$. Since $T_{FM} = \Theta S \Theta$, all in all we have

Theorem 7.3.17
Torus T–duality induces a duality transformation Θ as given by (7.3.23) and (7.3.24) on the subspace of $\mathcal{M}^{K3}$ of theories associated to Kummer surfaces in the orbifold limit (see also [RW98]). The Fourier–Mukai transform T_{FM} which exchanges $\widehat{\upsilon}$ with $\widehat{\upsilon}^0$ is conjugate to a classical symmetry S by the image Θ of the T-duality map on theories associated to the torus.

Note that by theorem 7.3.17 we can prove Aspinwall's and Morrison's description (7.1.8) of the moduli space $\mathcal{M}^{K3}$ purely within conformal field theory without recourse to Landau–Ginzburg arguments. Namely, as explained in section 7.1, the group $\mathcal{G}^{(16)}$ needed to project from the Teichmüller space (7.1.2) to the component $\mathcal{M}^{K3}$ of the moduli space contains the group $O^+(H^2(X, \mathbb{Z}))$ of classical symmetries which fix the vectors $\widehat{\upsilon}, \widehat{\upsilon}^0$ determining our geometric interpretation.

Moreover, for any primitive null vector $\widetilde{v}^0$ with $\langle \widehat{v}, \widetilde{v}^0 \rangle = 1$ there exists an element $\widetilde{g} \in \mathcal{G}^{(16)}$ such that $\widetilde{g}\widehat{v} = \widehat{v}$ and $\widetilde{g}\widehat{v}^0 = \widetilde{v}^0$. By theorem 7.3.17 the symmetry $T_{FM} \in O^+(H^{even}(X,\mathbb{Z}))$ which exchanges $\widehat{v}$ and $\widehat{v}^0$ and leaves x invariant also is an element of $\mathcal{G}^{(16)}$, thus $O^+(H^{even}(X,\mathbb{Z})) \subset \mathcal{G}^{(16)}$ and $O^+(H^{even}(X,\mathbb{Z})) = \mathcal{G}^{(16)}$ under the assumption that $\mathcal{M}^{K3}$ is Hausdorff, as argued in section 7.1.
The Fourier–Mukai transform $v \leftrightarrow v^0$ that on the torus is called T–duality is well known to mathematicians by now. Namely, elements of the integer cohomology of X are used to encode invariants of vector bundles, or rather coherent sheaves on X; then the Fourier–Mukai transform gives a duality between the moduli space of coherent sheaves on X and that on its dual $\widehat{X}$.
To be more precise, we discuss the Fourier–Mukai transform on four–tori, following [Nah82, Nah84, Sch88, BvB89, HO97]. Let $X = T = \mathbb{R}^4/\Lambda$ and $\widehat{X} = \widehat{T} = \mathbb{R}^4/\Lambda^*$ with coordinates x_μ, y_μ. The dual torus $\widehat{X}$ is interpreted as moduli space of flat $U(1)$ bundles or equivalently line bundles on X. We will make use of the Poincaré bundle $\boxdot \to T \times \widehat{T}$ which is defined by

$$\begin{array}{ccc} p^*\mathcal{L} & \xrightarrow{\ \widetilde{g}\ } & \boxdot \\ \downarrow & & \downarrow \\ T \times \mathbb{R}^4 & \xrightarrow{\ g\ } & T \times \widehat{T} \end{array}$$

Here, $\mathcal{L} \to T$ denotes a trivial line bundle, and $p : T \times \mathbb{R}^4 \to T$ is the restriction to T. The map g is the projection $\mathbb{R}^4 \to \widehat{T} = \mathbb{R}^4/\Lambda^*$, and $\widetilde{g}$ denotes the projection corresponding to $\forall\, \lambda \in \Lambda :\ (x, y, v) \sim (x + \lambda, y, e^{-i\lambda y} v)$ for $(x, y) \in T \times \mathbb{R}^4$ and $(x, y, v) \in (p^*\mathcal{L})_{(x,y)}$. The restriction of $\boxdot$ to $T \times \{y\}$ is the line bundle $L_y \to T$ associated to the character $\chi_y(x) = e^{ixy}$. The Poincaré bundle is equipped with the connection $\omega(x, y) = 2\pi i \sum_\mu y_\mu dx_\mu$ and has curvature $\Omega(x, y) = 2\pi i \sum_\mu dy_\mu \wedge dx_\mu$. Consider a vector bundle $E \to T, E = Q \times_{U(n)} \mathbb{C}^n$, where Q is a principal $U(n)$ bundle with anti–self–dual connection A. Then $A_y = A + 2\pi i\, \mathbb{1}_n \omega(x, y)$ defines a family D_A of Dirac operators on E,

$$D^{\pm}_{A_y} :\ \Gamma(T, S^{\pm} \otimes E) \longrightarrow \Gamma(T, S^{\mp} \otimes E),$$

where $S^{\pm} \to T$ denote the spin bundles on T. Observe that A_y is the restriction of the connection $\mathcal{A} := A \otimes 1 + \mathbb{1}_n \otimes \omega$ on $p^*(E \otimes \mathcal{L})$ to $T \times \{y\}$.
Now the Nahm transform $(\widehat{T}; \widehat{E}, \widehat{A})$ of the data $(T; E, A)$ can be defined if A is a 1-irreducible connection, i.e. E has no covariantly constant sections. Namely, $\widehat{E}$ is the index bundle $\widehat{E} = \ker D^-_A = \text{ind } D_A$ and carries the connection $\widehat{A}$ induced by the horizontal component of the connection $\mathcal{A}$ on $p^*(E \otimes \mathcal{L})$ as above [Ati79]. One can show that under the above assumptions $\widehat{A}$ is anti–self–dual, too. The proof goes back to W. Nahm [Nah84], where the Nahm transform was introduced for instantons on $\mathbb{R}^4$ as well as monopoles and calorons. Instantons on T comprise the simplest special case of those on $\mathbb{R}^4$ and are therefore included in Nahm's investigation. The holomorphic version of the Nahm transform on T is known to algebraic geometers as Fourier–Mukai transform and was discovered by S. Mukai [Muk81, Muk85]. Here, a holomorphic structure on T has to be

chosen, and the above condition of 1–irreducibility translates into stability of the (holomorphic) vector bundle E [Don85, UY86]. In this case the connection $\widehat{A}$ is just the Bismut–Freed connection on the index bundle $\widehat{E}$ [BvB89, BGS88].
In [Sch88, BvB89], the crucial observation was made that one can use the Atiyah–Singer index theorem for families

$$\mathrm{ch}(\mathrm{ind}\ D_A) = \frac{1}{(2\pi i)^2}\int_T \widehat{A}(T)\,\mathrm{ch}(\boxdot \otimes E)$$

[AS71] to relate the invariants of $(\widehat{T}; \widehat{E}, \widehat{A})$ to those of $(T; E, A)$:

$$\begin{aligned} \mathrm{rk}\,(\widehat{E}) &= c_2(E) - \tfrac{1}{2}c_1^2(E), \\ c_1(\widehat{E}) &= \tfrac{1}{4\pi^2}\,\Omega \cup \Omega \cup c_1(E)[T], \\ c_2(\widehat{E}) &= \mathrm{rk}\,(E) + \tfrac{1}{2}c_1^2(E). \end{aligned} \tag{7.3.25}$$

The above data can be encoded in the MUKAI VECTOR $M(E) \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ of E,

$$M(E) := \mathrm{ch}(E)\sqrt{\widehat{A}(X)} = \left(\mathrm{rk}\,(E), c_1(E), -\tfrac{1}{2}c_1^2(E) + c_2(E) + \tfrac{\mathrm{rk}\,(E)}{48}p_1(X)\right). \tag{7.3.26}$$

For later convenience, (7.3.26) contains the general definition of the Mukai vector for a complex surface X, where the Pontrjagin class $p_1(X)$ vanishes on the torus but gives -48 when evaluated on $K3$. Note that compared to the standard conventions we have flipped the sign of the last component of $M(E)$, since we always choose generators υ^0, υ of $H^0(X, \mathbb{Z}), H^4(X, \mathbb{Z})$ with $\langle \upsilon^0, \upsilon \rangle = 1$ as opposed to the Mukai intersection product -1 [Muk87]. For any vector $M = (Q_4, Q_2, Q_0) \in H^{even}(X, \mathbb{Z})$ as above with $Q_4 > 1$ one can construct a $U(Q_4)$ bundle $E \to T$ such that $M = M(E)$ [BvB89, p.269]. For $Q_4 = 1$ we have to assume $Q_0 = 0$ to find a vector bundle E with $M = M(E)$. If we extend the consideration to the category of coherent sheaves on X, $(1, 0, n)$ is the Mukai vector of the sheaf of holomorphic functions on T that vanish at n points.
By (7.3.25) the Nahm transform maps the moduli space $\widehat{T}$ of flat $U(1)$ bundles on T, associated to the Mukai vector $\upsilon^0 := (1, 0, 0)$, to the moduli space of skyscraper sheaves on T, which is isomorphic to T itself and is associated to the Mukai vector $\upsilon := (0, 0, 1)$. This confirms our remarks above, by which the Nahm (or Fourier–Mukai) transform induces the automorphism $\upsilon^0 \leftrightarrow \upsilon$ on $H^{even}(T, \mathbb{Z})$.
S. Mukai has generalized the Nahm transform to $K3$ surfaces X in the language of algebraic geometry [Muk84, Muk87]. K. Hori and Y. Oz have generalized P.J. Braam's and P. van Baal's construction with the Poincaré bundle and the Atiyah–Singer index theorem for families to the $K3$ case [HO97]. Again, the transformation is encoded in the Mukai vector (7.3.26), where now due to the nonvanishing Pontrjagin class for a coherent sheaf $E \to X$

$$\mathrm{rk}\,(\widehat{E}) = c_2(E) - \mathrm{rk}\,(E) - \tfrac{1}{2}c_1^2(E), \quad c_1(\widehat{E}) \sim c_1(E), \quad c_2(\widehat{E}) = c_2(E)$$

[HO97]. Hence the Fourier–Mukai transform on $K3$ maps flat $U(N)$ vector bundles with instanton number k onto flat $U(k-N)$ vector bundles with instanton number k. The $K3$ surface X is isomorphic to the moduli space of skyscraper sheaves on X, associated to the Mukai vector $\widehat{v} := (0,0,1)$, and is mapped to the moduli space of sheaves on X with Mukai vector $\widehat{v}^0 := (1,0,0)$. By (7.3.26) with $p_1(X) = -48$ this is the sheaf of holomorphic functions on X that are vanishing in one point and thus is a $K3$ surface $\widehat{X}$ again, the dual of X. From (7.1.6) one finds that the Fourier–Mukai transform $\widehat{v} \leftrightarrow \widehat{v}^0$ induces $(\Sigma, V, 0) \to (\Sigma, 1/V, 0)$ for a geometric interpretation with vanishing B–field. But analogously to the Nahm transform on tori, the transformation of the Einstein metric from X to $\widehat{X}$ will in general be more complicated than just an inversion of the volume. We stress that $\widehat{v} \leftrightarrow \widehat{v}^0$ is a genuinely nonclassical symmetry. The notation in terms of geometric interpretations is misleading, since Σ is determined by its relative position to the lattice $H^*(X,\mathbb{Z})$, in particular the embedding of $H^2(X,\mathbb{Z})$ in $H^*(X,\mathbb{Z})$.
It is very fashionable to formulate all that has been said above in the language of "brane physics" [HO97]. Then the Mukai vector (7.3.26) is interpreted as charge vector of a system of D4,D2,D0 branes. A D0 brane can be described as instanton of zero size ("small instanton") on a D4 brane wrapping X and has Mukai vector $\widehat{v} = (0,0,1)$. By (7.3.26), a D4 brane has Mukai vector $(1,0,-1)$. Hence the Fourier–Mukai transform $\widehat{v} \leftrightarrow \widehat{v}^0$ interchanges D0 branes with a D4 plus a D0 brane.
This language is also used in [RW98], where elements of $H^*(X,\mathbb{Z})$ are interpreted as labelling BPS states in a given type IIA string theory on $K3$ specified by a four–plane $x \in \mathcal{M}^{K3}$. As above, $M = (Q_4, Q_2, Q_0) \in H^{even}(X,\mathbb{Z})$ corresponds to a combination of Q_2 membranes and Q_4 fourbranes, whereas Q_0 counts the number of D0 minus D4 branes. Under the conjectured heterotic–type IIA duality, M must correspond to a supersymmetric state of fixed winding and momentum. The momentum is given by the projection of M onto x, and the mass of the BPS state is the length of that vector in x. The resulting mass formula is compared to the mass formula on the type IIA side, and Q_4, Q_2, Q_0 can be interpreted in terms of the gauge theory description of the physical Ramond–Ramond charges in the presence of a B–field on a curved manifold [RW98].

7.3.4 $\mathbb{Z}_M$ Orbifolds within the moduli space

This section is devoted to the study of $\mathbb{Z}_M$ orbifolds in the moduli space $\mathcal{M}^{K3}$, where $M \in \{3,4,6\}$. We will use the notations introduced in section 7.3.1 in the context of geometric $\mathbb{Z}_M$ orbifolds, see in particular lemmata 7.3.7–7.3.9.
We first consider some features of the $\mathbb{Z}_M$ orbifold construction on the conformal field theory side which need further discussion. The $\mathbb{Z}_M$ action on a toroidal nonlinear σ model $\mathcal{T}(\Lambda, B_T), T = \mathbb{R}^4/\Lambda$, is given by (5.2.2). From (7.3.17) we readily read off that there always is a surviving $su(2)_1 \oplus u(1)$ subalgebra of the holomorphic W-algebra generated by $J, J^\pm, A$.
$\mathbb{Z}_4$ orbifold conformal field theories will be studied in a little more detail than the other two cases below, basically in view of the application in section 7.3.5.

The partition function of the $\mathbb{Z}_4$ orbifold conformal field theory of $\mathcal{T}(\Lambda, B_T)$ was given in corollary 5.2.3. There we also noted that it coincides with that of the $\mathbb{Z}_2$ orbifold of a theory whose NS-partition function is the expression in curly brackets in (5.2.10). Indeed, the partition function of $SU(2)_1^4/\mathbb{Z}_4$, i.e. of the $\mathbb{Z}_4$ orbifold of $T = \mathbb{R}^4/\mathbb{Z}^4$ with $B_T = 0$, agrees with that of the $\mathbb{Z}_2$ orbifold $\mathcal{K}(D_4, 0)$ [EOTY89]. In section 7.3.2 we observed that every $\mathbb{Z}_2$ orbifold conformal field theory has an $su(2)_1^2$ subalgebra (7.3.17) of the holomorphic W-algebra. On the other hand, as demonstrated above, the $\mathbb{Z}_4$ orbifold generically only possesses an $su(2)_1 \oplus u(1)$ current algebra. For $SU(2)_1^4/\mathbb{Z}_4$ this is enhanced to $su(2)_1 \oplus u(1)^3$ which still does not agree with the one for Kummer surfaces. Hence although the theories have the same partition function, they are not isomorphic.
Similarly, the partition function of the $\mathbb{Z}_4$ orbifold of the torus model with $SO(8)_1$ symmetry agrees with that of $\mathcal{K}(\mathbb{Z}^4, 0)$ as can be seen from (7.3.31). In this case the theories indeed are the same (theorem 7.3.31).
For a $\mathbb{Z}_M$ orbifold limit X of $K3$, $M \in \{3, 4, 6\}$, we have determined the analogs $\{\,\overline{\mathrm{III}},\ \overline{\mathrm{IIII}},\ \overline{\mathrm{VI}}\,\}$ of the Kummer lattice Π in lemmata 7.3.7–7.3.9. The embeddings of the moduli spaces of $\mathbb{Z}_M$ orbifolds in $\mathcal{M}^{K3}$ are now obtained analogously to that of $\mathbb{Z}_2$ orbifold conformal field theories as described in section 7.3.2. Namely, the embeddings $\pi_* : H^2(T, \mathbb{Z})^{\mathbb{Z}_M} \hookrightarrow H^2(X, \mathbb{Z})$ of lemmata 7.3.7–7.3.9 have to be lifted to $\widehat{\pi}_* : H^{even}(T, \mathbb{Z})^{\mathbb{Z}_M} \hookrightarrow H^{even}(X, \mathbb{Z})$. The image is denoted by $\widehat{K}_M$. Apart from twoforms, $H^{even}(T, \mathbb{Z})^{\mathbb{Z}_M}$ has generators υ, υ^0, and their images in $\widehat{K}_M$ are denoted $\sqrt{M}\upsilon, \sqrt{M}\upsilon^0$. Note that $\widehat{K}_M \cong K_M \oplus U(M)$. The four–plane $x_T \in \mathcal{M}^{tori}$ that determines a $\mathbb{Z}_M$ symmetric toroidal conformal field theory is mapped onto $x = \widehat{\pi}_* x_T$ by the $\mathbb{Z}_M$ orbifold procedure. We are looking for a geometric interpretation (Σ, V, B) of x, fixed by $\widehat{\upsilon}, \widehat{\upsilon}^0 \in H^{even}(X, \mathbb{Z})$, such that the data read off from Σ correspond to an orbifold limit of $K3$. In other words, $\Sigma^\perp \cap H^2(X, \mathbb{Z})$ contains a primitive sublattice $\widetilde{P}_M \in \{\,\overline{\mathrm{III}},\ \overline{\mathrm{IIII}},\ \overline{\mathrm{VI}}\,\}$, and $K_M \subset H^2(X, \mathbb{Z})$ is embedded primitively. All calculations below will be carried out in $H^{even}(X, \mathbb{Q})$; by pr we denote the orthogonal projection onto $\upsilon^\perp \cap (\upsilon^0)^\perp$ and set $P_M := pr(\widetilde{P}_M)$. Then as for the $\mathbb{Z}_2$ case in (7.3.19) we can fix $\widehat{\upsilon}$ and use the ansatz

$$\widehat{\upsilon} := \sqrt{M}\,\upsilon, \quad \widehat{\upsilon}^0 := \tfrac{1}{\sqrt{M}}\upsilon^0 - \tfrac{1}{M}\check{B}_M - \tfrac{\|\check{B}_M\|^2}{2M^2}\sqrt{M}\upsilon \tag{7.3.27}$$

with $\check{B}_M \in pr(H^{even}(X, \mathbb{Q}))$ to be determined. Then we have $P_M \cong \widetilde{P}_M$. We will now find an embedding $\widehat{\pi}_*$ such that $\widehat{\upsilon}, \widehat{\upsilon}^0$ give the desired geometric interpretation.

Lemma 7.3.18
Suppose that $\widehat{\upsilon}, \widehat{\upsilon}^0$ have the form (7.3.27) and are primitive vectors in $H^{even}(X, \mathbb{Z})$. Assume that for any $\mathbb{Z}_M$ orbifold conformal field theory $x = \widehat{\pi}_* x_T$ they give the geometric interpretation on the corresponding $\mathbb{Z}_M$ orbifold limit of $K3$ as described above. Then

$$\check{B}_M \in P_M, \quad \frac{\|\check{B}_M\|^2}{2M^2} \in \mathbb{Z}, \quad \langle \check{B}_M, E \rangle \equiv 1 \bmod M \text{ for some } E \in P_M.$$

Set $\widehat{M}_M := M_M \cup \{\widehat{\upsilon}, \widehat{\upsilon}^0\}$ *with* M_M *as defined in lemmata 7.3.7–7.3.9, and*

$$\forall\, E \in P_M : \quad \widehat{E} := E - \langle E, \widehat{\upsilon}^0 \rangle\, \widehat{\upsilon} = E + \tfrac{1}{\sqrt{M}} \langle \check{B}_M, E \rangle\, \upsilon.$$

Then $\widehat{M}_M$ *and* $\widehat{\mathcal{E}}_M := \{\widehat{E} \mid E \in P_M\}$ *generate* $H^{even}(X, \mathbb{Z}) \cong \Gamma^{4,20}$.

Proof:
We will use lemma 7.3.6, in other words, we need to find $L := (\widehat{K}_M)^\perp \cap H^{even}(X, \mathbb{Z})$, then by [Nik80b, Prop.1.6.1] $L^*/L \cong \widehat{K}_M^*/\widehat{K}_M$, with isomorphism denoted by γ, and the induced quadratic forms agree up to a sign. This will give

$$H^{even}(X, \mathbb{Z}) \cong \left\{ (x, y) \in \widehat{K}_M^* \oplus L^* \,|\gamma(\overline{x}) = \overline{y} \right\}. \tag{7.3.28}$$

We claim that $L = \widehat{P}_M$ with

$$\widehat{P}_M := \left\{ p \in P_M \mid \langle \widehat{\upsilon}^0, p \rangle \in \mathbb{Z} \right\}.$$

Namely, for $p \in \widehat{P}_M$ by construction we can find $\widetilde{p} \in H^{even}(X, \mathbb{Z})$ such that $\widetilde{p} - p = a\upsilon$, $a \in \mathbb{R}$. Since $\mathbb{Z} \ni \langle \widetilde{p}, \widehat{\upsilon}^0 \rangle = \frac{a}{\sqrt{M}} + \langle p, \widehat{\upsilon}^0 \rangle$, a is an integer multiple of $\sqrt{M}$, and therefore $\widehat{P}_M \subset H^{even}(X, \mathbb{Z})$. Since $L \otimes \mathbb{R} = \widehat{P}_M \otimes \mathbb{R}$ is clear on dimensional grounds, $L = \widehat{P}_M$ since both are primitive sublattices of $H^{even}(X, \mathbb{Z})$ by construction.
From Nikulin's results we find that $\check{B}_M$ must be chosen such that $\widehat{P}_M^*/\widehat{P}_M \cong \widehat{K}_M^*/\widehat{K}_M$ with quadratic forms of opposite sign. Because $\widehat{P}_M \subset P_M \subset P_M^* \subset \widehat{P}_M^*$, we can use the decomposition

$$\widehat{P}_M^*/\widehat{P}_M \cong \widehat{P}_M^*/P_M^* \times P_M^*/P_M \times P_M/\widehat{P}_M,$$

and since $\widehat{K}_M^*/\widehat{K}_M \cong K_M^*/K_M \times (\mathbb{Z}_M)^2$ with $K_M^*/K_M \cong P_M^*/P_M$ by lemmata 7.3.7–7.3.9 we find

$$P_M/\widehat{P}_M \cong \widehat{P}_M^*/P_M^* \cong \mathbb{Z}_M.$$

Moreover, $\frac{1}{M}\check{B}_M$ generates $\widehat{P}_M^*/P_M^*$, so $\check{B}_M \in P_M^*$. Since the quadratic forms of $\widehat{P}_M^*/\widehat{P}_M$ and $\widehat{K}_M^*/\widehat{K}_M$ agree up to a sign as forms with values in $\mathbb{Q}/2\mathbb{Z}$, we find $\frac{\|\check{B}_M\|^2}{2M^2} \in \mathbb{Z}$, and there exists an $E \in P_M$ which generates $P_M/\widehat{P}_M$ such that $\langle \check{B}_M, E \rangle \equiv 1 \bmod M$. Hence by (7.3.27) $\check{B}_M \in L = \widehat{P}_M \subset P_M$. The generators of $H^{even}(X, \mathbb{Z})$ are now read off from (7.3.28) with the results of lemmata 7.3.7–7.3.9.

□

To explicitly determine the embedding of $\mathbb{Z}_M$ orbifolds in $\mathcal{M}^{K3}$ we only need to find the correct vector $\check{B}_M$ in (7.3.27). In general, its properties listed in lemma 7.3.18 do not determine $\check{B}_M$ uniquely, but since a shift of $\widehat{\upsilon}^0$ by an element of $H^{even}(X, \mathbb{Z})$ corresponds to an integer shift of the B–field in the geometric interpretation and thus is irrelevant to our discussion, we can restrict ourselves to a finite number of candidates for $\check{B}_M$. A lot of them will be equivalent by lattice automorphisms in $O^+(H^2(X, \mathbb{Z}))$. As for the embedding of $\mathbb{Z}_2$ orbifold conformal field theories in theorem 7.3.16, the lift $B_{\mathbb{Z}}^{(M)}$ of $\check{B}_M \in P_M$ to $\widetilde{P}_M$ will determine the offset

$\frac{1}{M}B_{\mathbb{Z}}^{(M)}$ of the B–field induced on the exceptional divisors of the blow up by the orbifold process. Since this is a local effect for each fixed point, for every $\mathbb{Z}_2$ type fixed point we obtain $\frac{1}{2}$ upon integration of the B–field over the corresponding exceptional divisor, as shown in theorem 7.3.16. Analogously, the result for $\mathbb{Z}_3$ type fixed points will apply both to the $\mathbb{Z}_3$ and the $\mathbb{Z}_6$ orbifold case.
Moreover, algebraic symmetries of the underlying toroidal conformal field theory induce symmetries of the orbifold conformal field theory that must not be destroyed by the B–field. In particular, $B_{\mathbb{Z}}^{(M)}$ is invariant under all algebraic automorphisms of the orbifold limit of $K3$. For $\mathbb{Z}_4$ orbifold conformal field theories we can use the result on algebraic automorphisms of $SU(2)_1^4/\mathbb{Z}_4$, theorem 7.3.12, to verify that all $\mathbb{Z}_4$ type fixed points are related by symmetries and therefore must give the same result. Moreover, all $E_i^{(\pm)}, i \in I^{(4)}$, must carry the same B–field flux. Analogous reasoning severely restricts the number of candidates for $\check{B}_M$ in all cases. Actually,

Lemma 7.3.19
With notations taken from lemmata 7.3.7–7.3.9, the list of all $\check{B}_M$ that obey the conditions listed in lemma 7.3.18 and are consistent with the symmetries of $\mathbb{Z}_M$ orbifold conformal field theories up to equivalence by lattice automorphisms in $O^+(H^2(X,\mathbb{Z}))$ is

$$\begin{aligned}
\check{B}_3 &= \sum_{t\in\mathbb{F}_3^2}(E_t^{(+)}+E_t^{(-)}),\\
\check{B}_4 &= \sum_{i\in I^{(2)}} E_i + \tfrac{1}{2}\sum_{i\in I^{(4)}}(3E_i^{(+)}+4E_i^{(0)}+3E_i^{(-)}),\\
\check{B}_6 &= \tfrac{3}{2}\sum_b E_b + 2\sum_t (E_t^{(+)}+E_t^{(-)})\\
&\quad +\tfrac{1}{2}(5E_0^{(1)}+8E_0^{(2)}+9E_0^{(3)}+8E_0^{(4)}+5E_0^{(5)}).
\end{aligned}$$

Proof:
We only give the proof for the $\mathbb{Z}_4$ case, since the others are obtained analogously, the $\mathbb{Z}_6$ case being particularly tedious. The most general ansatz for $\check{B}_4 \in \Pi\Pi$ that is consistent with the symmetries of $SU(2)_1^4/\mathbb{Z}_4$ and our knowledge of the B–field on the $\mathbb{Z}_2$ fixed points is

$$\check{B}_4 = \sum_{i\in I^{(2)}} E_i + \tfrac{\alpha}{2}\sum_{i\in I^{(4)}}(E_i^{(+)}+E_i^{(-)}) + \beta\sum_{i\in I^{(4)}} E_i^{(0)},$$

where we can restrict to $\alpha, \beta \in \{0,\ldots,3\}$. Then $\frac{\|\check{B}_4\|^2}{32} \in \mathbb{Z}$, which must hold by lemma 7.3.18, iff $(\alpha,\beta) \in \{(1,2),(1,3),(3,1),(3,2)\}$. For $(\alpha,\beta)=(3,1)$ there is no $E \in \Pi\Pi$ with $\langle \check{B}_4, E\rangle \equiv 1 \bmod 4$. We claim that the remaining three cases are equivalent by lattice automorphisms in $O^+(H^2(X,\mathbb{Z}))$. Indeed, $(\alpha,\beta)=(1,3)$ turns into $(\alpha,\beta)=(1,2)$ by

$$E_i^{(\pm)} \longmapsto -E_i^{(\pm)} - E_i^{(0)}, \qquad E_i^{(0)} \longmapsto E_i^{(+)} + E_i^{(0)} + E_i^{(-)},$$

and $(\alpha,\beta)=(1,2)$ turns into $(\alpha,\beta)=(3,2)$ by

$$E_i^{(\pm)}\longmapsto -E_i^{(\pm)},\qquad E_i^{(0)}\longmapsto E_i^{(+)}+E_i^{(0)}+E_i^{(-)}.$$

□

For $M\in\{3,4,6\}$ we can now use lemma 7.3.18 with $\check{B}_M$ taken from lemma 7.3.19 to define $\widehat{\upsilon},\widehat{\upsilon}^0,\widehat{E}$ as above and give a consistent embedding $\widehat{\pi}_*:\; H^{even}(T,\mathbb{Z})^{\mathbb{Z}_M}\hookrightarrow H^{even}(X,\mathbb{Z})$. We use

$$B_{\mathbb{Z}}^{(M)}:=\check{B}_M+\tfrac{\|\check{B}_M\|^2}{\sqrt{M}}\upsilon\;\in H^{even}(X,\mathbb{Z}) \tag{7.3.29}$$

to find

$$\begin{aligned}
M\left(\sigma_i-\langle\sigma_i,B_T\rangle\upsilon\right) &= M\sigma_i-\langle M\sigma_i,\tfrac{1}{\sqrt{M}}B_T\rangle\widehat{\upsilon}\\
\tfrac{1}{\sqrt{M}}\left(\upsilon^0+B_T+\left(V-\tfrac{\|B_T\|^2}{2}\right)\upsilon\right) &= \widehat{\upsilon}^0+\tfrac{1}{\sqrt{M}}B_T+\tfrac{1}{M}B_{\mathbb{Z}}^{(M)}\\
&\quad+\left(\tfrac{V_T}{M}-\tfrac12\left\|\tfrac12 B_T+\tfrac1M B_{\mathbb{Z}}^{(M)}\right\|^2\right)\widehat{\upsilon}
\end{aligned}$$

which proves

Theorem 7.3.20
Let (Σ_T,V_T,B_T) denote a geometric interpretation of the nonlinear sigma model $\mathcal{T}(\Lambda,B_T)$, and $M\in\{3,4,6\}$. Assume that Λ is generated by $\Lambda_k, k=1,2$, with $\mathbb{Z}_M$ symmetric $T_{(k)}=\mathbb{R}^2/\Lambda_k$, and $B_T\in H^2(T,\mathbb{Z})^{\mathbb{Z}_M}$ such that a $\mathbb{Z}_M$ action is well defined on $\mathcal{T}(\Lambda,B_T)$. Then the image $x\in\mathcal{T}^{4,20}$ under the $\mathbb{Z}_M$ orbifold procedure has geometric interpretation (Σ,V,B) where $\Sigma\in\mathcal{T}^{3,19}$ is found as described in lemmata 7.3.7–7.3.9, $V=\frac{V_T}{M}$, and $B=\frac{1}{\sqrt{M}}B_T+\frac1M B_{\mathbb{Z}}^{(M)}$, $B_{\mathbb{Z}}^{(M)}\in H^{even}(X,\mathbb{Z})$ as in (7.3.29) with $\check{B}_M$ as given in lemma 7.3.19. Thus we find B–field flux $-\frac1m$ on each exceptional divisor that corresponds to a $\mathbb{Z}_m$ fixed point.
In particular, the moduli space of superconformal field theories admitting an interpretation as $\mathbb{Z}_M$ orbifold is a quaternionic submanifold of $\mathcal{M}^{K3}$. Moreover, $x^\perp\cap H^{even}(X,\mathbb{Z})$ does not contain vectors of length squared -2.

Theorem 7.3.20 shows that $\mathbb{Z}_M$ orbifold conformal field theories do not correspond to string compactifications of the type IIA string on $K3$ with enhanced gauge symmetry.
We remark that the results of theorem 7.3.20 agree with those obtained previously in [Dou97] and [BI97] in the context of "brane physics". Also, the discussion of (7.3.20) seems to lead to a simpler proof in terms of quantum cohomology: The entire reasoning presented there translates to $M\in\{3,4,6\}$. Note that e.g. for $\mathbb{Z}_2$ type fixed points $i\in I^{(2)}$ on the $\mathbb{Z}_4$ orbifold, $f^*(2e_i)=4C_i$ as opposed to $f^*(e_j)=4C_j$ for $j\in I^{(4)}$. In particular, if $\sqrt{M}\upsilon^0-\check{B}_M$ denotes the Poincaré dual of $f_*([\widetilde{T}]-\sum_i C_i)$, then $\langle\check{B}_M,E\rangle=-\frac{M}{m}$ for each E corresponding to a $\mathbb{Z}_m$ fixed point. Since f is $M:1$ and unbranched on $([\widetilde{T}]-\sum_i C_i)$, $\frac1M(\sqrt{M}\upsilon^0-\check{B}_M)\in H^{even}(X,\mathbb{Z})$, justifying the ansatz (7.3.27). Everything else follows from lemma 7.3.18, and we

confirm B–field flux $-\frac{1}{m}$ through each of the exceptional divisors corresponding to a $\mathbb{Z}_m$ fixed point. The fact that this argument correctly predicts the B–field fluxes supports our belief that it should be possible to put it onto a solid mathematical footing, too.

7.3.5 Application: Fermat's description for three orbifold models

Let us now apply our knowledge on geometric interpretations of models in $\mathcal{M}^{K3}$ gathered so far, in particular of orbifold conformal field theories, to perform changes in geometric interpretations.

Theorem 7.3.21
The $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\mathbb{Z}^4, 0)$ admits a geometric interpretation on the Fermat quartic $\mathcal{Q} = \mathcal{Q}_0$ in $\mathbb{CP}^3$ (7.3.8) with volume $V_\mathcal{Q} = \frac{1}{2}$ and B–field $B_\mathcal{Q} = -\frac{1}{2}\sigma_1^{(\mathcal{Q})}$ up to a shift in $H^2(X, \mathbb{Z})$, where $\sigma_1^{(\mathcal{Q})}$ denotes the Kähler class of $\mathcal{Q}$.

Proof:
Let $e_1, \dots, e_4$ denote the standard basis of $\mathbb{Z}^4$. Then $\mu_i = e_i$, and by theorem 7.3.20 with $\|B_\mathbb{Z}^{(4)}\|^2 = -32$ the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\mathbb{Z}^4, 0)$ is described by the four–plane $x \in \mathcal{T}^{4,20}$ spanned by

$$\begin{aligned} \xi_1 &= \mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2, & \xi_2 &= \mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3, \\ \xi_3 &= 2(\mu_1 \wedge \mu_2 + \mu_3 \wedge \mu_4), & \xi_4 &= 4\widehat{\upsilon}^0 + B_\mathbb{Z}^{(4)} + 5\widehat{\upsilon}. \end{aligned}$$

To read off a different geometric interpretation, we define

$$\begin{aligned} \upsilon_\mathcal{Q} &:= \tfrac{1}{2}\left(\mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 - \mu_1 \wedge \mu_4 - \mu_2 \wedge \mu_3\right) \\ &\qquad + \tfrac{1}{2}\left(\widehat{E}_{(0,1,1,0)} - \widehat{E}_{(1,0,1,0)}\right), \\ \upsilon_\mathcal{Q}^0 &:= \mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 + \mu_1 \wedge \mu_2 \\ &\qquad + \tfrac{1}{2}\left(\widehat{E}_{(0,0,0,1)} + \widehat{E}_{(1,1,0,1)} - \widehat{E}_{(0,1,1,0)} - \widehat{E}_{(1,0,1,0)}\right). \end{aligned} \tag{7.3.30}$$

One checks $\upsilon_\mathcal{Q}, \upsilon_\mathcal{Q}^0 \in H^{even}(X, \mathbb{Z})$ as given in lemma 7.3.18, $\|\upsilon_\mathcal{Q}\|^2 = \|\upsilon_\mathcal{Q}^0\|^2 = 0$ and $\langle \upsilon_\mathcal{Q}, \upsilon_\mathcal{Q}^0 \rangle = 1$ to show that $\upsilon_\mathcal{Q}, \upsilon_\mathcal{Q}^0$ is an admissible choice for null vectors in (7.1.6). For the corresponding geometric interpretation $(\Sigma_\mathcal{Q}, V_\mathcal{Q}, B_\mathcal{Q})$ we find that $\Sigma_\mathcal{Q}$ is spanned by

$$\begin{aligned} \sigma_1^{(\mathcal{Q})} &= \mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 + \mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_3 - 2\upsilon_\mathcal{Q}, \\ \sigma_2^{(\mathcal{Q})} &= 2(\mu_1 \wedge \mu_2 + \mu_3 \wedge \mu_4) - 2\upsilon_\mathcal{Q}, \\ \sigma_3^{(\mathcal{Q})} &= 4\widehat{\upsilon}^0 + B_\mathbb{Z}^{(4)} + 5\widehat{\upsilon}. \end{aligned}$$

As complex structure $\Omega_\mathcal{Q} \subset \Sigma_\mathcal{Q}$ we pick the twoplane spanned by $\sigma_2^{(\mathcal{Q})}$ and $\sigma_3^{(\mathcal{Q})}$. Note that this plane is generated by lattice vectors, so the Picard number of the corresponding geometric interpretation X is 20, i.e. X is singular (definitions

4.6.1 and 4.6.3). Because $\sigma_2^{(\mathcal{Q})}, \sigma_3^{(\mathcal{Q})}$ are primitive lattice vectors, one now easily checks that X equipped with the complex structure given by $\Omega_{\mathcal{Q}}$ has quadratic form diag$(8,8)$ on the transcendental lattice. This uniquely fixes the complex structure on X by theorem 4.6.6, which thus agrees with that of $\mathcal{Q}$ by [Ino76] (see (7.3.12)). Volume and B-field can now be read off using (7.1.6) and noting that in our geometric interpretation

$$\mu_1 \wedge \mu_3 + \mu_4 \wedge \mu_2 - \mu_1 \wedge \mu_4 - \mu_2 \wedge \mu_3 = \xi_4^{(\mathcal{Q})} \sim \upsilon_{\mathcal{Q}}^0 + B_{\mathcal{Q}} + \left(V_{\mathcal{Q}} - \tfrac{1}{2}\|B_{\mathcal{Q}}\|^2\right)\upsilon_{\mathcal{Q}}.$$

$\square$

We remark that the results of theorem 7.3.21 admit a couple of cross–checks of our approach: In theorem 7.3.27 we will show that $SU(2)_1^4/\mathbb{Z}_4$ agrees with the Gepner model $(2)^4$ with algebraic symmetry group $\mathcal{G}^{alg} = (\mathbb{Z}_4)^2 \rtimes \mathcal{S}_4$, the algebraic automorphism group of the quartic (see the discussion around (7.3.10)), and with minimal Mukai number $\mu(\mathcal{G}^{alg}) = 5$. The $\mathcal{G}^{alg}$ invariant Kähler class on the quartic is that of the Fubini–Study metric on $\mathbb{CP}^3$ with Kähler form ω_{FS}. So from the algebraic symmetries of $(2)^4$ and since the $\sigma_k^{(\mathcal{Q})}$ in theorem 7.3.21 were chosen as primitive lattice vectors we must expect $\sigma_1^{(\mathcal{Q})} = \omega_{FS}$ and $B_{\mathcal{Q}} \in \Sigma_{\mathcal{Q}}$. The latter is confirmed by theorem 7.3.21. Secondly, one checks that $\|\sigma_1^{(\mathcal{Q})}\|^2 = \|\omega_{FS}\|^2 = -4$, since ω_{FS} corresponds to the hypersurface divisor of $\mathbb{CP}^3$.
In a similar fashion to theorem 7.3.21 we find quartic interpretations for two special $\mathbb{Z}_2$ orbifold conformal field theories:

Theorem 7.3.22
The $\mathbb{Z}_2$ orbifold conformal field theories $\mathcal{K}(D_4, 0)$ and $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ admit geometric interpretations on the Fermat quartic $\mathcal{Q} = \mathcal{Q}_0$ in $\mathbb{CP}^3$, where B^* is the B–field (7.3.40) for which the latter theory has enhanced symmetry by the Frenkel–Kac mechanism.

The proof is obtained analogously to that of theorem 7.3.21, but now with

$$\begin{aligned}
\upsilon_{\mathcal{Q}} &:= \sqrt{2}\mu_2 \wedge \mu_4, \\
\upsilon_{\mathcal{Q}}^0 &:= \sqrt{2}(\mu_1 \wedge \mu_4 + \mu_2 \wedge \mu_4) \\
&\quad + \tfrac{1}{2}\left(\sqrt{2}\mu_3 \wedge \mu_1 + \widehat{E}_{(0,0,0,0)} - \widehat{E}_{(0,0,1,0)} - \widehat{E}_{(1,0,1,0)} + \widehat{E}_{(1,0,0,0)}\right) \\
&\quad + \tfrac{1}{2}\left(\sqrt{2}\mu_2 \wedge \mu_3 + \widehat{E}_{(0,0,0,0)} + \widehat{E}_{(0,0,1,0)} - \widehat{E}_{(0,1,0,0)} - \widehat{E}_{(0,1,1,0)}\right).
\end{aligned}$$

The results on $SU(2)_1^4/\mathbb{Z}_4$ and $\mathcal{K}(D_4, 0)$ of theorems 7.3.21 and 7.3.22 are particularly interesting in view of the observation made in [EOTY89] that the partition functions of these models agree. The theories are not isomorphic, since they have nonisomorphic W–algebras, as was explained at the beginning of section 7.3.4. The more striking it is that both of them admit geometric interpretations on the Fermat quartic hypersurface $\mathcal{Q}$. The deeper reason for the agreement of the partition functions remains a mystery, though. In particular, though on the level of (7.1.6) it is not hard to give explicit expressions for B–field and Kähler class as well as the volume of the quartic interpretations of $\mathcal{K}(D_4, 0)$ and $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$, a technique to translate this into coordinate expressions for the Kähler form on $\mathcal{Q}$ is lacking.

7.3.6 Special points in moduli space: Gepner and Gepner type models

In this section, we explicitly locate the Gepner model $(2)^4$ and some of its orbifolds within the moduli space $\mathcal{M}^{K3}$. This is achieved by giving σ model descriptions of these models in terms of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifolds which we know how to locate in moduli space by the results of sections 7.3.2 and 7.3.4.

