June 1997

Renormalization of the Electroweak Standard Model to All Orders [*]

Elisabeth Kraus [†]

Physikalisches Institut, Universität Bonn
Nußallee 12, D-53115 Bonn, Germany

Abstract

We give the renormalization of the standard model of electroweak interactions to all orders of perturbation theory by using the method of algebraic renormalization, which is based on general properties of renormalized perturbation theory and not on a specific regularization scheme. The Green functions of the standard model are uniquely constructed to all orders, if one defines the model by the Slavnov-Taylor identity, Ward-identities of rigid symmetry and a specific form of the abelian local gauge Ward-identity, which continues the Gell-Mann Nishijima relation to higher orders. Special attention is directed to the mass diagonalization of massless and massive neutral vectors and ghosts. For obtaining off-shell infrared finite expressions it is required to take into account higher order corrections into the functional symmetry operators. It is shown, that the normalization conditions of the on-shell schemes are in agreement with the most general symmetry transformations allowed by the algebraic constraints.

[*] to be published in Ann. Phys. (NY) 1997

[†] Supported by Deutsche Forschungsgemeinschaft

Contents

1. Introduction

The standard model of electroweak interaction has been tested in the last few years with precision experiments of remarkable accuracy [1]. Theoretical predictions are based on the consistent perturbative formulation of the standard model of electroweak interaction as a renormalizable and unitary quantum field theory, which allows the derivation of unambiguous results for physical scattering processes order by order in perturbation theory. In order to match the level of accuracy given by experiments, it is also necessary to take into account also higher order quantum corrections to the different processes considered. Conversely, present experiments enable the standard model to be tested beyond tree approximation. Higher order corrections to the electroweak processes have been computed and evaluated quite systematically by several groups. (For a review see [2] and references listed below.) The agreement between experimental results and theoretical predictions is quite impressive and by now there is no evidence — either from theoretical arguments or from experiments — for physics beyond the standard model.

The evaluation of higher order corrections in the standard model is quite an involved task. First, one has to remove the divergencies which appear in the naive perturbative expansion of Green functions in the course of renormalization and one has to establish the defining symmetries of the theory. At the same time, the independent parameters of the standard model have to be specified and fixed by normalization conditions in such a way that the remaining undetermined constants, such as masses and the coupling strength, can be taken as input parameters from experiment. Finally, there remains the explicit evaluation of higher order loop diagrams. Nearly all calculations have been carried out in the framework of dimensional regularization. There the one-loop order has been studied quite systematically (see [2]) and the computations have reached a high field-theoretic standard. However, there is no abstract approach which analyses the renormalization of the electroweak standard model to higher orders. With the present article we fill this gap, with special attention being paid to the symmetries, normalization conditions and infrared-finiteness of off-shell Green functions. In particular, the analysis does not refer to invariance properties of a scheme, but is based on properties of finite renormalized perturbation theory. (For a review of algebraic renormalization see [3].) The purpose of a systematic analysis is twofold. First, it is evident that such an analysis will support explicit calculations by allowing symmetries to be established and possible breakings of the symmetries to be characterised quite systematically. In particular, if one wants to take into account higher order corrections in theoretical predictions, either by summing up one loop induced higher order corrections or by explicit evaluation of some higher order diagrams, it has to be ensured that the defining symmetries are not violated at any stage of the calculations and that Green functions exist to higher orders, once they are specified in 1-loop order. Higher order existence of Green functions can be destroyed in the standard model due to off-shell infrared-divergencies, whenever a photon mass counterterm is enforced by symmetries and by lower order normalization conditions. In fact it appears that infrared finiteness and the establishment of symmetries cannot be considered as separate from each other. Apart from these practical reasons the analysis is also important in its own right. Since the standard model has been so successful by now, we are convinced that electroweak interaction can only be embedded into a more complete theory of fundamental interactions, once one understands their structure in its quantized version as prescribed by the standard model.

From the algebraic point of view, the abstract approach to the quantized version of the standard model is similar to the construction of the classical Lagrangian [4, 5, 6]. If one takes the charged currents of weak interactions as given in the lepton sector, it is seen that the algebra is closed, when one includes the weak and electromagnetic currents into the group structure. Coupling these currents to the gauge bosons of weak interactions and to the photon, and requiring local $SU(2) \times U(1)$ gauge invariance, the algebra, and at the same time the classical action, is uniquely determined and the transformation of all further fields is restricted [4]. Introducing a complex scalar doublet with one physical Higgs field, one generates all masses by the mechanism of spontaneous symmetry breaking and the final standard model Lagrangian is invariant under spontaneously broken $SU(2) \times U(1)$ symmetry, which is a natural algebraic generalization of unbroken symmetry. The implementation of symmetries is also the main ingredient of abstract renormalization.

At an early stage it was observed in the framework of dimensional regularization that gauge theories are renormalizable [7], in the sense that divergencies can be absorbed into a redefinition of coupling constants, mass parameters and fields. If one uses the renormalizable 't Hooft gauges, the divergence structure of a spontaneously broken theory is seen to be no worse than that of unbroken theories [8, 9].

The main advances in the systematic definition of renormalizable gauge theories occurred, when it was observed that gauge theories, including the gauge fixing and Faddeev-Popov part [10], are invariant under nonlinear symmetry transformations, the Becchi-Rouet-Stora (BRS) transformations [11, 12]. It is then possible to derive and postulate the Slavnov-Taylor identities, which are the functional version of BRS-symmetry, as expressing the defining symmetries of gauge theories in the quantized version. In particular, the program of algebraic renormalization has been applied to the abelian Higgs-Kibble model [13] and spontaneously broken gauge theories with semisimple gauge groups [14]. With the help of the action principle in its quantized version [15, 16], and algebraic consistency it was shown that the Green functions are completely characterized by normalization conditions on the mass and coupling constants and the Slavnov-Taylor identity. If the Adler-Bardeen anomalies [17, 18, 19] are absent, the Slavnov-Taylor identity can be established to all orders for off-shell Green functions. Then one is finally able to prove unitarity of the physical S-matrix, i.e. compensation of unphysical fields in physical scattering processes and gauge parameter independence of the physical S-matrix [13, 14, 20]. Algebraic renormalization yields finite Green functions by requiring invariance under symmetry transformations instead of defining them by an invariant scheme. In gauge theories without parity violation a specification by an invariant scheme is quite satisfactory. However, if anomalies are not forbidden for reasons of symmetry, then the algebraic renormalization becomes important, if one wants to formulate the theory consistently to all orders of perturbation theory.

In the early papers only gauge theories with a semisimple gauge structure were considered. Later on the renormalization procedure was extended to non-semisimple groups with several abelian factors [21]. But the analysis, as it is carried out there, is not immediately applicable to the standard model, due to the restriction to massive fields and due to the fact that the Green functions are not specified for on-shell fields. However, we shall use some technical components of this paper such as the form of the Slavnov-Taylor identity and the use of the Callan-Symanzik operator for solving the cohomology.

In the remainder of this introduction we shall outline the procedure of renormalizing the standard model, as it is presented in the paper. As the first step we have to specify all the symmetry transformations which characterize the tree approximation and higher order Green functions. It is important to note that the weak hypercharges are determined by requiring electromagnetic current conservation according to the Gell-Mann Nishijima formula. In the procedure of quantization electromagnetic gauge invariance is replaced by BRS-symmetry and the Slavnov-Taylor identity. For deriving the analog of the Gell-Mann Nishijima relation, we have to establish the local $U(1)$ Ward identity in addition to the Slavnov-Taylor identity. For specifying the abelian factor, however, it is necessary to have invariance under rigid $SU(2) \times U(1)$ Ward identities. This constraint restricts order by order the independent parameters of the gauge fixing functions, but rigid invariance is immediately established on the matter and Yang-Mills parts of the action. Only if one includes all these symmetry transformations, are the finite Green functions uniquely specified as being those of weak and electromagnetic interactions.

The symmetry invariants are free parameters and have to be fixed by normalization conditions. Here, the abstract analysis benefits from the fact, that different parameterizations have been considered for one-loop calculations and have been discussed quite extensively in the past (for a review see [22]), since their definition also enters the theoretical predictions of higher orders. It has been pointed out that those schemes are adequate, which allow the computation of different processes without switching to different parameter sets [23]. On-shell schemes which specify the mass parameters as physical masses on the 2-point functions [24, 25, 26, 27, 29, 28, 30, 31, 32] are certainly a safe choice, because all S-matrix elements are computed without adjusting further parameters when taking the LSZ-limit. Throughout this paper, we adopt an on-shell definition for the masses and in particular require mass diagonalization for massive/massless particles on-shell. In the abstract approach, such on-shell conditions are crucial, not only for physical particles but also for unphysical fields, when one finally wants to prove unitarity of the physical S-matrix [14, 20]. As far as the residua are concerned, we remain quite general in the construction, and do not specify special conditions. We finally see that some of the normalization conditions of residua can be eliminated by requiring a simple form of rigid Ward identities, but this is not essential at any stage of the procedure. The critical point in the analysis is the observation that on-shell conditions indeed fix more parameters than there are naive invariants. Requiring symmetries in their explicit tree form, one is unable simultaneously to adjust the W-mass and to diagonalize the neutral mass matrix at the mass of the Z-boson and at $p^2 = 0$. These normalization conditions are also deeply connected with off-shell infrared divergencies to higher orders. It has been pointed out already in [28] that complete on-shell schemes are compatible with the Slavnov-Taylor identity, and there are scattered remarks in the literature that on-shell schemes are in agreement with the symmetries if the transformations are themselves subject to renormalization (see e.g. [33]). But neither the Slavnov-Taylor identity nor rigid or local Ward-identities have been given in an explicit form valid for the Green functions of the standard model. The Slavnov-Taylor identity in its homogeneous form as given in [28] is not quite an adequate choice for the $SU(2) \times U(1)$-symmetry of the standard model, since one has to split off the abelian factor explicitly as done in [21]. In terms of on-shell fields, all symmetry transformations depend on the weak mixing angle in the tree approximation, and it is seen, that due to off-shell infrared divergencies, the symmetry operators have to be modified order by order in perturbation theory. For this reason we start the analysis by characterizing the symmetry transformations by algebra and field content, and find in this way all higher order deformations which are compatible with the symmetries. These general symmetry operators finally allow us to construct unique Green functions in the on-shell schemes, without introducing off-shell infrared divergencies.

For the present paper, we restrict ourselves to a diagonal quark mass matrix, because we are mainly interested in the renormalization of the vector sector. Apart from this we stay quite general and proceed as far as possible along the lines of concrete calculations. In particular, we use the general R_ξ-gauges, although in a modified form with an auxiliary field which couples to the gauge fixing functions. Particular attention is paid to the solution of the classical approximation, which gives the local four-dimensional invariants of symmetry transformations. In the higher order construction of finite Green functions we use the BPHZL scheme [34, 35]. In this scheme, massless particles are treated quite systematically by establishing those normalization conditions in the scheme which are necessary for the computation of finite Green functions to all orders. These normalization conditions are essentially the conditions for mass diagonalization of massless/massive field at $p^2 = 0$ (i.e. for the Z-boson and photon and the respective Faddeev Popov fields) and are established in the above-cited on-shell schemes by adjustment of counterterms.