Ideas of proof: An example with $c = 3$

We start with a survey on the steps of proof we will perform to show equivalences between Gepner or Gepner type models and nonlinear σ models. We have learned this technique from Werner Nahm. As an illustration we then prove the *well known* fact that Gepner's model $(2)^2$ admits a nonlinear σ model description on the torus associated to the $\mathbb{Z}^2$ lattice that was already used in theorem 5.6.4.
Given two $N = 2$ superconformal field theories $\mathcal{C}^1, \mathcal{C}^2$ with central charge $c = 3d/2$ ($d = 2$ or $d = 4$) and spaces of states $\mathcal{H}^1, \mathcal{H}^2$, to prove their equivalence we show the following:

i. The partition functions of the two theories agree sector by sector in the sense of (3.1).

ii. The fields of dimensions $(h;\overline{h}) = (1;0)$ in the two theories generate the same algebra $\mathcal{A} = \mathcal{A}_f \oplus \mathcal{A}_b$, where $\mathcal{A}_f = u(1)$ for $d = 2$, $\mathcal{A}_f = su(2)_1^2$ for $d = 4$, and $u(1)^d \subset \mathcal{A}_b$. In particular, $u(1)^c \subset \mathcal{A}$. $\mathcal{A}_f$ contains the $U(1)$-current $J^{(1)} = J$ of the $N = 2$ superconformal algebra, and a second $U(1)$-generator $J^{(2)}$ if $d = 4$. Furthermore, the fields of dimensions $(h;\overline{h}) = (0;1)$ in both theories generate algebras isomorphic to $\mathcal{A}$ as well, such that each of the left moving $U(1)$-currents j has a right moving partner $\overline{\jmath}$.

iii. For $i = 1, 2$ define

$$\mathcal{H}_b^i := \left\{ |\varphi\rangle \in \mathcal{H}^i \;\middle|\; J^{(k)}|\varphi\rangle = 0, \quad k \in \{1, \tfrac{d}{2}\} \right\}$$

and denote the $U(1)$-currents in $u(1)^d \subset \mathcal{A}_b$ by $j^1, \dots, j^d$. We normalize them as in (4.1.1). Let $j^{d+k} \sim J^{(k)}, k \in \{1, \frac{d}{2}\}$, denote the remaining $U(1)$-currents when normalized to (4.1.1), too, and set $\mathcal{J} := (j^1, \dots, j^d; \overline{\jmath}^1, \dots, \overline{\jmath}^d)$. The charge lattices

$$\Gamma_b^i := \left\{ \gamma \in \mathbb{R}^{d;d} \;\middle|\; \exists\, |\varphi\rangle \in \mathcal{H}_b^i : \mathcal{J}|\varphi\rangle = \gamma|\varphi\rangle \right\}$$

of $\mathcal{H}_b^1$ and $\mathcal{H}_b^2$ with respect to $\mathcal{J}$ are isomorphic to the same self dual lattice $\Gamma_b \subset \mathbb{R}^{d;d}$; because the states in $\mathcal{H}_b^i$ are pairwise local, in order to prove this it suffices to show agreement of the $\mathcal{J}$-action on a set of states whose charge vectors generate a self dual lattice Γ_b.

Theorem 7.3.23
If i.-iii. are true then theories $\mathcal{C}^1$ and $\mathcal{C}^2$ are isomorphic.

Proof:
By ii. and iii. the field $T := \frac{1}{2}\sum_{k=1}^{c}(j^k)^2$ acts as Virasoro field T^i on each of the theories (check that $T-T^i$ has dimensions $h=\overline{h}=0$ with respect to T^i). Thus the restriction of T^i to $\mathcal{H}_b^i$ is given by $T_b^i := \frac{1}{2}\sum_{k=1}^{d}(j^k)^2$. Moreover, $\mathcal{H}_b^i$ is closed under OPE by construction, so the $\mathcal{H}_b^i$ are spaces of states of conformal field theories with central charge d. Since by ii. and iii. the respective holomorphic W–algebras contain d Abelian currents $j^1,\dots,j^d$, and analogously on the right handed side, $\mathcal{H}_b^i$ correspond to toroidal conformal field theories in the sense of definition 4.1.1. By iii. their charge lattices agree, so these toroidal theories are isomorphic by theorem 4.1.2, and $\mathcal{H}_b^1\cong\mathcal{H}_b^2\cong\mathcal{H}_b$.
Because Γ_b is self dual, for any state $|\varphi\rangle\in\mathcal{H}^i$ carrying charge γ with respect to $\mathcal{J}$ we have $\gamma\in\Gamma_b$ and thus find primary fields $V^i[\pm\gamma]\in\mathcal{H}_b^i$ of charge $\pm\gamma$. With suitable combinations P of descendants j_{-n}^k and $\widetilde{P}$ of ascendants $j_n^k, n\geq 0, k\in\{1,\dots,d\}$, we find $|\psi\rangle := P\,V^i[-\gamma]|\varphi\rangle$ such that

$$|\varphi\rangle = |\psi\rangle\otimes V^i[\gamma]\,\widetilde{P}\,|0\rangle_b \quad\text{ and }\quad |\psi\rangle\in\mathcal{H}_f^i := \left\{\,|\chi\rangle\in\mathcal{H}^i \;\mid\; T_b^i|\chi\rangle = 0\right\}.$$

This shows $\mathcal{H}^i\cong\mathcal{H}_f^i\otimes\mathcal{H}_b$ for $i=1,2$. $\mathcal{H}_f^1$ and $\mathcal{H}_f^2$ are representations of $\mathcal{A}_f=u(1)$ (for $d=2$) or $\mathcal{A}_f=su(2)_1^2$ (for $d=4$) which are completely determined by charge and dimension of the lowest weight states. Because by ii. $\mathcal{A}_f$ contains the $U(1)$-current J of the total $N=2$ superconformal algebra, the partition functions of our theories agree by i., and we already know $\mathcal{H}^i\cong\mathcal{H}_f^i\otimes\mathcal{H}_b$ for $i=1,2$, we may conclude $\mathcal{H}_f^1\cong\mathcal{H}_f^2$. $\square$

Let's watch the procedure described above at work:

Theorem 7.3.24
Gepner's model $\mathcal{C}^1=(2)^2$ has a nonlinear σ model description $\mathcal{C}^2$ on the two dimensional torus $T_{SU(2)_1^2}$ with $SU(2)_1^2$ lattice $\Lambda=\mathbb{Z}^2$ and B-field $B=0$.

Proof:
If we can prove i.-iii. in the above list, by theorem 7.3.23 we are done.

i. By the flow invariant orbit technique described below theorem 3.1.16 one can compute the partition function of $(2)^2$. With (4.1.11) it is determined for the σ model on $T_{SU(2)_1^2}$, and we find

$$Z_{NS}(\tau,z) = \frac{1}{2}\left[\left|\frac{\vartheta_2}{\eta}\right|^4 + \left|\frac{\vartheta_3}{\eta}\right|^4 + \left|\frac{\vartheta_4}{\eta}\right|^4\right]\left|\frac{\vartheta_3(z)}{\eta}\right|^2$$

for both theories.

ii. The nonlinear σ model on $T_{SU(2)_1^2}$ has two rightmoving Abelian currents j_1, j_2 which we normalize to

$$j_\alpha(z)\; j_\beta(w) \sim \frac{\frac{1}{2}\delta_{\alpha\beta}}{(z-w)^2}.$$

Their superpartners are free Majorana fermions ψ_1, ψ_2 with coupled boundary conditions. By e_1, e_2 we denote the generators of the lattice $\Lambda = \Lambda^* = \mathbb{Z}^2$ which defines our torus. Then the $(1,0)$-fields in the nonlinear σ model are given by the three Abelian currents $J = i\psi_2\psi_1$ (the $U(1)$ current of the $N=2$ superconformal algebra), $Q = j_1 + j_2$, $R = j_1 - j_2$, and the four vertex operators $V_{\pm e_i, \pm e_i}$, $i = 1, 2$.

In the Gepner model $(2)^2$ we have an Abelian current j, j' from each minimal model factor along with Majorana fermions ψ, ψ', where by (3.1.21) $\psi\psi' = \Phi^0_{4,2;0,0} \otimes \Phi^0_{4,2;0,0}$. The $U(1)$ current of the total $N=2$ superconformal algebra is $J = j + j'$, and comparing J, Q, R-charges we can make the following identifications:

$$\begin{aligned}
i\psi_2\psi_1 \;=\; J \;&=\; j + j', \quad j_1 + j_2 \;=\; Q \;=\; j - j', \quad j_1 - j_2 \;=\; R \;=\; i\psi\psi', \\
V_{e_1,e_1} \;&=\; \Phi^0_{2,0;0,0} \otimes \Phi^0_{2,2;0,0} + \Phi^0_{-2,0;0,0} \otimes \Phi^0_{-2,2;0,0}, \\
V_{e_2,e_2} \;&=\; \Phi^0_{2,0;0,0} \otimes \Phi^0_{2,2;0,0} - \Phi^0_{-2,0;0,0} \otimes \Phi^0_{-2,2;0,0}, \\
V_{-e_1,-e_1} \;&=\; \Phi^0_{2,2;0,0} \otimes \Phi^0_{2,0;0,0} + \Phi^0_{-2,2;0,0} \otimes \Phi^0_{-2,0;0,0}, \\
V_{-e_2,-e_2} \;&=\; \Phi^0_{2,2;0,0} \otimes \Phi^0_{2,0;0,0} - \Phi^0_{-2,2;0,0} \otimes \Phi^0_{-2,0;0,0}.
\end{aligned}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A} = u(1) \oplus su(2)_1^2 = \mathcal{A}_f \oplus \mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. The space $\mathcal{H}_b^1$ for the σ model is just the bosonic part of the theory. The charge lattice Γ_b with respect to the currents $\mathcal{J} := (Q, R; \overline{Q}, \overline{R}) = (j_1+j_2, j_1-j_2; \overline{\jmath}_1 + \overline{\jmath}_2, \overline{\jmath}_1 - \overline{\jmath}_2)$ thus contains the charges $M := \left\{\frac{1}{2}(\varepsilon; \pm\varepsilon), \varepsilon \in \{\pm 1\}^2\right\}$, carried by vertex operators $V_{\pm e_i,0}, V_{0,\pm e_i}$, $i = 1, 2$. M generates the self dual lattice $\left\{\frac{1}{2}(a+b; a-b) \mid a, b \in \mathbb{Z}^2, \sum_{k=1}^2 a_k \equiv \sum_{k=1}^2 b_k \equiv 0\,(2)\right\} = \Gamma_b$.

To complete the proof of iii. we observe that in the Gepner model the fields $\Phi^1_{n,0;n,0} \otimes \Phi^1_{-n,0;-n,0} \pm \Phi^1_{-3n,2;n,0} \otimes \Phi^1_{3n,2;-n,0}$ and $\Phi^1_{n,0;-n,0} \otimes \Phi^1_{3n,2;n,0}$ $\pm \Phi^1_{-3n,2;-n,0} \otimes \Phi^1_{-n,0;n,0}$, $n \in \{\pm 1\}$, are uncharged with respect to J and carry $\mathcal{J} = (j - j', i\psi\psi'; \overline{\jmath} - \overline{\jmath}', i\overline{\psi}\,\overline{\psi}')$-charges $M = \left\{\frac{1}{2}(\varepsilon; \pm\varepsilon), \varepsilon \in \{\pm 1\}^2\right\}$ generating the lattice Γ_b.

□

Gepner type description of $SU(2)_1^4/\mathbb{Z}_2$

Theorem 7.3.25
Let $\mathcal{C}^1 = \widehat{(2)}^4$ denote the Gepner type model obtained by enhancement of $\mathcal{W}_{Gepner}$ of $(2)^4$ with the simple currents $\widetilde{J}_{12}, \widetilde{J}_{34}$ of theorem 3.1.18 or equivalently as orbifold of $(2)^4$ by the group $\mathbb{Z}_2 \cong \langle [2,2,0,0] \rangle \subset \mathcal{G}^{alg}_{ab}$. Then $\mathcal{C}^1 = \mathcal{K}(\mathbb{Z}^4, 0)$ admits a nonlinear σ model description $\mathcal{C}^2$ on the Kummer surface associated to the torus $T_{SU(2)_1^4}$ with $SU(2)_1^4$ lattice $\Lambda = \mathbb{Z}^4$ and vanishing B-field.

Proof:
We prove conditions i.-iii. of theorem 7.3.23.

i. From (4.1.7) one finds

$$Z_{\Lambda=\mathbb{Z}^4,B_T=0}(\tau)=\left[\frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^4+\left|\frac{\vartheta_3}{\eta}\right|^4+\left|\frac{\vartheta_4}{\eta}\right|^4\right)\right]^2. \tag{7.3.31}$$

Applying the orbifold procedure for the $\mathbb{Z}_2$-action of $[2,2,0,0]\in\mathcal{G}_{ab}^{alg}$ to the partition function of the Gepner model $(2)^4$, as can be computed by the methods described in section 3.1.3, one checks that $\mathcal{C}^1$ and $\mathcal{C}^2$ have the same partition function obtained by inserting (7.3.31) into (5.2.9).

ii. In the nonlinear σ model $\mathcal{C}^2$ the current algebra (7.3.17) is enhanced to $u(1)^4\oplus su(2)_1^2$. The additional $U(1)$-currents are $U_i:=V_{e_i,e_i}+V_{-e_i,-e_i}$, $i=1,\dots,4$, where the e_i are the standard generators of $\Lambda=\Lambda^*=\mathbb{Z}^4$.

In the Gepner type model $\mathcal{C}^1=(\widehat{2})^4$, apart from the $U(1)$-currents $J_1,\dots,J_4$ from the factor theories, where $J=J_1+\dots+J_4$, we find four additional fields with dimensions $(h;\overline{h})=(1;0)$; by comparison of the respective OPEs the following identifications can be made:

$$\begin{array}{rclrcl}
J &=& J_1+J_2+J_3+J_4, & J^\pm &=& \left(\Phi^0_{\mp2,2;0,0}\right)^{\otimes4};\\
A &=& J_1+J_2-J_3-J_4, & A^\pm &=& \left(\Phi^0_{\mp2,2;0,0}\right)^{\otimes2}\otimes\left(\Phi^0_{\pm2,2;0,0}\right)^{\otimes2};\\
\hline
\end{array}$$

$$\begin{array}{rclcl}
\frac{1}{2}(U_1+U_2) &=& P &=& J_1-J_2;\\
\frac{1}{2}(U_3+U_4) &=& Q &=& J_3-J_4;\\
\frac{1}{2}(U_1-U_2) &=& R &=& i\left(\Phi^0_{4,2;0,0}\right)^{\otimes2}\otimes\left(\Phi^0_{0,0;0,0}\right)^{\otimes2};\\
\frac{1}{2}(U_3-U_4) &=& S &=& i\left(\Phi^0_{0,0;0,0}\right)^{\otimes2}\otimes\left(\Phi^0_{4,2;0,0}\right)^{\otimes2}.
\end{array} \tag{7.3.32}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A}=su(2)_1^2\oplus u(1)^4=\mathcal{A}_f\oplus\mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. We show that $\mathcal{H}_b^1$ and $\mathcal{H}_b^2$ both have self dual $\mathcal{J}:=(P,Q,R,S;\overline{P},\overline{Q},\overline{R},\overline{S})$-charge lattice

$$\Gamma_b=\left\{(x+y;x-y)\,\middle|\,x\in\tfrac{1}{2}D_4,y\in D_4^*\right\}, \tag{7.3.33}$$

generated by

$$\begin{array}{rcl}
M_{tw} &:=& \left\{\frac{1}{2}(x;x)\in\mathbb{R}^{4,4}\,\middle|\,x\in\{(\varepsilon_1,\varepsilon_2,0,0),(0,0,\varepsilon_1,\varepsilon_2),\right.\\
&& \left.\qquad(0,\varepsilon_1,\varepsilon_2,0),(\varepsilon_1,0,0,\varepsilon_2),\varepsilon_i\in\{\pm1\}\}\right\}\\
\text{and } M_{inv} &:=& \left\{(\varepsilon;0)\,\middle|\,\varepsilon\in\{\pm1\}^4\right\}.
\end{array}$$

In the σ model $\mathcal{C}^2$ we denote by Σ_δ, $\delta \in \mathbb{F}_2^4$, the twist field corresponding to the fixed point $\alpha_\delta = \frac{1}{2}\sum_{i=1}^4 \delta_i e_i$ of the $\mathbb{Z}_2$ orbifold. Action of a vertex operator with winding mode λ will shift the constant mode α_δ of each twisted field (see section 5.2) by $\frac{\lambda}{2}$ [HV87]. Hence,

$$U_i(z)\,\Sigma_\delta(w) \sim \frac{1/2}{z-w}\,\Sigma_{\delta+e_i}(w), \tag{7.3.34}$$

where the factor $\frac{1}{2}$ is determined up to phases by observing $T_f^2|\Sigma_\delta\rangle = 0, T_b^2 = \frac{1}{4}\sum_{i=1}^4 (U_i)^2$, and $h = \overline{h} = \frac{1}{4}$ for twist fields. The phases are fixed by appropriately normalizing the twist fields. One now checks that

$$\forall\, \varepsilon \in \{\pm 1\}^4: \quad s_\varepsilon := \sum_{\delta\in\mathbb{F}_2^4}\prod_{i=1}^4 (\varepsilon_i)^{\delta_i}\,\Sigma_\delta$$

are uncharged under $(J;\overline{J})$ and $(A;\overline{A})$ and carry $\mathcal{J}$-charges M_{tw}. For $\varepsilon,\delta \in \{\pm 1\}$ and $k,l \in \{1,\dots,4\}$ we define

$$E_{kl}^{\varepsilon\delta} \;:=\; \left(j_k - \tfrac{\delta}{2}\left(V_{e_k,e_k} - V_{-e_k,-e_k}\right)\right)\left(j_l - \tfrac{\varepsilon}{2}\left(V_{e_l,e_l} - V_{-e_l,-e_l}\right)\right).$$

Then $E_{13}^{\varepsilon\delta}, E_{14}^{\varepsilon\delta}, E_{23}^{\varepsilon\delta}, E_{24}^{\varepsilon\delta}$ are $(J,A;\overline{J},\overline{A})$-uncharged and carry $\mathcal{J}$-charges M_{inv}.

In the Gepner model, with the shorthand notation $\mathcal{O}(n_1) := \left(\Phi^1_{2,1;2n_1,n_1}\right)^{\otimes 2}$, $\mathcal{P}(n_2) := \Phi^0_{n_2,n_2;n_2,n_2} \otimes \Phi^0_{-n_2,-n_2;-n_2,-n_2}$ ($n_i \in \{\pm 1\}$) we find $(J,A;\overline{J},\overline{A})$-uncharged fields $\mathcal{O}(n_1)\otimes\mathcal{O}(n_2)$, $\mathcal{O}(n_1)\otimes\mathcal{P}(n_2)$, $\mathcal{P}(n_1)\otimes\mathcal{O}(n_2)$, $\mathcal{P}(n_1)\otimes\mathcal{P}(n_2)$, which after diagonalization with respect to the $\mathcal{J}$-action carry the charges M_{tw}.

Similarly, with $\mathcal{Q}(n,s) := \Phi^0_{2n,s;0,0}\otimes\Phi^0_{2n,s+2;0,0}$, the fields $\mathcal{Q}(n_1,s_1)\otimes\mathcal{Q}(n_2,s_2)$, $n_i \in \{\pm 1\}, s_i \in \{0,2\}$, after diagonalization have charges M_{inv}.

For later reference we note that by what was said in section 7.1 there are eight more fields in the Ramond sector with dimensions $h = \overline{h} = \frac{1}{4}$. Each of them is uncharged under $\mathcal{J}$ and either $(A;\overline{A})$ or $(J;\overline{J})$. We denote by $W^J_{\varepsilon_1,\varepsilon_2}, W^A_{\varepsilon_1,\varepsilon_2}, \varepsilon_i \in \{\pm 1\}$ the fields corresponding to the lowest weight states of $su(2)_1 \cong \langle J, J^\pm\rangle$ or $su(2)_1 \cong \langle A, A^\pm\rangle$, with $(J;\overline{J})$ or $(A;\overline{A})$-charge $(\varepsilon_1;\varepsilon_2)$ respectively and identify

$$\begin{aligned} W^J_{\varepsilon_1,\varepsilon_2} &= \left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 4}, \\ W^A_{\varepsilon_1,\varepsilon_2} &= \left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 2}\otimes\left(\Phi^0_{\varepsilon_1,\varepsilon_1;\varepsilon_2,\varepsilon_2}\right)^{\otimes 2}. \end{aligned} \tag{7.3.35}$$

In σ model language and by the discussion in section 7.1, application of left and right handed spectral flow to the J-uncharged $W^A_{\varepsilon_1,\varepsilon_2}$ yields $(\frac{1}{2},\frac{1}{2})$-fields in $\mathcal{F}_{1/2}$, the real and imaginary parts of whose $(1,1)$-superpartners describe infinitesimal deformations of the torus $T_{SU(2)_1^4}$ our Kummer surface is associated to.
Summarizing, we can now obtain a list of all fields needed to generate $\mathcal{H}^1$ and $\mathcal{H}^2$ as well as a complete field by field identification by comparison of charges; for the resulting list of $(\frac{1}{4},\frac{1}{4})$-fields see appendix B. □

Note that because $D_4 \cong \sqrt{2}D_4^*$ for the $\mathcal{J}$-charge lattice (7.3.33)

$$\Gamma_b \cong \left\{ \tfrac{1}{\sqrt{2}}(\mu+\lambda, \mu-\lambda) \,\middle|\, \mu \in D_4^*, \lambda \in D_4 \right\}.$$

Thus Γ_b is the charge lattice of the bosonic part of the σ model $\mathcal{C}^3 = \mathcal{T}(D_4, 0)$. Theory $\mathcal{C}^1$ was obtained by taking the ordinary $\mathbb{Z}_2$ orbifold of the torus model on $T_{SU(2)_1^4}$, but as pointed out in [KS88], for the bosonic part of the theory this is equivalent to taking the $\mathbb{Z}_2$ orbifold associated to a shift $\delta = \frac{1}{2\sqrt{2}}(\mu_0; \mu_0)$, $\mu_0 = \sum_i e_i \in \Lambda^*$, on the charge lattice of $T_{SU(2)_1^4}$. Under this shift orbifold, by (5.1.6) the lattices $\Lambda = \Lambda^* = \mathbb{Z}^4$ are transformed by

$$\Lambda^* \mapsto \Lambda^* + \left(\Lambda^* + \tfrac{1}{2}\mu_0\right) = D_4^*, \quad \Lambda \mapsto \{\lambda \in \Lambda \,|\langle \mu_0, \lambda\rangle \equiv 0\,(2)\} = D_4,$$

so the bosonic part of the resulting theory indeed is that of $\mathcal{C}^3$. The entire bosonic subtheory of $\mathcal{C}^1 = \mathcal{C}^2$ in the sense of property 11 in section 3 agrees with that of $\mathcal{C}^3$, because the shift acts trivially on fermions, and the ordinary $\mathbb{Z}_2$ orbifold just interchanges twisted and untwisted boundary conditions of the fermions in time direction. The difference between the theories in the bosonic sector merely amounts in opposite assignments of Ramond and Neveu-Schwarz sector on the twisted states. So, on the level of bosonic conformal field theories:

Remark 7.3.26
The Gepner type model $\mathcal{C}^1 = (\widehat{2})^4$ viewed as nonlinear σ model $\mathcal{C}^2 = \mathcal{K}(\mathbb{Z}^4, 0)$ on the Kummer surface $\mathcal{K}(\mathbb{Z}^4)$ is located at a meeting point of the moduli spaces of theories associated to $K3$ surfaces and tori, respectively. Namely, its bosonic sector is identical with that of the nonlinear σ model $\mathcal{C}^3 = \mathcal{T}(D_4, 0)$.

This property does not translate to the stringy interpretation of our conformal field theories, though. When we take external degrees of freedom into account, the spin statistics theorem dictates in which representations of $SO(4)$ the external free fields may couple to internal Neveu-Schwarz or Ramond fields, respectively. The theories $\mathcal{C}^1 = \mathcal{C}^2$ and $\mathcal{C}^3$ therefore correspond to different compactifications of the type IIA string.

Gepner's description for $SU(2)_1^4/\mathbb{Z}_4$

Theorem 7.3.27
The Gepner model $\mathcal{C}^I = (2)^4$ admits a nonlinear σ model description $\mathcal{C}^{II}$ on the $\mathbb{Z}_4$ orbifold of the torus $T_{SU(2)_1^4}$ with $SU(2)_1^4$-lattice $\Lambda = \mathbb{Z}^4$ and with vanishing B-field.

Proof:
It is clear that $\mathcal{C}^I = (2)^4$ can be obtained from $\mathcal{C}^1 = (\widehat{2})^4$, for which we already have a σ model description by theorem 7.3.25, by the $\mathbb{Z}_2$ orbifold procedure which revokes the orbifold used to construct $\mathcal{C}^1$. The corresponding action is multiplication by -1 on $\langle[2,2,0,0]\rangle$-twisted states, i.e.

$$[2', 2', 0, 0]: \quad \bigotimes_{i=1}^{4} \Phi^{l_i}_{m_i,s_i;\overline{m}_i,\overline{s}_i} \longmapsto e^{\frac{2\pi i}{8}(\overline{m}_1 - m_1 - \overline{m}_3 + m_3)} \bigotimes_{i=1}^{4} \Phi^{l_i}_{m_i,s_i;\overline{m}_i,\overline{s}_i}. \qquad (7.3.36)$$

Among the $(1,0)$-fields the following are invariant under $[2',2',0,0]$ (use (7.3.17) and (7.3.32)):

$$\begin{array}{rclrclrcl} J &=& \psi_+^{(1)}\psi_-^{(1)} + \psi_+^{(2)}\psi_-^{(2)}, & J^+ &=& \psi_+^{(1)}\psi_+^{(2)}, & J^- &=& \psi_-^{(2)}\psi_-^{(1)}; \\ A &=& \psi_+^{(1)}\psi_-^{(1)} - \psi_+^{(2)}\psi_-^{(2)}; & P &=& \frac{1}{2}(U_1+U_2); & Q &=& \frac{1}{2}(U_3+U_4). \end{array} \tag{7.3.37}$$

Hence we have a surviving $su(2)_1 \oplus u(1)^3$ subalgebra of our holomorphic W-algebra. In appendix B we give a list of all $(\frac{1}{4},\frac{1}{4})$-fields in $\mathcal{C}^1 = (\widehat{2})^4$ together with their description in the σ model $\mathcal{C}^2$ on the $\mathbb{Z}_2$ orbifold $\mathcal{K}(\mathbb{Z}^4,0)$. A similar list can be obtained for the $(2,0)$-fields as discussed in the proof of theorem 7.3.25. From these lists and (7.3.37) one readily reads off that the states invariant under (7.3.36) coincide with those invariant under the automorphism r_{12} on $\mathcal{K}(\mathbb{Z}^4,0)$ (see theorem 7.3.12) which is induced by the $\mathbb{Z}_4$ action $(j_1,j_2,j_3,j_4) \mapsto (-j_2,j_1,j_4,-j_3)$, i.e. $(\psi_\pm^{(1)},\psi_\pm^{(2)}) \mapsto (\pm i\psi_\pm^{(1)}, \mp i\psi_\pm^{(2)})$ on the underlying torus $T_{SU(2)_1^4}$. The appertaining permutation of exceptional divisors in the $\mathbb{Z}_2$ fixed points has been depicted in figure 7.3.1 (section 7.3.1). The action of r_{12} and that induced by (7.3.36) agree on the algebra $\mathcal{A}$ of $(1,0)$-fields and a set of states generating the entire space of states, thus they are the same. Because of $\mathcal{C}^1 = \mathcal{C}^2$ (theorem 7.3.25) and the fact that $\mathcal{C}^I = (2)^4$ is obtained from $\mathcal{C}^1$ by modding out (7.3.36), it is clear that modding out $\mathcal{K}(\mathbb{Z}^4,0)$ by the algebraic automorphism r_{12} will lead to a σ model description of $(2)^4$. As shown in theorem 7.3.13 the result is the $\mathbb{Z}_4$ orbifold $\mathcal{C}^{II}$ of $T_{SU(2)_1^4}$. □

Theorem 7.3.27 has been conjectured in [EOTY89] because of agreement of the partition functions of $\mathcal{C}^I$ and $\mathcal{C}^{II}$. This of course is only part of the proof as can be seen from our argumentation at the beginning of section 7.3.4. There we showed that $SU(2)_1^4/\mathbb{Z}_4$ does not admit a σ model description on a Kummer surface although its partition function by [EOTY89] agrees with that of $\mathcal{K}(D_4,0)$, too.
From theorems 7.3.21 and 7.3.27 we conclude:

Corollary 7.3.28
The Gepner type models $(2)^4$ admits a geometric interpretation on the Fermat quartic (7.3.8) in $\mathbb{CP}^3$.

Let (Σ, V, B) denote the geometric interpretation of $(2)^4$ we gain from theorem 7.3.27. By the proof of theorem 7.3.25 we know the moduli $V_{\delta,\varepsilon}^\pm + V_{-\delta,-\varepsilon}^\pm$ and $i(V_{\delta,\varepsilon}^\pm - V_{-\delta,-\varepsilon}^\pm), \delta,\varepsilon \in \{\pm 1\}$, for volume and B-field deformation in direction of Σ of the underlying torus $T_{SU(2)_1^4}$ of our $\mathbb{Z}_4$ orbifold: We apply left and right handed spectral flows to $W_{1,1}^A, W_{-1,-1}^A$ as given in (7.3.35) and then compute the corresponding $(1,1)$-superpartners. In terms of Gepner fields this means

$$\begin{array}{rcl} V_{\delta,\varepsilon}^+ &=& \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \otimes \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \\ && +\Phi^2_{2\delta,0;2\varepsilon,0} \otimes \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2}, \\ V_{\delta,\varepsilon}^- &=& \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \otimes \Phi^2_{2\delta,2;2\varepsilon,2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \\ && + \left(\Phi^0_{0,0;0,0}\right)^{\otimes 2} \otimes \Phi^2_{2\delta,0;2\varepsilon,0} \otimes \Phi^2_{2\delta,2;2\varepsilon,2}. \end{array} \tag{7.3.38}$$

Indeed, $V^{\pm}_{\delta,\varepsilon}$ are uncharged under J and A as they should, because both $U(1)$-currents must survive deformations within the moduli space of $\mathbb{Z}_4$ orbifold conformal field theories. On the other hand by our discussion below (7.3.10) the $(1,1)$-superpartners of $(\Phi^1_{\pm1,0;\mp3,2})^{\otimes 4}$, $(\Phi^1_{\pm1,0;\mp1,0})^{\otimes 4}$ which carry $(A;\overline{A})$-charges $\mp(1;1)$, give the moduli of volume and corresponding B-field deformation if we choose the quartic hypersurface (7.3.8) as geometric interpretation of Gepner's model $(2)^4$. Hence along the "quartic line" we generically only have an $su(2)_1$-algebra of $(1,0)$-fields. This agrees with the analogous picture for $c=9$ and the Gepner model $(3)^5$ where all additional $U(1)$-currents vanish upon deformation along the quintic line [DG88].

Symmetries and algebraic automorphisms revised: $(2)^4$ and $\widehat{(2)}^4$

Among the algebraic symmetries $\mathbb{Z}_4^2 \rtimes \mathcal{S}_4$ of the Gepner model $(2)^4$ all the phase symmetries $\mathbb{Z}_4^2$ commute with the action of $[2,2,0,0]$ which we mod out to obtain $\widehat{(2)}^4$. The residual $\mathbb{Z}_2 \times \mathbb{Z}_4$ has a straightforward continuation to $\widehat{(2)}^4$ (i.e. to the twisted states). Moreover, $[2',2',0,0]$ as given in (7.3.36) which reverts the orbifold with respect to $[2,2,0,0]$ must belong to the algebraic symmetry group $\widehat{\mathcal{G}}^{alg}$ of $\widehat{(2)}^4$. Nevertheless, one notices that $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle [2',2',0,0],[1,3,0,0]\rangle$ leaves $6 \neq 8 = \mu(\mathbb{Z}_2\times\mathbb{Z}_2)-4$ states invariant and thus does not act algebraically by (7.3.4). We temporarily leave the symmetry $[1,3,0,0]$ out of discussion, because then by the methods described below (7.3.10) we find a consistent algebraic action of $(\mathbb{Z}_2\times\mathbb{Z}_4)\rtimes D_4$ on $\widehat{(2)}^4$, where $\mathbb{Z}_2\times\mathbb{Z}_4 = \langle [2',2',0,0],[1,0,3,0]\rangle$ and $D_4 = \langle (12),(13)(24)\rangle \subset \mathcal{S}_4$ is the commutant of $[2,2,0,0]$.
Let us compare to the σ model description $\mathcal{K}(\mathbb{Z}^4,0)$ of $\widehat{(2)}^4$: In theorem 7.3.12 the group of algebraic automorphisms of $\mathcal{K}(\mathbb{Z}^4,0)$ which leave the orbifold singular metric invariant was determined to $\mathcal{G}^+_{Kummer} = \mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4$. Although it is isomorphic to the algebraic symmetry group $(\mathbb{Z}_2\times\mathbb{Z}_4)\rtimes D_4$ of $\widehat{(2)}^4$ found so far, $\mathcal{G}^+_{Kummer}$ must act differently on $\widehat{(2)}^4$. Namely, from the proof of theorem 7.3.27 we know that the σ model equivalent of $[2',2',0,0]$ is $r_{12}\in\mathcal{G}^+_{Kummer}$. Thus only the commutant $\mathcal{H}\subset\mathcal{G}^+_{Kummer}$ of r_{12} can comprise residual symmetries descending from the $\mathbb{Z}_4$ orbifold description on $(2)^4$. This is no contradiction, because by the discussion in section 7.3.1 different subgroups of the entire algebraic symmetry group of $\widehat{(2)}^4$ may leave the respective null vector v invariant which defines the geometric interpretation. By what was said in section 7.1 it is actually no surprise to find symmetries of conformal field theories which do not descend to classical symmetries of a given geometric interpretation. The Gepner type model $\widehat{(2)}^4$ is an example where the existence of such symmetries can be checked explicitly.
By the results of section 7.3.1 we find $\mathcal{H} = \mathbb{Z}_2\times D_4 = \langle r_{12}, r_{13}, t_{1100}\rangle$ (see also theorem 7.3.14). We now use our state by state identification obtained in the proof of theorem 7.3.25 (see appendix B) to determine the corresponding elements of $\widehat{\mathcal{G}}^{alg}$ and find

$$\begin{array}{rcl} r_{13} &=& (13)(24) \qquad\in\; \mathcal{S}_4\\ t_{1100} &=& \zeta\circ[1,3,0,0] \;=:\; [1',3',0,0]. \end{array} \tag{7.3.39}$$

Here ζ acts by multiplication with -1 on those Gepner states corresponding to the

16 twist fields Σ_δ of the Kummer surface and trivially on all the other generating fields of the space of states we discussed in the proof of theorem 7.3.25. Note that ζ is a symmetry of the theory because by the selection rules for amplitudes of twist fields any n point function containing an odd number of twist fields will vanish. The geometric interpretation tells us that modding out $\widehat{(2)}^4$ by ζ will revoke the ordinary $\mathbb{Z}_2$ orbifold procedure i.e. produce $\mathcal{T}(\mathbb{Z}^4, 0)$. We conclude with the remark that by the modification (7.3.39) of the $[1,3,0,0]$-action the full group $\widehat{\mathcal{G}}^{alg} = (\mathbb{Z}_2^2 \times \mathbb{Z}_4) \rtimes D_4$ acts algebraically on $\widehat{(2)}^4$. The subgroup $\mathcal{H}$ consists of all the residual symmetries of $(2)^4$ that survive both deformations along the quartic and the $\mathbb{Z}_4$ orbifold line and act classically in both geometric interpretations of $(2)^4$ known so far, the $\mathbb{Z}_4$ orbifold and the quartic one.

Gepner type description of $SO(8)_1/\mathbb{Z}_2$

Theorem 7.3.29
Let $\widetilde{\mathcal{C}}^1 = \widetilde{(2)}^4$ denote the Gepner type model discussed in theorem 3.1.18, i.e. the orbifold of $(2)^4$ by the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle [2,2,0,0],[2,0,2,0]\rangle \subset \mathcal{G}^{alg}_{ab}$. This model admits a nonlinear σ model description $\widetilde{\mathcal{C}}^2$ on the Kummer surface $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ associated to the torus $T_{SO(8)_1}$ with $SO(8)_1$-lattice $\Lambda = \frac{1}{\sqrt{2}}D_4$ and B-field value B^* for which the theory has enhanced symmetry by the Frenkel-Kac mechanism.

Proof:
Let $e_1, \dots, e_4$ denote the standard basis of $\mathbb{Z}^4$. With respect to this basis the B-field which leads to a full $SO(8)_1$ symmetry for the σ model on $T_{SO(8)_1}$ is

$$B^* = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array}\right) : \Lambda \otimes \mathbb{R} \longrightarrow \Lambda^* \otimes \mathbb{R}\,, \tag{7.3.40}$$

a twotorsion point in $H^2(T_{SO(8)_1}, \mathbb{R})/H^2(T_{SO(8)_1}, \mathbb{Z})$.
We are now ready to use theorem 7.3.23 if we can prove conditions i.-iii.

i. From (4.1.7) we find

$$Z_{\frac{1}{\sqrt{2}}D_4, B^*}(\tau) = \frac{1}{2}\left(\left|\frac{\vartheta_2}{\eta}\right|^8 + \left|\frac{\vartheta_3}{\eta}\right|^8 + \left|\frac{\vartheta_4}{\eta}\right|^8 \right). \tag{7.3.41}$$

With theorem 3.1.18 one checks that $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ have the same partition function obtained by inserting (7.3.41) into (5.2.9).

ii. We have an enhancement of the current algebra (7.3.17) of the nonlinear σ model $\widetilde{\mathcal{C}}^2$ to $su(2)_1^6$. The 12 additional $(1,0)$-fields are $U_\alpha := \frac{1}{\sqrt{2}}(V_{\alpha,\alpha+B^*\alpha} + V_{-\alpha,-\alpha-B^*\alpha})$, where α belongs to the D_4 root system $\{\pm\frac{1}{\sqrt{2}}e_i \pm \frac{1}{\sqrt{2}}e_j\}$. We set

$$W^{\pm}_{i,j} := \tfrac{1}{2}\left(U_{\frac{1}{\sqrt{2}}(e_i+e_j)} \pm U_{\frac{1}{\sqrt{2}}(e_i-e_j)}\right)$$

to see that upon a consistent choice of cocycle factors for the vertex operators these fields indeed comprise an extra $su(2)_1^4$:

$$\begin{array}{rclrcl} P &:=& W^+_{1,4}+W^+_{2,3}, & P^\pm &:=& \frac{1}{\sqrt2}\left(W^+_{1,2}+W^+_{3,4}\right)\pm\frac{1}{\sqrt2}\left(W^+_{2,4}+W^+_{1,3}\right),\\ Q &:=& W^+_{1,2}-W^+_{3,4}, & Q^\pm &:=& \frac{1}{\sqrt2}\left(W^+_{1,3}-W^+_{2,4}\right)\pm\frac{1}{\sqrt2}\left(W^+_{1,4}-W^+_{2,3}\right),\\ R &:=& iW^-_{2,4}-iW^-_{1,3}, & R^\pm &:=& \frac{1}{\sqrt2}\left(W^-_{1,4}-W^-_{2,3}\right)\pm\frac{1}{\sqrt2}\left(W^-_{1,2}-W^-_{3,4}\right),\\ S &:=& W^-_{1,4}+W^-_{2,3}, & S^\pm &:=& \frac{1}{\sqrt2}\left(W^-_{1,2}+W^-_{3,4}\right)\pm\frac{1}{\sqrt2}\left(W^-_{2,4}+W^-_{1,3}\right). \end{array} \tag{7.3.42}$$

For the Gepner type model $\widetilde{\mathcal{C}}^2=(\widetilde{2})^4$ we use X_{ij} as a shorthand notation for the field having factors $\Phi^0_{4,2;0,0}$ in the ith and jth position and factors $\Phi^0_{0,0;0,0}$ otherwise, and Y_{ij} for the field having factors $\Phi^0_{-2,2;0,0}$ in the ith and jth position and factors $\Phi^0_{2,2;0,0}$ otherwise. By comparison of OPEs one then checks that the following identifications can be made:

$$\begin{array}{rclrcl} J &=& J_1+J_2+J_3+J_4, & J^\pm &=& \left(\Phi^0_{\mp2,2;0,0}\right)^{\otimes4};\\ A &=& J_1+J_2-J_3-J_4, & A^+ &=& Y_{12},\quad A^- \;=\; Y_{34};\\ \hline P &=& \frac{1}{\sqrt2}\left(J_1-J_2+J_3-J_4\right), & P^+ &=& Y_{13},\quad P^- \;=\; Y_{24};\\ Q &=& \frac{1}{\sqrt2}\left(J_1-J_2-J_3+J_4\right), & Q^+ &=& Y_{14},\quad Q^- \;=\; Y_{23};\\ R &=& \frac{i}{\sqrt2}\left(X_{13}-X_{24}\right), & R^\pm &=& \mp\frac12\left(X_{12}+X_{34}\right)+\frac{i}{2}\left(X_{14}+X_{23}\right);\\ S &=& \frac{i}{\sqrt2}\left(X_{13}+X_{24}\right), & S^\pm &=& \pm\frac12\left(X_{12}-X_{34}\right)+\frac{i}{2}\left(X_{14}-X_{23}\right). \end{array}$$

Thus the $(1,0)$-fields in the two theories generate the same algebra $\mathcal{A}=su(2)_1^2\oplus su(2)_1^4=\mathcal{A}_f\oplus\mathcal{A}_b$. Obviously, the same structure arises on the right handed sides.

iii. We will show that the spaces of states $\widetilde{\mathcal{H}}_b^1$ and $\widetilde{\mathcal{H}}_b^2$ of $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ both have self dual $\mathcal{J}:=(P,Q,R,S;\overline{P},\overline{Q},\overline{R},\overline{S})$-charge lattice

$$\widetilde{\Gamma}_b=\left\{\tfrac{1}{\sqrt2}(x+y;x-y)\,\middle|\,x,y\in\mathbb{Z}^4\right\}. \tag{7.3.43}$$

In the Gepner type model $\widetilde{\mathcal{C}}^1=(\widetilde{2})^4$ we find 16 fields with dimensions $h=\overline{h}=\frac14$ which are uncharged under $(J,A;\overline{J},\overline{A})$; diagonalizing them with respect to the $\mathcal{J}$-action for $j\in\{P,Q,R,S\}$ we obtain fields $E_j^\pm,F_j^\pm$ uncharged under all $U(1)$-currents apart from j and with $(j,\overline{\jmath})$-charge $\frac{1}{\sqrt2}(\pm1,\pm1)$ and $\frac{1}{\sqrt2}(\pm1,\mp1)$, respectively. Namely,

$$\begin{array}{rcl} E^\pm_P &=& \Phi^0_{\mp1,\mp1;\mp1,\mp1}\otimes\Phi^0_{\pm1,\pm1;\pm1,\pm1}\otimes\Phi^0_{\mp1,\mp1;\mp1,\mp1}\otimes\Phi^0_{\pm1,\pm1;\pm1,\pm1},\\ F^\pm_P &=& \Phi^0_{\mp1,\mp1;\pm1,\pm1}\otimes\Phi^0_{\pm1,\pm1;\mp1,\mp1}\otimes\Phi^0_{\mp1,\mp1;\pm1,\pm1}\otimes\Phi^0_{\pm1,\pm1;\mp1,\mp1},\\ E^\pm_Q &=& \Phi^0_{\mp1,\mp1;\mp1,\mp1}\otimes\Phi^0_{\pm1,\pm1;\pm1,\pm1}\otimes\Phi^0_{\pm1,\pm1;\pm1,\pm1}\otimes\Phi^0_{\mp1,\mp1;\mp1,\mp1},\\ F^\pm_Q &=& \Phi^0_{\mp1,\mp1;\pm1,\pm1}\otimes\Phi^0_{\pm1,\pm1;\mp1,\mp1}\otimes\Phi^0_{\pm1,\pm1;\mp1,\mp1}\otimes\Phi^0_{\mp1,\mp1;\pm1,\pm1}, \end{array}$$

and with $\varepsilon_R := -1, \varepsilon_S := 1$ for $j \in \{R, S\}$

$$\begin{aligned}
E_j^\pm &= \left(\Phi^1_{2,1;2,1}\right)^{\otimes 4} + \varepsilon_j \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 4} \\
&\quad \pm \left[\Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1}\right. \\
&\qquad \left. + \varepsilon_j \, \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1}\right], \\
F_j^\pm &= \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} + \varepsilon_j \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \\
&\quad \pm \left[\Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1}\right. \\
&\qquad \left. + \varepsilon_j \, \Phi^1_{2,1;2,1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;-2,-1} \otimes \Phi^1_{2,1;2,1}\right].
\end{aligned}$$

Among the corresponding charges under $\mathcal{J}$ we find $\frac{1}{\sqrt{2}}(e_i; \pm e_i)$ generating $\widetilde{\Gamma}_b$. In the sigma model $\widetilde{\mathcal{C}}^1$ we set

$$\begin{aligned}
\alpha_1 &:= \tfrac{1}{\sqrt{2}}\left(e_1 + e_2\right), & \alpha_2 &:= \tfrac{1}{\sqrt{2}}\left(e_2 - e_1\right), \\
\alpha_3 &:= \tfrac{1}{\sqrt{2}}\left(e_1 + e_3\right), & \alpha_4 &:= \tfrac{1}{\sqrt{2}}\left(e_4 - e_2\right).
\end{aligned}$$

Let Σ_δ with $\delta \in \mathbb{F}_2^4$ denote the twist field corresponding to the fixed point $\frac{1}{2}\sum_{i=1}^4 \delta_i \alpha_i$. The action of P, Q, R, S and their right handed partners is determined as in (7.3.34). Then by normalizing appropriately and matching $(\mathcal{J}; \overline{\mathcal{J}})$-charges we find that the following identifications can be made (sums run over $\delta \in \mathbb{F}_2^4$ with the indicated restrictions):

$$\begin{aligned}
E_P^\pm &= \sum_{\delta_1=\delta_2, \delta_3=\delta_4} \Sigma_\delta \pm \sum_{\delta_1\neq\delta_2, \delta_3\neq\delta_4} \Sigma_\delta, \\
F_P^\pm &= \sum_{\delta_1\neq\delta_2, \delta_3=\delta_4} \Sigma_\delta \pm \sum_{\delta_1=\delta_2, \delta_3\neq\delta_4} \Sigma_\delta, \\
E_Q^\pm &= \sum_{\delta_1=\delta_2, \delta_3=\delta_4} (-1)^{\delta_4} \Sigma_\delta \pm \sum_{\delta_1\neq\delta_2, \delta_3=\delta_4} (-1)^{\delta_4} \Sigma_\delta, \\
F_Q^\pm &= \sum_{\delta_1\neq\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_3} \Sigma_\delta \pm \sum_{\delta_1=\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_3} \Sigma_\delta, \\
E_R^\pm &= \sum_{\delta_1=\delta_2, \delta_3=\delta_4} (-1)^{\delta_1} \Sigma_\delta \pm \sum_{\delta_1=\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_1} \Sigma_\delta, \\
F_R^\pm &= \sum_{\delta_1\neq\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_2} \Sigma_\delta \pm \sum_{\delta_1\neq\delta_2, \delta_3=\delta_4} (-1)^{\delta_2} \Sigma_\delta, \\
E_S^\pm &= \sum_{\delta_1=\delta_2, \delta_3=\delta_4} (-1)^{\delta_2+\delta_3} \Sigma_\delta \pm \sum_{\delta_1\neq\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_2+\delta_3} \Sigma_\delta, \\
F_S^\pm &= \sum_{\delta_1\neq\delta_2, \delta_3=\delta_4} (-1)^{\delta_2+\delta_3} \Sigma_\delta \pm \sum_{\delta_1=\delta_2, \delta_3\neq\delta_4} (-1)^{\delta_2+\delta_3} \Sigma_\delta.
\end{aligned}$$

In particular, the corresponding $(\mathcal{J}; \overline{\mathcal{J}})$–charges generate $\widetilde{\Gamma}_b$.

$\square$

Recall from theorem 5.6.3 the Greene-Plesser construction for mirror symmetry to observe that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold $(\widetilde{2})^4$ of $(2)^4$ is invariant under mirror symmetry. This can be regarded as an explanation for the high degree of symmetry found for $(\widetilde{2})^4 = \widetilde{\mathcal{C}}^1$.
In view of (7.3.43) it is clear that the same phenomenon as described in remark 7.3.26 appears for the theory discussed above:

Remark 7.3.30
The Gepner type model $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$, or equivalently the nonlinear σ model $\widetilde{\mathcal{C}}^2 = \mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$, B^* given by (7.3.40), is located at a meeting point of the moduli spaces of theories associated to $K3$ surfaces and tori, respectively. Namely, its bosonic sector is identical with that of the nonlinear σ model $\widetilde{\mathcal{C}}^3$ on the $SU(2)_1^4$-torus with vanishing B-field.

This again can be deduced from the results in [KS88] once one observes that the lattice denoted by $\Lambda_{O(n)\times O(n)}$ there in the case $n = 4$ is isomorphic to $\widetilde{\Gamma}_b$ as defined in (7.3.43). The relation between the two meeting points $(2)^4 = \mathcal{C}^1 = \mathcal{C}^2 \cong \mathcal{C}^3$ and $(\widetilde{2})^4 = \widetilde{\mathcal{C}}^1 = \widetilde{\mathcal{C}}^2 \cong \widetilde{\mathcal{C}}^3$ of the moduli spaces found so far is best understood from the fact that $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ can be constructed from $\mathcal{C}^1 = (\widehat{2})^4$ by modding out $\mathbb{Z}_2 \cong \langle[2,0,2,0]\rangle \subset \mathcal{G}^{ab}_{alg}$. If we formulate the orbifold procedure in terms of the charge lattice Γ_b of $\mathcal{C}^1 = (\widehat{2})^4$ as described in [GP90], this amounts to a shift orbifold by the vector $\delta = \frac{1}{2}(-1,1,0,0;1,-1,0,0)$ on Γ_b. Indeed, this shift simply reverts the shift we used to explain remark 7.3.26 and brings us back onto the torus $T_{SU(2)_1^4}$. But as for $\mathcal{C}^1 = \mathcal{C}^2$ and $\mathcal{C}^3$, $\widetilde{\mathcal{C}}^1 = \widetilde{\mathcal{C}}^2$ and $\widetilde{\mathcal{C}}^3$ will correspond to different compactifications of the type IIA string.
From (7.3.39) we are able to determine the geometric counterpart of $[2,0,2,0]$ on $\mathcal{K}(\mathbb{Z}^4, 0)$: It is the unique nontrivial central element t_{1111} of the algebraic automorphism group $\mathcal{G}^+_{Kummer}$ depicted in figure 7.3.2. Hence the commutant of t_{1111} is the

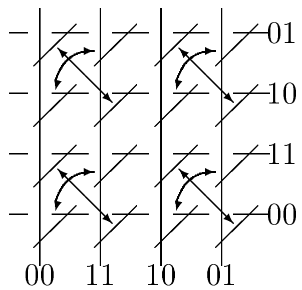

Figure 7.3.2: Action of the algebraic automorphism t_{1111} on the Kummer lattice Π.

entire $\mathcal{G}^+_{Kummer}$, but it is not clear so far how to continue the residual $\mathcal{G}^+_{Kummer}/\mathbb{Z}_2$ algebraically to the twisted sectors in $(\widetilde{2})^4$ with respect to the t_{1111} orbifold.
We remark that conformal field theory also helps us to draw conclusions on the geometry of the Kummer surfaces under inspection: $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*)$ is obtained from $\mathcal{K}(\mathbb{Z}^4, 0)$ by modding out the classical symmetry t_{1111}, so in terms of the decomposition (7.1.5) we stay in the same "chart" of $\mathcal{M}^{K3}$, i.e. choose the same null

vector v for both theories. This means that we can explicitly relate the respective geometric data. For both Kummer surfaces we choose the complex structures induced by the $N = (2,2)$ algebra in the corresponding Gepner models $(\widehat{2})^4$ and $(\widetilde{2})^4$. Thus we identify $J^\pm = \left(\Phi^0_{\mp 2,2;\mp 2,2}\right)^{\otimes 4}$ in both theories with the twoforms $\pi_*(dz_1 \wedge dz_2), \pi_*(d\overline{z}_1 \wedge d\overline{z}_2)$ defining the complex structure of $\mathcal{K}(\Lambda)$. Here $\pi : T_\Lambda \to \mathcal{K}(\Lambda)$ is the rational map of degree two, $\Lambda = \mathbb{Z}^4$ or $\Lambda = \frac{1}{\sqrt{2}} D_4$, respectively. Then both $\mathcal{K}(\Lambda)$ are singular $K3$ surfaces (see definitions 4.6.3). Given the lattices of the underlying tori one can compute the intersection form for real and imaginary part of the above twoforms defining the complex structure. One finds that they span sublattices of the transcendental lattices with forms diag$(4,4)$ for $\mathcal{K}(\mathbb{Z}^4)$ and diag$(8,8)$ for $\mathcal{K}(\frac{1}{\sqrt{2}} D_4)$, respectively. The factor of two difference was to be expected, because t_{1111} has degree two. Nevertheless, by (4.6.2) the transcendental lattices themselves for both surfaces have quadratic form diag$(4,4)$. Note that for a given algebraic automorphism in general it is hard to decide how the transcendental lattice transforms under modding out [Ino76, Cor. 1.3.3]. In our case, we could read it off thanks to the Gepner type descriptions of our conformal field theories.

Gepner type description of $SO(8)_1/\mathbb{Z}_4$

Theorem 7.3.31
The Gepner type model $\mathcal{C}^1 = (\widehat{2})^4$ which agrees with $\mathcal{C}^2 = \mathcal{K}(\mathbb{Z}^4, 0)$ by theorem 7.3.25 admits a nonlinear σ model description as $\mathbb{Z}_4$ orbifold of the torus model $\mathcal{T}(\frac{1}{\sqrt{2}} D_4, B^*)$ with $SO(8)_1$ symmetry.

Proof:
The proof works analogously to that of theorem 7.3.27. From theorem 7.3.13 it follows that the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}} D_4, B^*)$ with B^* defined by (7.3.40) is obtained from $\widetilde{\mathcal{C}}^2 = \mathcal{K}(\frac{1}{\sqrt{2}} D_4, B^*)$ by modding out the automorphism r_{12} as depicted in figure 7.3.1. Thus we should work with the models $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ and $\widetilde{\mathcal{C}}^2 = \mathcal{K}(\frac{1}{\sqrt{2}} D_4, B^*)$ which are isomorphic by theorem 7.3.29. We use the notations introduced there. Then r_{12} is induced by $e_1 \mapsto e_2, e_2 \mapsto -e_1, e_3 \mapsto -e_4, e_4 \mapsto e_3$. Of the $su(2)_1^6$ current algebra of $\widetilde{\mathcal{C}}^2$ we find a surviving $su(2)_1^2 \oplus u(1)^4$ current algebra on the $\mathbb{Z}_4$ orbifold generated by $J, J^\pm, A; P, P^\pm, Q, R, S$ (see equations (7.3.17) and (7.3.42)). The action on the generators $E_j^\pm, F_j^\pm, j \in \{P, Q, R, S\}$, is already diagonalized. All the $E_j^\pm$ are invariant as well as $F_P^\pm$. On the fermionic part of the space of states of $\widetilde{\mathcal{C}}^2$ the identifications (7.3.35) hold. The fields $W^J_{\varepsilon_1,\varepsilon_2}$ and $W^A_{\varepsilon_1,\varepsilon_1}, \varepsilon_i \in \{\pm 1\}$, are those invariant under the $\mathbb{Z}_4$ action. Our field by field identifications of theorem 7.3.29 now allow us to read off the induced action on the Gepner type model $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$. One checks that it agrees with the symmetry $[2', 2', 0, 0]$ defined in (7.3.36) which revokes the orbifold by the $\mathbb{Z}_2$ action of $[2,2,0,0]$. Because $\widetilde{\mathcal{C}}^1 = (\widetilde{2})^4$ was constructed from the Gepner model $(2)^4$ by modding out $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle [2,2,0,0], [2,0,2,0] \rangle \subset \mathcal{G}_{ab}^{alg}$, it follows that the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}} D_4, B^*)$ agrees with the Gepner type model obtained from $(2)^4$ by modding out $\mathbb{Z}_2 \cong \langle [2,0,2,0] \rangle$. This clearly is isomorphic to $(\widehat{2})^4$ by a permutation of the minimal model factors. $\square$

7.4 Summary: A hiker's view of $K3$

We conclude by joining the information we have gathered so far to a panoramic picture of those strata of the moduli space we have fully under control now (figure 7.4.1). The rest of this section is devoted to a summary of what we have learned

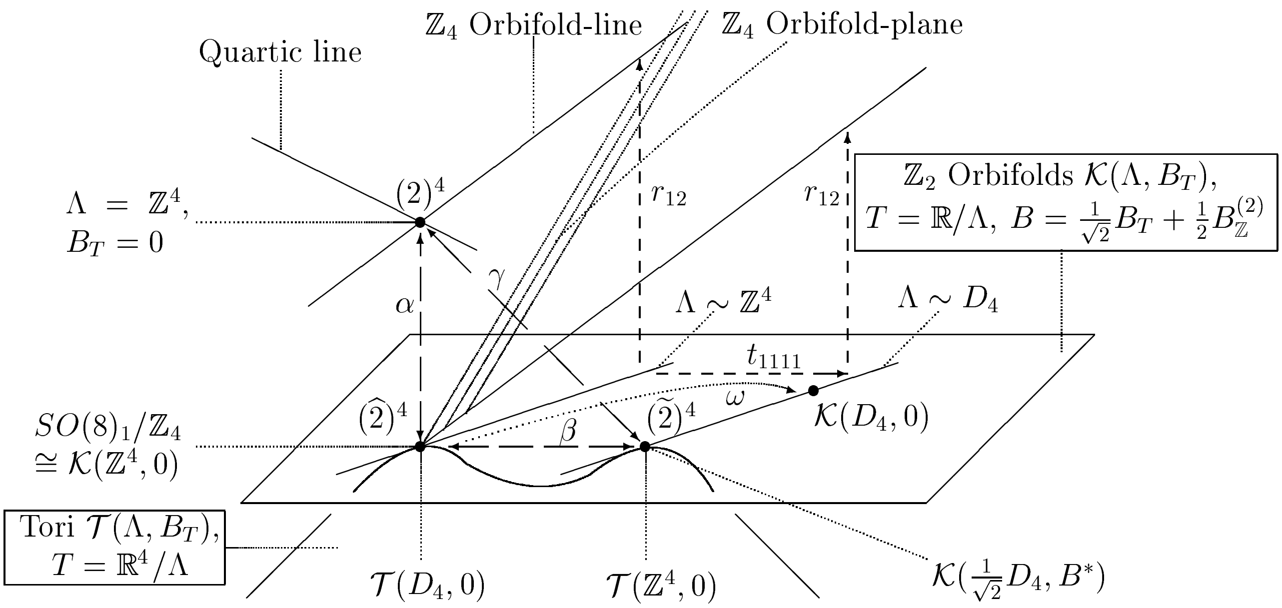

Figure 7.4.1: Strata of the moduli space.

about the various components depicted in figure 7.4.1. All the strata are defined as quaternionic submanifolds of the moduli space $\mathcal{M}^{K3}$ consisting of theories which admit certain restricted geometric interpretations. In other words, a suitable choice of υ as described in section 7.1 yields (Σ, V, B) such that Σ, B have the respective properties. In the following we will always tacitly assume that an appropriate choice of υ has been performed already.

Figure 7.4.1 contains two strata of real dimension 16, depicted as a horizontal plane and a mexican hat like object, respectively. The horizontal plane is the *Kummer stratum*, the subspace of the moduli space consisting of all theories which admit a geometric interpretation on a Kummer surface X in the orbifold limit. In other words, it is the 16 dimensional moduli space of all theories $\mathcal{K}(\Lambda, B_T)$ obtained from a nonlinear σ model on a torus $T = \mathbb{R}^4/\Lambda$ by applying the ordinary $\mathbb{Z}_2$ orbifold procedure; the B-field takes values $B = \frac{1}{\sqrt{2}}B_T + \frac{1}{2}B^{(2)}_{\mathbb{Z}}$, where $B_T \in H^2(T, \mathbb{R}) \hookrightarrow H^2(X, \mathbb{R})$ (see the explanation after theorem 7.3.5), and $B^{(2)}_{\mathbb{Z}} \in H^{even}(X, \mathbb{Z})$ as described in theorem 7.3.16. We have an embedding $\mathcal{M}^{tori} \hookrightarrow \mathcal{M}^{K3}$ as quaternionic submanifold, and we know how to locate this stratum within $\mathcal{M}^{K3}$. Kummer surfaces in the orbifold limit have a generic group $\mathbb{F}_2^4$ of algebraic automorphisms

which leave the metric invariant. Any conformal field theory associated to such a Kummer surface possesses an $su(2)_1^2$ subalgebra (7.3.17) of the holomorphic W-algebra.
The mexican hat like object in figure 7.4.1 depicts the moduli space (4.2.6) of theories associated to tori. Two meeting points with the Kummer stratum have been determined so far, namely $(\widehat{2})^4$ and $(\widetilde{2})^4$ (see remarks 7.3.26 and 7.3.30). We found $(\widehat{2})^4 = \mathcal{K}(\mathbb{Z}^4, 0) = \mathcal{T}(D_4, 0)$ and $(\widetilde{2})^4 = \mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*) = \mathcal{T}(\mathbb{Z}^4, 0)$, where B^* was defined in (7.3.40).
The vertical plane in figure 7.4.1 depicts a stratum of real dimension 8, namely the moduli space of theories admitting a geometric interpretation as $\mathbb{Z}_4$ orbifold of a nonlinear σ model on $T = \mathbb{R}^4/\Lambda$. In order for the orbifold procedure to be well defined we assume Λ to be generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$, ($\Lambda_1$ is not necessarily orthogonal to Λ_2) and $B_T \in H^2(T, \mathbb{R})^{\mathbb{Z}_4} \hookrightarrow H^2(X, \mathbb{R})$ (see lemma 7.3.8). The B-field then takes values $B = \frac{1}{2}B_T + \frac{1}{4}B_{\mathbb{Z}}^{(4)}$ as described in theorem 7.3.20, where the embedding of this stratum in $\mathcal{M}^{K3}$ is also explained. The generic group of algebraic automorphisms for $\mathbb{Z}_4$ orbifolds is $\mathbb{Z}_2 \ltimes \mathbb{F}_2^4$. By theorem 7.3.31 there is a meeting point with the Kummer stratum in the $\mathbb{Z}_4$ orbifold of $\mathcal{T}(\frac{1}{\sqrt{2}}D_4, B^*)$, where B^* is given by (7.3.40), which agrees with $\mathcal{K}(\mathbb{Z}^4, 0) = (\widehat{2})^4$.
The four lines in figure 7.4.1 are strata of real dimension 4 which are defined by restriction to theories admitting a geometric interpretation (Σ, V, B) with fixed Σ and allowed B-field values $B \in \Sigma$. Thus the volume is the only geometric parameter along the lines and we can associate a fixed hyperkähler structure on $K3$ to each of them. For all four lines it turns out that one can choose a complex structure such that the respective $K3$ surface is singular. Hence Σ can be described by giving the quadratic form on the transcendental lattice and the Kähler class for this choice of complex structure. Specifically we have:

- $\mathbb{Z}^4$*-line:* The subspace of the Kummer stratum given by theories $\mathcal{K}(\Lambda, B_T)$ with $\Lambda \sim \mathbb{Z}^4$ and $B_T \in \Sigma$, which is marked by $\Lambda \sim \mathbb{Z}^4$ in figure 7.4.1.

- $\mathbb{Z}_4$ *Orbifold-line:* The moduli space of all theories which admit a geometric interpretation on a $K3$ surface obtained from the nonlinear σ model on a torus $T = \mathbb{R}^4/\Lambda$, $\Lambda \sim \mathbb{Z}^4$, with B-field B_T commuting with the automorphisms listed in (7.3.16).

- *Quartic line:* Though well established in the context of Landau-Ginzburg theories, this stratum has been somewhat conjectural up to now. We describe it as the moduli space of theories admitting a geometric interpretation $(\Sigma_{\mathcal{Q}}, V_{\mathcal{Q}}, B_{\mathcal{Q}})$ on the Fermat quartic (7.3.8) equipped with a Kähler metric in the class of the Fubini-Study metric, in order for $\Sigma_{\mathcal{Q}}$ to be invariant under the algebraic automorphism group $G = \mathbb{Z}_4^2 \rtimes \mathcal{S}_4$. The B-field is restricted to values $B_{\mathcal{Q}} \in \Sigma_{\mathcal{Q}}$, because $\mu(G) = 5$ and therefore $H^2(X, \mathbb{R})^G = \Sigma_{\mathcal{Q}}$.

- D_4*-line:* The moduli space of $\mathbb{Z}_2$ orbifold theories admitting as geometric interpretation a Kummer surface $\mathcal{K}(\Lambda), \Lambda \sim D_4$, and with $B_T \in \Sigma$. This line is labelled by $\Lambda \sim D_4$ in figure 7.4.1.

The four lines are characterized by the following data:

name of line	associated form on the transcendental lattice	group of algebraic automorphisms leaving the metric invariant	generic $(1,0)$–current algebra
$\mathbb{Z}^4$-line	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\mathcal{G}^+_{Kummer} = \mathbb{Z}_2^2 \ltimes \mathbb{F}_2^4 \cong (\mathbb{Z}_2 \times \mathbb{Z}_4) \rtimes D_4$	$su(2)_1^2$
$\mathbb{Z}_4$ orbifold-line	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	D_4	$su(2)_1 \oplus u(1)$
quartic line	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$	$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathcal{S}_4$	$su(2)_1$
D_4-line	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\mathbb{Z}_2 \ltimes \mathbb{F}_2^4$	$su(2)_1^2$

In figure 7.4.1 we have two different shortdashed arrows indicating relations between lines. Consider the Kummer surface $\mathcal{K}(\mathbb{Z}^4)$ associated to the $\mathbb{Z}^4$-line. As demonstrated in theorem 7.3.13, the group $\mathcal{G}^+_{Kummer}$ of algebraic automorphisms of $\mathcal{K}(\mathbb{Z}^4)$ which leave the metric invariant contains the automorphism r_{12} of order two (see figure 7.3.1 in section 7.3.1) which upon modding out produces the $\mathbb{Z}_4$ orbifold-line. The entire moduli space of $\mathbb{Z}_4$ orbifold conformal field theories is obtained this way from $\mathbb{Z}_2$ orbifold theories $\mathcal{K}(\Lambda, B_T)$, where Λ is generated by $\Lambda_i \cong R_i\mathbb{Z}^2, R_i \in \mathbb{R}, i = 1, 2$, and $B_T \in H^2(T, \mathbb{R})^{\mathbb{Z}_4}$.
Modding out $t_{1111} \in \mathcal{G}^+_{Kummer}$ (see figure 7.3.2 in section 7.3.6) on the $\mathbb{Z}^4$-line produces the D_4-line, as argued before theorem 7.3.31. Note that the $K3$ surfaces associated to $\mathbb{Z}^4$- and D_4-lines have the same quadratic form on their transcendental lattices and hence are identical as algebraic varieties. Still, the corresponding lines in moduli space are different because different Kähler classes are fixed. In our terminology this is expressed by the change of lattices of the underlying tori on transition from one line to the other. The D_4-line can also be viewed as the image of the $\mathbb{Z}^4$-line upon shift orbifold on the underlying torus.
Finally, we list the zero dimensional strata shown in figure 7.4.1.
To construct $\mathcal{K}(D_4, 0)$ on the D_4-line, we may as well apply the ordinary $\mathbb{Z}_2$ orbifold procedure to the D_4-torus theory in the meeting point $\widehat{(2)}^4$ (the arrow with label ω in figure 7.4.1). We stress that in contrast to what was conjectured in [EOTY89] this is not a meeting point with the $\mathbb{Z}_4$ orbifold-plane.
As demonstrated in theorem 7.3.27 and also conjectured in [EOTY89], Gepner's model $(2)^4$ is the point of enhanced symmetry $\Lambda = \mathbb{Z}^4, B_T = 0$ on the $\mathbb{Z}_4$ orbifold-line. In section 7.3.1 we have studied the algebraic symmetry group of $(2)^4$ and in corollary 7.3.28 proved that it admits a geometric interpretation with Fermat quartic target space, too. In terms of the Gepner model, the moduli of infinitesimal deformation along the $\mathbb{Z}_4$ orbifold and the quartic line are real and imaginary parts of $V^{\pm}_{\delta,\varepsilon}(\delta, \varepsilon \in \{\pm 1\})$ as in (7.3.38) and of the $(1,1)$-superpartners of $(\Phi^1_{\pm 1,0;\mp 3,2})^{\otimes 4}$, $(\Phi^1_{\pm 1,0;\mp 1,0})^{\otimes 4}$, respectively (see section 7.3.6). The models $\mathcal{K}(D_4, 0)$

and $\mathcal{K}(\frac{1}{\sqrt{2}}D_4, B^*) = \widetilde{(2)}^4$ have geometric interpretations on the Fermat quartic as well by theorem 7.3.22. Since it is unlikely that the corresponding Kähler class agrees with that of the quartic interpretation of $(2)^4$, we have not depicted these models on the quartic line.
The Gepner type models $\widehat{(2)}^4$ and $\widetilde{(2)}^4$ which are meeting points of torus and $K3$ moduli spaces have been mentioned above. For all the longdash arrowed correspondences $(2)^4 \stackrel{\alpha}{\longleftrightarrow} \widehat{(2)}^4 \stackrel{\beta}{\longleftrightarrow} \widetilde{(2)}^4 \stackrel{\gamma}{\longleftrightarrow} (2)^4$ in figure 7.4.1 we explicitly know the symmetries to be modded out from the Gepner (type) model as well as the corresponding algebraic automorphisms on the geometric interpretations. For instance, with respect to the appropriate geometric interpretations $\widetilde{(2)}^4 \stackrel{r_{12}}{\longrightarrow} \widehat{(2)}^4 \stackrel{r_{12}}{\longrightarrow} (2)^4$, where r_{12} transforms a $\mathbb{Z}_2$ into a $\mathbb{Z}_4$ orbifold by theorem 7.3.13. Hence for these examples we know precisely how to continue geometric symmetries to the quantum level.
Figure 7.4.1 does not show the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ orbifold strata of $\mathcal{M}^{K3}$, though in section 7.3.4 their embedding was determined. We have not studied their relative position to the strata discussed above.

Chapter 8

Conclusions and Outlook

Let us summarize what has been achieved in this work:

We have studied **rational conformal field theories** and have proven that a toroidal conformal field theory with central charge $c = d \in \mathbb{N}$ is rational iff it possesses a geometric interpretation $\mathcal{T}(\Lambda, B)$ such that $B \in Skew(d) \cap \mathrm{Mat}(d, \mathbb{Q})$ and $T^d = \mathbb{R}^d/\Lambda$ has a finite cover which is the product of $\frac{d}{2}$ rational CM tori. In particular, rational conformal field theories are dense in the moduli space of toroidal conformal field theories with $c = d$.
Moreover, if $T^4 = \mathbb{R}^4/\Lambda$ is a singular torus with Kähler form $\omega = \delta\omega_0$ and $\omega_0 \in H^2(T^4, \mathbb{Z}) \otimes \mathbb{Q}, \delta^2 \in \mathbb{Q}$, then there is an $\varepsilon \in \mathbb{R}$ with $\varepsilon^4 \in \mathbb{Q}$, such that $\mathcal{T}(\varepsilon V \Lambda, B)$ is a rational superconformal field theory for all $V \in \mathbb{R}$, $V^2 \in \mathbb{Q}$, and all $B \in Skew(4) \cap \mathrm{Mat}(4, \mathbb{Q})$.
Our results show that in general, rationality of a conformal field theory is a much coarser condition than that of the corresponding torus to be singular.

We have given a detailed account on **orbifold conformal field theories**. In particular, we have presented a geometric interpretation of the $\mathbb{Z}_M$ orbifold procedure for theories with central charge $c = 3$ in the context of singularity theory. We have explicitly constructed the one loop partition functions for $\mathbb{Z}_M$ orbifolds of toroidal conformal field theories in arbitrary dimensions, for orbifolds involving the fermion number operator, and* for all crystallographic orbifolds with central charge $c = 2$. For the latter, some unexpected effects of the B–field have occurred which might lead to a better understanding of its properties, also for higher dimensional cases. The generalized GSO construction was used to achieve first results on a classification of unitary conformal field theories with $c = 3$ and to give a new interpretation for tensor products of minimal models with $c = 3$ as ordinary $\mathbb{Z}_M$ orbifold conformal field theories of toroidal models. A construction of Gepner (type) models was worked out that does not make use of the orbifold procedure, and its equivalence to the Gepner construction was shown.

For **unitary conformal field theories with** $c = 2$, we have given a complete description of those nonisolated parts of the moduli space $\mathcal{M}^2$ of unitary conformal

*joint work with Sayipjamal Dulat

field theories with $c = 2$ that can be constructed by an orbifold procedure from toroidal theories and are nonexceptional. All the nonexceptional cases are obtained as orbifolds with geometric interpretation by crystallographic symmetries. We have determined all multicritical points* and lines, and have proven multicriticality on the level of the operator algebra for all of them.
By a study of tensor products of theories with $c < 2$, we have related our results to those on $c = 3/2$ superconformal field theories [DGH88], and showed agreement as long as their bosonic subtheories are concerned only. In particular, this gives geometric interpretations to all nonisolated orbifolds discussed in [DGH88] in terms of crystallographic orbifolds. We have also corrected the statements in [DGH88] on multicritical points of the moduli space of superconformal theories, and showed that our corrections are in full agreement with our picture of $\mathcal{M}^2$.

For **the moduli space $\mathcal{M}$ of $N = (4,4)$ superconformal field theories with** $c = 6$, after a slight emendation of its global description†, we have found an expression for generic parts of the partition function for theories in $\mathcal{M}$. We have given a detailed description of algebraic automorphisms on $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold limits of $K3$ and have generalized† Nikulin's method for the determination of the Kummer lattice for $\mathbb{Z}_2$ orbifold limits to arbitrary $\mathbb{Z}_M$ orbifold limits of $K3$, $M \in \{3,4,6\}$. The respective lattices have been calculated explicitly. These results were used to find the locations of orbifold conformal field theories‡ in the moduli space $\mathcal{M}$. In particular, the correct values of the B–field in direction of the exceptional divisors gained from the orbifold procedure could be determined. With these results, we have shown† that for $\mathbb{Z}_2$ orbifold conformal field theories the Fourier–Mukai transform is conjugate to a classical symmetry by the image of torus T–duality under $\mathbb{Z}_2$ orbifolding. This proves T–duality for the $\mathbb{Z}_2$ orbifolds, and we can use it to derive the form of $\mathcal{M}$ purely within conformal field theory. We have explicitly found three orbifold models that admit a geometric interpretation on the Fermat quartic hypersurface. For the Gepner model $(2)^4$ and some of its orbifolds we have determined the locations in $\mathcal{M}$ and proved isomorphisms to nonlinear σ models. In particular, we have proven that the Gepner model $(2)^4$ as well as one of its orbifolds have a geometric interpretation with Fermat quartic target space. We have also found a meeting point of the moduli spaces of $\mathbb{Z}_2$ and $\mathbb{Z}_4$ orbifold conformal field theories different from the one conjectured in [EOTY89].

Our work leaves many open questions and interesting problems to study. Firstly, of course, there is the classification of all $N = (2,2)$ superconformal field theories with central charge $c = 3$, which seems not far out of reach, taken all the results presented in the text. Namely, a first step for a classification follows from theorems 5.3.3 and 5.3.4. The difficult part is to prove that the assumptions made there are true in general for $N = (2,2)$ superconformal field theories with central charge $c = 3$. It then remains to be shown that all possible orbifold conformal field theories with $c = 3$ have been determined in sections 5.2 and 5.4, compare to section 6.1 for ideas of proof. It is not hard to find all intersection points of the

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†joint work with Werner Nahm
‡for $M = 2,4$, joint work with Werner Nahm

components of the moduli space obtained that way in analogy to [DGH88] and our discussion in section 6.2.
For conformal field theories with central charge $c = 2$ a complete classification would be desirable. Apart from the exceptional components of the moduli space, which have not been studied in this work, isolated points deserve further investigation, too. In both cases, asymmetric orbifolds might play a rôle. Moreover, it is unclear whether other components of the moduli space exist that are not given by orbifold conformal field theories of torus models.
The most interesting questions come about in the context of conformal field theories on $K3$, though. Concerning rational theories, it would be favorable to translate the results on toroidal theories to the $K3$ case. In particular, one can suspect that rational conformal field theories are dense within the entire moduli space $\mathcal{M}$ of superconformal field theories. Note that complex structures corresponding to singular $K3$ surfaces are dense in the moduli space of complex structures on $K3$ only with respect to a non–Hausdorff topology. On the other hand, we have stressed that the notion of rationality already for toroidal theories is a coarser one than that of the corresponding torus being singular. The rôle of integral cohomology in conformal field theory has not been clarified yet and poses an important, perhaps related problem. At best the situation could turn out to be comparable to the geometric one on $K3$, where density of singular $K3$ surfaces within the moduli space of complex surfaces is a basic ingredient for the proof of the global Torelli theorem.
Our results on the generic part of the partition functions for theories associated to $K3$ are footed on a conjecture concerning Mordell's function. Here we have left an interesting open problem for number theorists. The determination of the generic fields that correspond to the states counted by our generic partition function is an intriguing open question. Do they generate a closed algebra, maybe a non–holomorphic generalization of a W–algebra? It should be possible to determine what kind of deformations of the conformal field theory these fields correspond to. This would also help to understand the correct translation of geometric data to conformal field theory and vice versa.
Our discussion of properties of $N = (4, 4)$ characters may also be driven on to get new insight in the representation theory of the $N = 4$ Supervirasoro algebra. In particular, further study of Mordell's function seems very promising, e.g. its relation to more tractable functions from number theory, as was started in section 3.2.2.
The discussion of orbifold conformal field theories on $K3$ presented in this work is not complete. Are there other orbifold constructions of $K3$ involving non Abelian groups? How do the corresponding conformal field theories look like? What is the rôle of discrete torsion in these cases?
Of course, to know the location of orbifold conformal field theories in the moduli space of theories associated to $K3$ can only be the starting point for a complete understanding of this space. It seems possible to get further information about other subspaces of the moduli space that are determined by a high amount of symmetry. Firstly, one can try to find the locations of all Gepner models within the

moduli space. Steps in this direction have already been taken in [EOTY89], where orbifold descriptions of all Gepner models with at least four minimal model factors have been conjectured. The analysis could be driven on along the lines we presented for $(2)^4$, but again, non Abelian orbifold constructions might be necessary. Another good candidate is the subvariety of theories that admit a geometric interpretation on the Fermat quartic hypersurface. Three points on this subvariety have been determined in the present work, but more general statements should be in reach. In particular, we suspect that the subvariety of real dimension four, where the geometric interpretation (Σ_Q, V_Q, B_Q) on the quartic has the pull back of the class of the Fubini–Study metric as Kähler class and $B_Q \in \Sigma_Q$, contains a conifold point that allows for a description on a degenerate Kummer surface. If so, further study might reveal interesting features of the conifold singularity in moduli space.

Another promising approach to the latter problem is to make use of Borcherds' automorphic forms. Namely, Borcherds has constructed and studied an automorphic function on $\mathcal{M}^{K3}$ which attains its singularities exactly in those points where the moduli space is expected to have conifold singularities [Bor98]. This function is also likely to be useful for the description of global properties of the moduli space, and we speculate that it can be interpreted in terms of the Weil–Petersson metric on $\mathcal{M}^{K3}$ [JT96, FS90].

Further interesting questions arise from our discussion of T–duality, Fourier–Mukai, and Nahm transform. How does the map explicitly act on the moduli space of Einstein metrics on $K3$? What is the relation to the MacKay correspondence [BKR99] and to mirror symmetry?

We could now start to raise questions from more fashionable parts of today's Physics in the context of heterotic–type IIA dualities, M–theory, D–branes or boundary conformal field theories, where our explicit results might at least give good toy models to play with. We have mentioned the close relation in the context of T–duality in section 7.3.3, see also [OP00a, OP00b, KOP00]. But going into details here we could never end this thesis, so we choose to resign at this point.

Appendix A

Theta functions

A1 Definition and first properties

In this section we list the most important definitions and properties of theta functions that are used within the text.