The plan of this paper is as follows: In section 2, we give the classical action in renormalizable gauges compatible with rigid symmetry and local $U(1)$-gauge symmetry. We also present the symmetry transformations of the tree approximation in a functional form. These are the Slavnov-Taylor identity, rigid Ward-identities and the local $U(1)$ Ward identity. In section 3, we outline the method of algebraic renormalization. In section 4 we solve the algebra of symmetry operators and obtain the general consistent symmetry operators of the standard model. Section 5 is devoted to solving the symmetries for the local four-dimensional field polynomials. This analysis allows us to give the free parameters of the model and also to list the invariant counterterms of higher orders. In section 5.4 a complete treatment of the ghost equations is also included. In section 6, we derive the Callan-Symanzik equation of 1-loop order. By means of symmetric differential operators, it is possible to characterize symmetric nonlocal contributions of higher orders and in particular to determine the independent parameters of the theory in a scheme-independent way. In section 7, we proceed to higher orders and prove that Green functions can be constructed in agreement with the infrared normalization conditions to all orders, if one takes into account the modifications of the symmetry operators to higher orders as suggested by the tree approximation.

2. The tree approximation of the standard model

2.1. The gauge invariant part of the action

The standard model of electroweak interactions is a non-abelian gauge theory with the non-semisimple gauge group $SU(2)\times U(1)$. The gauge structure is essentially determined in the matter sector: It is seen, that the matter currents of weak interactions, the charged current J^μ_{CC} and the neutral current J^μ_{NC}, together with the electromagnetic current j^μ_{em} form a closed representation with respect to $SU(2)\times U(1)$ [4]. In order to embed these currents into a gauge theory, one groups the fermions into left-handed doublets, which transform under the fundamental representation of $SU(2)\times U(1)$, and right handed singlets, which only transform with respect to the abelian subgroup. The decomposition of the Dirac spinors into left and right handed fields is defined by the following projections:

$$\begin{aligned} f^L &= \tfrac{1}{2}(1-\gamma_5)f & \qquad f^R &= \tfrac{1}{2}(1+\gamma_5)f \\ \overline{f^L} &= \bar f\tfrac{1}{2}(1+\gamma_5) & \qquad \overline{f^R} &= \bar f\tfrac{1}{2}(1-\gamma_5) \end{aligned} \tag{2.1}$$

The fermions appear in families: Each family consists of a neutrino ν_i, a charged lepton e_i with electric charge $Q_e=-1$, and the up and down-type quarks u_i and d_i with charge $Q_u=\frac{2}{3}$ and $Q_d=-\frac{1}{3}$. For simplicity we suppress the colour index of the quarks throughout the paper. The lepton doublets $F^L_{l_i}$ and quark doublets $F^L_{q_i}, i=1,2,3$, are given by

$$F^L_{l_i}=\begin{pmatrix}\nu^L_i\\ e^L_i\end{pmatrix} = \begin{pmatrix}\nu^L_e\\ e^L\end{pmatrix}\begin{pmatrix}\nu^L_\mu\\ \mu^L\end{pmatrix}\begin{pmatrix}\nu^L_\tau\\ \tau^L\end{pmatrix} \tag{2.2}$$

$$F^L_{q_i}=\begin{pmatrix}u^L_i\\ d^L_i\end{pmatrix} = \begin{pmatrix}u^L\\ d^L\end{pmatrix}\begin{pmatrix}c^L\\ s^L\end{pmatrix}\begin{pmatrix}t^L\\ b^L\end{pmatrix} \tag{2.3}$$

The singlets only comprise the charged fermions:

$$f^R_i = e^R_i, u^R_i, d^R_i \tag{2.4}$$

The $SU(2)$ and $U(1)$ gauge transformations ($\alpha=+,-,3$):

$$\begin{aligned} \epsilon_\alpha(x)\delta_\alpha F^L_{\delta_i} &= i\epsilon_\alpha(x)\frac{\tau^T_\alpha}{2}F^L_{\delta_i} & \qquad \epsilon_\alpha(x)\delta_\alpha f^R_i &= 0 \\ \epsilon_4(x)\delta_4 F^L_{\delta_i} &= -i\epsilon_4(x)\frac{Y^\delta_W}{2}F^L_{\delta_i} & \qquad \epsilon_4(x)\delta_4 f^R_i &= -iQ_f f^R_i \end{aligned} \tag{2.5}$$

give rise to the matter currents of electroweak interactions

$$
\begin{aligned}
J_+^\mu &= -\frac{1}{2}\sum_{\delta_i}\overline{F_{\delta_i}^L}\gamma^\mu\tau_- F_{\delta_i}^L & J_-^\mu &= -\frac{1}{2}\sum_{\delta_i}\overline{F_{\delta_i}^L}\gamma^\mu\tau_+ F_{\delta_i}^L \\
J_3^\mu &= -\frac{1}{2}\sum_{i}\overline{F_{\delta_i}^L}\gamma^\mu\tau_3 F_{\delta_i}^L & J_4^\mu &= \frac{1}{2}\sum_{\delta_i}Y_W^\delta\overline{F_{\delta_i}^L}\gamma^\mu F_{\delta_i}^L + \sum_{f_i} Q_f\overline{f_i^R}\gamma^\mu f_i^R
\end{aligned}
\tag{2.6}
$$

with $\delta = l, q$. If one identifies out of the neutral currents the electromagnetic current

$$
j_{em}^\mu = \sum_{i,f} Q_f \bar{f}_i\gamma^\mu f_i = J_4^\mu - J_3^\mu \tag{2.7}
$$

the weak hypercharge and the electric charge are related according to the Gell-Mann Nishijima formula:

$$
\frac{1}{2}(\tau_3 + Y_W) = Q \tag{2.8}
$$

which means explicitly

$$
Y_W^l = -1 \quad \text{and} \quad Y_W^q = \frac{1}{3} \tag{2.9}
$$

In (2.5) and (2.6) $\tau_\alpha, \alpha = +,-3$, denote the generators of the charged fundamental representation of $SU(2)$. They are defined by

$$
\tau_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \quad \tau_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.10}
$$

and satisfy the following commutation relations:

$$
[\tau_\alpha, \tau_\beta] = 2i\epsilon_{\alpha\beta\gamma}\tau_\gamma^T \tag{2.11}
$$

The structure constants $\epsilon_{\alpha\beta\gamma}$ are imaginary and completely antisymmetric in all three indices:

$$
\epsilon_{+-3} = -i \tag{2.12}
$$

According to the Noether construction of gauge theories the matter action consists of the kinetic terms and the currents coupled to a $SU(2)$-triplet of vector fields $W_\alpha^\mu, \alpha = +,-,3$, and an abelian vector field W_4^μ

$$
\begin{aligned}
\Gamma_{matter} &= \sum_{i=1}^{N_F}\int\Big(i\overline{F_{l_i}^L}\partial\!\!\!/ F_{l_i}^L + i\overline{F_{q_i}^L}\partial\!\!\!/ F_{q_i}^L + i\overline{f_i^R}\partial\!\!\!/ f_i^R \\
&\qquad - g_2(W_+^\mu J_{\mu-} + W_-^\mu J_{\mu+} + W_3^\mu J_{\mu 3}) - g_1 W_4^\mu J_{\mu 4}\Big) \\
&\equiv \sum_{i=1}^{N_F}\int\Big(\overline{F_{l_i}^L} iD\!\!\!\!/\, F_{l_i}^L + \overline{F_{q_i}^L} iD\!\!\!\!/\, F_{q_i}^L + \overline{f_i^R} iD\!\!\!\!/\, f_i^R\Big)
\end{aligned}
\tag{2.13}
$$

The covariant derivatives are therefore given by:

$$\begin{aligned} D^\mu F^L_{\delta_i} &= \partial^\mu F^L_{\delta_i} - i g_2 \frac{\tau_\alpha}{2} F^L_{\delta_i} W^\mu_\alpha + i g_1 \frac{Y^\delta_W}{2} F^L_{\delta_i} W^\mu_4 \qquad \delta = l, q \\ D^\mu f^R_i &= \partial^\mu f^R_i + i g_1 Q_f f^R_i W^\mu_4 \end{aligned} \tag{2.14}$$

The gauge transformations of the vectors are uniquely determined from gauge invariance of the matter action:

$$\begin{aligned} \epsilon_\alpha(x)\delta_\alpha W^\mu_\beta &= (\tilde{I}_{\alpha\beta}\tfrac{1}{g_2}\partial^\mu + W^\mu_\gamma \tilde{I}_{\gamma\gamma'}\epsilon_{\gamma'\beta\alpha})\epsilon_\alpha(x) & \epsilon_4(x)\delta_4 W^\mu_\alpha &= 0 \\ \epsilon_\alpha(x)\delta_\alpha W^\mu_4 &= 0 & \epsilon_4(x)\delta_4 W^\mu_4 &= \tfrac{1}{g_1}\partial^\mu \epsilon_4(x) \end{aligned} \tag{2.15}$$

The matrix $\tilde{I}_{\alpha\beta}$ is the charge conjugation matrix:

$$\tilde{I} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} &\tilde{I}_{+-} = \tilde{I}_{-+} = \tilde{I}_{33} = \tilde{I}_{44} = 1 \\ &\tilde{I}_{\alpha\beta} = 0, \quad \text{else} \end{aligned} \tag{2.16}$$

From (2.15) the Yang-Mills part which involves the kinetic terms of the vectors is determined

$$\Gamma_{YM} = -\frac{1}{4}\int \left(G^{\mu\nu}_\alpha \tilde{I}_{\alpha\alpha'} G_{\mu\nu\alpha'} + F^{\mu\nu}F_{\mu\nu} \right) \tag{2.17}$$

with

$$G^{\mu\nu}_\alpha = \partial^\mu W^\nu_\alpha - \partial^\nu W^\mu_\alpha + g_2 \tilde{I}_{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma} W^\mu_\beta W^\nu_\gamma \tag{2.18}$$

$$F^{\mu\nu} = \partial^\mu W^\nu_4 - \partial^\nu W^\mu_4 \tag{2.19}$$

The bosons of weak interactions as well as the charged fermions are massive. The mass terms break chiral gauge invariance and have to be generated by the spontaneous breaking of the gauge symmetry. In the standard model all the masses are generated by introducing one complex scalar doublet Φ and its complex conjugate $\tilde{\Phi}$:

$$\Phi \equiv \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(H(x) + i\chi(x)) \end{pmatrix} \qquad \tilde{\Phi} \equiv i\tau_2 \Phi^* = \begin{pmatrix} \frac{1}{\sqrt{2}}(H(x) - i\chi(x)) \\ -\phi^-(x) \end{pmatrix} \tag{2.20}$$

$\phi^\pm$ are charged, H and χ neutral scalar fields. The doublet transforms under the fundamental representation and includes in its transformation a constant shift v into the direction of the neutral component of the scalar doublet:

$$\begin{aligned} \epsilon_\alpha(x)\delta_\alpha \Phi &= i\epsilon_\alpha(x)\frac{\tau^T_\alpha}{2}(\Phi + \mathrm{v}) \\ \epsilon_4(x)\delta_4 \Phi &= -i\epsilon_4(x)\frac{Y^s_W}{2}(\Phi + \mathrm{v}) \end{aligned} \tag{2.21}$$

with

$$\mathrm{v} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \tag{2.22}$$

The weak hypercharge is determined from (2.8)

$$Y_W^s = 1 \tag{2.23}$$

As a response to the transformation (2.21), the gauge invariant parts of the action Γ_{scalar} and Γ_{Yuk}, depend on the shift:

$$\begin{aligned} \Gamma_{scalar} &= \int \Big((D^\mu \Phi)^\dagger D_\mu \Phi - \frac{1}{2} \frac{m_H^2}{v^2} (\Phi^\dagger \Phi + \mathrm{v}^\dagger \Phi + \Phi^\dagger \mathrm{v})^2 \Big) && (2.24) \\ \Gamma_{Yuk} &= -\sum_i^{N_F} \int \frac{\sqrt{2}}{v} (m_{e_i} \overline{F_{l_i}^L} (\Phi + \mathrm{v}) e_i^R \\ & \qquad + m_{u_i} \overline{F_{q_i}^L} (\Phi + \mathrm{v}) u_i^R + m_{d_i} \overline{F_{q_i}^L} (\tilde{\Phi} + \tilde{\mathrm{v}}) d_i^R + \mathrm{h.c.}) && (2.25) \end{aligned}$$

The Yukawa interaction contains via the shift all mass terms of the fermions m_{f_i}. We have chosen the couplings of the Yukawa interactions in such a form, that the mass terms are parametrized by the mass of the respective fermions. For the purpose of this paper we forbid mixing between different families and especially assume CP-invariance throughout the paper.