Definition A1.1
A function $T : \mathbb{C} \to \mathbb{C}$ is called THETA FUNCTION WITH PERIOD σ AND CHARACTERISTIC $(a_1, b_1; a_2, b_2)$, *if*

$$T(z+1) = e^{a_1 z + b_1} T(z), \quad T(z+\sigma) = e^{a_2 z + b_2} T(z).$$

A theta function T is called TRIVIAL, *if the corresponding divisor (T) is the null divisor. The* DEGREE *of a theta function is given by $\frac{1}{2\pi i}(a_1\sigma - a_2)$.*

If T is a theta function with period σ, then $\left(\frac{T'}{T}\right)'$ is an elliptic function with the same period. The residual theorem for elliptic functions shows that the degree of T is given by

$$\frac{1}{2\pi i}(a_1\sigma - a_2) = \sum_{z \in \mathbb{C}/\mathbb{Z}\langle 1, \sigma\rangle} \operatorname{ord}_z(T) = \deg((T)).$$

The degree therefore always is an integer. Moreover, the trivial theta functions are exactly those with degree 0. Characteristics add up if we multiply theta functions, and every theta function can be multiplied by an appropriate trivial theta function such that for the characteristic of the result $a_1 = b_1 = 0$. Modulo multiplication by trivial theta functions we can therefore restrict ourselves to the discussion of the spaces $\mathcal{T}_n(e^b)$ of theta functions of degree n and characteristic $(0, 0; -2\pi i\, n, b)$. Let T denote a representative of an element of $\mathcal{T}_n(e^b)$. Then by definition T is an entire function with exactly n zeros. Therefore $\dim \mathcal{T}_n(e^b) = n$, and $T, \widetilde{T}$ with $[T], [\widetilde{T}] \in \mathcal{T}_n(e^b)$ agree iff they have the same zeros and the same normalization.
The following theta functions of degree 1 and period σ are particularly important,

where generally $q = exp(2\pi i\sigma)$ and $y = exp(2\pi iz)$:

theta function	e^b	zero
$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} y^n$	-1	0
$\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)} y^n$	1	$\frac{1}{2}$
$\sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n$	$q^{-\frac{1}{2}}$	$\frac{1+\sigma}{2}$
$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n$	$-q^{-\frac{1}{2}}$	$\frac{\sigma}{2}$

In general the following slightly different functions are used (note that ϑ_1 and ϑ_2 are no theta functions in the above sense). The product formulas are derived via Poisson resummation:

$$
\begin{aligned}
\vartheta_1(\sigma, z) &= -\vartheta_{11}(\sigma, z) \\
&:= i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} \\
&= i q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-1}y)(1-q^n y^{-1}) \\
\vartheta_2(\sigma, z) &= \vartheta_{10}(\sigma, z) \\
&:= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} \\
&= q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-1}y)(1+q^n y^{-1}) \\
\vartheta_3(\sigma, z) &= \vartheta_{00}(\sigma, z) \\
&:= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n \\
&= \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}} y^{-1}) \\
\vartheta_4(\sigma, z) &= \vartheta_{01}(\sigma, z) \\
&:= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n \\
&= \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}}y)(1-q^{n-\frac{1}{2}} y^{-1})
\end{aligned}
\tag{A1.1}
$$

We always use the shorthand $\vartheta_i(\sigma) := \vartheta_i(\sigma, 0)$. The following transformation laws and special values are obtained directly from the definition or by Poisson resummation:

Operation	$\vartheta_1(\sigma,z)$	$\vartheta_2(\sigma,z)$	$\vartheta_3(\sigma,z)$	$\vartheta_4(\sigma,z)$
$\sigma\mapsto\sigma+1$	$e^{\frac{\pi i}{4}}\vartheta_1(\sigma,z)$	$e^{\frac{\pi i}{4}}\vartheta_2(\sigma,z)$	$\vartheta_4(\sigma,z)$	$\vartheta_3(\sigma,z)$
$\sigma\mapsto-\frac{1}{\sigma}$, $z\mapsto\frac{z}{\sigma}$	$(-i)(-i\sigma)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\sigma}}\cdot$ $\cdot\vartheta_1(\sigma,z)$	$(-i\sigma)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\sigma}}\cdot$ $\cdot\vartheta_4(\sigma,z)$	$(-i\sigma)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\sigma}}\cdot$ $\cdot\vartheta_3(\sigma,z)$	$(-i\sigma)^{\frac{1}{2}}e^{\frac{\pi i z^2}{\sigma}}\cdot$ $\cdot\vartheta_2(\sigma,z)$
$z\mapsto z+1$	$-\vartheta_1(\sigma,z)$	$-\vartheta_2(\sigma,z)$	$\vartheta_3(\sigma,z)$	$\vartheta_4(\sigma,z)$
$z\mapsto z+\sigma$	$-q^{-\frac{1}{2}}y^{-1}\cdot$ $\cdot\vartheta_1(\sigma,z)$	$q^{-\frac{1}{2}}y^{-1}\cdot$ $\cdot\vartheta_2(\sigma,z)$	$q^{-\frac{1}{2}}y^{-1}\cdot$ $\cdot\vartheta_3(\sigma,z)$	$-q^{-\frac{1}{2}}y^{-1}\cdot$ $\cdot\vartheta_4(\sigma,z)$
$z=\frac{1}{2}$	$\vartheta_2(\sigma)$	0	$\vartheta_4(\sigma)$	$\vartheta_3(\sigma)$
$z=\frac{\sigma}{2}$	$iq^{-\frac{1}{8}}\vartheta_4(\sigma)$	$q^{-\frac{1}{8}}\vartheta_3(\sigma)$	$q^{-\frac{1}{8}}\vartheta_2(\sigma)$	0
$z=\frac{\sigma+1}{2}$	$q^{-\frac{1}{8}}\vartheta_3(\sigma)$	$-iq^{-\frac{1}{8}}\vartheta_4(\sigma)$	0	$q^{-\frac{1}{8}}\vartheta_2(\sigma)$
$z\mapsto z+\frac{1}{2}$	$\vartheta_2(\sigma,z)$	$-\vartheta_1(\sigma,z)$	$\vartheta_4(\sigma,z)$	$\vartheta_3(\sigma,z)$
$z\mapsto z+\frac{\sigma}{2}$	$iq^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_4(\sigma,z)$	$q^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_3(\sigma,z)$	$q^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_2(\sigma,z)$	$iq^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_1(\sigma,z)$
$z\mapsto z+\frac{\sigma+1}{2}$	$q^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_3(\sigma,z)$	$-iq^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_4(\sigma,z)$	$iq^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_1(\sigma,z)$	$q^{-\frac{1}{8}}y^{-\frac{1}{2}}\cdot$ $\cdot\vartheta_2(\sigma,z)$

(A1.2)

A2 Known product and doubling formulas

By using the Jacobi tripel identity one can prove the following product formulas, where $\eta(\sigma)$ is the Dedekind eta function:

$$\vartheta_2(\sigma)\vartheta_3(\sigma)\vartheta_4(\sigma) = 2\eta(\sigma)^3 \tag{A2.1}$$

$$\vartheta_2(\sigma)^4-\vartheta_3(\sigma)^4+\vartheta_4(\sigma)^4 = 0. \tag{A2.2}$$

We moreover have the doubling formulas

$$\begin{aligned}
2\vartheta_2(2\sigma)^2 &= \vartheta_3(\sigma)^2-\vartheta_4(\sigma)^2, \quad 2\vartheta_3(2\sigma)^2 = \vartheta_3(\sigma)^2+\vartheta_4(\sigma)^2\\
\vartheta_4(2\sigma)^2 &= \vartheta_3(\sigma)\vartheta_4(\sigma)\\
\vartheta_2\left(\frac{\sigma}{2}\right)^2 &= 2\vartheta_2(\sigma)\vartheta_3(\sigma)\\
\vartheta_3\left(\frac{\sigma}{2}\right)^2 &= \vartheta_3(\sigma)^2+\vartheta_2(\sigma)^2, \quad \vartheta_4\left(\frac{\sigma}{2}\right)^2 = \vartheta_3(\sigma)^2-\vartheta_2(\sigma)^2.
\end{aligned} \tag{A2.3}$$

A3 Generalizations

To prove the following generalized formulas, one first shows that both sides are theta functions of the same degree and with same characteristic. Then, by what was said in A1, it suffices to show that zeros and normalizations agree. Some of the below formulas may also be found in Tölke's collection [Töl66].
Formula (A2.2) can be generalized in the following way:

$$\begin{aligned}
\vartheta_2(\sigma)^2\vartheta_2(\sigma,z)^2 - \vartheta_3(\sigma)^2\vartheta_3(\sigma,z)^2 + \vartheta_4(\sigma)^2\vartheta_4(\sigma,z)^2 &= 0 \\
\vartheta_2(\sigma)^2\vartheta_1(\sigma,z)^2 + \vartheta_4(\sigma)^2\vartheta_3(\sigma,z)^2 - \vartheta_3(\sigma)^2\vartheta_4(\sigma,z)^2 &= 0 \\
\vartheta_3(\sigma)^2\vartheta_1(\sigma,z)^2 + \vartheta_4(\sigma)^2\vartheta_2(\sigma,z)^2 - \vartheta_2(\sigma)^2\vartheta_4(\sigma,z)^2 &= 0 \\
\vartheta_4(\sigma)^2\vartheta_1(\sigma,z)^2 + \vartheta_3(\sigma)^2\vartheta_2(\sigma,z)^2 - \vartheta_2(\sigma)^2\vartheta_3(\sigma,z)^2 &= 0.
\end{aligned} \tag{A3.1}$$

Writing $\overline{\vartheta}_i(\sigma,z) = \vartheta_i(\overline{\sigma},\overline{z})$, we have

$$\begin{aligned}
\left(\vartheta_2(\sigma)^2\,\overline{\vartheta}_4(\sigma)^2 - \vartheta_4(\sigma)^2\,\overline{\vartheta}_2(\sigma)^2\right)^2 &= \left(|\vartheta_2(\sigma)|^4 + |\vartheta_3(\sigma)|^4 + |\vartheta_4(\sigma)|^4\right)\cdot \\
&\quad \cdot\left(|\vartheta_3(\sigma)|^4 - |\vartheta_2(\sigma)|^4 - |\vartheta_4(\sigma)|^4\right) \\
\left(\vartheta_2(\sigma)^2\,\overline{\vartheta}_4(\sigma)^2 - \vartheta_4(\sigma)^2\,\overline{\vartheta}_2(\sigma)^2\right) &\cdot \left(\vartheta_2(\sigma,z)^2\,\overline{\vartheta}_4(\sigma,z)^2 + \vartheta_1(\sigma,z)^2\,\overline{\vartheta}_3(\sigma,z)^2\right. \\
&\quad \left. -\vartheta_4(\sigma,z)^2\,\overline{\vartheta}_2(\sigma,z)^2 - \vartheta_3(\sigma,z)^2\,\overline{\vartheta}_1(\sigma,z)^2\right) \\
&= \left(|\vartheta_3(\sigma)|^4 - |\vartheta_2(\sigma)|^4 - |\vartheta_4(\sigma)|^4\right)\sum_{i=1}^{4}|\vartheta_i(\sigma,z)|^4 \\
\vartheta_2(\sigma)^4\,\overline{\vartheta}_4(\sigma)^4 + \vartheta_4(\sigma)^4\,\overline{\vartheta}_2(\sigma)^4 &= |\vartheta_3(\sigma)|^8 - |\vartheta_2(\sigma)|^8 - |\vartheta_4(\sigma)|^8 \\
\vartheta_3(\sigma)^4\,\overline{\vartheta}_2(\sigma)^4 + \vartheta_2(\sigma)^4\,\overline{\vartheta}_3(\sigma)^4 &= |\vartheta_2(\sigma)|^8 + |\vartheta_3(\sigma)|^8 - |\vartheta_4(\sigma)|^8 \\
\vartheta_3(\sigma)^4\,\overline{\vartheta}_4(\sigma)^4 + \vartheta_4(\sigma)^4\,\overline{\vartheta}_3(\sigma)^4 &= |\vartheta_3(\sigma)|^8 + |\vartheta_4(\sigma)|^8 - |\vartheta_2(\sigma)|^8 .
\end{aligned} \tag{A3.2}$$

The doubling formulas (A2.3) can be generalized in various manners:
Doubling of σ:

$$\begin{aligned}
2\vartheta_1(2\sigma,z)^2 &= \vartheta_3(\sigma)\vartheta_4(\sigma,z) - \vartheta_4(\sigma)\vartheta_3(\sigma,z) \\
2\vartheta_2(2\sigma,z)^2 &= \vartheta_3(\sigma)\vartheta_3(\sigma,z) - \vartheta_4(\sigma)\vartheta_4(\sigma,z) \\
2\vartheta_3(2\sigma,z)^2 &= \vartheta_3(\sigma)\vartheta_3(\sigma,z) + \vartheta_4(\sigma)\vartheta_4(\sigma,z) \\
2\vartheta_4(2\sigma,z)^2 &= \vartheta_3(\sigma)\vartheta_4(\sigma,z) + \vartheta_4(\sigma)\vartheta_3(\sigma,z) \\
2\vartheta_1(2\sigma,z)\vartheta_4(2\sigma,z) &= \vartheta_2(\sigma)\vartheta_1(\sigma,z) \\
2\vartheta_2(2\sigma,z)\vartheta_3(2\sigma,z) &= \vartheta_2(\sigma)\vartheta_2(\sigma,z)
\end{aligned} \tag{A3.3}$$

Doubling of z:

$$\begin{aligned}
\vartheta_2(\sigma,2z)\vartheta_2(\sigma)\vartheta_3(\sigma)^2 &= \vartheta_2(\sigma,z)^2\vartheta_3(\sigma,z)^2 - \vartheta_1(\sigma,z)^2\vartheta_4(\sigma,z)^2 \\
\vartheta_4(\sigma,2z)\vartheta_4(\sigma)\vartheta_3(\sigma)^2 &= \vartheta_3(\sigma,z)^2\vartheta_4(\sigma,z)^2 + \vartheta_1(\sigma,z)^2\vartheta_2(\sigma,z)^2
\end{aligned} \tag{A3.4}$$

Doubling of σ and z:

$$\begin{aligned}
\vartheta_2(2\sigma, 2z) &= \frac{\vartheta_2(2\sigma)}{\vartheta_3(\sigma)^2}\,\vartheta_3(\sigma, z)^2 - \frac{\vartheta_3(2\sigma)}{\vartheta_3(\sigma)^2}\,\vartheta_1(\sigma, z)^2 \\
\vartheta_3(2\sigma, 2z) &= \frac{\vartheta_3(2\sigma)}{\vartheta_3(\sigma)^2}\,\vartheta_3(\sigma, z)^2 + \frac{\vartheta_2(2\sigma)}{\vartheta_3(\sigma)^2}\,\vartheta_1(\sigma, z)^2 \\
\vartheta_4(2\sigma, 2z)\vartheta_4(2\sigma) &= \vartheta_3(\sigma, z)\vartheta_4(\sigma, z)
\end{aligned} \tag{A3.5}$$

General product formulas:

$$\begin{aligned}
\vartheta_4(2\sigma, 2a)\vartheta_4(2\sigma, 2b) + \vartheta_1(2\sigma, 2a)\vartheta_1(2\sigma, 2b) &= \vartheta_4(\sigma, a+b)\vartheta_3(\sigma, a-b) \\
\vartheta_4(2\sigma, 2a)\vartheta_4(2\sigma, 2b) - \vartheta_1(2\sigma, 2a)\vartheta_1(2\sigma, 2b) &= \vartheta_3(\sigma, a+b)\vartheta_4(\sigma, a-b) \\
\vartheta_3(2\sigma, 2a)\vartheta_3(2\sigma, 2b) + \vartheta_2(2\sigma, 2a)\vartheta_2(2\sigma, 2b) &= \vartheta_3(\sigma, a+b)\vartheta_3(\sigma, a-b) \\
\vartheta_3(2\sigma, 2a)\vartheta_3(2\sigma, 2b) - \vartheta_2(2\sigma, 2a)\vartheta_2(2\sigma, 2b) &= \vartheta_4(\sigma, a+b)\vartheta_4(\sigma, a-b) \\
\vartheta_3(\sigma, a)\vartheta_3(\sigma, b) + \vartheta_4(\sigma, a)\vartheta_4(\sigma, b) &= 2\vartheta_3(2\sigma, a+b)\vartheta_3(2\sigma, a-b).
\end{aligned} \tag{A3.6}$$

Appendix B

Explicit field identifications: $\widehat{(2)}^4 = \mathcal{K}(\mathbb{Z}^4, 0)$

In this appendix, we give a complete list of $(\frac{1}{4}, \frac{1}{4})$-fields in $\widehat{(2)}^4$ (see theorem 7.3.25) together with their equivalents in the nonlinear σ model on $\mathcal{K}(\mathbb{Z}^4, 0)$. As usual, $\varepsilon, \varepsilon_i \in \{\pm 1\}$ and we use notations as in (7.3.34) and (7.3.35).

Untwisted $(\frac{1}{4}, \frac{1}{4})$-fields with respect to the $\langle[2,2,0,0]\rangle$-orbifold:

$$\begin{aligned}
\left(\Phi^0_{-\varepsilon_1,-\varepsilon_1;-\varepsilon_2,-\varepsilon_2}\right)^{\otimes 4} &= W^J_{\varepsilon_1,\varepsilon_2} \\
\left(\Phi^0_{-\varepsilon,-\varepsilon;-\varepsilon,-\varepsilon}\right)^{\otimes 2} \otimes \left(\Phi^0_{\varepsilon,\varepsilon;\varepsilon,\varepsilon}\right)^{\otimes 2} &= W^A_{\varepsilon,\varepsilon} \\
\left(\Phi^1_{2,1;2,1}\right)^{\otimes 4} &= \Sigma_{0000} - \Sigma_{1100} + \Sigma_{1111} - \Sigma_{0011} \\
\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 4} &= \Sigma_{1010} + \Sigma_{0101} - \Sigma_{0110} - \Sigma_{1001}
\end{aligned}$$

$$\begin{aligned}
&\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \\
&\qquad = \Sigma_{0000} - \Sigma_{1100} - \Sigma_{1111} + \Sigma_{0011} + \Sigma_{0010} + \Sigma_{0001} - \Sigma_{1101} - \Sigma_{1110} \\
&\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \\
&\qquad = \Sigma_{0000} - \Sigma_{1100} - \Sigma_{1111} + \Sigma_{0011} - \Sigma_{0010} - \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110} \\
&\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \\
&\qquad = \Sigma_{0000} + \Sigma_{1100} - \Sigma_{1111} - \Sigma_{0011} + \Sigma_{1000} + \Sigma_{0100} - \Sigma_{1011} - \Sigma_{0111} \\
&\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \\
&\qquad = \Sigma_{0000} + \Sigma_{1100} - \Sigma_{1111} - \Sigma_{0011} - \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} + \Sigma_{0111} \\
&\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \\
&\qquad = (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) + (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011}) \\
&\qquad \quad + (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) + (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})
\end{aligned}$$

$$\begin{aligned}
&\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \\
&\qquad = (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) + (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011}) \\
&\qquad - (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) - (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001}) \\
&\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \\
&\qquad = (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) - (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011}) \\
&\qquad - (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) + (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001}) \\
&\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \\
&\qquad = (\Sigma_{0000} + \Sigma_{1100} + \Sigma_{1111} + \Sigma_{0011}) - (\Sigma_{1000} + \Sigma_{0100} + \Sigma_{0111} + \Sigma_{1011}) \\
&\qquad + (\Sigma_{0010} + \Sigma_{0001} + \Sigma_{1101} + \Sigma_{1110}) - (\Sigma_{1010} + \Sigma_{0101} + \Sigma_{0110} + \Sigma_{1001})
\end{aligned}$$

Twisted $(\frac{1}{4}, \frac{1}{4})$-fields with respect to the $\langle[2,2,0,0]\rangle$-orbifold:

$$\begin{aligned}
\left(\Phi^0_{-\varepsilon,-\varepsilon;\varepsilon,\varepsilon}\right)^{\otimes 2} \otimes \left(\Phi^0_{\varepsilon,\varepsilon;-\varepsilon,-\varepsilon}\right)^{\otimes 2} &= W^A_{\varepsilon,-\varepsilon} \\
\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} &= \Sigma_{1000} - \Sigma_{0100} + \Sigma_{0111} - \Sigma_{1011} \\
\left(\Phi^1_{2,1;2,1}\right)^{\otimes 2} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} &= \Sigma_{0010} - \Sigma_{0001} + \Sigma_{1101} - \Sigma_{1110}
\end{aligned}$$

$$\begin{aligned}
&\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \\
&\qquad = \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} - \Sigma_{0111} + \Sigma_{1010} - \Sigma_{0101} + \Sigma_{1001} - \Sigma_{0110} \\
&\left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \otimes \Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \\
&\qquad = \Sigma_{1000} - \Sigma_{0100} + \Sigma_{1011} + \Sigma_{0111} - \Sigma_{1010} + \Sigma_{0101} - \Sigma_{1001} + \Sigma_{0110} \\
&\Phi^0_{-1,-1;-1,-1} \otimes \Phi^0_{1,1;1,1} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \\
&\qquad = \Sigma_{0010} - \Sigma_{0001} - \Sigma_{1101} + \Sigma_{1110} + \Sigma_{1010} - \Sigma_{0101} - \Sigma_{1001} + \Sigma_{0110} \\
&\Phi^0_{1,1;1,1} \otimes \Phi^0_{-1,-1;-1,-1} \otimes \left(\Phi^1_{2,1;-2,-1}\right)^{\otimes 2} \\
&\qquad = \Sigma_{0010} - \Sigma_{0001} - \Sigma_{1101} + \Sigma_{1110} - \Sigma_{1010} + \Sigma_{0101} + \Sigma_{1001} - \Sigma_{0110}
\end{aligned}$$

Appendix C

Partition functions and vacuum characters of the Gepner models with central charge $c = 6$

In this appendix, we list our numerical results for vacuum characters and partition functions of the Gepner models with $c = 6$. The code was written in C^{++}; the classes for symbolical calculations of power series and their print out were written by F. Rohsiepe.
It is a simple combinatorial task to determine all Gepner models (here we always use the A–invariant) with central charge $c = 6$. We have checked that for all these models the elliptic genus (section 3.1.2) allows to assign them either to the four–torus or to $K3$. For the $K3$ cases, we have used the flow invariant orbit technique described in section 3.1.3 to determine the vacuum characters.

```
Vacuum character of (1)(1)(1)(1)(1)(1):
1 + (1 y^(-2) + 6 + 1 y^2) q + (6 y^(-2) + 57 + 6 y^2) q^2 + (57 y^(-2) + 308 + 57 y^2) q^3
+ (1 y^(-4) + 308 y^(-2) + 1305 + 308 y^2 + 1 y^4) q^4 + (6 y^(-4) + 1305 y^(-2) + 4800 + 1305 y^2 + 6 y^4) q^5
+ (57 y^(-4) + 4800 y^(-2) + 15764 + 4800 y^2 + 57 y^4) q^6
+ (308 y^(-4) + 15764 y^(-2) + 47466 + 15764 y^2 + 308 y^4) q^7
+ (1305 y^(-4) + 47466 y^(-2) + 133461 + 47466 y^2 + 1305 y^4) q^8

Vacuum character of (1)(1)(1)(1)(4):
1 + (1 y^(-2) + 6 + 1 y^2) q + (6 y^(-2) + 57 + 6 y^2) q^2 + (57 y^(-2) + 308 + 57 y^2) q^3
+ (1 y^(-4) + 308 y^(-2) + 1305 + 308 y^2 + 1 y^4) q^4 + (6 y^(-4) + 1305 y^(-2) + 4800 + 1305 y^2 + 6 y^4) q^5
+ (57 y^(-4) + 4800 y^(-2) + 15764 + 4800 y^2 + 57 y^4) q^6
+ (308 y^(-4) + 15764 y^(-2) + 47466 + 15764 y^2 + 308 y^4) q^7
+ (1305 y^(-4) + 47466 y^(-2) + 133461 + 47466 y^2 + 1305 y^4) q^8

Vacuum character of (2)(2)(2)(2):
1 + (1 y^(-2) + 4 + 1 y^2) q + (4 y^(-2) + 31 + 4 y^2) q^2 + (31 y^(-2) + 172 + 31 y^2) q^3
+ (1 y^(-4) + 172 y^(-2) + 737 + 172 y^2 + 1 y^4) q^4 + (4 y^(-4) + 737 y^(-2) + 2700 + 737 y^2 + 4 y^4) q^5
+ (31 y^(-4) + 2700 y^(-2) + 8862 + 2700 y^2 + 31 y^4) q^6
+ (172 y^(-4) + 8862 y^(-2) + 26704 + 8862 y^2 + 172 y^4) q^7
+ (737 y^(-4) + 26704 y^(-2) + 75075 + 26704 y^2 + 737 y^4) q^8

Vacuum character of (1)(2)(2)(4):
1 + (1 y^(-2) + 9 + 1 y^2) q + (9 y^(-2) + 73 + 9 y^2) q^2 + (73 y^(-2) + 398 + 73 y^2) q^3
+ (1 y^(-4) + 398 y^(-2) + 1700 + 398 y^2 + 1 y^4) q^4 + (9 y^(-4) + 1700 y^(-2) + 6234 + 1700 y^2 + 9 y^4) q^5
+ (73 y^(-4) + 6234 y^(-2) + 20471 + 6234 y^2 + 73 y^4) q^6
+ (398 y^(-4) + 20471 y^(-2) + 61671 + 20471 y^2 + 398 y^4) q^7
+ (1700 y^(-4) + 61671 y^(-2) + 173370 + 61671 y^2 + 1700 y^4) q^8

Vacuum character of (1)(1)(4)(4):
1 + (1 y^(-2) + 4 + 1 y^2) q + (4 y^(-2) + 31 + 4 y^2) q^2 + (31 y^(-2) + 160 + 31 y^2) q^3
+ (1 y^(-4) + 160 y^(-2) + 665 + 160 y^2 + 1 y^4) q^4 + (4 y^(-4) + 665 y^(-2) + 2424 + 665 y^2 + 4 y^4) q^5
+ (31 y^(-4) + 2424 y^(-2) + 7926 + 2424 y^2 + 31 y^4) q^6
+ (160 y^(-4) + 7926 y^(-2) + 23812 + 7926 y^2 + 160 y^4) q^7
+ (665 y^(-4) + 23812 y^(-2) + 66867 + 23812 y^2 + 665 y^4) q^8

Vacuum character of (1)(1)(2)(10):
1 + (1 y^(-2) + 4 + 1 y^2) q + (4 y^(-2) + 32 + 4 y^2) q^2 + (32 y^(-2) + 172 + 32 y^2) q^3
+ (1 y^(-4) + 172 y^(-2) + 724 + 172 y^2 + 1 y^4) q^4 + (4 y^(-4) + 724 y^(-2) + 2646 + 724 y^2 + 4 y^4) q^5
+ (32 y^(-4) + 2646 y^(-2) + 8676 + 2646 y^2 + 32 y^4) q^6
+ (172 y^(-4) + 8676 y^(-2) + 26104 + 8676 y^2 + 172 y^4) q^7
```

```
+ (724 y^(-4) + 26104 y^(-2) + 73341 + 26104 y^2 + 724 y^4) q^8

Vacuum character of (4)(4)(4):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 18 + 3 y^2) q^2 + (18 y^(-2) + 86 + 18 y^2) q^3
+ (1 y^(-4) + 86 y^(-2) + 345 + 86 y^2 + 1 y^4) q^4 + (3 y^(-4) + 345 y^(-2) + 1236 + 345 y^2 + 3 y^4) q^5
+ (18 y^(-4) + 1236 y^(-2) + 4007 + 1236 y^2 + 18 y^4) q^6
+ (86 y^(-4) + 4007 y^(-2) + 11985 + 4007 y^2 + 86 y^4) q^7
+ (345 y^(-4) + 11985 y^(-2) + 33570 + 11985 y^2 + 345 y^4) q^8

Vacuum character of (3)(3)(8):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 19 + 3 y^2) q^2 + (19 y^(-2) + 97 + 19 y^2) q^3
+ (1 y^(-4) + 97 y^(-2) + 406 + 97 y^2 + 1 y^4) q^4 + (3 y^(-4) + 406 y^(-2) + 1484 + 406 y^2 + 3 y^4) q^5
+ (19 y^(-4) + 1484 y^(-2) + 4859 + 1484 y^2 + 19 y^4) q^6
+ (97 y^(-4) + 4859 y^(-2) + 14613 + 4859 y^2 + 97 y^4) q^7
+ (406 y^(-4) + 14613 y^(-2) + 41062 + 14613 y^2 + 406 y^4) q^8

Vacuum character of (2)(6)(6):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 18 + 3 y^2) q^2 + (18 y^(-2) + 87 + 18 y^2) q^3
+ (1 y^(-4) + 87 y^(-2) + 345 + 87 y^2 + 1 y^4) q^4 + (3 y^(-4) + 345 y^(-2) + 1213 + 345 y^2 + 3 y^4) q^5
+ (18 y^(-4) + 1213 y^(-2) + 3880 + 1213 y^2 + 18 y^4) q^6
+ (87 y^(-4) + 3880 y^(-2) + 11496 + 3880 y^2 + 87 y^4) q^7
+ (345 y^(-4) + 11496 y^(-2) + 31969 + 11496 y^2 + 345 y^4) q^8

Vacuum character of (2)(4)(10):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 18 + 3 y^2) q^2 + (18 y^(-2) + 91 + 18 y^2) q^3
+ (1 y^(-4) + 91 y^(-2) + 373 + 91 y^2 + 1 y^4) q^4 + (3 y^(-4) + 373 y^(-2) + 1344 + 373 y^2 + 3 y^4) q^5
+ (18 y^(-4) + 1344 y^(-2) + 4376 + 1344 y^2 + 18 y^4) q^6
+ (91 y^(-4) + 4376 y^(-2) + 13120 + 4376 y^2 + 91 y^4) q^7
+ (373 y^(-4) + 13120 y^(-2) + 36789 + 13120 y^2 + 373 y^4) q^8

Vacuum character of (2)(3)(18):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 18 + 3 y^2) q^2 + (18 y^(-2) + 88 + 18 y^2) q^3
+ (1 y^(-4) + 88 y^(-2) + 351 + 88 y^2 + 1 y^4) q^4 + (3 y^(-4) + 351 y^(-2) + 1240 + 351 y^2 + 3 y^4) q^5
+ (18 y^(-4) + 1240 y^(-2) + 3967 + 1240 y^2 + 18 y^4) q^6
+ (88 y^(-4) + 3967 y^(-2) + 11736 + 3967 y^2 + 88 y^4) q^7
+ (351 y^(-4) + 11736 y^(-2) + 32589 + 11736 y^2 + 351 y^4) q^8

Vacuum character of (1)(10)(10):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 17 + 3 y^2) q^2 + (17 y^(-2) + 72 + 17 y^2) q^3
+ (1 y^(-4) + 72 y^(-2) + 258 + 72 y^2 + 1 y^4) q^4 + (3 y^(-4) + 258 y^(-2) + 846 + 258 y^2 + 3 y^4) q^5
+ (17 y^(-4) + 846 y^(-2) + 2555 + 846 y^2 + 17 y^4) q^6
+ (72 y^(-4) + 2555 y^(-2) + 7223 + 2555 y^2 + 72 y^4) q^7
+ (258 y^(-4) + 7223 y^(-2) + 19350 + 7223 y^2 + 258 y^4) q^8

Vacuum character of (1)(8)(13):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 18 + 3 y^2) q^2 + (18 y^(-2) + 90 + 18 y^2) q^3
+ (1 y^(-4) + 90 y^(-2) + 366 + 90 y^2 + 1 y^4) q^4 + (3 y^(-4) + 366 y^(-2) + 1318 + 366 y^2 + 3 y^4) q^5
+ (18 y^(-4) + 1318 y^(-2) + 4286 + 1318 y^2 + 18 y^4) q^6
+ (90 y^(-4) + 4286 y^(-2) + 12826 + 4286 y^2 + 90 y^4) q^7
+ (366 y^(-4) + 12826 y^(-2) + 35924 + 12826 y^2 + 366 y^4) q^8

Vacuum character of (1)(7)(16):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 17 + 3 y^2) q^2 + (17 y^(-2) + 72 + 17 y^2) q^3
+ (1 y^(-4) + 72 y^(-2) + 259 + 72 y^2 + 1 y^4) q^4 + (3 y^(-4) + 259 y^(-2) + 859 + 259 y^2 + 3 y^4) q^5
+ (17 y^(-4) + 859 y^(-2) + 2628 + 859 y^2 + 17 y^4) q^6
+ (72 y^(-4) + 2628 y^(-2) + 7528 + 2628 y^2 + 72 y^4) q^7
+ (259 y^(-4) + 7528 y^(-2) + 20436 + 7528 y^2 + 259 y^4) q^8

Vacuum character of (1)(6)(22):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 17 + 3 y^2) q^2 + (17 y^(-2) + 72 + 17 y^2) q^3
+ (1 y^(-4) + 72 y^(-2) + 258 + 72 y^2 + 1 y^4) q^4 + (3 y^(-4) + 258 y^(-2) + 851 + 258 y^2 + 3 y^4) q^5
+ (17 y^(-4) + 851 y^(-2) + 2590 + 851 y^2 + 17 y^4) q^6
+ (72 y^(-4) + 2590 y^(-2) + 7389 + 2590 y^2 + 72 y^4) q^7
+ (258 y^(-4) + 7389 y^(-2) + 19979 + 7389 y^2 + 258 y^4) q^8

Vacuum character of (1)(5)(40):
1 + (1 y^(-2) + 3 + 1 y^2) q + (3 y^(-2) + 17 + 3 y^2) q^2 + (17 y^(-2) + 72 + 17 y^2) q^3
+ (1 y^(-4) + 72 y^(-2) + 258 + 72 y^2 + 1 y^4) q^4 + (3 y^(-4) + 258 y^(-2) + 847 + 258 y^2 + 3 y^4) q^5
+ (17 y^(-4) + 847 y^(-2) + 2561 + 847 y^2 + 17 y^4) q^6
+ (72 y^(-4) + 2561 y^(-2) + 7234 + 2561 y^2 + 72 y^4) q^7
+ (258 y^(-4) + 7234 y^(-2) + 19329 + 7234 y^2 + 258 y^4) q^8
```

The partition functions of the Gepner models were obtained by the usual Gepner construction; in the print outs below, $r := \overline{q}$ and $v := \overline{y}$.

```
Partition function for (1)(1)(1)(1)(1)(1):

1
+  [ (1 v^(-2) + 6 + 1 v^2) ] r
+  [ (6 v^(-2) + 57 + 6 v^2) ] r^2
+  [ (57 v^(-2) + 308 + 57 v^2) ] r^3

+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (6 v^(-1) + 6 v) y^(-1) + (20 v^(-2) + 240 + 20 v^2) + (6 v^(-1) + 6 v) y ] r^(5/4)
+  [ (1 v^(-3) + 57 v^(-1) + 57 v + 1 v^3) y^(-1) + (240 v^(-2) + 1800 + 240 v^2)
+ (1 v^(-3) + 57 v^(-1) + 57 v + 1 v^3) y ] r^(9/4)
 ] q^(1/4)
+  [
 [ 30 ] r^(1/3)
+  [ (30 v^(-2) + 450 + 30 v^2) ] r^(4/3)
+  [ (450 v^(-2) + 3090 + 450 v^2) ] r^(7/3)
 ] q^(1/3)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(3/2)
+  [ (20 v^(-3) + 1800 v^(-1) + 1800 v + 20 v^3) y^(-1) + (20 v^(-3) + 1800 v^(-1) + 1800 v + 20 v^3) y ] r^(5/2)
 ] q^(1/2)
+  [
 [ (30 v^(-1) + 30 v) y^(-1) + (120) + (30 v^(-1) + 30 v) y ] r^(7/12)
+  [ (450 v^(-1) + 450 v) y^(-1) + (120 v^(-2) + 1560 + 120 v^2) + (450 v^(-1) + 450 v) y ] r^(19/12)
+  [ (30 v^(-3) + 3090 v^(-1) + 3090 v + 30 v^3) y^(-1) + (1560 v^(-2) + 9360 + 1560 v^2)
+ (30 v^(-3) + 3090 v^(-1) + 3090 v + 30 v^3) y ] r^(31/12)
 ] q^(7/12)
+  [
 [ (270) ] r^(2/3)
+  [ (270 v^(-2) + 2700 + 270 v^2) ] r^(5/3)
+  [ (2700 v^(-2) + 16200 + 2700 v^2) ] r^(8/3)
 ] q^(2/3)
+  [
 [ (120 v^(-1) + 120 v) y^(-1) + (120 v^(-1) + 120 v) y ] r^(5/6)
+  [ (1560 v^(-1) + 1560 v) y^(-1) + (1560 v^(-1) + 1560 v) y ] r^(11/6)
+  [ (120 v^(-3) + 9360 v^(-1) + 9360 v + 120 v^3) y^(-1) + (120 v^(-3) + 9360 v^(-1) + 9360 v + 120 v^3) y ] r^(17/6)
 ] q^(5/6)
+  [
 [ (270 v^(-1) + 270 v) y^(-1) + (1080) + (270 v^(-1) + 270 v) y ] r^(11/12)
+  [ (2700 v^(-1) + 2700 v) y^(-1) + (1080 v^(-2) + 8640 + 1080 v^2) + (2700 v^(-1) + 2700 v) y ] r^(23/12)
+  [ (270 v^(-3) + 16200 v^(-1) + 16200 v + 270 v^3) y^(-1) + (8640 v^(-2) + 48600 + 8640 v^2)

+ (270 v^(-3) + 16200 v^(-1) + 16200 v + 270 v^3) y ] r^(35/12)
 ] q^(11/12)
+  [
1 y^(-2) + 6 + 1 y^2
+  [ (1 v^(-2) + 6 + 1 v^2) y^(-2) + (6 v^(-2) + 1016 + 6 v^2) + (1 v^(-2) + 6 + 1 v^2) y^2 ] r
+  [ (6 v^(-2) + 57 + 6 v^2) y^(-2) + (1016 v^(-2) + 8322 + 1016 v^2) + (6 v^(-2) + 57 + 6 v^2) y^2 ] r^2
+  [ (57 v^(-2) + 308 + 57 v^2) y^(-2) + (8322 v^(-2) + 44688 + 8322 v^2) + (57 v^(-2) + 308 + 57 v^2) y^2 ] r^3
 ] q
+  [
 [ (1080 v^(-1) + 1080 v) y^(-1) + (1080 v^(-1) + 1080 v) y ] r^(7/6)
+  [ (8640 v^(-1) + 8640 v) y^(-1) + (8640 v^(-1) + 8640 v) y ] r^(13/6)
 ] q^(7/6)
+  [
 [ (20) y^(-2) + (6 v^(-1) + 6 v) y^(-1) + (240) + (6 v^(-1) + 6 v) y + (20) y^2 ] r^(1/4)
+  [ (20 v^(-2) + 240 + 20 v^2) y^(-2) + (1016 v^(-1) + 1016 v) y^(-1) + (240 v^(-2) + 3024 + 240 v^2)
+ (1016 v^(-1) + 1016 v) y + (20 v^(-2) + 240 + 20 v^2) y^2 ] r^(5/4)
+  [ (240 v^(-2) + 1800 + 240 v^2) y^(-2) + (6 v^(-3) + 8322 v^(-1) + 8322 v + 6 v^3) y^(-1)
+ (3024 v^(-2) + 22704 + 3024 v^2) + (6 v^(-3) + 8322 v^(-1) + 8322 v + 6 v^3) y + (240 v^(-2) + 1800 + 240 v^2) y^2 ] r^(9/4)
 ] q^(5/4)
+  [
 [ (30) y^(-2) + (450) + (30) y^2 ] r^(1/3)
+  [ (30 v^(-2) + 450 + 30 v^2) y^(-2) + (450 v^(-2) + 6750 + 450 v^2) + (30 v^(-2) + 450 + 30 v^2) y^2 ] r^(4/3)
+  [ (450 v^(-2) + 3090 + 450 v^2) y^(-2) + (6750 v^(-2) + 46350 + 6750 v^2) + (450 v^(-2) + 3090 + 450 v^2) y^2 ] r^(7/3)
 ] q^(4/3)
+  [
 [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(1/2)
+  [ (3024 v^(-1) + 3024 v) y^(-1) + (3024 v^(-1) + 3024 v) y ] r^(3/2)
+  [ (240 v^(-3) + 22704 v^(-1) + 22704 v + 240 v^3) y^(-1) + (240 v^(-3) + 22704 v^(-1) + 22704 v + 240 v^3) y ] r^(5/2)
 ] q^(3/2)
+  [
 [ (120) y^(-2) + (450 v^(-1) + 450 v) y^(-1) + (1560) + (450 v^(-1) + 450 v) y + (120) y^2 ] r^(7/12)
+  [ (120 v^(-2) + 1560 + 120 v^2) y^(-2)
+ (6750 v^(-1) + 6750 v) y^(-1) + (1560 v^(-2) + 20280 + 1560 v^2) + (6750 v^(-1) + 6750 v) y
+ (120 v^(-2) + 1560 + 120 v^2) y^2 ] r^(19/12)
+  [ (1560 v^(-2) + 9360 + 1560 v^2) y^(-2) + (450 v^(-3) + 46350 v^(-1) + 46350 v + 450 v^3) y^(-1)
+ (20280 v^(-2) + 121680 + 20280 v^2) + (450 v^(-3) + 46350 v^(-1) + 46350 v + 450 v^3) y
+ (1560 v^(-2) + 9360 + 1560 v^2) y^2 ] r^(31/12)
 ] q^(19/12)
+  [
 [ (270) y^(-2) + (2700) + (270) y^2 ] r^(2/3)
+  [ (270 v^(-2) + 2700 + 270 v^2) y^(-2) + (2700 v^(-2) + 27000 + 2700 v^2) + (270 v^(-2) + 2700 + 270 v^2) y^2 ] r^(5/3)
+  [ (2700 v^(-2) + 16200 + 2700 v^2) y^(-2) + (27000 v^(-2) + 162000 + 27000 v^2)
+ (2700 v^(-2) + 16200 + 2700 v^2) y^2 ] r^(8/3)
 ] q^(5/3)
```

```
+  [
 [ (1560 v^(-1) + 1560 v) y^(-1) + (1560 v^(-1) + 1560 v) y ] r^(5/6)
+  [ (20280 v^(-1) + 20280 v) y^(-1) + (20280 v^(-1) + 20280 v) y ] r^(11/6)
+  [ (1560 v^(-3) + 121680 v^(-1) + 121680 v + 1560 v^3) y^(-1)
+ (1560 v^(-3) + 121680 v^(-1) + 121680 v + 1560 v^3) y ] r^(17/6)
 ] q^(11/6)
+  [
 [ (1080) y^(-2) + (2700 v^(-1) + 2700 v) y^(-1) + (8640) + (2700 v^(-1) + 2700 v) y + (1080) y^2 ] r^(11/12)
+  [ (1080 v^(-2) + 8640 + 1080 v^2) y^(-2) + (27000 v^(-1) + 27000 v) y^(-1) + (8640 v^(-2) + 69120 + 8640 v^2)
+ (27000 v^(-1) + 27000 v) y
+ (1080 v^(-2) + 8640 + 1080 v^2) y^2 ] r^(23/12)
+  [ (8640 v^(-2) + 48600 + 8640 v^2) y^(-2) + (2700 v^(-3) + 162000 v^(-1) + 162000 v + 2700 v^3) y^(-1)
+ (69120 v^(-2) + 388800 + 69120 v^2) + (2700 v^(-3) + 162000 v^(-1) + 162000 v + 2700 v^3) y
+ (8640 v^(-2) + 48600 + 8640 v^2) y^2 ] r^(35/12)
 ] q^(23/12)
+  [
6 y^(-2) + 57 + 6 y^2
+  [ (6 v^(-2) + 1016 + 6 v^2) y^(-2) + (57 v^(-2) + 8322 + 57 v^2) + (6 v^(-2) + 1016 + 6 v^2) y^2 ] r
+  [ (1016 v^(-2) + 8322 + 1016 v^2) y^(-2) + (8322 v^(-2) + 68229 + 8322 v^2) + (1016 v^(-2) + 8322 + 1016 v^2) y^2 ] r^2
+  [ (8322 v^(-2) + 44688 + 8322 v^2) y^(-2) + (68229 v^(-2) + 366396 + 68229 v^2) + (8322 v^(-2) + 44688 + 8322 v^2) y^2 ] r^3
 ] q^2
+  [
 [ (8640 v^(-1) + 8640 v) y^(-1) + (8640 v^(-1) + 8640 v) y ] r^(7/6)
+  [ (69120 v^(-1) + 69120 v) y^(-1) + (69120 v^(-1) + 69120 v) y ] r^(13/6)
 ] q^(13/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-3) + (240) y^(-2) + (57 v^(-1) + 57 v) y^(-1) + (1800) + (57 v^(-1) + 57 v) y + (240) y^2
+ (1 v^(-1) + 1 v) y^3 ] r^(1/4)
+  [ (6 v^(-1) + 6 v) y^(-3) + (240 v^(-2) + 3024 + 240 v^2) y^(-2) + (8322 v^(-1) + 8322 v) y^(-1)
+ (1800 v^(-2) + 22704 + 1800 v^2) + (8322 v^(-1) + 8322 v) y + (240 v^(-2) + 3024 + 240 v^2) y^2
+ (6 v^(-1) + 6 v) y^3 ] r^(5/4)
+  [ (1 v^(-3) + 57 v^(-1) + 57 v + 1 v^3) y^(-3) + (3024 v^(-2) + 22704 + 3024 v^2) y^(-2)
+ (57 v^(-3) + 68229 v^(-1) + 68229 v + 57 v^3) y^(-1) + (22704 v^(-2) + 170464 + 22704 v^2)
+ (57 v^(-3) + 68229 v^(-1) + 68229 v + 57 v^3) y + (3024 v^(-2) + 22704 + 3024 v^2) y^2
+ (1 v^(-3) + 57 v^(-1) + 57 v + 1 v^3) y^3 ] r^(9/4)
 ] q^(9/4)
+  [
 [ (450) y^(-2) + (3090) + (450) y^2 ] r^(1/3)
+  [ (450 v^(-2) + 6750 + 450 v^2) y^(-2) + (3090 v^(-2) + 46350 + 3090 v^2) + (450 v^(-2) + 6750 + 450 v^2) y^2 ] r^(4/3)
+  [ (6750 v^(-2) + 46350 + 6750 v^2) y^(-2) + (46350 v^(-2) + 318270 + 46350 v^2) + (6750 v^(-2) + 46350 + 6750 v^2) y^2 ] r^(7/3)
 ] q^(7/3)
+  [
 [ (20 v^(-1) + 20 v) y^(-3) + (1800 v^(-1) + 1800 v) y^(-1) + (1800 v^(-1) + 1800 v) y + (20 v^(-1) + 20 v) y^3 ] r^(1/2)
+  [ (240 v^(-1) + 240 v) y^(-3) + (22704 v^(-1) + 22704 v) y^(-1) + (22704 v^(-1) + 22704 v) y
+ (240 v^(-1) + 240 v) y^3 ] r^(3/2)
+  [ (20 v^(-3) + 1800 v^(-1) + 1800 v + 20 v^3) y^(-3) + (1800 v^(-3) + 170464 v^(-1) + 170464 v + 1800 v^3) y^(-1)
+ (1800 v^(-3) + 170464 v^(-1) + 170464 v + 1800 v^3) y + (20 v^(-3) + 1800 v^(-1) + 1800 v + 20 v^3) y^3 ] r^(5/2)
 ] q^(5/2)
+  [
 [ (30 v^(-1) + 30 v) y^(-3) + (1560) y^(-2) + (3090 v^(-1) + 3090 v) y^(-1) + (9360) + (3090 v^(-1) + 3090 v) y + (1560) y^2
+ (30 v^(-1) + 30 v) y^3 ] r^(7/12)
+  [ (450 v^(-1) + 450 v) y^(-3) + (1560 v^(-2) + 20280 + 1560 v^2) y^(-2) + (46350 v^(-1) + 46350 v) y^(-1)
+ (9360 v^(-2) + 121680 + 9360 v^2) + (46350 v^(-1) + 46350 v) y + (1560 v^(-2) + 20280 + 1560 v^2) y^2
+ (450 v^(-1) + 450 v) y^3 ] r^(19/12)
+  [ (30 v^(-3) + 3090 v^(-1) + 3090 v + 30 v^3) y^(-3) + (20280 v^(-2) + 121680 + 20280 v^2) y^(-2)
+ (3090 v^(-3) + 318270 v^(-1) + 318270 v + 3090 v^3) y^(-1) + (121680 v^(-2) + 730080 + 121680 v^2)
+ (3090 v^(-3) + 318270 v^(-1) + 318270 v + 3090 v^3) y + (20280 v^(-2) + 121680 + 20280 v^2) y^2
+ (30 v^(-3) + 3090 v^(-1) + 3090 v + 30 v^3) y^3 ] r^(31/12)
 ] q^(31/12)
+  [
 [ (2700) y^(-2) + (16200) + (2700) y^2 ] r^(2/3)
+  [ (2700 v^(-2) + 27000 + 2700 v^2) y^(-2) + (16200 v^(-2) + 162000 + 16200 v^2)
+ (2700 v^(-2) + 27000 + 2700 v^2) y^2 ] r^(5/3)
+  [ (27000 v^(-2) + 162000 + 27000 v^2) y^(-2) + (162000 v^(-2) + 972000 + 162000 v^2)
+ (27000 v^(-2) + 162000 + 27000 v^2) y^2 ] r^(8/3)
 ] q^(8/3)
+  [
 [ (120 v^(-1) + 120 v) y^(-3) + (9360 v^(-1) + 9360 v) y^(-1) + (9360 v^(-1) + 9360 v) y + (120 v^(-1) + 120 v) y^3 ] r^(5/6)
+  [ (1560 v^(-1) + 1560 v) y^(-3) + (121680 v^(-1) + 121680 v) y^(-1) + (121680 v^(-1) + 121680 v) y
+ (1560 v^(-1) + 1560 v) y^3 ] r^(11/6)
+  [ (120 v^(-3) + 9360 v^(-1) + 9360 v + 120 v^3) y^(-3) + (9360 v^(-3) + 730080 v^(-1) + 730080 v + 9360 v^3) y^(-1)
+ (9360 v^(-3) + 730080 v^(-1) + 730080 v + 9360 v^3) y + (120 v^(-3) + 9360 v^(-1) + 9360 v + 120 v^3) y^3 ] r^(17/6)
 ] q^(17/6)
+  [
 [ (270 v^(-1) + 270 v) y^(-3) + (8640) y^(-2) + (16200 v^(-1) + 16200 v) y^(-1) + (48600)
+ (16200 v^(-1) + 16200 v) y + (8640) y^2 + (270 v^(-1) + 270 v) y^3 ] r^(11/12)
+  [ (2700 v^(-1) + 2700 v) y^(-3) + (8640 v^(-2) + 69120 + 8640 v^2) y^(-2) + (162000 v^(-1) + 162000 v) y^(-1)
+ (48600 v^(-2) + 388800 + 48600 v^2) + (162000 v^(-1) + 162000 v) y + (8640 v^(-2) + 69120 + 8640 v^2) y^2
+ (2700 v^(-1) + 2700 v) y^3 ] r^(23/12)
+  [ (270 v^(-3) + 16200 v^(-1) + 16200 v + 270 v^3) y^(-3) + (69120 v^(-2) + 388800 + 69120 v^2) y^(-2)
+ (16200 v^(-3) + 972000 v^(-1) + 972000 v + 16200 v^3) y^(-1) + (388800 v^(-2) + 2.187e+006 + 388800 v^2)
+ (16200 v^(-3) + 972000 v^(-1) + 972000 v + 16200 v^3) y + (69120 v^(-2) + 388800 + 69120 v^2) y^2
+ (270 v^(-3) + 16200 v^(-1) + 16200 v + 270 v^3) y^3 ] r^(35/12)
 ] q^(35/12)
+  [
57 y^(-2) + 308 + 57 y^2
+  [ (57 v^(-2) + 8322 + 57 v^2) y^(-2) + (308 v^(-2) + 44688 + 308 v^2) + (57 v^(-2) + 8322 + 57 v^2) y^2 ] r
+  [ (8322 v^(-2) + 68229 + 8322 v^2) y^(-2) + (44688 v^(-2) + 366396 + 44688 v^2) + (8322 v^(-2) + 68229 + 8322 v^2) y^2 ] r^2
+  [ (68229 v^(-2) + 366396 + 68229 v^2) y^(-2) + (366396 v^(-2) + 1.96758e+006 + 366396 v^2)
+ (68229 v^(-2) + 366396 + 68229 v^2) y^2 ] r^3
 ] q^3
```

Partition function for (1)(1)(1)(1)(4):

```
1
+ (1 v^(-2) + 6 + 1 v^2) r
+ (6 v^(-2) + 57 + 6 v^2) r^2

+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (6 v^(-1) + 6 v) y^(-1) + (20 v^(-2) + 240 + 20 v^2) + (6 v^(-1) + 6 v) y ] r^(5/4)
 ] q^(1/4)
+  [
30 r^(1/3)
+ (30 v^(-2) + 450 + 30 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (30 v^(-1) + 30 v) y^(-1) + 120 + (30 v^(-1) + 30 v) y ] r^(7/12)
+  [ (450 v^(-1) + 450 v) y^(-1) + (120 v^(-2) + 1560 + 120 v^2) + (450 v^(-1) + 450 v) y ] r^(19/12)
 ] q^(7/12)
+  [
270 r^(2/3)
+ (270 v^(-2) + 2700 + 270 v^2) r^(5/3)
 ] q^(2/3)
+  [
 [ (120 v^(-1) + 120 v) y^(-1) + (120 v^(-1) + 120 v) y ] r^(5/6)
+  [ (1560 v^(-1) + 1560 v) y^(-1) + (1560 v^(-1) + 1560 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (270 v^(-1) + 270 v) y^(-1) + 1080 + (270 v^(-1) + 270 v) y ] r^(11/12)
+  [ (2700 v^(-1) + 2700 v) y^(-1) + (1080 v^(-2) + 8640 + 1080 v^2) + (2700 v^(-1) + 2700 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 6 + 1 y^2
+  [ (1 v^(-2) + 6 + 1 v^2) y^(-2) + (6 v^(-2) + 1016 + 6 v^2) + (1 v^(-2) + 6 + 1 v^2) y^2 ] r
+  [ (6 v^(-2) + 57 + 6 v^2) y^(-2) + (1016 v^(-2) + 8322 + 1016 v^2) + (6 v^(-2) + 57 + 6 v^2) y^2 ] r^2
 ] q
+
 [ (1080 v^(-1) + 1080 v) y^(-1) + (1080 v^(-1) + 1080 v) y ] r^(7/6)
 q^(7/6)
+  [
 [ 20 y^(-2) + (6 v^(-1) + 6 v) y^(-1) + 240 + (6 v^(-1) + 6 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 240 + 20 v^2) y^(-2) + (1016 v^(-1) + 1016 v) y^(-1) + (240 v^(-2) + 3024 + 240 v^2)
+ (1016 v^(-1) + 1016 v) y + (20 v^(-2) + 240 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 30 y^(-2) + 450 + 30 y^2 ] r^(1/3)
+  [ (30 v^(-2) + 450 + 30 v^2) y^(-2) + (450 v^(-2) + 6750 + 450 v^2) + (30 v^(-2) + 450 + 30 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(1/2)
+  [ (3024 v^(-1) + 3024 v) y^(-1) + (3024 v^(-1) + 3024 v) y ] r^(3/2)
 ] q^(3/2)
+  [
 [ 120 y^(-2) + (450 v^(-1) + 450 v) y^(-1) + 1560 + (450 v^(-1) + 450 v) y + 120 y^2 ] r^(7/12)
+  [ (120 v^(-2) + 1560 + 120 v^2) y^(-2) + (6750 v^(-1) + 6750 v) y^(-1) + (1560 v^(-2) + 20280 + 1560 v^2)
+ (6750 v^(-1) + 6750 v) y + (120 v^(-2) + 1560 + 120 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 270 y^(-2) + 2700 + 270 y^2 ] r^(2/3)
+  [ (270 v^(-2) + 2700 + 270 v^2) y^(-2) + (2700 v^(-2) + 27000 + 2700 v^2) + (270 v^(-2) + 2700 + 270 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ (1560 v^(-1) + 1560 v) y^(-1) + (1560 v^(-1) + 1560 v) y ] r^(5/6)
+  [ (20280 v^(-1) + 20280 v) y^(-1) + (20280 v^(-1) + 20280 v) y ] r^(11/6)
 ] q^(11/6)
+  [
 [ 1080 y^(-2) + (2700 v^(-1) + 2700 v) y^(-1) + 8640 + (2700 v^(-1) + 2700 v) y + 1080 y^2 ] r^(11/12)
+  [ (1080 v^(-2) + 8640 + 1080 v^2) y^(-2) + (27000 v^(-1) + 27000 v) y^(-1) + (8640 v^(-2) + 69120 + 8640 v^2)
+ (27000 v^(-1) + 27000 v) y + (1080 v^(-2) + 8640 + 1080 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
6 y^(-2) + 57 + 6 y^2
+  [ (6 v^(-2) + 1016 + 6 v^2) y^(-2) + (57 v^(-2) + 8322 + 57 v^2) + (6 v^(-2) + 1016 + 6 v^2) y^2 ] r
+  [ (1016 v^(-2) + 8322 + 1016 v^2) y^(-2) + (8322 v^(-2) + 68229 + 8322 v^2) + (1016 v^(-2) + 8322 + 1016 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(1)(1)(2)(2):

```
1
+ (1 v^(-2) + 18 + 1 v^2) r
+ (18 v^(-2) + 147 + 18 v^2) r^2

+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 12 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (18 v^(-1) + 18 v) y^(-1) + (12 v^(-2) + 192 + 12 v^2) + (18 v^(-1) + 18 v) y ] r^(5/4)
 ] q^(1/4)
+  [
18 r^(1/3)
+ (18 v^(-2) + 234 + 18 v^2) r^(4/3)
```

```
 ] q^(1/3)
+  [
 [ (12 v^(-1) + 12 v) y^(-1) + 48 + (12 v^(-1) + 12 v) y ] r^(1/2)
+  [ (192 v^(-1) + 192 v) y^(-1) + (48 v^(-2) + 672 + 48 v^2) + (192 v^(-1) + 192 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (18 v^(-1) + 18 v) y^(-1) + 216 + (18 v^(-1) + 18 v) y ] r^(7/12)
+  [ (234 v^(-1) + 234 v) y^(-1) + (216 v^(-2) + 2376 + 216 v^2) + (234 v^(-1) + 234 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (48 v^(-1) + 48 v) y^(-1) + 64 + (48 v^(-1) + 48 v) y ] r^(3/4)
+  [ (672 v^(-1) + 672 v) y^(-1) + (64 v^(-2) + 768 + 64 v^2) + (672 v^(-1) + 672 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (216 v^(-1) + 216 v) y^(-1) + 864 + (216 v^(-1) + 216 v) y ] r^(5/6)
+  [ (2376 v^(-1) + 2376 v) y^(-1) + (864 v^(-2) + 7776 + 864 v^2) + (2376 v^(-1) + 2376 v) y ] r^(11/6)
 ] q^(5/6)
+  [
1 y^(-2) + 18 + 1 y^2
+  [ (1 v^(-2) + 18 + 1 v^2) y^(-2) + (64 v^(-1) + 64 v) y^(-1) + (18 v^(-2) + 324 + 18 v^2)
+ (64 v^(-1) + 64 v) y + (1 v^(-2) + 18 + 1 v^2) y^2 ] r
+  [ (18 v^(-2) + 147 + 18 v^2) y^(-2) + (768 v^(-1) + 768 v) y^(-1) + (324 v^(-2) + 2646 + 324 v^2)
+ (768 v^(-1) + 768 v) y + (18 v^(-2) + 147 + 18 v^2) y^2 ] r^2
 ] q
+
 [ (864 v^(-1) + 864 v) y^(-1) + 1152 + (864 v^(-1) + 864 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 12 y^(-2) + (18 v^(-1) + 18 v) y^(-1) + 192 + (18 v^(-1) + 18 v) y + 12 y^2 ] r^(1/4)
+  [ (12 v^(-2) + 192 + 12 v^2) y^(-2) + (324 v^(-1) + 324 v) y^(-1) + (192 v^(-2) + 3072 + 192 v^2)
+ (324 v^(-1) + 324 v) y + (12 v^(-2) + 192 + 12 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 18 y^(-2) + 234 + 18 y^2 ] r^(1/3)
+  [ (18 v^(-2) + 234 + 18 v^2) y^(-2) + (1152 v^(-1) + 1152 v) y^(-1) + (234 v^(-2) + 3042 + 234 v^2)
+ (1152 v^(-1) + 1152 v) y + (18 v^(-2) + 234 + 18 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 48 y^(-2) + (192 v^(-1) + 192 v) y^(-1) + 672 + (192 v^(-1) + 192 v) y + 48 y^2 ] r^(1/2)
+  [ (48 v^(-2) + 672 + 48 v^2) y^(-2) + (3072 v^(-1) + 3072 v) y^(-1) + (672 v^(-2) + 9408 + 672 v^2)
+ (3072 v^(-1) + 3072 v) y + (48 v^(-2) + 672 + 48 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 216 y^(-2) + (234 v^(-1) + 234 v) y^(-1) + 2376 + (234 v^(-1) + 234 v) y + 216 y^2 ] r^(7/12)
+  [ (216 v^(-2) + 2376 + 216 v^2) y^(-2) + (3042 v^(-1) + 3042 v) y^(-1) + (2376 v^(-2) + 26136 + 2376 v^2)
+ (3042 v^(-1) + 3042 v) y + (216 v^(-2) + 2376 + 216 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 64 y^(-2) + (672 v^(-1) + 672 v) y^(-1) + 768 + (672 v^(-1) + 672 v) y + 64 y^2 ] r^(3/4)
+  [ (64 v^(-2) + 768 + 64 v^2) y^(-2) + (9408 v^(-1) + 9408 v) y^(-1) + (768 v^(-2) + 9216 + 768 v^2)
+ (9408 v^(-1) + 9408 v) y + (64 v^(-2) + 768 + 64 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 864 y^(-2) + (2376 v^(-1) + 2376 v) y^(-1) + 7776 + (2376 v^(-1) + 2376 v) y + 864 y^2 ] r^(5/6)
+  [ (864 v^(-2) + 7776 + 864 v^2) y^(-2) + (26136 v^(-1) + 26136 v) y^(-1) + (7776 v^(-2) + 69984 + 7776 v^2)
+ (26136 v^(-1) + 26136 v) y + (864 v^(-2) + 7776 + 864 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
18 y^(-2) + 147 + 18 y^2
+  [ (18 v^(-2) + 324 + 18 v^2) y^(-2) + (768 v^(-1) + 768 v) y^(-1) + (147 v^(-2) + 2646 + 147 v^2)
+ (768 v^(-1) + 768 v) y + (18 v^(-2) + 324 + 18 v^2) y^2 ] r
+  [ (324 v^(-2) + 2646 + 324 v^2) y^(-2) + (9216 v^(-1) + 9216 v) y^(-1) + (2646 v^(-2) + 21609 + 2646 v^2)
+ (9216 v^(-1) + 9216 v) y + (324 v^(-2) + 2646 + 324 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (2)(2)(2)(2):

```
1
+  [ (1 v^(-2) + 4 + 1 v^2) ] r
+  [ (4 v^(-2) + 31 + 4 v^2) ] r^2
+  [ (31 v^(-2) + 172 + 31 v^2) ] r^3

+  [
 [ (1 v^(-1) + 1 v) y^(-1) + (32) + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (4 v^(-1) + 4 v) y^(-1) + (32 v^(-2) + 400 + 32 v^2) + (4 v^(-1) + 4 v) y ] r^(5/4)
+  [ (1 v^(-3) + 31 v^(-1) + 31 v + 1 v^3) y^(-1) + (400 v^(-2) + 2912 + 400 v^2)
+ (1 v^(-3) + 31 v^(-1) + 31 v + 1 v^3) y ] r^(9/4)
 ] q^(1/4)
+  [
 [ (32 v^(-1) + 32 v) y^(-1) + (136) + (32 v^(-1) + 32 v) y ] r^(1/2)
+  [ (400 v^(-1) + 400 v) y^(-1) + (136 v^(-2) + 1632 + 136 v^2) + (400 v^(-1) + 400 v) y ] r^(3/2)
+  [ (32 v^(-3) + 2912 v^(-1) + 2912 v + 32 v^3) y^(-1) + (1632 v^(-2) + 10472 + 1632 v^2)
+ (32 v^(-3) + 2912 v^(-1) + 2912 v + 32 v^3) y ] r^(5/2)
 ] q^(1/2)
+  [
 [ (136 v^(-1) + 136 v) y^(-1) + (544) + (136 v^(-1) + 136 v) y ] r^(3/4)
+  [ (1632 v^(-1) + 1632 v) y^(-1) + (544 v^(-2) + 5440 + 544 v^2) + (1632 v^(-1) + 1632 v) y ] r^(7/4)
+  [ (136 v^(-3) + 10472 v^(-1) + 10472 v + 136 v^3) y^(-1) + (5440 v^(-2) + 31552 + 5440 v^2)
+ (136 v^(-3) + 10472 v^(-1) + 10472 v + 136 v^3) y ] r^(11/4)
 ] q^(3/4)
+  [
1 y^(-2) + 4 + 1 y^2
```

```
+  [ (1 v^(-2) + 4 + 1 v^2) y^(-2) + (544 v^(-1) + 544 v) y^(-1) + (4 v^(-2) + 1920 + 4 v^2)
+ (544 v^(-1) + 544 v) y + (1 v^(-2) + 4 + 1 v^2) y^2 ] r
+  [ (4 v^(-2) + 31 + 4 v^2) y^(-2) + (5440 v^(-1) + 5440 v) y^(-1) + (1920 v^(-2) + 15356 + 1920 v^2)
+ (5440 v^(-1) + 5440 v) y + (4 v^(-2) + 31 + 4 v^2) y^2 ] r^2
+  [ (31 v^(-2) + 172 + 31 v^2) y^(-2) + (544 v^(-3) + 31552 v^(-1) + 31552 v + 544 v^3) y^(-1)
+ (15356 v^(-2) + 82560 + 15356 v^2) + (544 v^(-3) + 31552 v^(-1) + 31552 v
+ 544 v^3) y + (31 v^(-2) + 172 + 31 v^2) y^2 ] r^3
 ] q
+  [
 [ (32) y^(-2) + (4 v^(-1) + 4 v) y^(-1) + (400) + (4 v^(-1) + 4 v) y + (32) y^2 ] r^(1/4)
+  [ (32 v^(-2) + 400 + 32 v^2) y^(-2) + (1920 v^(-1) + 1920 v) y^(-1) + (400 v^(-2) + 5952 + 400 v^2)
+ (1920 v^(-1) + 1920 v) y + (32 v^(-2) + 400 + 32 v^2) y^2 ] r^(5/4)
+  [ (400 v^(-2) + 2912 + 400 v^2) y^(-2) + (4 v^(-3) + 15356 v^(-1) + 15356 v + 4 v^3) y^(-1)
+ (5952 v^(-2) + 42112 + 5952 v^2) + (4 v^(-3) + 15356 v^(-1) + 15356 v + 4 v^3) y
+ (400 v^(-2) + 2912 + 400 v^2) y^2 ] r^(9/4)
 ] q^(5/4)
+  [
 [ (136) y^(-2) + (400 v^(-1) + 400 v) y^(-1) + (1632) + (400 v^(-1) + 400 v) y + (136) y^2 ] r^(1/2)
+  [ (136 v^(-2) + 1632 + 136 v^2) y^(-2) + (5952 v^(-1) + 5952 v) y^(-1) + (1632 v^(-2) + 19584 + 1632 v^2)
+ (5952 v^(-1) + 5952 v) y + (136 v^(-2) + 1632 + 136 v^2) y^2 ] r^(3/2)
+  [ (1632 v^(-2) + 10472 + 1632 v^2) y^(-2) + (400 v^(-3) + 42112 v^(-1) + 42112 v + 400 v^3) y^(-1)
+ (19584 v^(-2) + 125664 + 19584 v^2) + (400 v^(-3) + 42112 v^(-1) + 42112 v + 400 v^3) y
+ (1632 v^(-2) + 10472 + 1632 v^2) y^2 ] r^(5/2)
 ] q^(3/2)
+  [
 [ (544) y^(-2) + (1632 v^(-1) + 1632 v) y^(-1) + (5440) + (1632 v^(-1) + 1632 v) y + (544) y^2 ] r^(3/4)
+  [ (544 v^(-2) + 5440 + 544 v^2) y^(-2) + (19584 v^(-1) + 19584 v) y^(-1) + (5440 v^(-2) + 54400 + 5440 v^2)
+ (19584 v^(-1) + 19584 v) y + (544 v^(-2) + 5440 + 544 v^2) y^2 ] r^(7/4)
+  [ (5440 v^(-2) + 31552 + 5440 v^2) y^(-2) + (1632 v^(-3) + 125664 v^(-1) + 125664 v + 1632 v^3) y^(-1)
+ (54400 v^(-2) + 315520 + 54400 v^2) + (1632 v^(-3) + 125664 v^(-1) + 125664 v + 1632 v^3) y
+ (5440 v^(-2) + 31552 + 5440 v^2) y^2 ] r^(11/4)
 ] q^(7/4)
+  [
4 y^(-2) + 31 + 4 y^2
+  [ (4 v^(-2) + 1920 + 4 v^2) y^(-2) + (5440 v^(-1) + 5440 v) y^(-1) + (31 v^(-2) + 15356 + 31 v^2)
+ (5440 v^(-1) + 5440 v) y + (4 v^(-2) + 1920 + 4 v^2) y^2 ] r
+  [ (1920 v^(-2) + 15356 + 1920 v^2) y^(-2) + (54400 v^(-1) + 54400 v) y^(-1) + (15356 v^(-2) + 122817 + 15356 v^2)
+ (54400 v^(-1) + 54400 v) y + (1920 v^(-2) + 15356 + 1920 v^2) y^2 ] r^2
+  [ (15356 v^(-2) + 82560 + 15356 v^2) y^(-2) + (5440 v^(-3) + 315520 v^(-1) + 315520 v + 5440 v^3) y^(-1)
+ (122817 v^(-2) + 660308 + 122817 v^2) + (5440 v^(-3) + 315520 v^(-1) + 315520 v + 5440 v^3) y
+ (15356 v^(-2) + 82560 + 15356 v^2) y^2 ] r^3
 ] q^2
+  [
 [ (1 v^(-1) + 1 v) y^(-3) + (400) y^(-2) + (31 v^(-1) + 31 v) y^(-1) + (2912) + (31 v^(-1) + 31 v) y
+ (400) y^2 + (1 v^(-1) + 1 v) y^3 ] r^(1/4)
+  [ (4 v^(-1) + 4 v) y^(-3) + (400 v^(-2) + 5952 + 400 v^2) y^(-2) + (15356 v^(-1) + 15356 v) y^(-1)
+ (2912 v^(-2) + 42112 + 2912 v^2) + (15356 v^(-1) + 15356 v) y
+ (400 v^(-2) + 5952 + 400 v^2) y^2 + (4 v^(-1) + 4 v) y^3 ] r^(5/4)
+  [ (1 v^(-3) + 31 v^(-1) + 31 v + 1 v^3) y^(-3) + (5952 v^(-2) + 42112 + 5952 v^2) y^(-2)
+ (31 v^(-3) + 122817 v^(-1) + 122817 v + 31 v^3) y^(-1) + (42112 v^(-2) + 299264 + 42112 v^2)
+ (31 v^(-3) + 122817 v^(-1) + 122817 v + 31 v^3) y + (5952 v^(-2) + 42112 + 5952 v^2) y^2
+ (1 v^(-3) + 31 v^(-1) + 31 v + 1 v^3) y^3 ] r^(9/4)
 ] q^(9/4)
+  [
 [ (32 v^(-1) + 32 v) y^(-3) + (1632) y^(-2) + (2912 v^(-1) + 2912 v) y^(-1) + (10472)
+ (2912 v^(-1) + 2912 v) y + (1632) y^2 + (32 v^(-1) + 32 v) y^3 ] r^(1/2)
+  [ (400 v^(-1) + 400 v) y^(-3) + (1632 v^(-2) + 19584 + 1632 v^2) y^(-2) + (42112 v^(-1) + 42112 v) y^(-1)
+ (10472 v^(-2) + 125664 + 10472 v^2) + (42112 v^(-1) + 42112 v) y + (1632 v^(-2) + 19584 + 1632 v^2) y^2
+ (400 v^(-1) + 400 v) y^3 ] r^(3/2)
+  [ (32 v^(-3) + 2912 v^(-1) + 2912 v + 32 v^3) y^(-3) + (19584 v^(-2) + 125664 + 19584 v^2) y^(-2)
+ (2912 v^(-3) + 299264 v^(-1) + 299264 v + 2912 v^3) y^(-1) + (125664 v^(-2) + 806344 + 125664 v^2)
+ (2912 v^(-3) + 299264 v^(-1) + 299264 v + 2912 v^3) y + (19584 v^(-2) + 125664 + 19584 v^2) y^2
+ (32 v^(-3) + 2912 v^(-1) + 2912 v + 32 v^3) y^3 ] r^(5/2)
 ] q^(5/2)
+  [
 [ (136 v^(-1) + 136 v) y^(-3) + (5440) y^(-2) + (10472 v^(-1) + 10472 v) y^(-1) + (31552)
+ (10472 v^(-1) + 10472 v) y + (5440) y^2 + (136 v^(-1) + 136 v) y^3 ] r^(3/4)
+  [ (1632 v^(-1) + 1632 v) y^(-3) + (5440 v^(-2) + 54400 + 5440 v^2) y^(-2) + (125664 v^(-1) + 125664 v) y^(-1)
+ (31552 v^(-2) + 315520 + 31552 v^2) + (125664 v^(-1) + 125664 v) y + (5440 v^(-2)
+ 54400 + 5440 v^2) y^2 + (1632 v^(-1) + 1632 v) y^3 ] r^(7/4)
+  [ (136 v^(-3) + 10472 v^(-1) + 10472 v + 136 v^3) y^(-3) + (54400 v^(-2) + 315520 + 54400 v^2) y^(-2)
+ (10472 v^(-3) + 806344 v^(-1) + 806344 v + 10472 v^3) y^(-1) + (315520 v^(-2) + 1.83002e+006 + 315520 v^2)
+ (10472 v^(-3) + 806344 v^(-1) + 806344 v + 10472 v^3) y + (54400 v^(-2) + 315520 + 54400 v^2) y^2
+ (136 v^(-3) + 10472 v^(-1) + 10472 v + 136 v^3) y^3 ] r^(11/4)
 ] q^(11/4)
+  [
31 y^(-2) + 172 + 31 y^2
+  [ (544 v^(-1) + 544 v) y^(-3) + (31 v^(-2) + 15356 + 31 v^2) y^(-2) + (31552 v^(-1) + 31552 v) y^(-1)
+ (172 v^(-2) + 82560 + 172 v^2) + (31552 v^(-1) + 31552 v) y
+ (31 v^(-2) + 15356 + 31 v^2) y^2 + (544 v^(-1) + 544 v) y^3 ] r
+  [ (5440 v^(-1) + 5440 v) y^(-3) + (15356 v^(-2) + 122817 + 15356 v^2) y^(-2) + (315520 v^(-1) + 315520 v) y^(-1)
+ (82560 v^(-2) + 660308 + 82560 v^2) + (315520 v^(-1) + 315520 v) y + (15356 v^(-2) + 122817 + 15356 v^2) y^2
+ (5440 v^(-1) + 5440 v) y^3 ] r^2
+  [ (544 v^(-3) + 31552 v^(-1) + 31552 v + 544 v^3) y^(-3) + (122817 v^(-2) + 660308 + 122817 v^2) y^(-2)
+ (31552 v^(-3) + 1.83002e+006 v^(-1) + 1.83002e+006 v + 31552 v^3) y^(-1) + (660308 v^(-2) + 3.55008e+006 + 660308 v^2)
+ (31552 v^(-3) + 1.83002e+006 v^(-1) + 1.83002e+006 v + 31552 v^3) y + (122817 v^(-2) + 660308 + 122817 v^2) y^2
+ (544 v^(-3) + 31552 v^(-1) + 31552 v + 544 v^3) y^3 ] r^3
 ] q^3
```

Partition function for (1)(2)(2)(4):

```
1
+ (1 v^(-2) + 9 + 1 v^2) r
+ (9 v^(-2) + 73 + 9 v^2) r^2

+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 24 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (9 v^(-1) + 9 v) y^(-1) + (24 v^(-2) + 316 + 24 v^2) + (9 v^(-1) + 9 v) y ] r^(5/4)
 ] q^(1/4)
+  [
9 r^(1/3)
+ (9 v^(-2) + 117 + 9 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + 88 + (24 v^(-1) + 24 v) y ] r^(1/2)
+  [ (316 v^(-1) + 316 v) y^(-1) + (88 v^(-2) + 1104 + 88 v^2) + (316 v^(-1) + 316 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (9 v^(-1) + 9 v) y^(-1) + 108 + (9 v^(-1) + 9 v) y ] r^(7/12)
+  [ (117 v^(-1) + 117 v) y^(-1) + (108 v^(-2) + 1188 + 108 v^2) + (117 v^(-1) + 117 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (88 v^(-1) + 88 v) y^(-1) + 288 + (88 v^(-1) + 88 v) y ] r^(3/4)
+  [ (1104 v^(-1) + 1104 v) y^(-1) + (288 v^(-2) + 2944 + 288 v^2) + (1104 v^(-1) + 1104 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (108 v^(-1) + 108 v) y^(-1) + 432 + (108 v^(-1) + 108 v) y ] r^(5/6)
+  [ (1188 v^(-1) + 1188 v) y^(-1) + (432 v^(-2) + 3888 + 432 v^2) + (1188 v^(-1) + 1188 v) y ] r^(11/6)
 ] q^(5/6)
+  [
1 y^(-2) + 9 + 1 y^2
+  [ (1 v^(-2) + 9 + 1 v^2) y^(-2) + (288 v^(-1) + 288 v) y^(-1) + (9 v^(-2) + 1186 + 9 v^2)
+ (288 v^(-1) + 288 v) y + (1 v^(-2) + 9 + 1 v^2) y^2 ] r
+  [ (9 v^(-2) + 73 + 9 v^2) y^(-2) + (2944 v^(-1) + 2944 v) y^(-1) + (1186 v^(-2) + 9515 + 1186 v^2)
+ (2944 v^(-1) + 2944 v) y + (9 v^(-2) + 73 + 9 v^2) y^2 ] r^2
 ] q
+
 [ (432 v^(-1) + 432 v) y^(-1) + 576 + (432 v^(-1) + 432 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 24 y^(-2) + (9 v^(-1) + 9 v) y^(-1) + 316 + (9 v^(-1) + 9 v) y + 24 y^2 ] r^(1/4)
+  [ (24 v^(-2) + 316 + 24 v^2) y^(-2) + (1186 v^(-1) + 1186 v) y^(-1) + (316 v^(-2) + 4680 + 316 v^2)
+ (1186 v^(-1) + 1186 v) y + (24 v^(-2) + 316 + 24 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 9 y^(-2) + 117 + 9 y^2 ] r^(1/3)
+  [ (9 v^(-2) + 117 + 9 v^2) y^(-2) + (576 v^(-1) + 576 v) y^(-1) + (117 v^(-2) + 1521 + 117 v^2)
+ (576 v^(-1) + 576 v) y + (9 v^(-2) + 117 + 9 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 88 y^(-2) + (316 v^(-1) + 316 v) y^(-1) + 1104 + (316 v^(-1) + 316 v) y + 88 y^2 ] r^(1/2)
+  [ (88 v^(-2) + 1104 + 88 v^2) y^(-2) + (4680 v^(-1) + 4680 v) y^(-1) + (1104 v^(-2) + 13920 + 1104 v^2)
+ (4680 v^(-1) + 4680 v) y + (88 v^(-2) + 1104 + 88 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 108 y^(-2) + (117 v^(-1) + 117 v) y^(-1) + 1188 + (117 v^(-1) + 117 v) y + 108 y^2 ] r^(7/12)
+  [ (108 v^(-2) + 1188 + 108 v^2) y^(-2) + (1521 v^(-1) + 1521 v) y^(-1)
+ (1188 v^(-2) + 13068 + 1188 v^2) + (1521 v^(-1) + 1521 v) y + (108 v^(-2) + 1188 + 108 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 288 y^(-2) + (1104 v^(-1) + 1104 v) y^(-1) + 2944 + (1104 v^(-1) + 1104 v) y + 288 y^2 ] r^(3/4)
+  [ (288 v^(-2) + 2944 + 288 v^2) y^(-2) + (13920 v^(-1) + 13920 v) y^(-1)
+ (2944 v^(-2) + 30208 + 2944 v^2) + (13920 v^(-1) + 13920 v) y + (288 v^(-2) + 2944 + 288 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 432 y^(-2) + (1188 v^(-1) + 1188 v) y^(-1) + 3888 + (1188 v^(-1) + 1188 v) y + 432 y^2 ] r^(5/6)
+  [ (432 v^(-2) + 3888 + 432 v^2) y^(-2) + (13068 v^(-1) + 13068 v) y^(-1)
+ (3888 v^(-2) + 34992 + 3888 v^2) + (13068 v^(-1) + 13068 v) y + (432 v^(-2) + 3888 + 432 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
9 y^(-2) + 73 + 9 y^2
+  [ (9 v^(-2) + 1186 + 9 v^2) y^(-2) + (2944 v^(-1) + 2944 v) y^(-1) + (73 v^(-2) + 9515 + 73 v^2)
+ (2944 v^(-1) + 2944 v) y + (9 v^(-2) + 1186 + 9 v^2) y^2 ] r
+  [ (1186 v^(-2) + 9515 + 1186 v^2) y^(-2) + (30208 v^(-1) + 30208 v) y^(-1) + (9515 v^(-2) + 76341 + 9515 v^2)
+ (30208 v^(-1) + 30208 v) y + (1186 v^(-2) + 9515 + 1186 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(1)(4)(4):

```
1
+ (1 v^(-2) + 4 + 1 v^2) r
+ (4 v^(-2) + 31 + 4 v^2) r^2

+  [
2 r^(1/6)
+ (2 v^(-2) + 32 + 2 v^2) r^(7/6)
 ] q^(1/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (4 v^(-1) + 4 v) y^(-1) + (20 v^(-2) + 232 + 20 v^2) + (4 v^(-1) + 4 v) y ] r^(5/4)
 ] q^(1/4)
```

```
+  [
24 r^(1/3)
+ (24 v^(-2) + 324 + 24 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (2 v^(-1) + 2 v) y^(-1) + 8 + (2 v^(-1) + 2 v) y ] r^(5/12)
+  [ (32 v^(-1) + 32 v) y^(-1) + (8 v^(-2) + 112 + 8 v^2) + (32 v^(-1) + 32 v) y ] r^(17/12)
 ] q^(5/12)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + 40 + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (232 v^(-1) + 232 v) y^(-1) + (40 v^(-2) + 456 + 40 v^2) + (232 v^(-1) + 232 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + 96 + (24 v^(-1) + 24 v) y ] r^(7/12)
+  [ (324 v^(-1) + 324 v) y^(-1) + (96 v^(-2) + 1104 + 96 v^2) + (324 v^(-1) + 324 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (8 v^(-1) + 8 v) y^(-1) + 168 + (8 v^(-1) + 8 v) y ] r^(2/3)
+  [ (112 v^(-1) + 112 v) y^(-1) + (168 v^(-2) + 1704 + 168 v^2) + (112 v^(-1) + 112 v) y ] r^(5/3)
 ] q^(2/3)
+  [
 [ (40 v^(-1) + 40 v) y^(-1) + 160 + (40 v^(-1) + 40 v) y ] r^(3/4)
+  [ (456 v^(-1) + 456 v) y^(-1) + (160 v^(-2) + 1504 + 160 v^2) + (456 v^(-1) + 456 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (96 v^(-1) + 96 v) y^(-1) + 72 + (96 v^(-1) + 96 v) y ] r^(5/6)
+  [ (1104 v^(-1) + 1104 v) y^(-1) + (72 v^(-2) + 720 + 72 v^2) + (1104 v^(-1) + 1104 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (168 v^(-1) + 168 v) y^(-1) + 672 + (168 v^(-1) + 168 v) y ] r^(11/12)
+  [ (1704 v^(-1) + 1704 v) y^(-1) + (672 v^(-2) + 5472 + 672 v^2) + (1704 v^(-1) + 1704 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 4 + 1 y^2
+  [ (1 v^(-2) + 4 + 1 v^2) y^(-2) + (160 v^(-1) + 160 v) y^(-1) + (4 v^(-2) + 1056 + 4 v^2)
+ (160 v^(-1) + 160 v) y + (1 v^(-2) + 4 + 1 v^2) y^2 ] r
+  [ (4 v^(-2) + 31 + 4 v^2) y^(-2) + (1504 v^(-1) + 1504 v) y^(-1) + (1056 v^(-2) + 8588 + 1056 v^2)
+ (1504 v^(-1) + 1504 v) y + (4 v^(-2) + 31 + 4 v^2) y^2 ] r^2
 ] q
+
 [ (72 v^(-1) + 72 v) y^(-1) + 288 + (72 v^(-1) + 72 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 2 y^(-2) + 32 + 2 y^2 ] r^(1/6)
+  [ (2 v^(-2) + 32 + 2 v^2) y^(-2) + (672 v^(-1) + 672 v) y^(-1) + (32 v^(-2) + 512 + 32 v^2)
+ (672 v^(-1) + 672 v) y + (2 v^(-2) + 32 + 2 v^2) y^2 ] r^(7/6)
 ] q^(7/6)
+  [
 [ 20 y^(-2) + (4 v^(-1) + 4 v) y^(-1) + 232 + (4 v^(-1) + 4 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 232 + 20 v^2) y^(-2) + (1056 v^(-1) + 1056 v) y^(-1) + (232 v^(-2) + 3216 + 232 v^2)
+ (1056 v^(-1) + 1056 v) y + (20 v^(-2) + 232 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 24 y^(-2) + 324 + 24 y^2 ] r^(1/3)
+  [ (24 v^(-2) + 324 + 24 v^2) y^(-2) + (288 v^(-1) + 288 v) y^(-1) + (324 v^(-2) + 4536 + 324 v^2)
+ (288 v^(-1) + 288 v) y + (24 v^(-2) + 324 + 24 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 8 y^(-2) + (32 v^(-1) + 32 v) y^(-1) + 112 + (32 v^(-1) + 32 v) y + 8 y^2 ] r^(5/12)
+  [ (8 v^(-2) + 112 + 8 v^2) y^(-2) + (512 v^(-1) + 512 v) y^(-1) + (112 v^(-2) + 1568 + 112 v^2)
+ (512 v^(-1) + 512 v) y + (8 v^(-2) + 112 + 8 v^2) y^2 ] r^(17/12)
 ] q^(17/12)
+  [
 [ 40 y^(-2) + (232 v^(-1) + 232 v) y^(-1) + 456 + (232 v^(-1) + 232 v) y + 40 y^2 ] r^(1/2)
+  [ (40 v^(-2) + 456 + 40 v^2) y^(-2) + (3216 v^(-1) + 3216 v) y^(-1) + (456 v^(-2) + 5256 + 456 v^2)
+ (3216 v^(-1) + 3216 v) y + (40 v^(-2) + 456 + 40 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 96 y^(-2) + (324 v^(-1) + 324 v) y^(-1) + 1104 + (324 v^(-1) + 324 v) y + 96 y^2 ] r^(7/12)
+  [ (96 v^(-2) + 1104 + 96 v^2) y^(-2) + (4536 v^(-1) + 4536 v) y^(-1) + (1104 v^(-2) + 13344 + 1104 v^2)
+ (4536 v^(-1) + 4536 v) y + (96 v^(-2) + 1104 + 96 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 168 y^(-2) + (112 v^(-1) + 112 v) y^(-1) + 1704 + (112 v^(-1) + 112 v) y + 168 y^2 ] r^(2/3)
+  [ (168 v^(-2) + 1704 + 168 v^2) y^(-2) + (1568 v^(-1) + 1568 v) y^(-1) + (1704 v^(-2) + 17376 + 1704 v^2)
+ (1568 v^(-1) + 1568 v) y + (168 v^(-2) + 1704 + 168 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 160 y^(-2) + (456 v^(-1) + 456 v) y^(-1) + 1504 + (456 v^(-1) + 456 v) y + 160 y^2 ] r^(3/4)
+  [ (160 v^(-2) + 1504 + 160 v^2) y^(-2) + (5256 v^(-1) + 5256 v) y^(-1)
+ (1504 v^(-2) + 14368 + 1504 v^2) + (5256 v^(-1) + 5256 v) y + (160 v^(-2) + 1504 + 160 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 72 y^(-2) + (1104 v^(-1) + 1104 v) y^(-1) + 720 + (1104 v^(-1) + 1104 v) y + 72 y^2 ] r^(5/6)
+  [ (72 v^(-2) + 720 + 72 v^2) y^(-2) + (13344 v^(-1) + 13344 v) y^(-1) + (720 v^(-2) + 7200 + 720 v^2)
+ (13344 v^(-1) + 13344 v) y + (72 v^(-2) + 720 + 72 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 672 y^(-2) + (1704 v^(-1) + 1704 v) y^(-1) + 5472 + (1704 v^(-1) + 1704 v) y + 672 y^2 ] r^(11/12)
+  [ (672 v^(-2) + 5472 + 672 v^2) y^(-2) + (17376 v^(-1) + 17376 v) y^(-1)
+ (5472 v^(-2) + 44928 + 5472 v^2) + (17376 v^(-1) + 17376 v) y + (672 v^(-2) + 5472 + 672 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
```