The scalar part consists of the kinetic terms of the scalars and the scalar potential, which includes the mass of the Higgs field m_H^2. In order to have a proper particle interpretation, we have arranged the terms such that the contributions linear in $H(x)$ drop out. Via the covariant derivative

$$D_\mu \Phi = \partial_\mu \Phi - i (g_2 \frac{\tau_\alpha}{2} W_{\mu\alpha} - g_1 \frac{Y_W^s}{2} W_{\mu 4})(\Phi + \mathrm{v}) \tag{2.26}$$

the masses of the gauge fields are generated by eating up the massless Goldstone bosons ϕ_+, ϕ_- and χ:

$$\frac{1}{2} \frac{g_2^2 v^2}{4} (2 W_+^\mu W_{\mu -} + W_3^\mu W_{\mu 3}) + \frac{g_2 g_1 v^2}{4} W_3^\mu W_{\mu 4} + \frac{1}{2} \frac{g_1^2 v^2}{4} W_4^\mu W_{\mu 4} \tag{2.27}$$

Physical fields are constructed by diagonalizing the mass matrix with an orthogonal matrix:

$$W_\alpha^\mu = O_{\alpha a}(\theta_W) V_a^\mu \qquad O_{\alpha a}(\theta_W) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_W & -\sin\theta_W \\ 0 & 0 & \sin\theta_W & \cos\theta_W \end{pmatrix} \tag{2.28}$$

The fields which are generated by the rotation are the physical on-shell fields $V_a^\mu = (W_+^\mu, W_-^\mu, Z^\mu, A^\mu)$. Throughout the paper roman indices a, b, c are reserved to on-shell indices $a, b, c = +, -, Z, A$, whereas Greek indices α, β, γ denote group indices of $SU(2)$ and $U(1)$ $\alpha, \beta, \gamma = +, -, 3, 4$. In the tree approximation one calculates the following relations between the ratio of the gauge parameters and the weak mixing angle θ_W:

$$\frac{g_1}{g_2} = \tan \theta_W \tag{2.29}$$

$W^\mu_\pm$ are the charged bosons of weak interactions with mass M^2_W, Z^μ is the neutral boson with mass M^2_Z, A^μ the massless photon:

$$M^2_W = \frac{g_2^2 v^2}{4} \qquad M^2_Z = \frac{g_2^2 v^2}{4\cos^2\theta_W} \tag{2.30}$$

If one eliminates the parameters θ_W and v in favour of the masses M_W and M_Z, one arrives at the on-shell parameter set

$$M_W,\ M_Z,\ m_{f_i},\ m_H \tag{2.31}$$

which specifies the particles by their masses and electric charge. The weak mixing angle is then defined by the mass ratio of the W- and Z-mass

$$\cos\theta_W \equiv \frac{M_W}{M_Z}\,. \tag{2.32}$$

If one chooses the on-shell set for parametrizing the free parameters of the standard model, then one remains with one coupling constant, which in the QED-like parametrizations is taken to be the coupling of the electromagnetic current to the photon:

$$e = g_2 \sin\theta_W \tag{2.33}$$

The gauge invariant part of the classical action Γ_{GSW} [4, 5, 6] is given by the sum of the gauge invariant parts (2.13) (2.17) (2.24) and (2.25):

$$\Gamma_{GSW} = \Gamma_{YM} + \Gamma_{scalar} + \Gamma_{matter} + \Gamma_{Yuk} \tag{2.34}$$

It is completely specified by the gauge transformations, the masses of the interacting particles, their electric charge and the electromagnetic coupling.

We want to summarize the gauge transformations of the on-shell fields within the QED-like on-shell parameter set (2.31) and (2.33). In the spirit of the subsequent considerations we express the gauge transformations thereby in a functional operator acting on Γ_{GSW}:

$$\begin{aligned}
\Big(-\mathbf{w}_+ - \frac{\sin\theta_W}{e}\partial^\mu \frac{\delta}{\delta W^\mu_-}\Big)\Gamma_{GSW} &= 0 &(2.35)\\
\Big(-\mathbf{w}_- - \frac{\sin\theta_W}{e}\partial^\mu \frac{\delta}{\delta W^\mu_+}\Big)\Gamma_{GSW} &= 0 &(2.36)\\
\Big(-\mathbf{w}_3 - \frac{\sin\theta_W}{e}\partial^\mu(\cos\theta_W \frac{\delta}{\delta Z^\mu} - \sin\theta_W\frac{\delta}{\delta A^\mu})\Big)\Gamma_{GSW} &= 0 &(2.37)\\
\Big(\ \mathbf{w}_4 - \frac{\cos\theta_W}{e}\partial^\mu(\sin\theta_W \frac{\delta}{\delta Z^\mu} + \cos\theta_W\frac{\delta}{\delta A^\mu})\Big)\Gamma_{GSW} &= 0 &(2.38)
\end{aligned}$$

The functional operators of $SU(2)$-transformations are given by $(\alpha = +, -, 3)$

$$
\begin{aligned}
\mathbf{w}_\alpha &= \tilde{I}_{\alpha\alpha'} \quad \Big(V_b^\mu O^T_{b\beta}(\theta_W)\hat{\varepsilon}_{\beta\gamma\alpha} O_{\gamma c}(\theta_W)\tilde{I}_{cc'}\frac{\delta}{\delta V^\mu_{c'}} \\
&\qquad + i(\Phi + \mathrm{v})^\dagger \frac{\tau_{\alpha'}}{2}\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger} - i\frac{\overleftarrow{\delta}}{\delta\Phi}\frac{\tau_{\alpha'}}{2}(\Phi + \mathrm{v}) \\
&\qquad + \sum_{i=1}^{N_F}\sum_{\delta=l,q}\Big(i\overline{F^L_{\delta_i}}\frac{\tau_{\alpha'}}{2}\frac{\overrightarrow{\delta}}{\delta\overline{F^L_{\delta_i}}} - i\frac{\overleftarrow{\delta}}{\delta F^L_{\delta_i}}\frac{\tau_{\alpha'}}{2}F^L_{\delta_i}\Big)\Big)
\end{aligned}
\tag{2.39}
$$

The transformation of the on-shell vectors depends on the weak mixing angle:

$$
O^T_{b\beta}(\theta_W)\varepsilon_{\beta\gamma\alpha}O_{\gamma c}(\theta_W) \equiv \hat{\varepsilon}_{bc,\alpha} = \begin{cases} \hat{\varepsilon}_{Z+,-} &= -i\cos\theta_W \\ \hat{\varepsilon}_{A+,-} &= i\sin\theta_W \\ \hat{\varepsilon}_{+-,3} &= -i \end{cases}
\tag{2.40}
$$

The abelian Ward operator is given by

$$
\begin{aligned}
\mathbf{w}_4 &= \frac{i}{2}(\Phi + \mathrm{v})^\dagger\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger} - \frac{i}{2}\frac{\overleftarrow{\delta}}{\delta\Phi}(\Phi + \mathrm{v}) \\
&\quad + \sum_{i=1}^{N_F}\Big(\sum_{\delta=l,q} Y^\delta_W\Big(\frac{i}{2}\overline{F^L_{\delta_i}}\frac{\overrightarrow{\delta}}{\delta\overline{F^L_{\delta_i}}} - \frac{i}{2}\frac{\overleftarrow{\delta}}{\delta F^L_{\delta_i}}F^L_{\delta_i}\Big) \\
&\qquad - \sum_{f^R} Q_f\Big(i\overline{f^R_i}\frac{\overrightarrow{\delta}}{\delta\overline{f^R_i}} - i\frac{\overleftarrow{\delta}}{\delta f^R_i}f^R_i\Big)\Big)
\end{aligned}
\tag{2.41}
$$

In the notation we understand summation over all fermion singlets and doublets. The Ward operators satisfy the local $SU(2) \times U(1)$-algebra:

$$
\begin{aligned}
[\mathbf{w}_\alpha(x), \mathbf{w}_\beta(y)] &= \delta(x-y)\varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathbf{w}_{\gamma'}(x) \\
[\mathbf{w}_\alpha(x), \mathbf{w}_4(y)] &= 0
\end{aligned}
\tag{2.42}
$$

It is obvious that Γ_{GSW} is also invariant with respect to rigid transformations which are obtained by taking the infinitesimal parameters ϵ_α as constants or, equivalently, by integrating the local Ward operators $(\alpha = +, -, 3, 4)$:

$$
\mathcal{W}_\alpha\Gamma_{GSW} = 0 \qquad \text{and} \quad \mathcal{W}_\alpha = \int \mathbf{w}_\alpha
\tag{2.43}
$$

Rigid symmetries can be established for off-shell Green functions to all orders of perturbation theory in a modified form and turn out together with the abelian Ward identity to be important ingredients for defining the standard model in its quantized version.

2.2. The gauge fixing and rigid transformations

The perturbative construction of Green functions and finally the S-matrix starts with the specification of the free fields and their respective propagators. In the standard model the scalars $\phi_\pm$ and χ are unphysical fields being absorbed into the longitudinal polarization of the massive vectors $W_\pm$ and Z. Eliminating them by a gauge transformation, however, leads to propagators with a bad ultraviolet behaviour, and renormalizability by power counting is not evident anymore. For a systematic treatment of higher orders one better uses the renormalizable gauges as the R_ξ-gauges. If one constructs the off-shell Green functions in the renormalizable gauges, one is able to refer to power counting properties of renormalized perturbation theory and, especially, to the quantum action principle. In the end one has then to prove unitarity of the physical S-matrix, i.e. it has to be shown that the unphysical fields, as the scalar component of the vectors and the Goldstone bosons, do not appear in physical scattering processes.

The free field propagators are calculated from the bilinear parts of the gauge invariant action Γ_{GSW} and the gauge fixing part $\Gamma_{g.f.}$:

$$\Gamma^{(bil)} = \Gamma^{(bil)}_{GSW} + \Gamma_{g.f.} \tag{2.44}$$

The gauge fixing in the R_ξ-gauges is given by:

$$\Gamma_{g.f.} = \int -\frac{1}{\xi_W} F_+ F_- - \frac{1}{2\xi_Z} F_Z F_Z - \frac{1}{2\xi_A} F_A F_A \tag{2.45}$$

with

$$\begin{aligned} F_\pm &\equiv \partial_\mu W^\mu_\pm \mp i M_W \zeta_W \phi_\pm \\ F_Z &\equiv \partial_\mu Z^\mu - M_Z \zeta_Z \chi \\ F_A &\equiv \partial_\mu A^\mu \end{aligned} \tag{2.46}$$

The free field propagators are seen to have a good UV-behaviour which guarantees renormalizability by power counting:

$$G_{\varphi_k \varphi_l} \longrightarrow p^{-2(2-d_{\varphi_k})} \quad \text{if} \quad p^2 \to \infty \tag{2.47}$$

where d_{φ_α} is the mass dimension of the field φ_k. They also have good infrared behaviour, i.e. they diverge for the massless particles not stronger than p^{-2} as for the photon field. p^{-4} infrared divergent terms are removed by introducing mass terms for the would-be Goldstone fields into the gauge fixing functions.

Adding such a gauge fixing with arbitrary gauge parameters to the action one does not keep any knowledge about the $SU(2) \times U(1)$ structure of the standard model in the free field propagators, but treats the bilinear action as if it were composed of several $U(1)$-factors. But as a consequence of the gauge construction $\Gamma^{(bil)}_{GSW}$ *has* definite transformation properties under rigid unbroken $SU(2)\times U(1)$: 4-dimensional terms are invariants; the 3-dimensional terms, which are the fermion mass terms and the mixed scalar-vector terms, together with their variations transform as a vector under unbroken $SU(2)\times U(1)$. The mass terms of the vectors are composed with their variations and second variations to a second rank tensor. In order not to spoil these transformation properties by the gauge fixing part, one has to choose:

$$\xi \equiv \xi_W = \xi_Z = \xi_A \qquad \text{and} \qquad \zeta \equiv \zeta_Z = \zeta_W \tag{2.48}$$

Instead of requiring the complicated transformation behaviour of the mass terms one can introduce an external scalar field $\hat{\Phi}$ and its complex conjugate, which couples to the masses and their variations (see also [36]):

$$\hat{\Phi} = \begin{pmatrix} \hat{\phi}^+ \\ \frac{1}{\sqrt{2}}(\hat{H} + i\hat{\chi}) \end{pmatrix} \quad \hat{\Phi}^\dagger = \begin{pmatrix} \hat{\phi}^- \\ \frac{1}{\sqrt{2}}(\hat{H} - i\hat{\chi}) \end{pmatrix} \tag{2.49}$$

Under rigid transformations it transforms in the same way as the scalar doublet Φ, but includes a different shift parameterized by $\zeta\mathrm{v}$ into the transformation ($\epsilon_\alpha, \epsilon_4 =$ const.):

$$\begin{aligned} \epsilon_\alpha \delta_\alpha \hat{\Phi} &= i\epsilon_\alpha \frac{\tau_\alpha^T}{2}(\hat{\Phi} + \zeta\mathrm{v}) \\ \epsilon_4 \delta_4 \hat{\Phi} &= -i\epsilon_4 \frac{Y_W^s}{2}(\hat{\Phi} + \zeta\mathrm{v}) \end{aligned} \tag{2.50}$$

Algebraically this is the same procedure as one carries out if one introduces the scalar doublet and the Higgs mechanism for generating the masses of the fermions and vectors, but the external field is required to be non-propagating and does not have physical meaning. The gauge fixing functions can be enlarged by the external field in such a way that they transform as a vector under the adjoint representation:

$$F_a \to \mathcal{F}_a = \partial_\mu V_a^\mu - i\frac{e}{\sin\theta_W}\Big((\hat{\Phi} + \zeta\mathrm{v})^\dagger \frac{\tau_a^T}{2}(\Phi + \mathrm{v}) - (\Phi + \mathrm{v})^\dagger \frac{\tau_a^T}{2}(\hat{\Phi} + \zeta\mathrm{v})\Big) \tag{2.51}$$

The gauge fixing part is then invariant under the rigid transformations, if one includes the transformations of the external fields:

$$\Gamma_{g.f.} = \int -\frac{1}{2\xi}\mathcal{F}_a \tilde{I}_{ab} \mathcal{F}_b \qquad \epsilon_\alpha \delta_\alpha \Gamma_{g.f.} = 0 \tag{2.52}$$

$V_a, a = +, -, Z, A$ are the on-shell fields, and the respective representation matrices τ_a are obtained by acting with the orthogonal matrix $O(\theta_W)$ (2.28) on τ_3 and $G\mathbf{1}$:

$$\tau_a(G) = O^T_{a\alpha}(\theta_W)\tau_\alpha + O^T_{a4} G\mathbf{1} \tag{2.53}$$

Explicitly they read:

$$\begin{aligned}\tau_Z(G) &= \cos\theta_W\tau_3 + G\sin\theta_W\mathbf{1}\\ \tau_A(G) &= -\sin\theta_W\tau_3 + G\cos\theta_W\mathbf{1}\end{aligned}\tag{2.54}$$

The abelian parameter G is not fixed by rigid invariance. Choosing it

$$G = -\frac{\sin\theta_W}{\cos\theta_W}\tag{2.55}$$

one obtains for vanishing external fields the original gauge fixing with the parameters according to (2.48). The masses of the would-be Goldstones are generated by the shift of the external field. The transformation properties of the trilinear and the mass terms are now governed by the transformation properties of the external field $\hat{\Phi}$.

Modifying the functional operators of rigid transformations (2.43) by the transformations of the external field according to (2.50)

$$\begin{aligned}\mathcal{W}_\alpha &\to \mathcal{W}_\alpha + \tilde{I}_{\alpha\alpha'}\int\Big(i(\hat{\Phi}+\zeta\mathrm{v})^\dagger\frac{\tau_{\alpha'}}{2}\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger} - i\frac{\overleftarrow{\delta}}{\delta\hat{\Phi}}\frac{\tau_{\alpha'}}{2}(\hat{\Phi}+\zeta\mathrm{v})\Big) \qquad \alpha = +,-,3\\ \mathbf{w}_4 &\to \mathbf{w}_4 + \frac{i}{2}(\hat{\Phi}+\zeta\mathrm{v})^\dagger\frac{\overrightarrow{\delta}}{\delta\hat{\Phi}^\dagger} - \frac{i}{2}\frac{\overleftarrow{\delta}}{\delta\hat{\Phi}}(\hat{\Phi}+\zeta\mathrm{v})\end{aligned}\tag{2.56}$$

we write the invariance properties of $\Gamma_{GSW}+\Gamma_{g.f.}$ in functional form:

$$\mathcal{W}_\alpha(\Gamma_{GSW}+\Gamma_{g.f.}) = 0\tag{2.57}$$

Furthermore it is seen that the gauge transformation of the abelian subgroup is broken linearly in propagating fields:

$$\Big(\frac{e}{\cos\theta_W}\mathbf{w}_4 - \partial^\mu\big(\sin\theta_W\frac{\delta}{\delta Z^\mu}+\cos\theta_W\frac{\delta}{\delta A^\mu}\big)\Big)(\Gamma_{GSW}+\Gamma_{g.f.}) = -\frac{1}{\xi}\Box(\sin\theta_W\mathcal{F}_Z+\cos\theta_W\mathcal{F}_A)\tag{2.58}$$

For this reason it is possible to extend and interpret (2.58) as a Ward identity for Green functions.

This construction of the gauge fixing sector is essential if one wants to proceed to higher orders perturbation theory. Especially it is seen that we need a local Ward identity of the form (2.58) for the Green functions in order to fix the weak hypercharge and electric charge in a scheme independent way.

Finally we want to mention that choosing the parameter G according to (2.55) is arbitrary and not related to any symmetries. It turns out that this parameter as well as an additional abelian gauge parameter are renormalized in higher orders of perturbation theory.

2.3. BRS-invariance and Faddeev-Popov ghosts

The linear R_ξ-gauges break also in their covariant form gauge invariance and especially bring about that the unphysical fields, the scalar components of the vectors and the would-be Goldstones, interact with the physical fields violating thereby unitarity of the physical S-matrix. For this reason one has to introduce the Faddeev-Popov ghosts $c_a, a = +, -, Z, A$ with ghost charge 1 and the respective antighosts $\bar{c}_a, a = +, -, Z, A$ with ghost charge -1. They are anticommuting scalars with negative norm and compensate the unphysical degrees of freedom introduced by the gauge fixing, if one adds the ghost action in such a way, that the complete action is invariant under BRS-transformations.

There a several approaches to introduce the Faddeev-Popov fields [10] into the perturbative formulation of gauge theories. One way to proceed is to consider BRS-transformations in a first step as an alternative way to characterize the Lie algebra of the gauge group. This approach is close to the algebraic analysis which we carry out in the higher order construction, and therefore we outline the procedure in the following: Starting from the gauge transformations of the fields as summarized in functional form in (2.39) and (2.41) one translates the infinitesimal parameters $\epsilon_\alpha(x)$ into anticommuting parameters $c_\alpha(x), \alpha = +, -, 3, 4$. Considering the gauge transformations on the on-shell fields (2.35) one is lead to carry out the orthogonal transformation $O(\theta_W)$ (2.28) on the ghosts as well

$$c_\alpha = O_{\alpha a}(\theta_W) c_a \qquad c_a = (c_+, c_-, c_Z, c_A) \tag{2.59}$$

In this procedure is quite some arbitrariness, which has to be exploited in higher order perturbation theory for a proper definition of massless ghost propagators (see section 5.4). The BRS-transformations [13] on the vector bosons $V^\mu_a = (W_+, W_-, Z, A)$, the scalar doublet Φ and the fermion doublets and singlets read in the physical on-shell parameterization:

$$\begin{aligned}
\mathrm{s} V_{\mu a} &= \partial_\mu c_a + \frac{e}{\sin\theta_W} \tilde{I}_{aa'} f_{a'bc} V_{\mu b} c_c \\
\mathrm{s}\Phi &= i \frac{e}{\sin\theta_W} \frac{\tau_a(G_s)}{2} (\Phi + \mathrm{v}) c_a \\
\mathrm{s} F^L_{\delta_i} &= i \frac{e}{\sin\theta_W} \frac{\tau_a(G_\delta)}{2} F^L_{\delta_i} c_a \qquad \delta = l, q \\
\mathrm{s} f^R_i &= -ieQ_f \frac{\sin\theta_W}{\cos\theta_W} f^R_i c_Z - ieQ_f f^R_i c_A
\end{aligned} \tag{2.60}$$