```
+  [
4 y^(-2) + 31 + 4 y^2
+  [ (4 v^(-2) + 1056 + 4 v^2) y^(-2) + (1504 v^(-1) + 1504 v) y^(-1) + (31 v^(-2) + 8588 + 31 v^2)
+ (1504 v^(-1) + 1504 v) y + (4 v^(-2) + 1056 + 4 v^2) y^2 ] r
+  [ (1056 v^(-2) + 8588 + 1056 v^2) y^(-2) + (14368 v^(-1) + 14368 v) y^(-1)
+ (8588 v^(-2) + 70113 + 8588 v^2) + (14368 v^(-1) + 14368 v) y + (1056 v^(-2) + 8588 + 1056 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(1)(2)(10):

```
1
+ (1 v^(-2) + 4 + 1 v^2) r
+ (4 v^(-2) + 32 + 4 v^2) r^2

+  [
1 r^(1/6)
+ (1 v^(-2) + 16 + 1 v^2) r^(7/6)
 ] q^(1/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 24 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (4 v^(-1) + 4 v) y^(-1) + (24 v^(-2) + 296 + 24 v^2) + (4 v^(-1) + 4 v) y ] r^(5/4)
 ] q^(1/4)
+  [
18 r^(1/3)
+ (18 v^(-2) + 234 + 18 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 4 + (1 v^(-1) + 1 v) y ] r^(5/12)
+  [ (16 v^(-1) + 16 v) y^(-1) + (4 v^(-2) + 56 + 4 v^2) + (16 v^(-1) + 16 v) y ] r^(17/12)
 ] q^(5/12)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + 44 + (24 v^(-1) + 24 v) y ] r^(1/2)
+  [ (296 v^(-1) + 296 v) y^(-1) + (44 v^(-2) + 564 + 44 v^2) + (296 v^(-1) + 296 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (18 v^(-1) + 18 v) y^(-1) + 144 + (18 v^(-1) + 18 v) y ] r^(7/12)
+  [ (234 v^(-1) + 234 v) y^(-1) + (144 v^(-2) + 1584 + 144 v^2) + (234 v^(-1) + 234 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 57 + (4 v^(-1) + 4 v) y ] r^(2/3)
+  [ (56 v^(-1) + 56 v) y^(-1) + (57 v^(-2) + 582 + 57 v^2) + (56 v^(-1) + 56 v) y ] r^(5/3)
 ] q^(2/3)
+  [
 [ (44 v^(-1) + 44 v) y^(-1) + 112 + (44 v^(-1) + 44 v) y ] r^(3/4)
+  [ (564 v^(-1) + 564 v) y^(-1) + (112 v^(-2) + 1136 + 112 v^2) + (564 v^(-1) + 564 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (144 v^(-1) + 144 v) y^(-1) + 468 + (144 v^(-1) + 144 v) y ] r^(5/6)
+  [ (1584 v^(-1) + 1584 v) y^(-1) + (468 v^(-2) + 4248 + 468 v^2) + (1584 v^(-1) + 1584 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (57 v^(-1) + 57 v) y^(-1) + 228 + (57 v^(-1) + 57 v) y ] r^(11/12)
+  [ (582 v^(-1) + 582 v) y^(-1) + (228 v^(-2) + 1872 + 228 v^2) + (582 v^(-1) + 582 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 4 + 1 y^2
+  [ (1 v^(-2) + 4 + 1 v^2) y^(-2) + (112 v^(-1) + 112 v) y^(-1) + (4 v^(-2) + 998 + 4 v^2)
+ (112 v^(-1) + 112 v) y + (1 v^(-2) + 4 + 1 v^2) y^2 ] r
+  [ (4 v^(-2) + 32 + 4 v^2) y^(-2) + (1136 v^(-1) + 1136 v) y^(-1) + (998 v^(-2) + 8068 + 998 v^2)
+ (1136 v^(-1) + 1136 v) y + (4 v^(-2) + 32 + 4 v^2) y^2 ] r^2
 ] q
+
 [ (468 v^(-1) + 468 v) y^(-1) + 720 + (468 v^(-1) + 468 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 1 y^(-2) + 16 + 1 y^2 ] r^(1/6)
+  [ (1 v^(-2) + 16 + 1 v^2) y^(-2) + (228 v^(-1) + 228 v) y^(-1) + (16 v^(-2) + 256 + 16 v^2)
+ (228 v^(-1) + 228 v) y + (1 v^(-2) + 16 + 1 v^2) y^2 ] r^(7/6)
 ] q^(7/6)
+  [
 [ 24 y^(-2) + (4 v^(-1) + 4 v) y^(-1) + 296 + (4 v^(-1) + 4 v) y + 24 y^2 ] r^(1/4)
+  [ (24 v^(-2) + 296 + 24 v^2) y^(-2) + (998 v^(-1) + 998 v) y^(-1) + (296 v^(-2) + 4008 + 296 v^2)
+ (998 v^(-1) + 998 v) y + (24 v^(-2) + 296 + 24 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 18 y^(-2) + 234 + 18 y^2 ] r^(1/3)
+  [ (18 v^(-2) + 234 + 18 v^2) y^(-2) + (720 v^(-1) + 720 v) y^(-1) + (234 v^(-2) + 3114 + 234 v^2)
+ (720 v^(-1) + 720 v) y + (18 v^(-2) + 234 + 18 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 4 y^(-2) + (16 v^(-1) + 16 v) y^(-1) + 56 + (16 v^(-1) + 16 v) y + 4 y^2 ] r^(5/12)
+  [ (4 v^(-2) + 56 + 4 v^2) y^(-2) + (256 v^(-1) + 256 v) y^(-1) + (56 v^(-2) + 784 + 56 v^2)
+ (256 v^(-1) + 256 v) y + (4 v^(-2) + 56 + 4 v^2) y^2 ] r^(17/12)
 ] q^(17/12)
+  [
 [ 44 y^(-2) + (296 v^(-1) + 296 v) y^(-1) + 564 + (296 v^(-1) + 296 v) y + 44 y^2 ] r^(1/2)
+  [ (44 v^(-2) + 564 + 44 v^2) y^(-2) + (4008 v^(-1) + 4008 v) y^(-1) + (564 v^(-2) + 7332 + 564 v^2)
+ (4008 v^(-1) + 4008 v) y + (44 v^(-2) + 564 + 44 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 144 y^(-2) + (234 v^(-1) + 234 v) y^(-1) + 1584 + (234 v^(-1) + 234 v) y + 144 y^2 ] r^(7/12)
+  [ (144 v^(-2) + 1584 + 144 v^2) y^(-2) + (3114 v^(-1) + 3114 v) y^(-1)
```

```
+ (1584 v^(-2) + 17712 + 1584 v^2) + (3114 v^(-1) + 3114 v) y + (144 v^(-2) + 1584 + 144 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 57 y^(-2) + (56 v^(-1) + 56 v) y^(-1) + 582 + (56 v^(-1) + 56 v) y + 57 y^2 ] r^(2/3)
+  [ (57 v^(-2) + 582 + 57 v^2) y^(-2) + (784 v^(-1) + 784 v) y^(-1) + (582 v^(-2) + 5988 + 582 v^2)
+ (784 v^(-1) + 784 v) y + (57 v^(-2) + 582 + 57 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 112 y^(-2) + (564 v^(-1) + 564 v) y^(-1) + 1136 + (564 v^(-1) + 564 v) y + 112 y^2 ] r^(3/4)
+  [ (112 v^(-2) + 1136 + 112 v^2) y^(-2) + (7332 v^(-1) + 7332 v) y^(-1)
+ (1136 v^(-2) + 11792 + 1136 v^2) + (7332 v^(-1) + 7332 v) y + (112 v^(-2) + 1136 + 112 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 468 y^(-2) + (1584 v^(-1) + 1584 v) y^(-1) + 4248 + (1584 v^(-1) + 1584 v) y + 468 y^2 ] r^(5/6)
+  [ (468 v^(-2) + 4248 + 468 v^2) y^(-2) + (17712 v^(-1) + 17712 v) y^(-1)
+ (4248 v^(-2) + 38592 + 4248 v^2) + (17712 v^(-1) + 17712 v) y + (468 v^(-2) + 4248 + 468 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 228 y^(-2) + (582 v^(-1) + 582 v) y^(-1) + 1872 + (582 v^(-1) + 582 v) y + 228 y^2 ] r^(11/12)
+  [ (228 v^(-2) + 1872 + 228 v^2) y^(-2) + (5988 v^(-1) + 5988 v) y^(-1)
+ (1872 v^(-2) + 15552 + 1872 v^2) + (5988 v^(-1) + 5988 v) y + (228 v^(-2) + 1872 + 228 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
4 y^(-2) + 32 + 4 y^2
+  [ (4 v^(-2) + 998 + 4 v^2) y^(-2) + (1136 v^(-1) + 1136 v) y^(-1) + (32 v^(-2) + 8068 + 32 v^2)
+ (1136 v^(-1) + 1136 v) y + (4 v^(-2) + 998 + 4 v^2) y^2 ] r
+  [ (998 v^(-2) + 8068 + 998 v^2) y^(-2) + (11792 v^(-1) + 11792 v) y^(-1)
+ (8068 v^(-2) + 65355 + 8068 v^2) + (11792 v^(-1) + 11792 v) y + (998 v^(-2) + 8068 + 998 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (4)(4)(4):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 18 + 3 v^2) r^2

+  [
6 r^(1/6)
+ (6 v^(-2) + 60 + 6 v^2) r^(7/6)
 ] q^(1/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (20 v^(-2) + 228 + 20 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
21 r^(1/3)
+ (21 v^(-2) + 261 + 21 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (6 v^(-1) + 6 v) y^(-1) + 24 + (6 v^(-1) + 6 v) y ] r^(5/12)
+  [ (60 v^(-1) + 60 v) y^(-1) + (24 v^(-2) + 192 + 24 v^2) + (60 v^(-1) + 60 v) y ] r^(17/12)
 ] q^(5/12)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + 60 + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (228 v^(-1) + 228 v) y^(-1) + (60 v^(-2) + 684 + 60 v^2) + (228 v^(-1) + 228 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (21 v^(-1) + 21 v) y^(-1) + 84 + (21 v^(-1) + 21 v) y ] r^(7/12)
+  [ (261 v^(-1) + 261 v) y^(-1) + (84 v^(-2) + 876 + 84 v^2) + (261 v^(-1) + 261 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + 117 + (24 v^(-1) + 24 v) y ] r^(2/3)
+  [ (192 v^(-1) + 192 v) y^(-1) + (117 v^(-2) + 1206 + 117 v^2) + (192 v^(-1) + 192 v) y ] r^(5/3)
 ] q^(2/3)
+  [
 [ (60 v^(-1) + 60 v) y^(-1) + 240 + (60 v^(-1) + 60 v) y ] r^(3/4)
+  [ (684 v^(-1) + 684 v) y^(-1) + (240 v^(-2) + 2256 + 240 v^2) + (684 v^(-1) + 684 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (84 v^(-1) + 84 v) y^(-1) + 108 + (84 v^(-1) + 84 v) y ] r^(5/6)
+  [ (876 v^(-1) + 876 v) y^(-1) + (108 v^(-2) + 1080 + 108 v^2) + (876 v^(-1) + 876 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (117 v^(-1) + 117 v) y^(-1) + 468 + (117 v^(-1) + 117 v) y ] r^(11/12)
+  [ (1206 v^(-1) + 1206 v) y^(-1) + (468 v^(-2) + 3888 + 468 v^2) + (1206 v^(-1) + 1206 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (240 v^(-1) + 240 v) y^(-1) + (3 v^(-2) + 1076 + 3 v^2)
+ (240 v^(-1) + 240 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 18 + 3 v^2) y^(-2) + (2256 v^(-1) + 2256 v) y^(-1) + (1076 v^(-2) + 8721 + 1076 v^2)
+ (2256 v^(-1) + 2256 v) y + (3 v^(-2) + 18 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (108 v^(-1) + 108 v) y^(-1) + 432 + (108 v^(-1) + 108 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 6 y^(-2) + 60 + 6 y^2 ] r^(1/6)
+  [ (6 v^(-2) + 60 + 6 v^2) y^(-2) + (468 v^(-1) + 468 v) y^(-1) + (60 v^(-2) + 816 + 60 v^2)
+ (468 v^(-1) + 468 v) y + (6 v^(-2) + 60 + 6 v^2) y^2 ] r^(7/6)
 ] q^(7/6)
+  [
```

```
 [ 20 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 228 + (3 v^(-1) + 3 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 228 + 20 v^2) y^(-2) + (1076 v^(-1) + 1076 v) y^(-1) + (228 v^(-2) + 3312 + 228 v^2)
+ (1076 v^(-1) + 1076 v) y + (20 v^(-2) + 228 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 21 y^(-2) + 261 + 21 y^2 ] r^(1/3)
+  [ (21 v^(-2) + 261 + 21 v^2) y^(-2) + (432 v^(-1) + 432 v) y^(-1) + (261 v^(-2) + 3429 + 261 v^2)
+ (432 v^(-1) + 432 v) y + (21 v^(-2) + 261 + 21 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 24 y^(-2) + (60 v^(-1) + 60 v) y^(-1) + 192 + (60 v^(-1) + 60 v) y + 24 y^2 ] r^(5/12)
+  [ (24 v^(-2) + 192 + 24 v^2) y^(-2) + (816 v^(-1) + 816 v) y^(-1) + (192 v^(-2) + 2400 + 192 v^2)
+ (816 v^(-1) + 816 v) y + (24 v^(-2) + 192 + 24 v^2) y^2 ] r^(17/12)
 ] q^(17/12)
+  [
 [ 60 y^(-2) + (228 v^(-1) + 228 v) y^(-1) + 684 + (228 v^(-1) + 228 v) y + 60 y^2 ] r^(1/2)
+  [ (60 v^(-2) + 684 + 60 v^2) y^(-2) + (3312 v^(-1) + 3312 v) y^(-1) + (684 v^(-2) + 7884 + 684 v^2)
+ (3312 v^(-1) + 3312 v) y + (60 v^(-2) + 684 + 60 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 84 y^(-2) + (261 v^(-1) + 261 v) y^(-1) + 876 + (261 v^(-1) + 261 v) y + 84 y^2 ] r^(7/12)
+  [ (84 v^(-2) + 876 + 84 v^2) y^(-2) + (3429 v^(-1) + 3429 v) y^(-1) + (876 v^(-2) + 9876 + 876 v^2)
+ (3429 v^(-1) + 3429 v) y + (84 v^(-2) + 876 + 84 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 117 y^(-2) + (192 v^(-1) + 192 v) y^(-1) + 1206 + (192 v^(-1) + 192 v) y + 117 y^2 ] r^(2/3)
+  [ (117 v^(-2) + 1206 + 117 v^2) y^(-2) + (2400 v^(-1) + 2400 v) y^(-1)
+ (1206 v^(-2) + 12564 + 1206 v^2) + (2400 v^(-1) + 2400 v) y + (117 v^(-2) + 1206 + 117 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 240 y^(-2) + (684 v^(-1) + 684 v) y^(-1) + 2256 + (684 v^(-1) + 684 v) y + 240 y^2 ] r^(3/4)
+  [ (240 v^(-2) + 2256 + 240 v^2) y^(-2) + (7884 v^(-1) + 7884 v) y^(-1)
+ (2256 v^(-2) + 21552 + 2256 v^2) + (7884 v^(-1) + 7884 v) y + (240 v^(-2) + 2256 + 240 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 108 y^(-2) + (876 v^(-1) + 876 v) y^(-1) + 1080 + (876 v^(-1) + 876 v) y + 108 y^2 ] r^(5/6)
+  [ (108 v^(-2) + 1080 + 108 v^2) y^(-2) + (9876 v^(-1) + 9876 v) y^(-1)
+ (1080 v^(-2) + 10800 + 1080 v^2) + (9876 v^(-1) + 9876 v) y + (108 v^(-2) + 1080 + 108 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 468 y^(-2) + (1206 v^(-1) + 1206 v) y^(-1) + 3888 + (1206 v^(-1) + 1206 v) y + 468 y^2 ] r^(11/12)
+  [ (468 v^(-2) + 3888 + 468 v^2) y^(-2) + (12564 v^(-1) + 12564 v) y^(-1)
+ (3888 v^(-2) + 32832 + 3888 v^2) + (12564 v^(-1) + 12564 v) y + (468 v^(-2) + 3888 + 468 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
3 y^(-2) + 18 + 3 y^2
+  [ (3 v^(-2) + 1076 + 3 v^2) y^(-2) + (2256 v^(-1) + 2256 v) y^(-1) + (18 v^(-2) + 8721 + 18 v^2)
+ (2256 v^(-1) + 2256 v) y + (3 v^(-2) + 1076 + 3 v^2) y^2 ] r
+  [ (1076 v^(-2) + 8721 + 1076 v^2) y^(-2) + (21552 v^(-1) + 21552 v) y^(-1)
+ (8721 v^(-2) + 71055 + 8721 v^2) + (21552 v^(-1) + 21552 v) y + (1076 v^(-2) + 8721 + 1076 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (3)(3)(8):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 19 + 3 v^2) r^2

+  [
7 r^(1/5)
+ (7 v^(-2) + 90 + 7 v^2) r^(6/5)
 ] q^(1/5)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (20 v^(-2) + 228 + 20 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
38 r^(2/5)
+ (38 v^(-2) + 472 + 38 v^2) r^(7/5)
 ] q^(2/5)
+  [
 [ (7 v^(-1) + 7 v) y^(-1) + 28 + (7 v^(-1) + 7 v) y ] r^(9/20)
+  [ (90 v^(-1) + 90 v) y^(-1) + (28 v^(-2) + 304 + 28 v^2) + (90 v^(-1) + 90 v) y ] r^(29/20)
 ] q^(9/20)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(3/2)
 ] q^(1/2)
+  [
112 r^(3/5)
+ (112 v^(-2) + 1216 + 112 v^2) r^(8/5)
 ] q^(3/5)
+  [
 [ (38 v^(-1) + 38 v) y^(-1) + 152 + (38 v^(-1) + 38 v) y ] r^(13/20)
+  [ (472 v^(-1) + 472 v) y^(-1) + (152 v^(-2) + 1584 + 152 v^2) + (472 v^(-1) + 472 v) y ] r^(33/20)
 ] q^(13/20)
+  [
 [ (28 v^(-1) + 28 v) y^(-1) + (28 v^(-1) + 28 v) y ] r^(7/10)
+  [ (304 v^(-1) + 304 v) y^(-1) + (304 v^(-1) + 304 v) y ] r^(17/10)
 ] q^(7/10)
+  [
143 r^(4/5)
```

```
+ (143 v^(-2) + 1431 + 143 v^2) r^(9/5)
 ] q^(4/5)
+  [
 [ (112 v^(-1) + 112 v) y^(-1) + 448 + (112 v^(-1) + 112 v) y ] r^(17/20)
+  [ (1216 v^(-1) + 1216 v) y^(-1) + (448 v^(-2) + 3968 + 448 v^2) + (1216 v^(-1) + 1216 v) y ] r^(37/20)
 ] q^(17/20)
+  [
 [ (152 v^(-1) + 152 v) y^(-1) + (152 v^(-1) + 152 v) y ] r^(9/10)
+  [ (1584 v^(-1) + 1584 v) y^(-1) + (1584 v^(-1) + 1584 v) y ] r^(19/10)
 ] q^(9/10)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (3 v^(-2) + 1110 + 3 v^2) + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 19 + 3 v^2) y^(-2) + (1110 v^(-2) + 9013 + 1110 v^2) + (3 v^(-2) + 19 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (143 v^(-1) + 143 v) y^(-1) + 572 + (143 v^(-1) + 143 v) y ] r^(21/20)
 q^(21/20)
+
 [ (448 v^(-1) + 448 v) y^(-1) + (448 v^(-1) + 448 v) y ] r^(11/10)
 q^(11/10)
+  [
 [ 7 y^(-2) + 90 + 7 y^2 ] r^(1/5)
+  [ (7 v^(-2) + 90 + 7 v^2) y^(-2) + (90 v^(-2) + 1300 + 90 v^2) + (7 v^(-2) + 90 + 7 v^2) y^2 ] r^(6/5)
 ] q^(6/5)
+  [
 [ 20 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 228 + (3 v^(-1) + 3 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 228 + 20 v^2) y^(-2) + (1110 v^(-1) + 1110 v) y^(-1) + (228 v^(-2) + 3448 + 228 v^2)
+ (1110 v^(-1) + 1110 v) y + (20 v^(-2) + 228 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+
 [ (572 v^(-1) + 572 v) y^(-1) + (572 v^(-1) + 572 v) y ] r^(13/10)
 q^(13/10)
+  [
 [ 38 y^(-2) + 472 + 38 y^2 ] r^(2/5)
+  [ (38 v^(-2) + 472 + 38 v^2) y^(-2) + (472 v^(-2) + 5968 + 472 v^2) + (38 v^(-2) + 472 + 38 v^2) y^2 ] r^(7/5)
 ] q^(7/5)
+  [
 [ 28 y^(-2) + (90 v^(-1) + 90 v) y^(-1) + 304 + (90 v^(-1) + 90 v) y + 28 y^2 ] r^(9/20)
+  [ (28 v^(-2) + 304 + 28 v^2) y^(-2) + (1300 v^(-1) + 1300 v) y^(-1) + (304 v^(-2) + 3872 + 304 v^2)
+ (1300 v^(-1) + 1300 v) y + (28 v^(-2) + 304 + 28 v^2) y^2 ] r^(29/20)
 ] q^(29/20)
+  [
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(1/2)
+  [ (3448 v^(-1) + 3448 v) y^(-1) + (3448 v^(-1) + 3448 v) y ] r^(3/2)
 ] q^(3/2)
+  [
 [ 112 y^(-2) + 1216 + 112 y^2 ] r^(3/5)
+  [ (112 v^(-2) + 1216 + 112 v^2) y^(-2) + (1216 v^(-2) + 13238 + 1216 v^2) + (112 v^(-2) + 1216 + 112 v^2) y^2 ] r^(8/5)
 ] q^(8/5)
+  [
 [ 152 y^(-2) + (472 v^(-1) + 472 v) y^(-1) + 1584 + (472 v^(-1) + 472 v) y + 152 y^2 ] r^(13/20)
+  [ (152 v^(-2) + 1584 + 152 v^2) y^(-2) + (5968 v^(-1) + 5968 v) y^(-1) + (1584 v^(-2) + 16928 + 1584 v^2)
+ (5968 v^(-1) + 5968 v) y + (152 v^(-2) + 1584 + 152 v^2) y^2 ] r^(33/20)
 ] q^(33/20)
+  [
 [ (304 v^(-1) + 304 v) y^(-1) + (304 v^(-1) + 304 v) y ] r^(7/10)
+  [ (3872 v^(-1) + 3872 v) y^(-1) + (3872 v^(-1) + 3872 v) y ] r^(17/10)
 ] q^(17/10)
+  [
 [ 143 y^(-2) + 1431 + 143 y^2 ] r^(4/5)
+  [ (143 v^(-2) + 1431 + 143 v^2) y^(-2) + (1431 v^(-2) + 14327 + 1431 v^2) + (143 v^(-2) + 1431 + 143 v^2) y^2 ] r^(9/5)
 ] q^(9/5)
+  [
 [ 448 y^(-2) + (1216 v^(-1) + 1216 v) y^(-1) + 3968 + (1216 v^(-1) + 1216 v) y + 448 y^2 ] r^(17/20)
+  [ (448 v^(-2) + 3968 + 448 v^2) y^(-2) + (13238 v^(-1) + 13238 v) y^(-1) + (3968 v^(-2) + 35288 + 3968 v^2)
+ (13238 v^(-1) + 13238 v) y + (448 v^(-2) + 3968 + 448 v^2) y^2 ] r^(37/20)
 ] q^(37/20)
+  [
 [ (1584 v^(-1) + 1584 v) y^(-1) + (1584 v^(-1) + 1584 v) y ] r^(9/10)
+  [ (16928 v^(-1) + 16928 v) y^(-1) + (16928 v^(-1) + 16928 v) y ] r^(19/10)
 ] q^(19/10)
+  [
3 y^(-2) + 19 + 3 y^2
+  [ (3 v^(-2) + 1110 + 3 v^2) y^(-2) + (19 v^(-2) + 9013 + 19 v^2) + (3 v^(-2) + 1110 + 3 v^2) y^2 ] r
+  [ (1110 v^(-2) + 9013 + 1110 v^2) y^(-2) + (9013 v^(-2) + 73397 + 9013 v^2) + (1110 v^(-2) + 9013 + 1110 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (2)(6)(6):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 18 + 3 v^2) r^2

+  [
2 r^(1/8)
+ (2 v^(-2) + 20 + 2 v^2) r^(9/8)
 ] q^(1/8)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 32 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (32 v^(-2) + 368 + 32 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
```

```
+  [
 [ (2 v^(-1) + 2 v) y^(-1) + 20 + (2 v^(-1) + 2 v) y ] r^(3/8)
+  [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-2) + 220 + 20 v^2) + (20 v^(-1) + 20 v) y ] r^(11/8)
 ] q^(3/8)
+  [
 [ (32 v^(-1) + 32 v) y^(-1) + 114 + (32 v^(-1) + 32 v) y ] r^(1/2)
+  [ (368 v^(-1) + 368 v) y^(-1) + (114 v^(-2) + 1212 + 114 v^2) + (368 v^(-1) + 368 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + 100 + (20 v^(-1) + 20 v) y ] r^(5/8)
+  [ (220 v^(-1) + 220 v) y^(-1) + (100 v^(-2) + 1076 + 100 v^2) + (220 v^(-1) + 220 v) y ] r^(13/8)
 ] q^(5/8)
+  [
 [ (114 v^(-1) + 114 v) y^(-1) + 384 + (114 v^(-1) + 114 v) y ] r^(3/4)
+  [ (1212 v^(-1) + 1212 v) y^(-1) + (384 v^(-2) + 3648 + 384 v^2) + (1212 v^(-1) + 1212 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (100 v^(-1) + 100 v) y^(-1) + 272 + (100 v^(-1) + 100 v) y ] r^(7/8)
+  [ (1076 v^(-1) + 1076 v) y^(-1) + (272 v^(-2) + 2352 + 272 v^2) + (1076 v^(-1) + 1076 v) y ] r^(15/8)
 ] q^(7/8)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (384 v^(-1) + 384 v) y^(-1) + (3 v^(-2) + 1570 + 3 v^2)
+ (384 v^(-1) + 384 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 18 + 3 v^2) y^(-2) + (3648 v^(-1) + 3648 v) y^(-1) + (1570 v^(-2) + 12615 + 1570 v^2)
+ (3648 v^(-1) + 3648 v) y + (3 v^(-2) + 18 + 3 v^2) y^2 ] r^2
 ] q
+  [
 [ 2 y^(-2) + 20 + 2 y^2 ] r^(1/8)
+  [ (2 v^(-2) + 20 + 2 v^2) y^(-2) + (272 v^(-1) + 272 v) y^(-1) + (20 v^(-2) + 506 + 20 v^2)
+ (272 v^(-1) + 272 v) y + (2 v^(-2) + 20 + 2 v^2) y^2 ] r^(9/8)
 ] q^(9/8)
+  [
 [ 32 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 368 + (3 v^(-1) + 3 v) y + 32 y^2 ] r^(1/4)
+  [ (32 v^(-2) + 368 + 32 v^2) y^(-2) + (1570 v^(-1) + 1570 v) y^(-1) + (368 v^(-2) + 5216 + 368 v^2)
+ (1570 v^(-1) + 1570 v) y + (32 v^(-2) + 368 + 32 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 20 y^(-2) + (20 v^(-1) + 20 v) y^(-1) + 220 + (20 v^(-1) + 20 v) y + 20 y^2 ] r^(3/8)
+  [ (20 v^(-2) + 220 + 20 v^2) y^(-2) + (506 v^(-1) + 506 v) y^(-1) + (220 v^(-2) + 2740 + 220 v^2)
+ (506 v^(-1) + 506 v) y + (20 v^(-2) + 220 + 20 v^2) y^2 ] r^(11/8)
 ] q^(11/8)
+  [
 [ 114 y^(-2) + (368 v^(-1) + 368 v) y^(-1) + 1212 + (368 v^(-1) + 368 v) y + 114 y^2 ] r^(1/2)
+  [ (114 v^(-2) + 1212 + 114 v^2) y^(-2) + (5216 v^(-1) + 5216 v) y^(-1)
+ (1212 v^(-2) + 13896 + 1212 v^2) + (5216 v^(-1) + 5216 v) y + (114 v^(-2) + 1212 + 114 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 100 y^(-2) + (220 v^(-1) + 220 v) y^(-1) + 1076 + (220 v^(-1) + 220 v) y + 100 y^2 ] r^(5/8)
+  [ (100 v^(-2) + 1076 + 100 v^2) y^(-2) + (2740 v^(-1) + 2740 v) y^(-1)
+ (1076 v^(-2) + 11716 + 1076 v^2) + (2740 v^(-1) + 2740 v) y + (100 v^(-2) + 1076 + 100 v^2) y^2 ] r^(13/8)
 ] q^(13/8)
+  [
 [ 384 y^(-2) + (1212 v^(-1) + 1212 v) y^(-1) + 3648 + (1212 v^(-1) + 1212 v) y + 384 y^2 ] r^(3/4)
+  [ (384 v^(-2) + 3648 + 384 v^2) y^(-2) + (13896 v^(-1) + 13896 v) y^(-1)
+ (3648 v^(-2) + 35200 + 3648 v^2) + (13896 v^(-1) + 13896 v) y + (384 v^(-2) + 3648 + 384 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 272 y^(-2) + (1076 v^(-1) + 1076 v) y^(-1) + 2352 + (1076 v^(-1) + 1076 v) y + 272 y^2 ] r^(7/8)
+  [ (272 v^(-2) + 2352 + 272 v^2) y^(-2) + (11716 v^(-1) + 11716 v) y^(-1)
+ (2352 v^(-2) + 20880 + 2352 v^2) + (11716 v^(-1) + 11716 v) y + (272 v^(-2) + 2352 + 272 v^2) y^2 ] r^(15/8)
 ] q^(15/8)
+  [
3 y^(-2) + 18 + 3 y^2
+  [ (3 v^(-2) + 1570 + 3 v^2) y^(-2) + (3648 v^(-1) + 3648 v) y^(-1) + (18 v^(-2) + 12615 + 18 v^2)
+ (3648 v^(-1) + 3648 v) y + (3 v^(-2) + 1570 + 3 v^2) y^2 ] r
+  [ (1570 v^(-2) + 12615 + 1570 v^2) y^(-2) + (35200 v^(-1) + 35200 v) y^(-1)
+ (12615 v^(-2) + 102701 + 12615 v^2) + (35200 v^(-1) + 35200 v) y + (1570 v^(-2) + 12615 + 1570 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (2)(4)(10):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 18 + 3 v^2) r^2

+  [
3 r^(1/6)
+ (3 v^(-2) + 37 + 3 v^2) r^(7/6)
 ] q^(1/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 26 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (26 v^(-2) + 300 + 26 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
9 r^(1/3)
+ (9 v^(-2) + 117 + 9 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (3 v^(-1) + 3 v) y^(-1) + 20 + (3 v^(-1) + 3 v) y ] r^(5/12)
+  [ (37 v^(-1) + 37 v) y^(-1) + (20 v^(-2) + 220 + 20 v^2) + (37 v^(-1) + 37 v) y ] r^(17/12)
 ] q^(5/12)
```

```
+  [
 [ (26 v^(-1) + 26 v) y^(-1) + 75 + (26 v^(-1) + 26 v) y ] r^(1/2)
+  [ (300 v^(-1) + 300 v) y^(-1) + (75 v^(-2) + 849 + 75 v^2) + (300 v^(-1) + 300 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (9 v^(-1) + 9 v) y^(-1) + 72 + (9 v^(-1) + 9 v) y ] r^(7/12)
+  [ (117 v^(-1) + 117 v) y^(-1) + (72 v^(-2) + 792 + 72 v^2) + (117 v^(-1) + 117 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + 93 + (20 v^(-1) + 20 v) y ] r^(2/3)
+  [ (220 v^(-1) + 220 v) y^(-1) + (93 v^(-2) + 933 + 93 v^2) + (220 v^(-1) + 220 v) y ] r^(5/3)
 ] q^(2/3)
+  [
 [ (75 v^(-1) + 75 v) y^(-1) + 220 + (75 v^(-1) + 75 v) y ] r^(3/4)
+  [ (849 v^(-1) + 849 v) y^(-1) + (220 v^(-2) + 2204 + 220 v^2) + (849 v^(-1) + 849 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (72 v^(-1) + 72 v) y^(-1) + 234 + (72 v^(-1) + 72 v) y ] r^(5/6)
+  [ (792 v^(-1) + 792 v) y^(-1) + (234 v^(-2) + 2124 + 234 v^2) + (792 v^(-1) + 792 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (93 v^(-1) + 93 v) y^(-1) + 316 + (93 v^(-1) + 93 v) y ] r^(11/12)
+  [ (933 v^(-1) + 933 v) y^(-1) + (316 v^(-2) + 2684 + 316 v^2) + (933 v^(-1) + 933 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (220 v^(-1) + 220 v) y^(-1) + (3 v^(-2) + 1143 + 3 v^2)
+ (220 v^(-1) + 220 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 18 + 3 v^2) y^(-2) + (2204 v^(-1) + 2204 v) y^(-1) + (1143 v^(-2) + 9291 + 1143 v^2)
+ (2204 v^(-1) + 2204 v) y + (3 v^(-2) + 18 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (234 v^(-1) + 234 v) y^(-1) + 360 + (234 v^(-1) + 234 v) y ] r^(13/12)
 q^(13/12)
+  [
 [ 3 y^(-2) + 37 + 3 y^2 ] r^(1/6)
+  [ (3 v^(-2) + 37 + 3 v^2) y^(-2) + (316 v^(-1) + 316 v) y^(-1) + (37 v^(-2) + 810 + 37 v^2)
+ (316 v^(-1) + 316 v) y + (3 v^(-2) + 37 + 3 v^2) y^2 ] r^(7/6)
 ] q^(7/6)
+  [
 [ 26 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 300 + (3 v^(-1) + 3 v) y + 26 y^2 ] r^(1/4)
+  [ (26 v^(-2) + 300 + 26 v^2) y^(-2) + (1143 v^(-1) + 1143 v) y^(-1) + (300 v^(-2) + 4236 + 300 v^2)
+ (1143 v^(-1) + 1143 v) y + (26 v^(-2) + 300 + 26 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 9 y^(-2) + 117 + 9 y^2 ] r^(1/3)
+  [ (9 v^(-2) + 117 + 9 v^2) y^(-2) + (360 v^(-1) + 360 v) y^(-1) + (117 v^(-2) + 1557 + 117 v^2)
+ (360 v^(-1) + 360 v) y + (9 v^(-2) + 117 + 9 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 20 y^(-2) + (37 v^(-1) + 37 v) y^(-1) + 220 + (37 v^(-1) + 37 v) y + 20 y^2 ] r^(5/12)
+  [ (20 v^(-2) + 220 + 20 v^2) y^(-2) + (810 v^(-1) + 810 v) y^(-1) + (220 v^(-2) + 2696 + 220 v^2)
+ (810 v^(-1) + 810 v) y + (20 v^(-2) + 220 + 20 v^2) y^2 ] r^(17/12)
 ] q^(17/12)
+  [
 [ 75 y^(-2) + (300 v^(-1) + 300 v) y^(-1) + 849 + (300 v^(-1) + 300 v) y + 75 y^2 ] r^(1/2)
+  [ (75 v^(-2) + 849 + 75 v^2) y^(-2) + (4236 v^(-1) + 4236 v) y^(-1) + (849 v^(-2) + 9983 + 849 v^2)
+ (4236 v^(-1) + 4236 v) y + (75 v^(-2) + 849 + 75 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 72 y^(-2) + (117 v^(-1) + 117 v) y^(-1) + 792 + (117 v^(-1) + 117 v) y + 72 y^2 ] r^(7/12)
+  [ (72 v^(-2) + 792 + 72 v^2) y^(-2) + (1557 v^(-1) + 1557 v) y^(-1) + (792 v^(-2) + 8856 + 792 v^2)
+ (1557 v^(-1) + 1557 v) y + (72 v^(-2) + 792 + 72 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 93 y^(-2) + (220 v^(-1) + 220 v) y^(-1) + 933 + (220 v^(-1) + 220 v) y + 93 y^2 ] r^(2/3)
+  [ (93 v^(-2) + 933 + 93 v^2) y^(-2) + (2696 v^(-1) + 2696 v) y^(-1) + (933 v^(-2) + 9402 + 933 v^2)
+ (2696 v^(-1) + 2696 v) y + (93 v^(-2) + 933 + 93 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 220 y^(-2) + (849 v^(-1) + 849 v) y^(-1) + 2204 + (849 v^(-1) + 849 v) y + 220 y^2 ] r^(3/4)
+  [ (220 v^(-2) + 2204 + 220 v^2) y^(-2) + (9983 v^(-1) + 9983 v) y^(-1)
+ (2204 v^(-2) + 22572 + 2204 v^2) + (9983 v^(-1) + 9983 v) y + (220 v^(-2) + 2204 + 220 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 234 y^(-2) + (792 v^(-1) + 792 v) y^(-1) + 2124 + (792 v^(-1) + 792 v) y + 234 y^2 ] r^(5/6)
+  [ (234 v^(-2) + 2124 + 234 v^2) y^(-2) + (8856 v^(-1) + 8856 v) y^(-1)
+ (2124 v^(-2) + 19296 + 2124 v^2) + (8856 v^(-1) + 8856 v) y + (234 v^(-2) + 2124 + 234 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 316 y^(-2) + (933 v^(-1) + 933 v) y^(-1) + 2684 + (933 v^(-1) + 933 v) y + 316 y^2 ] r^(11/12)
+  [ (316 v^(-2) + 2684 + 316 v^2) y^(-2) + (9402 v^(-1) + 9402 v) y^(-1)
+ (2684 v^(-2) + 23176 + 2684 v^2) + (9402 v^(-1) + 9402 v) y + (316 v^(-2) + 2684 + 316 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
3 y^(-2) + 18 + 3 y^2
+  [ (3 v^(-2) + 1143 + 3 v^2) y^(-2) + (2204 v^(-1) + 2204 v) y^(-1) + (18 v^(-2) + 9291 + 18 v^2)
+ (2204 v^(-1) + 2204 v) y + (3 v^(-2) + 1143 + 3 v^2) y^2 ] r
+  [ (1143 v^(-2) + 9291 + 1143 v^2) y^(-2) + (22572 v^(-1) + 22572 v) y^(-1)
+ (9291 v^(-2) + 75834 + 9291 v^2) + (22572 v^(-1) + 22572 v) y + (1143 v^(-2) + 9291 + 1143 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (2)(3)(18):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 18 + 3 v^2) r^2

+  [
1 r^(1/10)
+ (1 v^(-2) + 9 + 1 v^2) r^(11/10)
 ] q^(1/10)
+  [
4 r^(1/5)
+ (4 v^(-2) + 46 + 4 v^2) r^(6/5)
 ] q^(1/5)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 28 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (28 v^(-2) + 316 + 28 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
1 r^(3/10)
+ (1 v^(-2) + 17 + 1 v^2) r^(13/10)
 ] q^(3/10)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 4 + (1 v^(-1) + 1 v) y ] r^(7/20)
+  [ (9 v^(-1) + 9 v) y^(-1) + (4 v^(-2) + 28 + 4 v^2) + (9 v^(-1) + 9 v) y ] r^(27/20)
 ] q^(7/20)
+  [
21 r^(2/5)
+ (21 v^(-2) + 255 + 21 v^2) r^(7/5)
 ] q^(2/5)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 24 + (4 v^(-1) + 4 v) y ] r^(9/20)
+  [ (46 v^(-1) + 46 v) y^(-1) + (24 v^(-2) + 264 + 24 v^2) + (46 v^(-1) + 46 v) y ] r^(29/20)
 ] q^(9/20)
+  [
 [ (28 v^(-1) + 28 v) y^(-1) + 62 + (28 v^(-1) + 28 v) y ] r^(1/2)
+  [ (316 v^(-1) + 316 v) y^(-1) + (62 v^(-2) + 625 + 62 v^2) + (316 v^(-1) + 316 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 4 + (1 v^(-1) + 1 v) y ] r^(11/20)
+  [ (17 v^(-1) + 17 v) y^(-1) + (4 v^(-2) + 60 + 4 v^2) + (17 v^(-1) + 17 v) y ] r^(31/20)
 ] q^(11/20)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 34 + (4 v^(-1) + 4 v) y ] r^(3/5)
+  [ (28 v^(-1) + 28 v) y^(-1) + (34 v^(-2) + 363 + 34 v^2) + (28 v^(-1) + 28 v) y ] r^(8/5)
 ] q^(3/5)
+  [
 [ (21 v^(-1) + 21 v) y^(-1) + 148 + (21 v^(-1) + 21 v) y ] r^(13/20)
+  [ (255 v^(-1) + 255 v) y^(-1) + (148 v^(-2) + 1524 + 148 v^2) + (255 v^(-1) + 255 v) y ] r^(33/20)
 ] q^(13/20)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + 81 + (24 v^(-1) + 24 v) y ] r^(7/10)
+  [ (264 v^(-1) + 264 v) y^(-1) + (81 v^(-2) + 825 + 81 v^2) + (264 v^(-1) + 264 v) y ] r^(17/10)
 ] q^(7/10)
+  [
 [ (62 v^(-1) + 62 v) y^(-1) + 120 + (62 v^(-1) + 62 v) y ] r^(3/4)
+  [ (625 v^(-1) + 625 v) y^(-1) + (120 v^(-2) + 1108 + 120 v^2) + (625 v^(-1) + 625 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 25 + (4 v^(-1) + 4 v) y ] r^(4/5)
+  [ (60 v^(-1) + 60 v) y^(-1) + (25 v^(-2) + 230 + 25 v^2) + (60 v^(-1) + 60 v) y ] r^(9/5)
 ] q^(4/5)
+  [
 [ (34 v^(-1) + 34 v) y^(-1) + 152 + (34 v^(-1) + 34 v) y ] r^(17/20)
+  [ (363 v^(-1) + 363 v) y^(-1) + (152 v^(-2) + 1324 + 152 v^2) + (363 v^(-1) + 363 v) y ] r^(37/20)
 ] q^(17/20)
+  [
 [ (148 v^(-1) + 148 v) y^(-1) + 365 + (148 v^(-1) + 148 v) y ] r^(9/10)
+  [ (1524 v^(-1) + 1524 v) y^(-1) + (365 v^(-2) + 3248 + 365 v^2) + (1524 v^(-1) + 1524 v) y ] r^(19/10)
 ] q^(9/10)
+  [
 [ (81 v^(-1) + 81 v) y^(-1) + 196 + (81 v^(-1) + 81 v) y ] r^(19/20)
+  [ (825 v^(-1) + 825 v) y^(-1) + (196 v^(-2) + 1372 + 196 v^2) + (825 v^(-1) + 825 v) y ] r^(39/20)
 ] q^(19/20)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (120 v^(-1) + 120 v) y^(-1) + (3 v^(-2) + 980 + 3 v^2)
+ (120 v^(-1) + 120 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 18 + 3 v^2) y^(-2) + (1108 v^(-1) + 1108 v) y^(-1) + (980 v^(-2) + 7859 + 980 v^2)
+ (1108 v^(-1) + 1108 v) y + (3 v^(-2) + 18 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (25 v^(-1) + 25 v) y^(-1) + 172 + (25 v^(-1) + 25 v) y ] r^(21/20)
 q^(21/20)
+  [
 [ 1 y^(-2) + 9 + 1 y^2 ] r^(1/10)
+  [ (1 v^(-2) + 9 + 1 v^2) y^(-2) + (152 v^(-1) + 152 v) y^(-1) + (9 v^(-2) + 266 + 9 v^2)
+ (152 v^(-1) + 152 v) y + (1 v^(-2) + 9 + 1 v^2) y^2 ] r^(11/10)
 ] q^(11/10)
+
 [ (365 v^(-1) + 365 v) y^(-1) + 436 + (365 v^(-1) + 365 v) y ] r^(23/20)
 q^(23/20)
```

```
+  [
 [ 4 y^(-2) + 46 + 4 y^2 ] r^(1/5)
+  [ (4 v^(-2) + 46 + 4 v^2) y^(-2) + (196 v^(-1) + 196 v) y^(-1) + (46 v^(-2) + 529 + 46 v^2)
+ (196 v^(-1) + 196 v) y + (4 v^(-2) + 46 + 4 v^2) y^2 ] r^(6/5)
 ] q^(6/5)
+  [
 [ 28 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 316 + (3 v^(-1) + 3 v) y + 28 y^2 ] r^(1/4)
+  [ (28 v^(-2) + 316 + 28 v^2) y^(-2) + (980 v^(-1) + 980 v) y^(-1) + (316 v^(-2) + 3904 + 316 v^2)
+ (980 v^(-1) + 980 v) y + (28 v^(-2) + 316 + 28 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 1 y^(-2) + 17 + 1 y^2 ] r^(3/10)
+  [ (1 v^(-2) + 17 + 1 v^2) y^(-2) + (172 v^(-1) + 172 v) y^(-1) + (17 v^(-2) + 577 + 17 v^2)
+ (172 v^(-1) + 172 v) y + (1 v^(-2) + 17 + 1 v^2) y^2 ] r^(13/10)
 ] q^(13/10)
+  [
 [ 4 y^(-2) + (9 v^(-1) + 9 v) y^(-1) + 28 + (9 v^(-1) + 9 v) y + 4 y^2 ] r^(7/20)
+  [ (4 v^(-2) + 28 + 4 v^2) y^(-2) + (266 v^(-1) + 266 v) y^(-1) + (28 v^(-2) + 680 + 28 v^2)
+ (266 v^(-1) + 266 v) y + (4 v^(-2) + 28 + 4 v^2) y^2 ] r^(27/20)
 ] q^(27/20)
+  [
 [ 21 y^(-2) + 255 + 21 y^2 ] r^(2/5)
+  [ (21 v^(-2) + 255 + 21 v^2) y^(-2) + (436 v^(-1) + 436 v) y^(-1) + (255 v^(-2) + 3242 + 255 v^2)
+ (436 v^(-1) + 436 v) y + (21 v^(-2) + 255 + 21 v^2) y^2 ] r^(7/5)
 ] q^(7/5)
+  [
 [ 24 y^(-2) + (46 v^(-1) + 46 v) y^(-1) + 264 + (46 v^(-1) + 46 v) y + 24 y^2 ] r^(9/20)
+  [ (24 v^(-2) + 264 + 24 v^2) y^(-2) + (529 v^(-1) + 529 v) y^(-1) + (264 v^(-2) + 3012 + 264 v^2)
+ (529 v^(-1) + 529 v) y + (24 v^(-2) + 264 + 24 v^2) y^2 ] r^(29/20)
 ] q^(29/20)
+  [
 [ 62 y^(-2) + (316 v^(-1) + 316 v) y^(-1) + 625 + (316 v^(-1) + 316 v) y + 62 y^2 ] r^(1/2)
+  [ (62 v^(-2) + 625 + 62 v^2) y^(-2) + (3904 v^(-1) + 3904 v) y^(-1) + (625 v^(-2) + 6458 + 625 v^2)
+ (3904 v^(-1) + 3904 v) y + (62 v^(-2) + 625 + 62 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 4 y^(-2) + (17 v^(-1) + 17 v) y^(-1) + 60 + (17 v^(-1) + 17 v) y + 4 y^2 ] r^(11/20)
+  [ (4 v^(-2) + 60 + 4 v^2) y^(-2) + (577 v^(-1) + 577 v) y^(-1) + (60 v^(-2) + 900 + 60 v^2)
+ (577 v^(-1) + 577 v) y + (4 v^(-2) + 60 + 4 v^2) y^2 ] r^(31/20)
 ] q^(31/20)
+  [
 [ 34 y^(-2) + (28 v^(-1) + 28 v) y^(-1) + 363 + (28 v^(-1) + 28 v) y + 34 y^2 ] r^(3/5)
+  [ (34 v^(-2) + 363 + 34 v^2) y^(-2) + (680 v^(-1) + 680 v) y^(-1) + (363 v^(-2) + 3969 + 363 v^2)
+ (680 v^(-1) + 680 v) y + (34 v^(-2) + 363 + 34 v^2) y^2 ] r^(8/5)
 ] q^(8/5)
+  [
 [ 148 y^(-2) + (255 v^(-1) + 255 v) y^(-1) + 1524 + (255 v^(-1) + 255 v) y + 148 y^2 ] r^(13/20)
+  [ (148 v^(-2) + 1524 + 148 v^2) y^(-2) + (3242 v^(-1) + 3242 v) y^(-1)
+ (1524 v^(-2) + 16280 + 1524 v^2) + (3242 v^(-1) + 3242 v) y + (148 v^(-2) + 1524 + 148 v^2) y^2 ] r^(33/20)
 ] q^(33/20)
+  [
 [ 81 y^(-2) + (264 v^(-1) + 264 v) y^(-1) + 825 + (264 v^(-1) + 264 v) y + 81 y^2 ] r^(7/10)
+  [ (81 v^(-2) + 825 + 81 v^2) y^(-2) + (3012 v^(-1) + 3012 v) y^(-1) + (825 v^(-2) + 8577 + 825 v^2)
+ (3012 v^(-1) + 3012 v) y + (81 v^(-2) + 825 + 81 v^2) y^2 ] r^(17/10)
 ] q^(17/10)
+  [
 [ 120 y^(-2) + (625 v^(-1) + 625 v) y^(-1) + 1108 + (625 v^(-1) + 625 v) y + 120 y^2 ] r^(3/4)
+  [ (120 v^(-2) + 1108 + 120 v^2) y^(-2) + (6458 v^(-1) + 6458 v) y^(-1)
+ (1108 v^(-2) + 10424 + 1108 v^2) + (6458 v^(-1) + 6458 v) y + (120 v^(-2) + 1108 + 120 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 25 y^(-2) + (60 v^(-1) + 60 v) y^(-1) + 230 + (60 v^(-1) + 60 v) y + 25 y^2 ] r^(4/5)
+  [ (25 v^(-2) + 230 + 25 v^2) y^(-2) + (900 v^(-1) + 900 v) y^(-1) + (230 v^(-2) + 2116 + 230 v^2)
+ (900 v^(-1) + 900 v) y + (25 v^(-2) + 230 + 25 v^2) y^2 ] r^(9/5)
 ] q^(9/5)
+  [
 [ 152 y^(-2) + (363 v^(-1) + 363 v) y^(-1) + 1324 + (363 v^(-1) + 363 v) y + 152 y^2 ] r^(17/20)
+  [ (152 v^(-2) + 1324 + 152 v^2) y^(-2) + (3969 v^(-1) + 3969 v) y^(-1)
+ (1324 v^(-2) + 11908 + 1324 v^2) + (3969 v^(-1) + 3969 v) y + (152 v^(-2) + 1324 + 152 v^2) y^2 ] r^(37/20)
 ] q^(37/20)
+  [
 [ 365 y^(-2) + (1524 v^(-1) + 1524 v) y^(-1) + 3248 + (1524 v^(-1) + 1524 v) y + 365 y^2 ] r^(9/10)
+  [ (365 v^(-2) + 3248 + 365 v^2) y^(-2) + (16280 v^(-1) + 16280 v) y^(-1) + (3248 v^(-2) + 29072 + 3248 v^2)
+ (16280 v^(-1) + 16280 v) y + (365 v^(-2) + 3248 + 365 v^2) y^2 ] r^(19/10)
 ] q^(19/10)
+  [
 [ 196 y^(-2) + (825 v^(-1) + 825 v) y^(-1) + 1372 + (825 v^(-1) + 825 v) y + 196 y^2 ] r^(19/20)
+  [ (196 v^(-2) + 1372 + 196 v^2) y^(-2) + (8577 v^(-1) + 8577 v) y^(-1)
+ (1372 v^(-2) + 9604 + 1372 v^2) + (8577 v^(-1) + 8577 v) y + (196 v^(-2) + 1372 + 196 v^2) y^2 ] r^(39/20)
 ] q^(39/20)
+  [
3 y^(-2) + 18 + 3 y^2
+  [ (3 v^(-2) + 980 + 3 v^2) y^(-2) + (1108 v^(-1) + 1108 v) y^(-1) + (18 v^(-2) + 7859 + 18 v^2)
+ (1108 v^(-1) + 1108 v) y + (3 v^(-2) + 980 + 3 v^2) y^2 ] r
+  [ (980 v^(-2) + 7859 + 980 v^2) y^(-2) + (10424 v^(-1) + 10424 v) y^(-1)
+ (7859 v^(-2) + 63465 + 7859 v^2) + (10424 v^(-1) + 10424 v) y + (980 v^(-2) + 7859 + 980 v^2) y^2 ] r^2
 ] q^2
```

```
Partition function for (1)(10)(10):

1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 17 + 3 v^2) r^2

+  [
2 r^(1/12)
+ (2 v^(-2) + 20 + 2 v^2) r^(13/12)
 ] q^(1/12)
+  [
4 r^(1/6)
+ (4 v^(-2) + 46 + 4 v^2) r^(7/6)
 ] q^(1/6)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 22 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (22 v^(-2) + 260 + 22 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
 [ (2 v^(-1) + 2 v) y^(-1) + 39 + (2 v^(-1) + 2 v) y ] r^(1/3)
+  [ (20 v^(-1) + 20 v) y^(-1) + (39 v^(-2) + 383 + 39 v^2) + (20 v^(-1) + 20 v) y ] r^(4/3)
 ] q^(1/3)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 28 + (4 v^(-1) + 4 v) y ] r^(5/12)
+  [ (46 v^(-1) + 46 v) y^(-1) + (28 v^(-2) + 284 + 28 v^2) + (46 v^(-1) + 46 v) y ] r^(17/12)
 ] q^(5/12)
+  [
 [ (22 v^(-1) + 22 v) y^(-1) + 50 + (22 v^(-1) + 22 v) y ] r^(1/2)
+  [ (260 v^(-1) + 260 v) y^(-1) + (50 v^(-2) + 558 + 50 v^2) + (260 v^(-1) + 260 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (39 v^(-1) + 39 v) y^(-1) + 176 + (39 v^(-1) + 39 v) y ] r^(7/12)
+  [ (383 v^(-1) + 383 v) y^(-1) + (176 v^(-2) + 1520 + 176 v^2) + (383 v^(-1) + 383 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (28 v^(-1) + 28 v) y^(-1) + 132 + (28 v^(-1) + 28 v) y ] r^(2/3)
+  [ (284 v^(-1) + 284 v) y^(-1) + (132 v^(-2) + 1266 + 132 v^2) + (284 v^(-1) + 284 v) y ] r^(5/3)
 ] q^(2/3)
+  [
 [ (50 v^(-1) + 50 v) y^(-1) + 206 + (50 v^(-1) + 50 v) y ] r^(3/4)
+  [ (558 v^(-1) + 558 v) y^(-1) + (206 v^(-2) + 1836 + 206 v^2) + (558 v^(-1) + 558 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (176 v^(-1) + 176 v) y^(-1) + 324 + (176 v^(-1) + 176 v) y ] r^(5/6)
+  [ (1520 v^(-1) + 1520 v) y^(-1) + (324 v^(-2) + 2648 + 324 v^2) + (1520 v^(-1) + 1520 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (132 v^(-1) + 132 v) y^(-1) + 336 + (132 v^(-1) + 132 v) y ] r^(11/12)
+  [ (1266 v^(-1) + 1266 v) y^(-1) + (336 v^(-2) + 2664 + 336 v^2) + (1266 v^(-1) + 1266 v) y ] r^(23/12)
 ] q^(11/12)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (206 v^(-1) + 206 v) y^(-1) + (3 v^(-2) + 1136 + 3 v^2)
+ (206 v^(-1) + 206 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 17 + 3 v^2) y^(-2) + (1836 v^(-1) + 1836 v) y^(-1) + (1136 v^(-2) + 9157 + 1136 v^2)
+ (1836 v^(-1) + 1836 v) y + (3 v^(-2) + 17 + 3 v^2) y^2 ] r^2
 ] q
+  [
 [ 2 y^(-2) + 20 + 2 y^2 ] r^(1/12)
+  [ (2 v^(-2) + 20 + 2 v^2) y^(-2) + (324 v^(-1) + 324 v) y^(-1) + (20 v^(-2) + 736 + 20 v^2)
+ (324 v^(-1) + 324 v) y + (2 v^(-2) + 20 + 2 v^2) y^2 ] r^(13/12)
 ] q^(13/12)
+  [
 [ 4 y^(-2) + 46 + 4 y^2 ] r^(1/6)
+  [ (4 v^(-2) + 46 + 4 v^2) y^(-2) + (336 v^(-1) + 336 v) y^(-1) + (46 v^(-2) + 652 + 46 v^2)
+ (336 v^(-1) + 336 v) y + (4 v^(-2) + 46 + 4 v^2) y^2 ] r^(7/6)
 ] q^(7/6)
+  [
 [ 22 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 260 + (3 v^(-1) + 3 v) y + 22 y^2 ] r^(1/4)
+  [ (22 v^(-2) + 260 + 22 v^2) y^(-2) + (1136 v^(-1) + 1136 v) y^(-1) + (260 v^(-2) + 3456 + 260 v^2)
+ (1136 v^(-1) + 1136 v) y + (22 v^(-2) + 260 + 22 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 39 y^(-2) + (20 v^(-1) + 20 v) y^(-1) + 383 + (20 v^(-1) + 20 v) y + 39 y^2 ] r^(1/3)
+  [ (39 v^(-2) + 383 + 39 v^2) y^(-2) + (736 v^(-1) + 736 v) y^(-1) + (383 v^(-2) + 4399 + 383 v^2)
+ (736 v^(-1) + 736 v) y + (39 v^(-2) + 383 + 39 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 28 y^(-2) + (46 v^(-1) + 46 v) y^(-1) + 284 + (46 v^(-1) + 46 v) y + 28 y^2 ] r^(5/12)
+  [ (28 v^(-2) + 284 + 28 v^2) y^(-2) + (652 v^(-1) + 652 v) y^(-1) + (284 v^(-2) + 3484 + 284 v^2)
+ (652 v^(-1) + 652 v) y + (28 v^(-2) + 284 + 28 v^2) y^2 ] r^(17/12)
 ] q^(17/12)
+  [
 [ 50 y^(-2) + (260 v^(-1) + 260 v) y^(-1) + 558 + (260 v^(-1) + 260 v) y + 50 y^2 ] r^(1/2)
+  [ (50 v^(-2) + 558 + 50 v^2) y^(-2) + (3456 v^(-1) + 3456 v) y^(-1) + (558 v^(-2) + 6442 + 558 v^2)
+ (3456 v^(-1) + 3456 v) y + (50 v^(-2) + 558 + 50 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 176 y^(-2) + (383 v^(-1) + 383 v) y^(-1) + 1520 + (383 v^(-1) + 383 v) y + 176 y^2 ] r^(7/12)
+  [ (176 v^(-2) + 1520 + 176 v^2) y^(-2) + (4399 v^(-1) + 4399 v) y^(-1)
+ (1520 v^(-2) + 14720 + 1520 v^2) + (4399 v^(-1) + 4399 v) y + (176 v^(-2) + 1520 + 176 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
```

```
+  [
 [ 132 y^(-2) + (284 v^(-1) + 284 v) y^(-1) + 1266 + (284 v^(-1) + 284 v) y + 132 y^2 ] r^(2/3)
+  [ (132 v^(-2) + 1266 + 132 v^2) y^(-2) + (3484 v^(-1) + 3484 v) y^(-1)
+ (1266 v^(-2) + 12708 + 1266 v^2) + (3484 v^(-1) + 3484 v) y + (132 v^(-2) + 1266 + 132 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 206 y^(-2) + (558 v^(-1) + 558 v) y^(-1) + 1836 + (558 v^(-1) + 558 v) y + 206 y^2 ] r^(3/4)
+  [ (206 v^(-2) + 1836 + 206 v^2) y^(-2) + (6442 v^(-1) + 6442 v) y^(-1)
+ (1836 v^(-2) + 17024 + 1836 v^2) + (6442 v^(-1) + 6442 v) y + (206 v^(-2) + 1836 + 206 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 324 y^(-2) + (1520 v^(-1) + 1520 v) y^(-1) + 2648 + (1520 v^(-1) + 1520 v) y + 324 y^2 ] r^(5/6)
+  [ (324 v^(-2) + 2648 + 324 v^2) y^(-2) + (14720 v^(-1) + 14720 v) y^(-1) + (2648 v^(-2) + 23232 + 2648 v^2)
+ (14720 v^(-1) + 14720 v) y + (324 v^(-2) + 2648 + 324 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 336 y^(-2) + (1266 v^(-1) + 1266 v) y^(-1) + 2664 + (1266 v^(-1) + 1266 v) y + 336 y^2 ] r^(11/12)
+  [ (336 v^(-2) + 2664 + 336 v^2) y^(-2) + (12708 v^(-1) + 12708 v) y^(-1)
+ (2664 v^(-2) + 21744 + 2664 v^2) + (12708 v^(-1) + 12708 v) y + (336 v^(-2) + 2664 + 336 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
3 y^(-2) + 17 + 3 y^2
+  [ (3 v^(-2) + 1136 + 3 v^2) y^(-2) + (1836 v^(-1) + 1836 v) y^(-1) + (17 v^(-2) + 9157 + 17 v^2)
+ (1836 v^(-1) + 1836 v) y + (3 v^(-2) + 1136 + 3 v^2) y^2 ] r
+  [ (1136 v^(-2) + 9157 + 1136 v^2) y^(-2) + (17024 v^(-1) + 17024 v) y^(-1) + (9157 v^(-2) + 74391 + 9157 v^2)
+ (17024 v^(-1) + 17024 v) y + (1136 v^(-2) + 9157 + 1136 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(8)(13):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 18 + 3 v^2) r^2

+  [
1 r^(2/15)
+ (1 v^(-2) + 9 + 1 v^2) r^(17/15)
 ] q^(2/15)
+  [
5 r^(1/5)
+ (5 v^(-2) + 70 + 5 v^2) r^(6/5)
 ] q^(1/5)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (20 v^(-2) + 228 + 20 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
13 r^(1/3)
+ (13 v^(-2) + 140 + 13 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 4 + (1 v^(-1) + 1 v) y ] r^(23/60)
+  [ (9 v^(-1) + 9 v) y^(-1) + (4 v^(-2) + 28 + 4 v^2) + (9 v^(-1) + 9 v) y ] r^(83/60)
 ] q^(23/60)
+  [
17 r^(2/5)
+ (17 v^(-2) + 223 + 17 v^2) r^(7/5)
 ] q^(2/5)
+  [
 [ (5 v^(-1) + 5 v) y^(-1) + 20 + (5 v^(-1) + 5 v) y ] r^(9/20)
+  [ (70 v^(-1) + 70 v) y^(-1) + (20 v^(-2) + 240 + 20 v^2) + (70 v^(-1) + 70 v) y ] r^(29/20)
 ] q^(9/20)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(3/2)
 ] q^(1/2)
+  [
36 r^(8/15)
+ (36 v^(-2) + 426 + 36 v^2) r^(23/15)
 ] q^(8/15)
+  [
 [ (13 v^(-1) + 13 v) y^(-1) + 52 + (13 v^(-1) + 13 v) y ] r^(7/12)
+  [ (140 v^(-1) + 140 v) y^(-1) + (52 v^(-2) + 456 + 52 v^2) + (140 v^(-1) + 140 v) y ] r^(19/12)
 ] q^(7/12)
+  [
83 r^(3/5)
+ (83 v^(-2) + 894 + 83 v^2) r^(8/5)
 ] q^(3/5)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + (4 v^(-1) + 4 v) y ] r^(19/30)
+  [ (28 v^(-1) + 28 v) y^(-1) + (28 v^(-1) + 28 v) y ] r^(49/30)
 ] q^(19/30)
+  [
 [ (17 v^(-1) + 17 v) y^(-1) + 68 + (17 v^(-1) + 17 v) y ] r^(13/20)
+  [ (223 v^(-1) + 223 v) y^(-1) + (68 v^(-2) + 756 + 68 v^2) + (223 v^(-1) + 223 v) y ] r^(33/20)
 ] q^(13/20)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(7/10)
+  [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(17/10)
 ] q^(7/10)
+  [
50 r^(11/15)
```

```
+ (50 v^(-2) + 450 + 50 v^2) r^(26/15)
 ] q^(11/15)
+  [
 [ (36 v^(-1) + 36 v) y^(-1) + 144 + (36 v^(-1) + 36 v) y ] r^(47/60)
+  [ (426 v^(-1) + 426 v) y^(-1) + (144 v^(-2) + 1416 + 144 v^2) + (426 v^(-1) + 426 v) y ] r^(107/60)
 ] q^(47/60)
+  [
29 r^(4/5)
+ (29 v^(-2) + 279 + 29 v^2) r^(9/5)
 ] q^(4/5)
+  [
 [ (52 v^(-1) + 52 v) y^(-1) + (52 v^(-1) + 52 v) y ] r^(5/6)
+  [ (456 v^(-1) + 456 v) y^(-1) + (456 v^(-1) + 456 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (83 v^(-1) + 83 v) y^(-1) + 332 + (83 v^(-1) + 83 v) y ] r^(17/20)
+  [ (894 v^(-1) + 894 v) y^(-1) + (332 v^(-2) + 2912 + 332 v^2) + (894 v^(-1) + 894 v) y ] r^(37/20)
 ] q^(17/20)
+  [
 [ (68 v^(-1) + 68 v) y^(-1) + (68 v^(-1) + 68 v) y ] r^(9/10)
+  [ (756 v^(-1) + 756 v) y^(-1) + (756 v^(-1) + 756 v) y ] r^(19/10)
 ] q^(9/10)
+  [
200 r^(14/15)
+ (200 v^(-2) + 1900 + 200 v^2) r^(29/15)
 ] q^(14/15)
+  [
 [ (50 v^(-1) + 50 v) y^(-1) + 200 + (50 v^(-1) + 50 v) y ] r^(59/60)
+  [ (450 v^(-1) + 450 v) y^(-1) + (200 v^(-2) + 1400 + 200 v^2) + (450 v^(-1) + 450 v) y ] r^(119/60)
 ] q^(59/60)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (3 v^(-2) + 996 + 3 v^2) + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 18 + 3 v^2) y^(-2) + (996 v^(-2) + 8042 + 996 v^2) + (3 v^(-2) + 18 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (144 v^(-1) + 144 v) y^(-1) + (144 v^(-1) + 144 v) y ] r^(31/30)
 q^(31/30)
+
 [ (29 v^(-1) + 29 v) y^(-1) + 116 + (29 v^(-1) + 29 v) y ] r^(21/20)
 q^(21/20)
+
 [ (332 v^(-1) + 332 v) y^(-1) + (332 v^(-1) + 332 v) y ] r^(11/10)
 q^(11/10)
+  [
 [ 1 y^(-2) + 9 + 1 y^2 ] r^(2/15)
+  [ (1 v^(-2) + 9 + 1 v^2) y^(-2) + (9 v^(-2) + 81 + 9 v^2) + (1 v^(-2) + 9 + 1 v^2) y^2 ] r^(17/15)
 ] q^(17/15)
+
 [ (200 v^(-1) + 200 v) y^(-1) + 800 + (200 v^(-1) + 200 v) y ] r^(71/60)
 q^(71/60)
+  [
 [ 5 y^(-2) + 70 + 5 y^2 ] r^(1/5)
+  [ (5 v^(-2) + 70 + 5 v^2) y^(-2) + (70 v^(-2) + 1000 + 70 v^2) + (5 v^(-2) + 70 + 5 v^2) y^2 ] r^(6/5)
 ] q^(6/5)
+
 [ (200 v^(-1) + 200 v) y^(-1) + (200 v^(-1) + 200 v) y ] r^(37/30)
 q^(37/30)
+  [
 [ 20 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 228 + (3 v^(-1) + 3 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 228 + 20 v^2) y^(-2) + (996 v^(-1) + 996 v) y^(-1) + (228 v^(-2) + 2992 + 228 v^2)
+ (996 v^(-1) + 996 v) y + (20 v^(-2) + 228 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+
 [ (116 v^(-1) + 116 v) y^(-1) + (116 v^(-1) + 116 v) y ] r^(13/10)
 q^(13/10)
+  [
 [ 13 y^(-2) + 140 + 13 y^2 ] r^(1/3)
+  [ (13 v^(-2) + 140 + 13 v^2) y^(-2) + (140 v^(-2) + 1517 + 140 v^2) + (13 v^(-2) + 140 + 13 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 4 y^(-2) + (9 v^(-1) + 9 v) y^(-1) + 28 + (9 v^(-1) + 9 v) y + 4 y^2 ] r^(23/60)
+  [ (4 v^(-2) + 28 + 4 v^2) y^(-2) + (81 v^(-1) + 81 v) y^(-1) + (28 v^(-2) + 196 + 28 v^2)
+ (81 v^(-1) + 81 v) y + (4 v^(-2) + 28 + 4 v^2) y^2 ] r^(83/60)
 ] q^(83/60)
+  [
 [ 17 y^(-2) + 223 + 17 y^2 ] r^(2/5)
+  [ (17 v^(-2) + 223 + 17 v^2) y^(-2) + (223 v^(-2) + 3187 + 223 v^2) + (17 v^(-2) + 223 + 17 v^2) y^2 ] r^(7/5)
 ] q^(7/5)
+
 [ (800 v^(-1) + 800 v) y^(-1) + (800 v^(-1) + 800 v) y ] r^(43/30)
 q^(43/30)
+  [
 [ 20 y^(-2) + (70 v^(-1) + 70 v) y^(-1) + 240 + (70 v^(-1) + 70 v) y + 20 y^2 ] r^(9/20)
+  [ (20 v^(-2) + 240 + 20 v^2) y^(-2) + (1000 v^(-1) + 1000 v) y^(-1) + (240 v^(-2) + 2960 + 240 v^2)
+ (1000 v^(-1) + 1000 v) y + (20 v^(-2) + 240 + 20 v^2) y^2 ] r^(29/20)
 ] q^(29/20)
+  [
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(1/2)
+  [ (2992 v^(-1) + 2992 v) y^(-1) + (2992 v^(-1) + 2992 v) y ] r^(3/2)
 ] q^(3/2)
+  [
```

```
 [ 36 y^(-2) + 426 + 36 y^2 ] r^(8/15)
+  [ (36 v^(-2) + 426 + 36 v^2) y^(-2) + (426 v^(-2) + 5041 + 426 v^2) + (36 v^(-2) + 426 + 36 v^2) y^2 ] r^(23/15)
 ] q^(23/15)
+  [
 [ 52 y^(-2) + (140 v^(-1) + 140 v) y^(-1) + 456 + (140 v^(-1) + 140 v) y + 52 y^2 ] r^(7/12)
+  [ (52 v^(-2) + 456 + 52 v^2) y^(-2) + (1517 v^(-1) + 1517 v) y^(-1) + (456 v^(-2) + 4036 + 456 v^2)
+ (1517 v^(-1) + 1517 v) y + (52 v^(-2) + 456 + 52 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 83 y^(-2) + 894 + 83 y^2 ] r^(3/5)
+  [ (83 v^(-2) + 894 + 83 v^2) y^(-2) + (894 v^(-2) + 9642 + 894 v^2) + (83 v^(-2) + 894 + 83 v^2) y^2 ] r^(8/5)
 ] q^(8/5)
+  [
 [ (28 v^(-1) + 28 v) y^(-1) + (28 v^(-1) + 28 v) y ] r^(19/30)
+  [ (196 v^(-1) + 196 v) y^(-1) + (196 v^(-1) + 196 v) y ] r^(49/30)
 ] q^(49/30)
+  [
 [ 68 y^(-2) + (223 v^(-1) + 223 v) y^(-1) + 756 + (223 v^(-1) + 223 v) y + 68 y^2 ] r^(13/20)
+  [ (68 v^(-2) + 756 + 68 v^2) y^(-2) + (3187 v^(-1) + 3187 v) y^(-1) + (756 v^(-2) + 9452 + 756 v^2)
+ (3187 v^(-1) + 3187 v) y + (68 v^(-2) + 756 + 68 v^2) y^2 ] r^(33/20)
 ] q^(33/20)
+  [
 [ (240 v^(-1) + 240 v) y^(-1) + (240 v^(-1) + 240 v) y ] r^(7/10)
+  [ (2960 v^(-1) + 2960 v) y^(-1) + (2960 v^(-1) + 2960 v) y ] r^(17/10)
 ] q^(17/10)
+  [
 [ 50 y^(-2) + 450 + 50 y^2 ] r^(11/15)
+  [ (50 v^(-2) + 450 + 50 v^2) y^(-2) + (450 v^(-2) + 4050 + 450 v^2) + (50 v^(-2) + 450 + 50 v^2) y^2 ] r^(26/15)
 ] q^(26/15)
+  [
 [ 144 y^(-2) + (426 v^(-1) + 426 v) y^(-1) + 1416 + (426 v^(-1) + 426 v) y + 144 y^2 ] r^(47/60)
+  [ (144 v^(-2) + 1416 + 144 v^2) y^(-2) + (5041 v^(-1) + 5041 v) y^(-1)
+ (1416 v^(-2) + 13924 + 1416 v^2) + (5041 v^(-1) + 5041 v) y + (144 v^(-2) + 1416 + 144 v^2) y^2 ] r^(107/60)
 ] q^(107/60)
+  [
 [ 29 y^(-2) + 279 + 29 y^2 ] r^(4/5)
+  [ (29 v^(-2) + 279 + 29 v^2) y^(-2) + (279 v^(-2) + 2693 + 279 v^2) + (29 v^(-2) + 279 + 29 v^2) y^2 ] r^(9/5)
 ] q^(9/5)
+  [
 [ (456 v^(-1) + 456 v) y^(-1) + (456 v^(-1) + 456 v) y ] r^(5/6)
+  [ (4036 v^(-1) + 4036 v) y^(-1) + (4036 v^(-1) + 4036 v) y ] r^(11/6)
 ] q^(11/6)
+  [
 [ 332 y^(-2) + (894 v^(-1) + 894 v) y^(-1) + 2912 + (894 v^(-1) + 894 v) y + 332 y^2 ] r^(17/20)
+  [ (332 v^(-2) + 2912 + 332 v^2) y^(-2) + (9642 v^(-1) + 9642 v) y^(-1)
+ (2912 v^(-2) + 25592 + 2912 v^2) + (9642 v^(-1) + 9642 v) y + (332 v^(-2) + 2912 + 332 v^2) y^2 ] r^(37/20)
 ] q^(37/20)
+  [
 [ (756 v^(-1) + 756 v) y^(-1) + (756 v^(-1) + 756 v) y ] r^(9/10)
+  [ (9452 v^(-1) + 9452 v) y^(-1) + (9452 v^(-1) + 9452 v) y ] r^(19/10)
 ] q^(19/10)
+  [
 [ 200 y^(-2) + 1900 + 200 y^2 ] r^(14/15)
+  [ (200 v^(-2) + 1900 + 200 v^2) y^(-2) + (1900 v^(-2) + 18050 + 1900 v^2) + (200 v^(-2) + 1900 + 200 v^2) y^2 ] r^(29/15)
 ] q^(29/15)
+  [
 [ 200 y^(-2) + (450 v^(-1) + 450 v) y^(-1) + 1400 + (450 v^(-1) + 450 v) y + 200 y^2 ] r^(59/60)
+  [ (200 v^(-2) + 1400 + 200 v^2) y^(-2) + (4050 v^(-1) + 4050 v) y^(-1)
+ (1400 v^(-2) + 9800 + 1400 v^2) + (4050 v^(-1) + 4050 v) y + (200 v^(-2) + 1400 + 200 v^2) y^2 ] r^(119/60)
 ] q^(119/60)
+  [
3 y^(-2) + 18 + 3 y^2
+  [ (3 v^(-2) + 996 + 3 v^2) y^(-2) + (18 v^(-2) + 8042 + 18 v^2) + (3 v^(-2) + 996 + 3 v^2) y^2 ] r
+  [ (996 v^(-2) + 8042 + 996 v^2) y^(-2) + (8042 v^(-2) + 65311 + 8042 v^2) + (996 v^(-2) + 8042 + 996 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(7)(16):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 17 + 3 v^2) r^2

+  [
3 r^(1/9)
+ (3 v^(-2) + 33 + 3 v^2) r^(10/9)
 ] q^(1/9)
+  [
3 r^(2/9)
+ (3 v^(-2) + 45 + 3 v^2) r^(11/9)
 ] q^(2/9)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (20 v^(-2) + 228 + 20 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
26 r^(1/3)
+ (26 v^(-2) + 282 + 26 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (3 v^(-1) + 3 v) y^(-1) + 12 + (3 v^(-1) + 3 v) y ] r^(13/36)
+  [ (33 v^(-1) + 33 v) y^(-1) + (12 v^(-2) + 108 + 12 v^2) + (33 v^(-1) + 33 v) y ] r^(49/36)
 ] q^(13/36)
```

```
+  [
30 r^(4/9)
+ (30 v^(-2) + 318 + 30 v^2) r^(13/9)
 ] q^(4/9)
+  [
 [ (3 v^(-1) + 3 v) y^(-1) + 12 + (3 v^(-1) + 3 v) y ] r^(17/36)
+  [ (45 v^(-1) + 45 v) y^(-1) + (12 v^(-2) + 156 + 12 v^2) + (45 v^(-1) + 45 v) y ] r^(53/36)
 ] q^(17/36)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(3/2)
 ] q^(1/2)
+  [
57 r^(5/9)
+ (57 v^(-2) + 585 + 57 v^2) r^(14/9)
 ] q^(5/9)
+  [
 [ (26 v^(-1) + 26 v) y^(-1) + 104 + (26 v^(-1) + 26 v) y ] r^(7/12)
+  [ (282 v^(-1) + 282 v) y^(-1) + (104 v^(-2) + 920 + 104 v^2) + (282 v^(-1) + 282 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (12 v^(-1) + 12 v) y^(-1) + (12 v^(-1) + 12 v) y ] r^(11/18)
+  [ (108 v^(-1) + 108 v) y^(-1) + (108 v^(-1) + 108 v) y ] r^(29/18)
 ] q^(11/18)
+  [
96 r^(2/3)
+ (96 v^(-2) + 987 + 96 v^2) r^(5/3)
 ] q^(2/3)
+  [
 [ (30 v^(-1) + 30 v) y^(-1) + 120 + (30 v^(-1) + 30 v) y ] r^(25/36)
+  [ (318 v^(-1) + 318 v) y^(-1) + (120 v^(-2) + 1032 + 120 v^2) + (318 v^(-1) + 318 v) y ] r^(61/36)
 ] q^(25/36)
+  [
 [ (12 v^(-1) + 12 v) y^(-1) + (12 v^(-1) + 12 v) y ] r^(13/18)
+  [ (156 v^(-1) + 156 v) y^(-1) + (156 v^(-1) + 156 v) y ] r^(31/18)
 ] q^(13/18)
+  [
75 r^(7/9)
+ (75 v^(-2) + 705 + 75 v^2) r^(16/9)
 ] q^(7/9)
+  [
 [ (57 v^(-1) + 57 v) y^(-1) + 228 + (57 v^(-1) + 57 v) y ] r^(29/36)
+  [ (585 v^(-1) + 585 v) y^(-1) + (228 v^(-2) + 1884 + 228 v^2) + (585 v^(-1) + 585 v) y ] r^(65/36)
 ] q^(29/36)
+  [
 [ (104 v^(-1) + 104 v) y^(-1) + (104 v^(-1) + 104 v) y ] r^(5/6)
+  [ (920 v^(-1) + 920 v) y^(-1) + (920 v^(-1) + 920 v) y ] r^(11/6)
 ] q^(5/6)
+  [
57 r^(8/9)
+ (57 v^(-2) + 582 + 57 v^2) r^(17/9)
 ] q^(8/9)
+  [
 [ (96 v^(-1) + 96 v) y^(-1) + 384 + (96 v^(-1) + 96 v) y ] r^(11/12)
+  [ (987 v^(-1) + 987 v) y^(-1) + (384 v^(-2) + 3180 + 384 v^2) + (987 v^(-1) + 987 v) y ] r^(23/12)
 ] q^(11/12)
+  [
 [ (120 v^(-1) + 120 v) y^(-1) + (120 v^(-1) + 120 v) y ] r^(17/18)
+  [ (1032 v^(-1) + 1032 v) y^(-1) + (1032 v^(-1) + 1032 v) y ] r^(35/18)
 ] q^(17/18)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (3 v^(-2) + 1018 + 3 v^2) + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 17 + 3 v^2) y^(-2) + (1018 v^(-2) + 8222 + 1018 v^2) + (3 v^(-2) + 17 + 3 v^2) y^2 ] r^2
 ] q
+
 [ (75 v^(-1) + 75 v) y^(-1) + 300 + (75 v^(-1) + 75 v) y ] r^(37/36)
 q^(37/36)
+
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(19/18)
 q^(19/18)
+  [
 [ 3 y^(-2) + 33 + 3 y^2 ] r^(1/9)
+  [ (3 v^(-2) + 33 + 3 v^2) y^(-2) + (33 v^(-2) + 435 + 33 v^2) + (3 v^(-2) + 33 + 3 v^2) y^2 ] r^(10/9)
 ] q^(10/9)
+
 [ (57 v^(-1) + 57 v) y^(-1) + 228 + (57 v^(-1) + 57 v) y ] r^(41/36)
 q^(41/36)
+
 [ (384 v^(-1) + 384 v) y^(-1) + (384 v^(-1) + 384 v) y ] r^(7/6)
 q^(7/6)
+  [
 [ 3 y^(-2) + 45 + 3 y^2 ] r^(2/9)
+  [ (3 v^(-2) + 45 + 3 v^2) y^(-2) + (45 v^(-2) + 729 + 45 v^2) + (3 v^(-2) + 45 + 3 v^2) y^2 ] r^(11/9)
 ] q^(11/9)
+  [
 [ 20 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 228 + (3 v^(-1) + 3 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 228 + 20 v^2) y^(-2) + (1018 v^(-1) + 1018 v) y^(-1) + (228 v^(-2) + 3080 + 228 v^2)
+ (1018 v^(-1) + 1018 v) y + (20 v^(-2) + 228 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+
 [ (300 v^(-1) + 300 v) y^(-1) + (300 v^(-1) + 300 v) y ] r^(23/18)
```

```
 q^(23/18)
+  [
 [ 26 y^(-2) + 282 + 26 y^2 ] r^(1/3)
+  [ (26 v^(-2) + 282 + 26 v^2) y^(-2) + (282 v^(-2) + 3354 + 282 v^2) + (26 v^(-2) + 282 + 26 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+  [
 [ 12 y^(-2) + (33 v^(-1) + 33 v) y^(-1) + 108 + (33 v^(-1) + 33 v) y + 12 y^2 ] r^(13/36)
+  [ (12 v^(-2) + 108 + 12 v^2) y^(-2) + (435 v^(-1) + 435 v) y^(-1) + (108 v^(-2) + 1260 + 108 v^2)
+ (435 v^(-1) + 435 v) y + (12 v^(-2) + 108 + 12 v^2) y^2 ] r^(49/36)
 ] q^(49/36)
+
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(25/18)
 q^(25/18)
+  [
 [ 30 y^(-2) + 318 + 30 y^2 ] r^(4/9)
+  [ (30 v^(-2) + 318 + 30 v^2) y^(-2) + (318 v^(-2) + 3675 + 318 v^2) + (30 v^(-2) + 318 + 30 v^2) y^2 ] r^(13/9)
 ] q^(13/9)
+  [
 [ 12 y^(-2) + (45 v^(-1) + 45 v) y^(-1) + 156 + (45 v^(-1) + 45 v) y + 12 y^2 ] r^(17/36)
+  [ (12 v^(-2) + 156 + 12 v^2) y^(-2) + (729 v^(-1) + 729 v) y^(-1) + (156 v^(-2) + 2244 + 156 v^2)
+ (729 v^(-1) + 729 v) y + (12 v^(-2) + 156 + 12 v^2) y^2 ] r^(53/36)
 ] q^(53/36)
+  [
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(1/2)
+  [ (3080 v^(-1) + 3080 v) y^(-1) + (3080 v^(-1) + 3080 v) y ] r^(3/2)
 ] q^(3/2)
+  [
 [ 57 y^(-2) + 585 + 57 y^2 ] r^(5/9)
+  [ (57 v^(-2) + 585 + 57 v^2) y^(-2) + (585 v^(-2) + 6075 + 585 v^2) + (57 v^(-2) + 585 + 57 v^2) y^2 ] r^(14/9)
 ] q^(14/9)
+  [
 [ 104 y^(-2) + (282 v^(-1) + 282 v) y^(-1) + 920 + (282 v^(-1) + 282 v) y + 104 y^2 ] r^(7/12)
+  [ (104 v^(-2) + 920 + 104 v^2) y^(-2) + (3354 v^(-1) + 3354 v) y^(-1) + (920 v^(-2) + 9320 + 920 v^2)
+ (3354 v^(-1) + 3354 v) y + (104 v^(-2) + 920 + 104 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ (108 v^(-1) + 108 v) y^(-1) + (108 v^(-1) + 108 v) y ] r^(11/18)
+  [ (1260 v^(-1) + 1260 v) y^(-1) + (1260 v^(-1) + 1260 v) y ] r^(29/18)
 ] q^(29/18)
+  [
 [ 96 y^(-2) + 987 + 96 y^2 ] r^(2/3)
+  [ (96 v^(-2) + 987 + 96 v^2) y^(-2) + (987 v^(-2) + 10374 + 987 v^2) + (96 v^(-2) + 987 + 96 v^2) y^2 ] r^(5/3)
 ] q^(5/3)
+  [
 [ 120 y^(-2) + (318 v^(-1) + 318 v) y^(-1) + 1032 + (318 v^(-1) + 318 v) y + 120 y^2 ] r^(25/36)
+  [ (120 v^(-2) + 1032 + 120 v^2) y^(-2) + (3675 v^(-1) + 3675 v) y^(-1) + (1032 v^(-2) + 10092 + 1032 v^2)
+ (3675 v^(-1) + 3675 v) y + (120 v^(-2) + 1032 + 120 v^2) y^2 ] r^(61/36)
 ] q^(61/36)
+  [
 [ (156 v^(-1) + 156 v) y^(-1) + (156 v^(-1) + 156 v) y ] r^(13/18)
+  [ (2244 v^(-1) + 2244 v) y^(-1) + (2244 v^(-1) + 2244 v) y ] r^(31/18)
 ] q^(31/18)
+  [
 [ 75 y^(-2) + 705 + 75 y^2 ] r^(7/9)
+  [ (75 v^(-2) + 705 + 75 v^2) y^(-2) + (705 v^(-2) + 6699 + 705 v^2) + (75 v^(-2) + 705 + 75 v^2) y^2 ] r^(16/9)
 ] q^(16/9)
+  [
 [ 228 y^(-2) + (585 v^(-1) + 585 v) y^(-1) + 1884 + (585 v^(-1) + 585 v) y + 228 y^2 ] r^(29/36)
+  [ (228 v^(-2) + 1884 + 228 v^2) y^(-2) + (6075 v^(-1) + 6075 v) y^(-1) + (1884 v^(-2) + 15852 + 1884 v^2)
+ (6075 v^(-1) + 6075 v) y + (228 v^(-2) + 1884 + 228 v^2) y^2 ] r^(65/36)
 ] q^(65/36)
+  [
 [ (920 v^(-1) + 920 v) y^(-1) + (920 v^(-1) + 920 v) y ] r^(5/6)
+  [ (9320 v^(-1) + 9320 v) y^(-1) + (9320 v^(-1) + 9320 v) y ] r^(11/6)
 ] q^(11/6)
+  [
 [ 57 y^(-2) + 582 + 57 y^2 ] r^(8/9)
+  [ (57 v^(-2) + 582 + 57 v^2) y^(-2) + (582 v^(-2) + 5988 + 582 v^2) + (57 v^(-2) + 582 + 57 v^2) y^2 ] r^(17/9)
 ] q^(17/9)
+  [
 [ 384 y^(-2) + (987 v^(-1) + 987 v) y^(-1) + 3180 + (987 v^(-1) + 987 v) y + 384 y^2 ] r^(11/12)
+  [ (384 v^(-2) + 3180 + 384 v^2) y^(-2) + (10374 v^(-1) + 10374 v) y^(-1) + (3180 v^(-2) + 27240 + 3180 v^2)
+ (10374 v^(-1) + 10374 v) y + (384 v^(-2) + 3180 + 384 v^2) y^2 ] r^(23/12)
 ] q^(23/12)
+  [
 [ (1032 v^(-1) + 1032 v) y^(-1) + (1032 v^(-1) + 1032 v) y ] r^(17/18)
+  [ (10092 v^(-1) + 10092 v) y^(-1) + (10092 v^(-1) + 10092 v) y ] r^(35/18)
 ] q^(35/18)
+  [
3 y^(-2) + 17 + 3 y^2
+  [ (3 v^(-2) + 1018 + 3 v^2) y^(-2) + (17 v^(-2) + 8222 + 17 v^2) + (3 v^(-2) + 1018 + 3 v^2) y^2 ] r
+  [ (1018 v^(-2) + 8222 + 1018 v^2) y^(-2) + (8222 v^(-2) + 66875 + 8222 v^2) + (1018 v^(-2) + 8222 + 1018 v^2) y^2 ] r^2
 ] q^2
```

```
Partition function for (1)(6)(22):

1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 17 + 3 v^2) r^2

+  [
1 r^(1/12)
+ (1 v^(-2) + 9 + 1 v^2) r^(13/12)
 ] q^(1/12)
+  [
2 r^(1/8)
+ (2 v^(-2) + 26 + 2 v^2) r^(9/8)
 ] q^(1/8)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 26 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (26 v^(-2) + 308 + 26 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 25 + (1 v^(-1) + 1 v) y ] r^(1/3)
+  [ (9 v^(-1) + 9 v) y^(-1) + (25 v^(-2) + 241 + 25 v^2) + (9 v^(-1) + 9 v) y ] r^(4/3)
 ] q^(1/3)
+  [
 [ (2 v^(-1) + 2 v) y^(-1) + 16 + (2 v^(-1) + 2 v) y ] r^(3/8)
+  [ (26 v^(-1) + 26 v) y^(-1) + (16 v^(-2) + 192 + 16 v^2) + (26 v^(-1) + 26 v) y ] r^(11/8)
 ] q^(3/8)
+  [
4 r^(11/24)
+ (4 v^(-2) + 56 + 4 v^2) r^(35/24)
 ] q^(11/24)
+  [
 [ (26 v^(-1) + 26 v) y^(-1) + 70 + (26 v^(-1) + 26 v) y ] r^(1/2)
+  [ (308 v^(-1) + 308 v) y^(-1) + (70 v^(-2) + 746 + 70 v^2) + (308 v^(-1) + 308 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (25 v^(-1) + 25 v) y^(-1) + 120 + (25 v^(-1) + 25 v) y ] r^(7/12)
+  [ (241 v^(-1) + 241 v) y^(-1) + (120 v^(-2) + 1038 + 120 v^2) + (241 v^(-1) + 241 v) y ] r^(19/12)
 ] q^(7/12)
+  [
 [ (16 v^(-1) + 16 v) y^(-1) + 64 + (16 v^(-1) + 16 v) y ] r^(5/8)
+  [ (192 v^(-1) + 192 v) y^(-1) + (64 v^(-2) + 704 + 64 v^2) + (192 v^(-1) + 192 v) y ] r^(13/8)
 ] q^(5/8)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + 80 + (4 v^(-1) + 4 v) y ] r^(17/24)
+  [ (56 v^(-1) + 56 v) y^(-1) + (80 v^(-2) + 768 + 80 v^2) + (56 v^(-1) + 56 v) y ] r^(41/24)
 ] q^(17/24)
+  [
 [ (70 v^(-1) + 70 v) y^(-1) + 254 + (70 v^(-1) + 70 v) y ] r^(3/4)
+  [ (746 v^(-1) + 746 v) y^(-1) + (254 v^(-2) + 2202 + 254 v^2) + (746 v^(-1) + 746 v) y ] r^(7/4)
 ] q^(3/4)
+  [
 [ (120 v^(-1) + 120 v) y^(-1) + 202 + (120 v^(-1) + 120 v) y ] r^(5/6)
+  [ (1038 v^(-1) + 1038 v) y^(-1) + (202 v^(-2) + 1730 + 202 v^2) + (1038 v^(-1) + 1038 v) y ] r^(11/6)
 ] q^(5/6)
+  [
 [ (64 v^(-1) + 64 v) y^(-1) + 128 + (64 v^(-1) + 64 v) y ] r^(7/8)
+  [ (704 v^(-1) + 704 v) y^(-1) + (128 v^(-2) + 1152 + 128 v^2) + (704 v^(-1) + 704 v) y ] r^(15/8)
 ] q^(7/8)
+  [
 [ (80 v^(-1) + 80 v) y^(-1) + 256 + (80 v^(-1) + 80 v) y ] r^(23/24)
+  [ (768 v^(-1) + 768 v) y^(-1) + (256 v^(-2) + 1792 + 256 v^2) + (768 v^(-1) + 768 v) y ] r^(47/24)
 ] q^(23/24)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (254 v^(-1) + 254 v) y^(-1) + (3 v^(-2) + 1246 + 3 v^2)
+ (254 v^(-1) + 254 v) y + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 17 + 3 v^2) y^(-2) + (2202 v^(-1) + 2202 v) y^(-1) + (1246 v^(-2) + 9882 + 1246 v^2)
+ (2202 v^(-1) + 2202 v) y + (3 v^(-2) + 17 + 3 v^2) y^2 ] r^2
 ] q
+  [
 [ 1 y^(-2) + 9 + 1 y^2 ] r^(1/12)
+  [ (1 v^(-2) + 9 + 1 v^2) y^(-2) + (202 v^(-1) + 202 v) y^(-1) + (9 v^(-2) + 482 + 9 v^2)
+ (202 v^(-1) + 202 v) y + (1 v^(-2) + 9 + 1 v^2) y^2 ] r^(13/12)
 ] q^(13/12)
+  [
 [ 2 y^(-2) + 26 + 2 y^2 ] r^(1/8)
+  [ (2 v^(-2) + 26 + 2 v^2) y^(-2) + (128 v^(-1) + 128 v) y^(-1) + (26 v^(-2) + 338 + 26 v^2)
+ (128 v^(-1) + 128 v) y + (2 v^(-2) + 26 + 2 v^2) y^2 ] r^(9/8)
 ] q^(9/8)
+
 [ (256 v^(-1) + 256 v) y^(-1) + (256 v^(-1) + 256 v) y ] r^(29/24)
 q^(29/24)
+  [
 [ 26 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 308 + (3 v^(-1) + 3 v) y + 26 y^2 ] r^(1/4)
+  [ (26 v^(-2) + 308 + 26 v^2) y^(-2) + (1246 v^(-1) + 1246 v) y^(-1) + (308 v^(-2) + 4012 + 308 v^2)
+ (1246 v^(-1) + 1246 v) y + (26 v^(-2) + 308 + 26 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+  [
 [ 25 y^(-2) + (9 v^(-1) + 9 v) y^(-1) + 241 + (9 v^(-1) + 9 v) y + 25 y^2 ] r^(1/3)
+  [ (25 v^(-2) + 241 + 25 v^2) y^(-2) + (482 v^(-1) + 482 v) y^(-1) + (241 v^(-2) + 3077 + 241 v^2)
+ (482 v^(-1) + 482 v) y + (25 v^(-2) + 241 + 25 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
```

```
+  [
 [ 16 y^(-2) + (26 v^(-1) + 26 v) y^(-1) + 192 + (26 v^(-1) + 26 v) y + 16 y^2 ] r^(3/8)
+  [ (16 v^(-2) + 192 + 16 v^2) y^(-2) + (338 v^(-1) + 338 v) y^(-1) + (192 v^(-2) + 2448 + 192 v^2)
+ (338 v^(-1) + 338 v) y + (16 v^(-2) + 192 + 16 v^2) y^2 ] r^(11/8)
 ] q^(11/8)
+  [
 [ 4 y^(-2) + 56 + 4 y^2 ] r^(11/24)
+  [ (4 v^(-2) + 56 + 4 v^2) y^(-2) + (56 v^(-2) + 784 + 56 v^2) + (4 v^(-2) + 56 + 4 v^2) y^2 ] r^(35/24)
 ] q^(35/24)
+  [
 [ 70 y^(-2) + (308 v^(-1) + 308 v) y^(-1) + 746 + (308 v^(-1) + 308 v) y + 70 y^2 ] r^(1/2)
+  [ (70 v^(-2) + 746 + 70 v^2) y^(-2) + (4012 v^(-1) + 4012 v) y^(-1) + (746 v^(-2) + 8596 + 746 v^2)
+ (4012 v^(-1) + 4012 v) y + (70 v^(-2) + 746 + 70 v^2) y^2 ] r^(3/2)
 ] q^(3/2)
+  [
 [ 120 y^(-2) + (241 v^(-1) + 241 v) y^(-1) + 1038 + (241 v^(-1) + 241 v) y + 120 y^2 ] r^(7/12)
+  [ (120 v^(-2) + 1038 + 120 v^2) y^(-2) + (3077 v^(-1) + 3077 v) y^(-1) + (1038 v^(-2) + 9278 + 1038 v^2)
+ (3077 v^(-1) + 3077 v) y + (120 v^(-2) + 1038 + 120 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 64 y^(-2) + (192 v^(-1) + 192 v) y^(-1) + 704 + (192 v^(-1) + 192 v) y + 64 y^2 ] r^(5/8)
+  [ (64 v^(-2) + 704 + 64 v^2) y^(-2) + (2448 v^(-1) + 2448 v) y^(-1) + (704 v^(-2) + 8256 + 704 v^2)
+ (2448 v^(-1) + 2448 v) y + (64 v^(-2) + 704 + 64 v^2) y^2 ] r^(13/8)
 ] q^(13/8)
+  [
 [ 80 y^(-2) + (56 v^(-1) + 56 v) y^(-1) + 768 + (56 v^(-1) + 56 v) y + 80 y^2 ] r^(17/24)
+  [ (80 v^(-2) + 768 + 80 v^2) y^(-2) + (784 v^(-1) + 784 v) y^(-1) + (768 v^(-2) + 7488 + 768 v^2)
+ (784 v^(-1) + 784 v) y + (80 v^(-2) + 768 + 80 v^2) y^2 ] r^(41/24)
 ] q^(41/24)
+  [
 [ 254 y^(-2) + (746 v^(-1) + 746 v) y^(-1) + 2202 + (746 v^(-1) + 746 v) y + 254 y^2 ] r^(3/4)
+  [ (254 v^(-2) + 2202 + 254 v^2) y^(-2) + (8596 v^(-1) + 8596 v) y^(-1) + (2202 v^(-2) + 20580 + 2202 v^2)
+ (8596 v^(-1) + 8596 v) y + (254 v^(-2) + 2202 + 254 v^2) y^2 ] r^(7/4)
 ] q^(7/4)
+  [
 [ 202 y^(-2) + (1038 v^(-1) + 1038 v) y^(-1) + 1730 + (1038 v^(-1) + 1038 v) y + 202 y^2 ] r^(5/6)
+  [ (202 v^(-2) + 1730 + 202 v^2) y^(-2) + (9278 v^(-1) + 9278 v) y^(-1) + (1730 v^(-2) + 15554 + 1730 v^2)
+ (9278 v^(-1) + 9278 v) y + (202 v^(-2) + 1730 + 202 v^2) y^2 ] r^(11/6)
 ] q^(11/6)
+  [
 [ 128 y^(-2) + (704 v^(-1) + 704 v) y^(-1) + 1152 + (704 v^(-1) + 704 v) y + 128 y^2 ] r^(7/8)
+  [ (128 v^(-2) + 1152 + 128 v^2) y^(-2) + (8256 v^(-1) + 8256 v) y^(-1) + (1152 v^(-2) + 10368 + 1152 v^2)
+ (8256 v^(-1) + 8256 v) y + (128 v^(-2) + 1152 + 128 v^2) y^2 ] r^(15/8)
 ] q^(15/8)
+  [
 [ 256 y^(-2) + (768 v^(-1) + 768 v) y^(-1) + 1792 + (768 v^(-1) + 768 v) y + 256 y^2 ] r^(23/24)
+  [ (256 v^(-2) + 1792 + 256 v^2) y^(-2) + (7488 v^(-1) + 7488 v) y^(-1) + (1792 v^(-2) + 12544 + 1792 v^2)
+ (7488 v^(-1) + 7488 v) y + (256 v^(-2) + 1792 + 256 v^2) y^2 ] r^(47/24)
 ] q^(47/24)
+  [
3 y^(-2) + 17 + 3 y^2
+  [ (3 v^(-2) + 1246 + 3 v^2) y^(-2) + (2202 v^(-1) + 2202 v) y^(-1) + (17 v^(-2) + 9882 + 17 v^2)
+ (2202 v^(-1) + 2202 v) y + (3 v^(-2) + 1246 + 3 v^2) y^2 ] r
+  [ (1246 v^(-2) + 9882 + 1246 v^2) y^(-2) + (20580 v^(-1) + 20580 v) y^(-1)
+ (9882 v^(-2) + 79115 + 9882 v^2) + (20580 v^(-1) + 20580 v) y + (1246 v^(-2) + 9882 + 1246 v^2) y^2 ] r^2
 ] q^2
```