The matrices $\tau_a, a = +, -, Z, A$ are given in (2.54) and satisfy the algebra

$$[\tau_a(G), \tau_b(G)] = i f_{abc} \tilde{I}_{cc'} \tau_{c'}(G) \tag{2.61}$$

with the structure constants

$$f_{abc} = O^T_{a\alpha}(\theta_W)O^T_{b\beta}(\theta_W)\epsilon_{\alpha\beta\gamma}O_{\gamma c}(\theta_W) = \begin{cases} f_{+-Z} & = & -i\cos\theta_W \\ f_{+-A} & = & i\sin\theta_W \end{cases} \tag{2.62}$$

The abelian parameter G appearing in the BRS-transformations is related to the weak hypercharge according to the Gell-Mann Nishijima relation, explicitly:

$$G_k = -Y^k_W \frac{\sin\theta_W}{\cos\theta_W} \qquad Y^k_W = \begin{cases} 1 & \text{for the scalar } (k = s) \\ \text{-1} & \text{for the lepton doublets } (k = l) \\ \frac{1}{3} & \text{for the quark doublets } (k = q) \end{cases} \tag{2.63}$$

The algebra of the functional operators (2.42), which contains the complete information about the group structure, is translated into the BRS-transformation of the ghosts

$$\mathrm{s}c_a \quad = \quad -\frac{e}{2\sin\theta_W}\tilde{I}_{aa'}f_{a'bc}c_b c_c \tag{2.64}$$

The representation equations and also the Jacobi identities are now encoded in the nilpotency of the BRS-transformations:

$$\mathrm{s}^2\varphi_k = 0 \quad \text{with} \quad \varphi_k = V_{\mu a}, \Phi, F^L_{\delta_i}, f^R_i, c_a \tag{2.65}$$

From the construction it is obvious that the gauge invariant part of the action (2.34) is BRS-invariant:

$$\mathrm{s}\Gamma_{GSW} = 0 \tag{2.66}$$

The gauge fixing (2.52) breaks gauge invariance; having introduced the anticommuting fields c_a this breaking is absorbed into the transformation of the antighosts:

$$\int -\frac{1}{\xi}\mathcal{F}_a\tilde{I}_{aa'}\mathrm{s}\mathcal{F}_{a'} - \mathrm{s}\bar{c}_a\tilde{I}_{aa'}\mathrm{s}\mathcal{F}_{a'} \stackrel{!}{=} 0 \tag{2.67}$$

Therefrom one obtains:

$$\mathrm{s}\bar{c}_a = -\frac{1}{\xi}\mathcal{F}_a \tag{2.68}$$

and

$$\mathrm{s}(\Gamma_{g.f.} + \Gamma_{ghost}) = 0 \quad \text{with} \quad \Gamma_{ghost} = -\int \bar{c}_a\tilde{I}_{ab}\mathrm{s}\mathcal{F}_b \tag{2.69}$$

The ghost action contains kinetic terms for the Faddeev-Popov fields, which allows to introduce them as dynamical fields into the theory.

The BRS-transformation of the anti-ghosts is not nilpotent. To remedy this situation one reformulates the gauge fixing part of the action by introducing the auxiliary fields $B_a, a = +, -, Z, A$

$$\Gamma_{g.f.} \quad = \quad \int \frac{1}{2}\xi B_a\tilde{I}_{ab}B_b + B_a\tilde{I}_{ab}\mathcal{F}_b \tag{2.70}$$

It can be transformed into the usual form of the R_ξ gauges by eliminating the B_a-fields via their equations of motions:

$$\frac{\delta\Gamma}{\delta B_a} = \tilde{I}_{ab}(\xi B_b + \mathcal{F}_b) \stackrel{*}{=} 0 \quad \Longrightarrow \quad B_a \stackrel{*}{=} -\frac{1}{\xi}\mathcal{F}_a \tag{2.71}$$

Therefore the propagators of vectors and scalars are not changed, but in addition one has mixed propagators between B_a-fields and vectors and B_a-fields and scalars. The ghost action is likewise determined from (2.69), but the BRS-transformations turn out to be nilpotent also on the antighosts:

$$\mathrm{s}\bar{c}_a = B_a \qquad \mathrm{s}B_a = 0 \tag{2.72}$$

For the algebraic characterization it is useful to have nilpotency of BRS-transformations throughout and we refer to this form of the gauge fixing in the algebraic proof of renormalizability in higher orders. Invariance under rigid transformation is maintained, if one transforms the B_a-fields according to the adjoint representation ($\epsilon_\alpha, \epsilon_4 = \text{const.}$)

$$\epsilon_\alpha \delta_\alpha B_b = B_c \tilde{I}_{cc'} \hat{\varepsilon}_{c'b,\alpha}(\theta_W)\epsilon_\alpha \qquad \epsilon_4 \delta_4 B_b = 0 \tag{2.73}$$

The tensor $\hat{\epsilon}_{bc,\alpha}(\theta_W)$ is defined in eq. (2.40)

The gauge fixing functions $\mathcal{F}_a$ depend on the external scalar doublet $\hat{\Phi}$ and we have to assign to them also definite transformation properties under BRS-transformations. Transforming $\hat{\Phi}$ into an external anticommuting scalar doublet $\hat{\mathbf{q}}$ with ghost charge 1

$$\mathrm{s}\hat{\Phi} = \hat{\mathbf{q}} \qquad \mathrm{s}\hat{\mathbf{q}} = 0 \tag{2.74}$$

does this job and allows to distinguish the propagating and external scalar fields algebraically.

Explicitly the ghost action is given by

$$\begin{aligned}\Gamma_{ghost} \;=\; & \int \Big(-\bar{c}_a \Box \tilde{I}_{ab} c_b - \frac{e}{\sin\theta_W} \bar{c}_a f_{abc} \partial(V_b c_c) \\ & + i\frac{e}{2\sin\theta_W}(\hat{\mathbf{q}}^\dagger \tau_a(G_s)(\Phi + \mathrm{v}) - (\Phi+\mathrm{v})^\dagger \tau_a(G_s)\hat{\mathbf{q}})\bar{c}_a \\ & - \frac{e}{4\sin\theta_W}\Big((\hat{\Phi} + \zeta \mathrm{v})^\dagger \tau_a(G_s)\tau_b(G_s)(\Phi + \mathrm{v}) \\ & \qquad + (\Phi + \mathrm{v})^\dagger \tau_b(G_s)\tau_a(G_s)(\hat{\Phi} + \zeta\mathrm{v})\Big)\bar{c}_a c_b\Big)\end{aligned} \tag{2.75}$$

G_s is related to the weak hypercharge of the scalar doublets according to (2.63). Via the shift of the external and the quantum scalar fields the charged ghosts as well as the neutral Z-ghost become massive, whereas the ghost associated with the photon field remains massless. The bilinear part of the ghost action

$$\Gamma^{(bil)}_{ghost} = \int\Big(-\bar c_a \Box \tilde I_{ab} c_b - \zeta M_W^2(\bar c_+ c_- + \bar c_- c_+) - \zeta M_Z^2 \bar c_Z c_Z\Big) \tag{2.76}$$

gives rise to free field propagators for the Faddeev Popov fields.

The ghost action is seen to be invariant under rigid transformations if one assigns the following transformations under $SU(2)\times U(1)$ ($\epsilon_\alpha, \epsilon_Y$ =const.)

$$\begin{array}{rclrcl} \epsilon_\alpha\delta_\alpha \bar c_b &=& \bar c_c \tilde I_{cc'}\hat\varepsilon_{c'b,\alpha}(\theta_W)\epsilon_\alpha & \quad \epsilon_\alpha\delta_\alpha c_a &=& c_c \tilde I_{cc'}\hat\varepsilon_{c'b,\alpha}(\theta_W)\epsilon_\alpha \\ \epsilon_4\delta_4 \bar c_a &=& 0 & \epsilon_4\delta_4 c_a &=& 0 \end{array} \tag{2.77}$$

In particular $\Gamma^{(bil)}_{ghost}$ transforms covariantly in the same way as $\Gamma^{(bil)}_{GSW}$.

2.4. The tree approximation: the Slavnov-Taylor identity

In the last sections we have derived the classical action of the standard model

$$\Gamma_{cl} = \Gamma_{GSW} + \Gamma_{g.f.} + \Gamma_{ghost} \tag{2.78}$$

in a way that is invariant under BRS-transformations

$$\mathrm{s}\Gamma_{cl} = 0 \tag{2.79}$$

Spontaneously broken rigid $SU(2)\times U(1)$-symmetry has been established by introducing an external scalar doublet $\hat\Phi$ into the gauge fixing part of the action.

In order to quantize the model in perturbation theory one has to construct the Green functions of the interacting theory according to the Gell-Mann Low formula.

$$\begin{aligned} G_{\varphi_{i_1}\ldots\varphi_{i_n}}(x_1,\ldots,x_n) &= \langle T\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n)\rangle \\ &= R\frac{\langle T\varphi^{(o)}_{i_1}(x_1)\ldots\varphi^{(o)}_{i_n}(x_n)e^{i\Gamma_{int}(\varphi^{(o)}_k,\hat\Phi,\hat{\mathbf{q}})}\rangle}{\langle T e^{i\Gamma_{int}(\varphi^{(o)}_k,\hat\Phi,\hat{\mathbf{q}})}\rangle}\Bigg|_{\substack{\hat\Phi=0\\ q=0}} \end{aligned} \tag{2.80}$$

where φ_k denotes the propagating fields of the standard model

$$\varphi_k = \begin{cases} V_a^\mu, B_a, c_a, \bar c_a & a = +,-,Z,A \\ \phi_\pm, H, \chi & \\ \nu_i^L, e_i, u_i, d_i & i = 1\ldots N_F \end{cases} \tag{2.81}$$

Γ_{int} includes all the interactions and the field polynomials depending on the external fields and is obtained by splitting off from the classical action the bilinear part:

$$\Gamma_{cl} = \Gamma^{(bil)} + \Gamma_{int} \tag{2.82}$$

with

$$\Gamma^{(bil)} = \Gamma^{(bil)}_{GSW} + \Gamma_{g.f.}|_{\substack{\dot\Phi=0\\ \dot q=0}} + \Gamma^{(bil)}_{ghost} \tag{2.83}$$

The index (o) stands for free fields.

The formal expansion of the exponential yields the Green functions of the interacting theory in expressions of time ordered vacuum expectation values of free fields. These expressions are decomposed into a sum of products of free field propagators and certain vertex factors according to Wick's theorem. The combinatorics and the vertex factors are summarized graphically in the Feynman rules. The free field propagators are determined from $\Gamma^{(bil)}$. The Feynman rules of the standard model are listed in the literature and are given e.g. in [37] according to the conventions we have adopted.

Due to the well-known ultraviolet divergencies the formal expansion of the Gell-Mann Low formula is not meaningful in higher orders of perturbation theory and has to be rendered meaningful in the course of renormalization. (This is the sense of R in eq. (2.80).) In the lowest order, the tree approximation, the Green functions are well-defined and it has to be shown, that the physical S-matrix, which is constructed from these Green functions according to the LSZ reduction formula, is unitary in the lowest order. This means, that one has to verify that unphysical particles do not contribute in physical scattering processes, and that they are canceled among each other. This cancellation mechanism is governed by the Slavnov-Taylor identity, which expresses consequences of the classical BRS-symmetry for the off-shell Green functions. In order to derive the Slavnov-Taylor identity in the tree approximation we introduce the generating functional of Green functions:

$$\begin{aligned} &Z(j^\mu_a, j^B_a \jmath_a, \bar\jmath_a, J, J^\dagger, \eta_i, \bar\eta_i) \qquad (2.84)\\ &= \Big\langle T exp\Big\{ i\int dx \Big(\tilde I_{ab}(j^\mu_a V_{\mu b} + j^B_a B_b + \bar\jmath_a c_b + \bar c_a \jmath_b) + \Phi^\dagger J + J^\dagger \Phi + \bar f_i \eta_i + \bar\eta_i f_i \Big) \Big\}\Big\rangle \end{aligned}$$

In (2.84) we understand summation over on-shell field indices $a, b = +, -, Z, A$ and summation over all fermions $f_i = \nu_i, e_i, u_i, d_i$. The source functions are commuting (j^μ_a, j^B_a, J) and anticommuting $(\jmath, \bar\jmath, \eta_i)$ test functions. Electric and $\phi\pi$-charge is assigned in such a way that the generating functional is neutral. The Green functions are obtained by differentiation with respect to the respective source functions.

Although the Green functions are the basic objects of the theory, for the purpose of renormalization one better refers to the building blocks composing them. These are the connected Green functions and the one-particle-irreducible (1PI) Green functions. The generating functional of connected Green functions

$$Z_c(j_k) \equiv Z_c(j^\mu_a, j^B_a, \jmath_a, \bar\jmath_a, J, J^\dagger, \eta_i, \bar\eta_i) \tag{2.85}$$

is defined by

$$Z(j_k) = e^{iZ_c(j_k)} \tag{2.86}$$

and one can show that the differentiation with respect to the sources yields the connected Green functions in the diagrammatic expansion. The generating functional of 1PI Green functions is obtained from $Z_c(j_k)$ by Legendre transformation. For this purpose one introduces the classical fields

$$\varphi_k^{cl}(x, j_i) = \frac{\delta Z_c(j_i)}{\delta j_k(x)} \qquad \varphi_k^{cl}(x, 0) = 0 \tag{2.87}$$

and defines the generating functional of 1PI Green functions $\Gamma(\varphi_k^{cl})$ according to

$$\begin{aligned} Z_c(j_a^\mu, j_a, \bar{j}_a, J, J^\dagger, \eta_i, \bar{\eta}_i) &= \Gamma(V_{\mu a}^{cl}, B_a^{cl}, c_a^{cl}, \bar{c}_a^{cl}, \Phi^{cl}, {\Phi^{cl}}^\dagger, f_i^{cl}, \bar{f}_i^{cl}) \\ &+ \int dx \Big(\tilde{I}_{ab}(j_a^\mu V_{\mu b}^{cl} + j_a^B B_b + \bar{j}_a c_b^{cl} + \bar{c}_a^{cl} j_b) + {\Phi^{cl}}^\dagger J + J^\dagger \Phi^{cl} + \bar{f}_i^{cl} \eta_i + \bar{\eta}_i f_i^{cl} \Big) \end{aligned} \tag{2.88}$$

Here the sources j_k are understood as solutions of (2.87)

$$j_k = j_k(x, \varphi_k^{cl}) \qquad j_k(x, 0) = 0 \tag{2.89}$$

The 1PI Green functions are obtained by differentiating the generating functional with respect to the classical fields φ_k^{cl}, and one can show, that they correspond to the 1PI diagrams in the diagrammatic expansion according to the Feynman rules.

The Slavnov-Taylor identity of the tree approximation can be derived most simply on the generating functional of 1PI Green functions, because its lowest order is seen to be the classical action:

$$\Gamma(\varphi_k^{cl}) = \Gamma_{cl}(\varphi_k^{cl})|_{\substack{\Phi=0 \\ q=0}} + O(\hbar) \tag{2.90}$$

Therefore we are able to write down the Ward identity of BRS-transformation as

$$\sum_{\varphi_k^{cl}} \int dx \, \mathrm{s}\varphi_k^{cl}(x) \frac{\delta \Gamma_{cl}(\varphi_i^{cl})}{\delta \varphi_k^{cl}(x)} = 0 \tag{2.91}$$

which is a well defined expression in the tree approximation. The BRS-transformations are non-linear symmetry transformation in propagating fields and it is seen that the non-linear symmetry transformations become insertions into (connected) Green functions, if one carries out the Legendre transformation. Roughly speaking one has to replace

$$\mathrm{s}\varphi_k^{cl} \longrightarrow [\mathrm{s}\varphi_k] \cdot Z_c(j_k) \tag{2.92}$$

where $[\mathrm{s}\varphi_k] \cdot Z_c(j_k)$ is the generating functional of BRS-inserted connected Green functions. The Green functions with insertions are defined according to

$$\begin{aligned} G_{\mathrm{s}\varphi_k;\varphi_{i_1}\ldots\varphi_{i_n}}(x; x_1, \ldots, x_n) &= \langle T : \mathrm{s}\varphi_k(x) : \varphi_{i_1}(x_1) \ldots \varphi_{i_n}(x_n) \rangle \\ &= R \frac{\langle T : \mathrm{s}\varphi_k^{(o)}(x) : \varphi_{i_1}^{(o)}(x_1) \ldots \varphi_{i_n}^{(o)}(x_n) e^{i\Gamma_{int}(\varphi_k^{(o)})} \rangle}{\langle T e^{i\Gamma_{int}(\varphi_k^{(o)})} \rangle} \end{aligned} \tag{2.93}$$

and are summarized in the functional of BRS-inserted (connected) Green functions according to the above definitions. For setting up the Slavnov-Taylor identity for off-shell Green functions one does not only have to consider the ordinary Green functions but also the ones with BRS-insertions. For defining the BRS-inserted as well as the ordinary Green functions consistently one enlarges the classical action by the external field part and couples the non-linear BRS-transformations to external fields:

$$\Gamma_{cl}(\varphi_k) \longrightarrow \Gamma_{cl}(\varphi_k, \Upsilon_k) = \Gamma_{cl}(\varphi_k) + \Gamma_{ext.f}(\varphi_k, \Upsilon_k) \tag{2.94}$$

$$\begin{aligned}
\Gamma_{ext.f.} \; = \; & \int \Big(\rho_+^\mu \mathrm{s} W_{\mu,-} + \rho_-^\mu \mathrm{s} W_{\mu,+} + \rho_3^\mu (\cos\theta_W \mathrm{s} Z_\mu - \sin\theta_W \mathrm{s} A_\mu) \\
& + \sigma_+ \mathrm{s} c_- + \sigma_- \mathrm{s} c_+ + \sigma_3 (\cos\theta_W \mathrm{s} c_Z - \sin\theta_W \mathrm{s} c_A) \\
& + Y^\dagger \mathrm{s}\Phi + (\mathrm{s}\Phi)^\dagger Y \\
& + \sum_{i=1}^{N_F} (\overline{\Psi^R_{\delta_i}} \mathrm{s} F^L_{\delta_i} + \overline{\psi^L_{f_i}} \mathrm{s} f^R_i + \mathrm{h.c.}) \Big)
\end{aligned} \tag{2.95}$$

The external fields ρ^μ_α and $\sigma_\alpha, \alpha = +,-,3$, are anticommuting and commuting $SU(2)$-triplets with ghost charge -1 and -2 respectively. The external field Y is a complex anticommuting scalar doublet with ghost charge -1, $\psi^L_{f_i}$ denotes external left-handed spinor singlets with ghost charge -1

$$\psi^L_{f_i} \equiv \psi^L_{e_i}, \psi^L_{u_i}, \psi^L_{d_i} \tag{2.96}$$

whereas $\Psi^R_{\delta_i}$ denotes external right-handed spinor doublets

$$\Psi^R_{\delta_i} \equiv \Psi^R_{l_i}, \Psi^R_{q_i} \qquad \Psi^R_{l_i} = \begin{pmatrix} \psi^R_{\nu_i} \\ \psi^R_{e_i} \end{pmatrix} \qquad \Psi^R_{q_i} = \begin{pmatrix} \psi^R_{u_i} \\ \psi^R_{d_i} \end{pmatrix} \tag{2.97}$$

The Green functions with insertions (2.93) are defined via the external field part:

$$G_{\mathrm{s}\varphi_k;\varphi_{i_1}\ldots\varphi_{i_n}}(x; x_1, ..., x_n) \; = \; \frac{\delta}{\delta \Upsilon_k(x)} \langle T \varphi_{i_1}(x_1) ... \varphi_{i_n}(x_n) \exp\{i\Gamma_{ext.f}\}\rangle\Big|_{\Upsilon_i=0} \tag{2.98}$$

The generating functional of Green functions

$$Z(j_k, \Upsilon_k) = \Big\langle T \exp\Big\{ i \int dx \Big(j_k \varphi_k + \Gamma_{ext.f.}(\varphi_k, \Upsilon_k) \Big) \Big\} \Big\rangle$$

includes ordinary and BRS-inserted Green functions, which are obtained by differentiation with respect to the external fields Υ_k. $\varphi_k j_k$ symbolically denotes the sum over the quantum fields coupled to their sources as explicitly given in (2.84). Therefrom the connected Green functions are obtained according to (2.86) and the 1PI Green functions by a Legendre transformation of the propagating fields as given in (2.87) and (2.88), where the classical fields depend on the sources and external fields:

$$Z_c(j_k,\Upsilon_k) = \Gamma(\varphi_k^{cl},\Upsilon_k) + \int dx\, \varphi_k^{cl} j_k \tag{2.99}$$

Differentiation with respect to external fields on $\Gamma(\varphi_k^{cl},\Upsilon_k)$ reproduces BRS-insertions into 1PI Green functions. Especially one verifies

$$\frac{\delta\Gamma(\varphi_i^{cl},\Upsilon_i)}{\delta\Upsilon_k(x)} = \frac{\delta Z_c(j_i,\Upsilon_i)}{\delta\Upsilon_k(x)} \tag{2.100}$$

With the help of the external field part one is now able to derive the Slavnov-Taylor identity in a way, that the non-linear BRS-transformations are properly defined as insertions into Green functions. Taking the external field as being invariant under classical BRS-transformations

$$\mathrm{s}\Upsilon_k = 0 \tag{2.101}$$

the enlarged classical action (2.94) is BRS-invariant due to the nilpotency of BRS-transformations.