Partition function for (1)(5)(40):

```
1
+ (1 v^(-2) + 3 + 1 v^2) r
+ (3 v^(-2) + 17 + 3 v^2) r^2

+  [
1 r^(1/21)
+ (1 v^(-2) + 9 + 1 v^2) r^(22/21)
 ] q^(1/21)
+  [
5 r^(1/7)
+ (5 v^(-2) + 63 + 5 v^2) r^(8/7)
 ] q^(1/7)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 20 + (1 v^(-1) + 1 v) y ] r^(1/4)
+  [ (3 v^(-1) + 3 v) y^(-1) + (20 v^(-2) + 228 + 20 v^2) + (3 v^(-1) + 3 v) y ] r^(5/4)
 ] q^(1/4)
+  [
6 r^(2/7)
+ (6 v^(-2) + 94 + 6 v^2) r^(9/7)
 ] q^(2/7)
+  [
 [ (1 v^(-1) + 1 v) y^(-1) + 4 + (1 v^(-1) + 1 v) y ] r^(25/84)
+  [ (9 v^(-1) + 9 v) y^(-1) + (4 v^(-2) + 28 + 4 v^2) + (9 v^(-1) + 9 v) y ] r^(109/84)
 ] q^(25/84)
+  [
29 r^(1/3)
+ (29 v^(-2) + 283 + 29 v^2) r^(4/3)
 ] q^(1/3)
+  [
 [ (5 v^(-1) + 5 v) y^(-1) + 20 + (5 v^(-1) + 5 v) y ] r^(11/28)
+  [ (63 v^(-1) + 63 v) y^(-1) + (20 v^(-2) + 212 + 20 v^2) + (63 v^(-1) + 63 v) y ] r^(39/28)
 ] q^(11/28)
+  [
```

```
29 r^(3/7)
+ (29 v^(-2) + 289 + 29 v^2) r^(10/7)
 ] q^(3/7)
+  [
9 r^(10/21)
+ (9 v^(-2) + 93 + 9 v^2) r^(31/21)
 ] q^(10/21)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(1/2)
+  [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(3/2)
 ] q^(1/2)
+  [
 [ (6 v^(-1) + 6 v) y^(-1) + 24 + (6 v^(-1) + 6 v) y ] r^(15/28)
+  [ (94 v^(-1) + 94 v) y^(-1) + (24 v^(-2) + 328 + 24 v^2) + (94 v^(-1) + 94 v) y ] r^(43/28)
 ] q^(15/28)
+  [
 [ (4 v^(-1) + 4 v) y^(-1) + (4 v^(-1) + 4 v) y ] r^(23/42)
+  [ (28 v^(-1) + 28 v) y^(-1) + (28 v^(-1) + 28 v) y ] r^(65/42)
 ] q^(23/42)
+  [
89 r^(4/7)
+ (89 v^(-2) + 895 + 89 v^2) r^(11/7)
 ] q^(4/7)
+  [
 [ (29 v^(-1) + 29 v) y^(-1) + 116 + (29 v^(-1) + 29 v) y ] r^(7/12)
+  [ (283 v^(-1) + 283 v) y^(-1) + (116 v^(-2) + 900 + 116 v^2) + (283 v^(-1) + 283 v) y ] r^(19/12)
 ] q^(7/12)
+  [
36 r^(13/21)
+ (36 v^(-2) + 390 + 36 v^2) r^(34/21)
 ] q^(13/21)
+  [
 [ (20 v^(-1) + 20 v) y^(-1) + (20 v^(-1) + 20 v) y ] r^(9/14)
+  [ (212 v^(-1) + 212 v) y^(-1) + (212 v^(-1) + 212 v) y ] r^(23/14)
 ] q^(9/14)
+  [
 [ (29 v^(-1) + 29 v) y^(-1) + 116 + (29 v^(-1) + 29 v) y ] r^(19/28)
+  [ (289 v^(-1) + 289 v) y^(-1) + (116 v^(-2) + 924 + 116 v^2) + (289 v^(-1) + 289 v) y ] r^(47/28)
 ] q^(19/28)
+  [
25 r^(5/7)
+ (25 v^(-2) + 243 + 25 v^2) r^(12/7)
 ] q^(5/7)
+  [
 [ (9 v^(-1) + 9 v) y^(-1) + 36 + (9 v^(-1) + 9 v) y ] r^(61/84)
+  [ (93 v^(-1) + 93 v) y^(-1) + (36 v^(-2) + 300 + 36 v^2) + (93 v^(-1) + 93 v) y ] r^(145/84)
 ] q^(61/84)
+  [
121 r^(16/21)
+ (121 v^(-2) + 1089 + 121 v^2) r^(37/21)
 ] q^(16/21)
+  [
 [ (24 v^(-1) + 24 v) y^(-1) + (24 v^(-1) + 24 v) y ] r^(11/14)
+  [ (328 v^(-1) + 328 v) y^(-1) + (328 v^(-1) + 328 v) y ] r^(25/14)
 ] q^(11/14)
+  [
 [ (89 v^(-1) + 89 v) y^(-1) + 356 + (89 v^(-1) + 89 v) y ] r^(23/28)
+  [ (895 v^(-1) + 895 v) y^(-1) + (356 v^(-2) + 2868 + 356 v^2) + (895 v^(-1) + 895 v) y ] r^(51/28)
 ] q^(23/28)
+  [
 [ (116 v^(-1) + 116 v) y^(-1) + (116 v^(-1) + 116 v) y ] r^(5/6)
+  [ (900 v^(-1) + 900 v) y^(-1) + (900 v^(-1) + 900 v) y ] r^(11/6)
 ] q^(5/6)
+  [
65 r^(6/7)
+ (65 v^(-2) + 677 + 65 v^2) r^(13/7)
 ] q^(6/7)
+  [
 [ (36 v^(-1) + 36 v) y^(-1) + 144 + (36 v^(-1) + 36 v) y ] r^(73/84)
+  [ (390 v^(-1) + 390 v) y^(-1) + (144 v^(-2) + 1272 + 144 v^2) + (390 v^(-1) + 390 v) y ] r^(157/84)
 ] q^(73/84)
+  [
57 r^(19/21)
+ (57 v^(-2) + 561 + 57 v^2) r^(40/21)
 ] q^(19/21)
+  [
 [ (116 v^(-1) + 116 v) y^(-1) + (116 v^(-1) + 116 v) y ] r^(13/14)
+  [ (924 v^(-1) + 924 v) y^(-1) + (924 v^(-1) + 924 v) y ] r^(27/14)
 ] q^(13/14)
+  [
 [ (25 v^(-1) + 25 v) y^(-1) + 100 + (25 v^(-1) + 25 v) y ] r^(27/28)
+  [ (243 v^(-1) + 243 v) y^(-1) + (100 v^(-2) + 772 + 100 v^2) + (243 v^(-1) + 243 v) y ] r^(55/28)
 ] q^(27/28)
+  [
 [ (36 v^(-1) + 36 v) y^(-1) + (36 v^(-1) + 36 v) y ] r^(41/42)
+  [ (300 v^(-1) + 300 v) y^(-1) + (300 v^(-1) + 300 v) y ] r^(83/42)
 ] q^(41/42)
+  [
1 y^(-2) + 3 + 1 y^2
+  [ (1 v^(-2) + 3 + 1 v^2) y^(-2) + (3 v^(-2) + 960 + 3 v^2) + (1 v^(-2) + 3 + 1 v^2) y^2 ] r
+  [ (3 v^(-2) + 17 + 3 v^2) y^(-2) + (960 v^(-2) + 7610 + 960 v^2) + (3 v^(-2) + 17 + 3 v^2) y^2 ] r^2
 ] q
```

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+
 [ (121 v^(-1) + 121 v) y^(-1) + 484 + (121 v^(-1) + 121 v) y ] r^(85/84)
 q^(85/84)
+  [
 [ 1 y^(-2) + 9 + 1 y^2 ] r^(1/21)
+  [ (1 v^(-2) + 9 + 1 v^2) y^(-2) + (9 v^(-2) + 81 + 9 v^2) + (1 v^(-2) + 9 + 1 v^2) y^2 ] r^(22/21)
 ] q^(22/21)
+
 [ (356 v^(-1) + 356 v) y^(-1) + (356 v^(-1) + 356 v) y ] r^(15/14)
 q^(15/14)
+
 [ (65 v^(-1) + 65 v) y^(-1) + 260 + (65 v^(-1) + 65 v) y ] r^(31/28)
 q^(31/28)
+
 [ (144 v^(-1) + 144 v) y^(-1) + (144 v^(-1) + 144 v) y ] r^(47/42)
 q^(47/42)
+  [
 [ 5 y^(-2) + 63 + 5 y^2 ] r^(1/7)
+  [ (5 v^(-2) + 63 + 5 v^2) y^(-2) + (63 v^(-2) + 830 + 63 v^2) + (5 v^(-2) + 63 + 5 v^2) y^2 ] r^(8/7)
 ] q^(8/7)
+
 [ (57 v^(-1) + 57 v) y^(-1) + 228 + (57 v^(-1) + 57 v) y ] r^(97/84)
 q^(97/84)
+
196 r^(25/21)
 q^(25/21)
+
 [ (100 v^(-1) + 100 v) y^(-1) + (100 v^(-1) + 100 v) y ] r^(17/14)
 q^(17/14)
+  [
 [ 20 y^(-2) + (3 v^(-1) + 3 v) y^(-1) + 228 + (3 v^(-1) + 3 v) y + 20 y^2 ] r^(1/4)
+  [ (20 v^(-2) + 228 + 20 v^2) y^(-2) + (960 v^(-1) + 960 v) y^(-1) + (228 v^(-2) + 2848 + 228 v^2)
+ (960 v^(-1) + 960 v) y + (20 v^(-2) + 228 + 20 v^2) y^2 ] r^(5/4)
 ] q^(5/4)
+
 [ (484 v^(-1) + 484 v) y^(-1) + (484 v^(-1) + 484 v) y ] r^(53/42)
 q^(53/42)
+  [
 [ 6 y^(-2) + 94 + 6 y^2 ] r^(2/7)
+  [ (6 v^(-2) + 94 + 6 v^2) y^(-2) + (94 v^(-2) + 1531 + 94 v^2) + (6 v^(-2) + 94 + 6 v^2) y^2 ] r^(9/7)
 ] q^(9/7)
+  [
 [ 4 y^(-2) + (9 v^(-1) + 9 v) y^(-1) + 28 + (9 v^(-1) + 9 v) y + 4 y^2 ] r^(25/84)
+  [ (4 v^(-2) + 28 + 4 v^2) y^(-2) + (81 v^(-1) + 81 v) y^(-1) + (28 v^(-2) + 196 + 28 v^2)
+ (81 v^(-1) + 81 v) y + (4 v^(-2) + 28 + 4 v^2) y^2 ] r^(109/84)
 ] q^(109/84)
+  [
 [ 29 y^(-2) + 283 + 29 y^2 ] r^(1/3)
+  [ (29 v^(-2) + 283 + 29 v^2) y^(-2) + (283 v^(-2) + 2771 + 283 v^2) + (29 v^(-2) + 283 + 29 v^2) y^2 ] r^(4/3)
 ] q^(4/3)
+
 [ (260 v^(-1) + 260 v) y^(-1) + (260 v^(-1) + 260 v) y ] r^(19/14)
 q^(19/14)
+  [
 [ 20 y^(-2) + (63 v^(-1) + 63 v) y^(-1) + 212 + (63 v^(-1) + 63 v) y + 20 y^2 ] r^(11/28)
+  [ (20 v^(-2) + 212 + 20 v^2) y^(-2) + (830 v^(-1) + 830 v) y^(-1) + (212 v^(-2) + 2392 + 212 v^2)
+ (830 v^(-1) + 830 v) y + (20 v^(-2) + 212 + 20 v^2) y^2 ] r^(39/28)
 ] q^(39/28)
+
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(59/42)
 q^(59/42)
+  [
 [ 29 y^(-2) + 289 + 29 y^2 ] r^(3/7)
+  [ (29 v^(-2) + 289 + 29 v^2) y^(-2) + (289 v^(-2) + 3049 + 289 v^2) + (29 v^(-2) + 289 + 29 v^2) y^2 ] r^(10/7)
 ] q^(10/7)
+
 [ (196 v^(-1) + 196 v) y^(-1) + 784 + (196 v^(-1) + 196 v) y ] r^(121/84)
 q^(121/84)
+  [
 [ 9 y^(-2) + 93 + 9 y^2 ] r^(10/21)
+  [ (9 v^(-2) + 93 + 9 v^2) y^(-2) + (93 v^(-2) + 1089 + 93 v^2) + (9 v^(-2) + 93 + 9 v^2) y^2 ] r^(31/21)
 ] q^(31/21)
+  [
 [ (228 v^(-1) + 228 v) y^(-1) + (228 v^(-1) + 228 v) y ] r^(1/2)
+  [ (2848 v^(-1) + 2848 v) y^(-1) + (2848 v^(-1) + 2848 v) y ] r^(3/2)
 ] q^(3/2)
+  [
 [ 24 y^(-2) + (94 v^(-1) + 94 v) y^(-1) + 328 + (94 v^(-1) + 94 v) y + 24 y^2 ] r^(15/28)
+  [ (24 v^(-2) + 328 + 24 v^2) y^(-2) + (1531 v^(-1) + 1531 v) y^(-1) + (328 v^(-2) + 4716 + 328 v^2)
+ (1531 v^(-1) + 1531 v) y + (24 v^(-2) + 328 + 24 v^2) y^2 ] r^(43/28)
 ] q^(43/28)
+  [
 [ (28 v^(-1) + 28 v) y^(-1) + (28 v^(-1) + 28 v) y ] r^(23/42)
+  [ (196 v^(-1) + 196 v) y^(-1) + (196 v^(-1) + 196 v) y ] r^(65/42)
 ] q^(65/42)
+  [
 [ 89 y^(-2) + 895 + 89 y^2 ] r^(4/7)
+  [ (89 v^(-2) + 895 + 89 v^2) y^(-2) + (895 v^(-2) + 9266 + 895 v^2) + (89 v^(-2) + 895 + 89 v^2) y^2 ] r^(11/7)
 ] q^(11/7)
+  [
 [ 116 y^(-2) + (283 v^(-1) + 283 v) y^(-1) + 900 + (283 v^(-1) + 283 v) y + 116 y^2 ] r^(7/12)
+  [ (116 v^(-2) + 900 + 116 v^2) y^(-2) + (2771 v^(-1) + 2771 v) y^(-1) + (900 v^(-2) + 7020 + 900 v^2)
```

```
+ (2771 v^(-1) + 2771 v) y + (116 v^(-2) + 900 + 116 v^2) y^2 ] r^(19/12)
 ] q^(19/12)
+  [
 [ 36 y^(-2) + 390 + 36 y^2 ] r^(13/21)
+  [ (36 v^(-2) + 390 + 36 v^2) y^(-2) + (390 v^(-2) + 4353 + 390 v^2) + (36 v^(-2) + 390 + 36 v^2) y^2 ] r^(34/21)
 ] q^(34/21)
+  [
 [ (212 v^(-1) + 212 v) y^(-1) + (212 v^(-1) + 212 v) y ] r^(9/14)
+  [ (2392 v^(-1) + 2392 v) y^(-1) + (2392 v^(-1) + 2392 v) y ] r^(23/14)
 ] q^(23/14)
+  [
 [ 116 y^(-2) + (289 v^(-1) + 289 v) y^(-1) + 924 + (289 v^(-1) + 289 v) y + 116 y^2 ] r^(19/28)
+  [ (116 v^(-2) + 924 + 116 v^2) y^(-2) + (3049 v^(-1) + 3049 v) y^(-1) + (924 v^(-2) + 8036 + 924 v^2)
+ (3049 v^(-1) + 3049 v) y + (116 v^(-2) + 924 + 116 v^2) y^2 ] r^(47/28)
 ] q^(47/28)
+
 [ (784 v^(-1) + 784 v) y^(-1) + (784 v^(-1) + 784 v) y ] r^(71/42)
 q^(71/42)
+  [
 [ 25 y^(-2) + 243 + 25 y^2 ] r^(5/7)
+  [ (25 v^(-2) + 243 + 25 v^2) y^(-2) + (243 v^(-2) + 2458 + 243 v^2) + (25 v^(-2) + 243 + 25 v^2) y^2 ] r^(12/7)
 ] q^(12/7)
+  [
 [ 36 y^(-2) + (93 v^(-1) + 93 v) y^(-1) + 300 + (93 v^(-1) + 93 v) y + 36 y^2 ] r^(61/84)
+  [ (36 v^(-2) + 300 + 36 v^2) y^(-2) + (1089 v^(-1) + 1089 v) y^(-1) + (300 v^(-2) + 3012 + 300 v^2)
+ (1089 v^(-1) + 1089 v) y + (36 v^(-2) + 300 + 36 v^2) y^2 ] r^(145/84)
 ] q^(145/84)
+  [
 [ 121 y^(-2) + 1089 + 121 y^2 ] r^(16/21)
+  [ (121 v^(-2) + 1089 + 121 v^2) y^(-2) + (1089 v^(-2) + 9801 + 1089 v^2) + (121 v^(-2) + 1089 + 121 v^2) y^2 ] r^(37/21)
 ] q^(37/21)
+  [
 [ (328 v^(-1) + 328 v) y^(-1) + (328 v^(-1) + 328 v) y ] r^(11/14)
+  [ (4716 v^(-1) + 4716 v) y^(-1) + (4716 v^(-1) + 4716 v) y ] r^(25/14)
 ] q^(25/14)
+  [
 [ 356 y^(-2) + (895 v^(-1) + 895 v) y^(-1) + 2868 + (895 v^(-1) + 895 v) y + 356 y^2 ] r^(23/28)
+  [ (356 v^(-2) + 2868 + 356 v^2) y^(-2) + (9266 v^(-1) + 9266 v) y^(-1) + (2868 v^(-2) + 24168 + 2868 v^2)
+ (9266 v^(-1) + 9266 v) y + (356 v^(-2) + 2868 + 356 v^2) y^2 ] r^(51/28)
 ] q^(51/28)
+  [
 [ (900 v^(-1) + 900 v) y^(-1) + (900 v^(-1) + 900 v) y ] r^(5/6)
+  [ (7020 v^(-1) + 7020 v) y^(-1) + (7020 v^(-1) + 7020 v) y ] r^(11/6)
 ] q^(11/6)
+  [
 [ 65 y^(-2) + 677 + 65 y^2 ] r^(6/7)
+  [ (65 v^(-2) + 677 + 65 v^2) y^(-2) + (677 v^(-2) + 7058 + 677 v^2) + (65 v^(-2) + 677 + 65 v^2) y^2 ] r^(13/7)
 ] q^(13/7)
+  [
 [ 144 y^(-2) + (390 v^(-1) + 390 v) y^(-1) + 1272 + (390 v^(-1) + 390 v) y + 144 y^2 ] r^(73/84)
+  [ (144 v^(-2) + 1272 + 144 v^2) y^(-2) + (4353 v^(-1) + 4353 v) y^(-1) + (1272 v^(-2) + 11748 + 1272 v^2)
+ (4353 v^(-1) + 4353 v) y + (144 v^(-2) + 1272 + 144 v^2) y^2 ] r^(157/84)
 ] q^(157/84)
+  [
 [ 57 y^(-2) + 561 + 57 y^2 ] r^(19/21)
+  [ (57 v^(-2) + 561 + 57 v^2) y^(-2) + (561 v^(-2) + 5553 + 561 v^2) + (57 v^(-2) + 561 + 57 v^2) y^2 ] r^(40/21)
 ] q^(40/21)
+  [
 [ (924 v^(-1) + 924 v) y^(-1) + (924 v^(-1) + 924 v) y ] r^(13/14)
+  [ (8036 v^(-1) + 8036 v) y^(-1) + (8036 v^(-1) + 8036 v) y ] r^(27/14)
 ] q^(27/14)
+  [
 [ 100 y^(-2) + (243 v^(-1) + 243 v) y^(-1) + 772 + (243 v^(-1) + 243 v) y + 100 y^2 ] r^(27/28)
+  [ (100 v^(-2) + 772 + 100 v^2) y^(-2) + (2458 v^(-1) + 2458 v) y^(-1) + (772 v^(-2) + 6344 + 772 v^2)
+ (2458 v^(-1) + 2458 v) y + (100 v^(-2) + 772 + 100 v^2) y^2 ] r^(55/28)
 ] q^(55/28)
+  [
 [ (300 v^(-1) + 300 v) y^(-1) + (300 v^(-1) + 300 v) y ] r^(41/42)
+  [ (3012 v^(-1) + 3012 v) y^(-1) + (3012 v^(-1) + 3012 v) y ] r^(83/42)
 ] q^(83/42)
+  [
3 y^(-2) + 17 + 3 y^2
+  [ (3 v^(-2) + 960 + 3 v^2) y^(-2) + (17 v^(-2) + 7610 + 17 v^2) + (3 v^(-2) + 960 + 3 v^2) y^2 ] r
+  [ (960 v^(-2) + 7610 + 960 v^2) y^(-2) + (7610 v^(-2) + 60585 + 7610 v^2) + (960 v^(-2) + 7610 + 960 v^2) y^2 ] r^2
 ] q^2
```

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