$$\mathrm{s}\Gamma_{cl}(\varphi_k,\Upsilon_k) = 0 \tag{2.102}$$

The lowest order of $\Gamma(\varphi_k^{cl},\Upsilon_k)$ is the classical action (cf. (2.90)) and the Ward identity of BRS-transformations (2.91) can be rewritten into the Slavnov-Taylor (ST) identity of 1PI Green functions in the tree approximation:

$$\begin{aligned}
\mathcal{S}(\Gamma_{cl}) \;=\; & \int\Big((\sin\theta_W\partial_\mu c_Z + \cos\theta_W\partial_\mu c_A)\Big(\sin\theta_W\frac{\delta\Gamma_{cl}}{\delta Z_\mu} + \cos\theta_W\frac{\delta\Gamma_{cl}}{\delta A_\mu}\Big) \\
& + \frac{\delta\Gamma_{cl}}{\delta\rho_3^\mu}\Big(\cos\theta_W\frac{\delta\Gamma_{cl}}{\delta Z_\mu} - \sin\theta_W\frac{\delta\Gamma_{cl}}{\delta A_\mu}\Big) + \frac{\delta\Gamma_{cl}}{\delta\sigma_3}\Big(\cos\theta_W\frac{\delta\Gamma_{cl}}{\delta c_Z} - \sin\theta_W\frac{\delta\Gamma_{cl}}{\delta c_A}\Big) \\
& + \frac{\delta\Gamma_{cl}}{\delta\rho_+^\mu}\frac{\delta\Gamma_{cl}}{\delta W_{\mu,-}} + \frac{\delta\Gamma_{cl}}{\delta\rho_-^\mu}\frac{\delta}{\delta W_{\mu,+}} + \frac{\delta\Gamma_{cl}}{\delta\sigma_+}\frac{\delta\Gamma_{cl}}{\delta c_-} + \frac{\delta\Gamma_{cl}}{\delta\sigma_-}\frac{\delta\Gamma_{cl}}{\delta c_+} + \frac{\delta\Gamma_{cl}}{\delta Y^\dagger}\frac{\delta\Gamma_{cl}}{\delta\Phi} + \frac{\delta\Gamma_{cl}}{\delta\Phi^\dagger}\frac{\delta\Gamma_{cl}}{\delta Y} \\
& + \sum_{i=1}^{N_F}\Big(\frac{\delta\Gamma_{cl}}{\delta\overline{\psi}_{f_i}^L}\frac{\Gamma_{cl}\delta}{\delta f_i^R} + \frac{\delta\Gamma_{cl}}{\delta\overline{\Psi}_{\delta_i}^R}\frac{\Gamma_{cl}\delta}{\delta F_{\delta_i}^L} + \text{h.c.}\Big) \\
& + B_a\frac{\delta\Gamma_{cl}}{\delta\bar{c}_a} + \hat{\mathbf{q}}\frac{\delta\Gamma_{cl}}{\delta\hat{\Phi}} + \frac{\delta\Gamma_{cl}}{\delta\hat{\Phi}^\dagger}\hat{\mathbf{q}}^\dagger\Big) = 0
\end{aligned} \tag{2.103}$$

There we have dropped the index classical for the classical fields appearing in the generating functional of 1PI Green functions

$$\Gamma(\varphi_k,\hat{\Phi},q,\Upsilon) = \Gamma_{cl}(\varphi_k,\hat{\Phi},q,\Upsilon) + O(\hbar) \tag{2.104}$$

Nonlinear BRS-transformations are now obtained by differentiating with respect to the external fields. We have included the external fields $\hat{\Phi}$ into the definition of the generating functional in order to be able to derive Ward identities of rigid symmetry for the Green functions. They produce by differentiation the mass insertions and their variations under rigid symmetry of Goldstone fields.

The algebraic properties of BRS-transformations are transfered to nilpotency properties of the Slavnov-Taylor operator:

$$
\begin{aligned}
\mathrm{s}_\Gamma\, \mathcal{S}(\Gamma) &= 0 \quad \text{for any functional}\, \Gamma \\
\mathrm{s}_\Gamma\, \mathrm{s}_\Gamma &= 0 \quad \text{if} \quad \mathcal{S}(\Gamma) = 0
\end{aligned}
\tag{2.105}
$$

The operator s_Γ is the linearized version of the ST identity and is defined by

$$
\begin{aligned}
\mathrm{s}_\Gamma = & \int \Big((\sin\theta_W \partial_\mu c_Z + \cos\theta_W \partial_\mu c_A)\Big(\sin\theta_W \frac{\delta}{\delta Z_\mu} + \cos\theta_W \frac{\delta}{\delta A_\mu}\Big) \\
& + B_a \frac{\delta}{\delta \bar{c}_a} + \hat{\mathbf{q}} \frac{\delta}{\delta \hat{\Phi}} + \frac{\delta}{\delta \hat{\Phi}^\dagger} \hat{\mathbf{q}}^\dagger \\
& + \sum_{\varphi_k, \Upsilon_k} u_k \Big(\frac{\delta\Gamma}{\delta \Upsilon_k} \frac{\delta}{\delta \varphi_k} + \frac{\delta\Gamma}{\delta \varphi_k} \frac{\delta}{\delta \Upsilon_k} \Big)\Big)
\end{aligned}
\tag{2.106}
$$

The sum is over all external and corresponding propagating fields which gave rise to bilinear appearance of Γ in the ST identity, u_k denotes the respective coefficients as $\cos\theta_W, \sin\theta_W$ and 1.

By Legendre transformation one is immediately able to give the ST identity for the functional of connected Green functions $Z_c \equiv Z_c(j_k, \hat{\Phi}, q, \Upsilon_k)$ in the tree approximation:

$$
\begin{aligned}
\mathcal{S}(Z_c) = & \int \Big((\sin\theta_W \partial_\mu j_Z^\mu + \cos\theta_W \partial_\mu j_A^\mu)\Big(\sin\theta_W \frac{\delta Z_c}{\delta \jmath_Z} + \cos\theta_W \frac{\delta Z_c}{\delta \jmath_A}\Big) \\
& + \Big(\cos\theta_W j_Z^\mu - \sin\theta_W j_A^\mu\Big) \frac{\delta Z_c}{\delta \rho_{\mu 3}} + \Big(\cos\theta_W \jmath_Z - \sin\theta_W \jmath_A\Big) \frac{\delta Z_c}{\delta \sigma_3} \\
& + j_{\mu+} \frac{\delta Z_c}{\delta \rho_+^\mu} + j_{\mu-} \frac{\delta Z_c}{\delta \rho_-^\mu} + \jmath_+ \frac{\delta Z_c}{\delta \sigma_+} + \jmath_- \frac{\delta Z_c}{\delta \sigma_-} + J^\dagger \frac{\delta Z_c}{\delta Y^\dagger} + \frac{\delta Z_c}{\delta Y} J \\
& + \sum_{i=1}^{N_F} \Big(\overline{\eta_i^L} \frac{\delta Z_c}{\delta \overline{\psi_{f_i}^L}} + \overline{\eta_i^R} \frac{\delta Z_c}{\delta \overline{\psi_{\delta_i}^R}} + \frac{\delta Z_c}{\delta \psi_{f_i}^L} \eta_i^L + \frac{\delta Z_c}{\delta \psi_{\delta_i}^R} \eta_i^R \Big) \\
& + \bar{\jmath}_a \frac{\delta Z_c}{\delta j_a^B} + \hat{\mathbf{q}} \frac{\delta Z_c}{\delta \hat{\Phi}} + \frac{\delta Z_c}{\delta \hat{\Phi}^\dagger} \hat{\mathbf{q}}^\dagger \Big) = 0
\end{aligned}
\tag{2.107}
$$

The ST identity of the connected Green functions is linear in contrast to the one of the 1PI Green functions. It is the starting point for proving unitarity of the physical S-matrix [13, 14, 20] . Although the proof of unitarity is beyond the scope of the paper we want to indicate how the cancellation mechanism works: Eliminating the B_a-fields and their sources in (2.107) by

$$
\frac{\delta}{\delta j_a^B} \longrightarrow -\frac{1}{\xi} \tilde{I}_{aa'} \mathcal{F}_{a'} \Big(\frac{\delta}{\delta j_b^\mu}, \frac{\delta}{\delta J}, \frac{\delta}{\delta J^\dagger}, \hat{\Phi} \Big)
\tag{2.108}
$$

with

$$\mathcal{F}_b\Big(\frac{\delta}{\delta j_a^\mu},\frac{\delta}{\delta J},\frac{\delta}{\delta J^\dagger},\hat\Phi\Big) = \tilde I_{bb'}\partial_\mu\frac{\delta}{\delta j_{b'}^\mu} - i\frac{e}{\sin\theta_W}\Big((\hat\Phi+\zeta \mathrm{v})^\dagger\frac{\tau_b^T}{2}(\frac{\delta}{\delta J}+\mathrm{v}) - (\frac{\delta}{\delta J^\dagger}+\mathrm{v}^\dagger)\frac{\tau_b^T}{2}(\hat\Phi+\zeta\mathrm{v})\Big) \tag{2.109}$$

it is seen that the ST identity indeed relates the Green functions of ghosts to the ones with longitudinal vector propagators and would-be Goldstones at the external legs. (Due to linear contributions the Green functions of external fields include 1-particle reducible ghost propagators.) Applying the S-matrix operator the corresponding unphysical contributions have to be shown to cancel in physical scattering processes.

Renormalization concerns the 1PI Green functions. Having these well-defined the connected Green functions exist and are also well-defined and can be obtained to all orders by the Legendre transformations (2.88). For this reason we are able to restrict all the further considerations to 1PI Green functions.

In the procedure of renormalization the ST identity is the defining symmetry of the theory, because it yields unitarity of the physical S-matrix as indicated above. Due to the abelian subgroup, however, the ST identity is not sufficient to fix uniquely the Green functions of higher orders. In addition we have to take into account the Ward identities of rigid $SU(2)\times U(1)$ invariance and especially the local $U(1)$ Ward identity for being able to fix the electric charges of the fermions. In the tree approximation the Ward identities of rigid symmetry are immediately derived according to the construction of the gauge fixing sector (cf. (2.43), (2.57), (2.73) and (2.77)). For consistency we have to assign definite transformation properties under rigid transformation to the external fields Υ_k in such a way, that the external field part $\Gamma_{ext.f.}$ (2.94) is rigid invariant. It is obvious, that the fields ρ_α and σ_α transform under the adjoint representation, whereas Y and Ψ^R under the fundamental representation of $SU(2)$. We thus arrive at

$$\mathcal{W}_\alpha\Gamma_{cl}(\varphi_k,\Upsilon_k,\hat\Phi,\hat{\mathbf{q}}) = 0 \quad \text{and} \quad \mathcal{W}_4\Gamma_{cl}(\varphi_k,\Upsilon_k,\hat\Phi,\hat{\mathbf{q}}) = 0 \tag{2.110}$$

where Γ_{cl} is understood to be the lowest order of the generating functional of 1PI Green functions.

$$\Gamma_{cl} = \Gamma_{GSW} + \Gamma_{g.f.} + \Gamma_{ghost} + \Gamma_{ext.f} \tag{2.111}$$

The Ward operators of rigid $SU(2)$-transformations include all the propagating and external fields we have introduced:

$$\begin{aligned}\mathcal{W}_\alpha = \tilde I_{\alpha\alpha'}\int\Bigg(& V_b^\mu O^T_{b\beta}(\theta_W)\hat\varepsilon_{\beta\gamma\alpha}O_{\gamma c}(\theta_W)\tilde I_{cc'}\frac{\delta}{\delta V^\mu_{c'}} + \{c_a, B_a, \bar c_a\} \\ & +\rho^\mu_\beta\varepsilon_{\beta\gamma\alpha'}\tilde I_{\gamma\gamma'}\frac{\delta}{\delta\rho^\mu_{\gamma'}} + \{\sigma_\alpha\}\end{aligned} \tag{2.112}$$

$$+i(\Phi+\mathrm{v})^\dagger\frac{\tau_{\alpha'}}{2}\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger}-i\frac{\overleftarrow{\delta}}{\delta\Phi}\frac{\tau_{\alpha'}}{2}(\Phi+\mathrm{v})+\{Y,\hat{\Phi}+\zeta\mathrm{v},\hat{\mathbf{q}}\}$$
$$+\sum_{i=1}^{N_F}\sum_{\delta=l,q}\Big(i\overline{F^L_{\delta_i}}\frac{\tau_{\alpha'}}{2}\frac{\overrightarrow{\delta}}{\delta\overline{F^L_{\delta_i}}}-i\frac{\overleftarrow{\delta}}{\delta F^L_{\delta_i}}\frac{\tau_{\alpha'}}{2}F^L_{\delta_i}+\{\Psi^R_{\delta_i}\}\Big)\Big)$$

The abelian Ward operator comprises the doublets and right-handed fermions together with the external fields coupled to their BRS-variations.

$$\begin{aligned}\mathcal{W}_4 \;=\;& \int\Big(\frac{i}{2}(\Phi+\mathrm{v})^\dagger\frac{\overrightarrow{\delta}}{\delta\Phi^\dagger}-\frac{i}{2}\frac{\overleftarrow{\delta}}{\delta\Phi}(\Phi+\mathrm{v})+\{Y,\hat{\Phi}+\zeta\mathrm{v},q\} \\ &+\sum_{i=1}^{N_F}\Big(\sum_{\delta=l,q}Y_W^\delta\Big(\frac{i}{2}\overline{F^L_{\delta_i}}\frac{\overrightarrow{\delta}}{\delta\overline{F^L_{\delta_i}}}-\frac{i}{2}\frac{\overleftarrow{\delta}}{\delta F^L_{\delta_i}}F^L_{\delta_i}+\{\Psi^R_{\delta_i}\}\Big) \\ &-\sum_f Q_f\Big(i\overline{f^R_i}\frac{\overrightarrow{\delta}}{\delta\overline{f^R_i}}-i\frac{\overleftarrow{\delta}}{\delta f^R_i}f^R_i+\{\psi^R_{f_i}\}\Big)\Big)\Big)\end{aligned} \tag{2.113}$$

The Pauli matrices τ_α are defined in (2.10), the antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$ in (2.12). The Ward operators of rigid symmetry satisfy the $SU(2)\times U(1)$ algebra:

$$\begin{aligned}[\mathcal{W}_\alpha,\mathcal{W}_\beta] &= \varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\mathcal{W}_{\gamma'} \\ [\mathcal{W}_\alpha,\mathcal{W}_4] &= 0\end{aligned} \tag{2.114}$$

In connection with the abelian Ward identity of rigid symmetry there exists also a local version (cf. (2.58)), which reads in B_a-gauges:

$$\mathbf{w}_4\Gamma_{cl}-\frac{1}{e}\cos\theta_W\Big(\sin\theta_W\partial\frac{\delta\Gamma_{cl}}{\delta Z}+\cos\theta_W\partial\frac{\delta\Gamma_{cl}}{\delta A}\Big)=\frac{1}{e}\cos\theta_W(\sin\theta_W\Box B_Z+\cos\theta_W\Box B_A) \tag{2.115}$$

The local operator $\mathbf{w}_4$ is defined by dropping the integration from the rigid operator:

$$\mathcal{W}_4=\int\mathbf{w}_4 \tag{2.116}$$

The ST identity (2.103), the Ward identities of rigid symmetry (2.110) and the local abelian Ward identity (2.115) are the algebraic symmetries of the standard model in the tree approximation. It has to be shown, that these symmetries can be continued to higher orders and that they together with appropriate normalization conditions uniquely define the Green functions of the standard model to all orders.

3. The construction of higher orders: The algebraic method

In the procedure of renormalization one has to make meaningful the undefined expressions, which are obtained in the formal expansion of the Gell-Mann Low formula according to Wick's theorem (cf. (2.80) and (2.93)). As we have already mentioned renormalization concerns the 1PI Green functions summarized in the generating functional (2.99)

$$\Gamma(\phi_k, \Upsilon, \hat{\Phi}, \hat{\mathbf{q}}) = \Gamma(V_a, \Phi, f_i, c_a, \bar{c}_a, \rho_\alpha, Y, \psi_i, \sigma_\alpha, \hat{\Phi}, \hat{\mathbf{q}}) \tag{3.1}$$

It depends on the external fields and the "classical" fields defined by the Legendre transformation (2.87). For simplification we have dropped the index 'classical'. The 1PI Green functions are divergent according to their degree of divergencies:

$$d_\Gamma = 4 - \sum_{ext.legs} d_E - \sum_{vertices} (d_i - 4) \tag{3.2}$$

Here d_E is the ultraviolet (UV) dimension of the fields appearing at the external (amputated) legs: They include propagating as well as external fields. d_i denotes the UV-dimension of the vertices. The UV-dimensions of the fields are listed in the appendix. There are different schemes, which remove the divergencies consistently. For practical calculations it is convenient to use dimensional regularization in connection with a prescription for subtracting the D-dimensional poles in the limit to 4 dimensions [38]. For abstract renormalization one better refers to the momentum subtraction scheme in the version of BPHZL [34, 35, 39].

For higher orders the Gell-Mann Low formula has to be modified taking in the interaction part not only the vertices of the tree approximation but also the counterterms of higher orders. In the QED-like on-shell schemes the counterterms are power series in the electromagnetic coupling e. All terms are collected in a Γ_{eff}:

$$\Gamma_{eff} = \Gamma_{cl} + O(\hbar) = \Gamma^{(bil)} + \Gamma^{(int)}_{eff} \tag{3.3}$$

The bilinear parts are defined in (2.83). At first Γ_{eff} contains all field polynomials in external and quantum fields which are compatible with the power counting analysis of renormalizable quantum field theory, i.e. they have UV-dimension less than of equal to 4. The explicit form of Γ_{eff} depends on the renormalization scheme one has used to remove the divergencies. Therefore rather than relying on properties of an explicit Γ_{eff} and a subtraction scheme, one deals in the construction of 1PI Green functions with finite renormalized Green functions and their properties with respect to the symmetries of the standard model. (For an introduction to algebraic renormalization see [3]

In the last sections we have given the tree approximation and the symmetries of the tree approximation, the Slavnov-Taylor identity (2.103), the Ward identities of rigid symmetry (2.110) and the local $U(1)$-Ward identity (2.115). Having readily defined the lowest order, the Green functions of the 1-loop approximation are calculated with a specific renormalization scheme leading to a finite result Γ_{ren}. Different schemes differently dispose of the local contributions of the next order, whereas the non-local contributions are uniquely defined. Therefore after subtraction the Green functions are well-defined up to local contributions. In order to determine these local contributions one has to adjust those which break the symmetry, in a way that the symmetries of the lowest order are restored in the 1-loop order. The remaining (symmetric) ones have to be fixed by normalization conditions. Then one is able to proceed to higher orders by induction repeating the above steps from order n to order $n+1$.

The symmetries of the lowest order can be also violated by anomalies. Anomalies arise from non-local contributions and cannot be removed by adjusting local contributions. They have then explicitly to be proven to be absent to all orders of perturbation theory.

In the standard model restoration of symmetries and the setting of proper normalization conditions are deeply connected with each other: It has to be shown that one is able to impose normalization conditions on the 2-point functions in such a way that the 2-point Green functions have one particle properties in the LSZ limit (apart from the problem of unstable particles). Thereby special attention has to be paid to the massless particles: In order not to introduce off-shell infrared divergent diagrams to the next order the 2-point functions of massless particles as well as also the mixed 2-point functions of massive and massless particles have to be required to vanish at $p^2 = 0$ to all orders of perturbation theory:

$$\begin{aligned} \Gamma_{ZA}(p^2=0) = \Gamma_{AA}(p^2=0) &= 0 \\ \Gamma_{\bar{c}_A c_Z}(p^2=0) = \Gamma_{\bar{c}_Z c_A}(p^2=0) = \Gamma_{\bar{c}_A c_A}(p^2=0) &= 0 \end{aligned} \tag{3.4}$$

These normalization conditions have to be proven to be in accordance with the symmetries of the standard model and will be shown to lead to higher order corrections of the weak mixing angle in the ST identity and the Ward identities.

The 1PI Green function of the standard model summarized in the generating functional are defined in order n by

$$\Gamma^{(\leq n)} = \Gamma^{(\leq n)}_{ren} + \Gamma^{(n)}_{inv} + \Gamma^{(n)}_{break} \tag{3.5}$$

and have to be shown to have well defined normalization properties and to satisfy the symmetries

$$\Big(\mathcal{S}(\Gamma)\Big)^{(\leq n)} = 0 \qquad \Big(\mathcal{W}_\alpha \Gamma\Big)^{(\leq n)} = 0 \tag{3.6}$$

and a local $U(1)$ Ward identity. The ST operator and the Ward operators are thereby established via their algebraic characterization (2.105) and (2.114). In (3.5) $\Gamma^{(n)}_{inv}$ and $\Gamma^{(n)}_{break}$ denote purely local field polynomials. They depend on propagating and external fields introduced in the classical approximation and constitute a complete basis of field polynomials with UV-dimension less than or equal 4. In a specific scheme the local contributions are governed by the counterterms appearing in a Γ_{eff}. Discrete symmetries are not affected by renormalization, we are therefore able to restrict the analysis to field polynomials which are neutral with respect to electric and Faddeev-Popov charge and are CP-even, due to the fact, that we did not introduce family mixing in the classical approximation. The quantum numbers of the fields under the discrete symmetries are listed in the appendix.

As indicated by the notation (3.5) local contributions are algebraically divided into two classes: the invariant and non-invariant field polynomials. The invariant field polynomials appearing in Γ_{inv} constitute together with Γ_{cl} the general classical solution Γ^{gen}_{cl}, i.e. the general field polynomials, which are solutions of the Slavnov-Taylor identity and are rigid invariant

$$\Gamma^{gen}_{cl} = \Gamma_{cl} + \sum_{n=1}^{\infty} \Gamma^{(n)}_{inv} \qquad \begin{array}{rcl} \mathcal{S}(\Gamma^{gen}_{cl}) & = & 0 \\ W_\alpha(\Gamma^{gen}_{cl}) & = & 0 \end{array} \tag{3.7}$$

The free parameters of Γ^{gen}_{cl} are determined by the normalization conditions and the local $U(1)$ Ward identity order by order in perturbation theory.

The non-invariant field polynomials Γ_{break} are used to remove the breakings of the symmetries, which have been introduced by an implicit scheme dependent adjustment of finite counterterms in the subtraction procedure. The abstract construction of Γ_{break} is carried out with the algebraic method which is based on the action principle in its quantized version valid for off-shell Green functions [16, 39]. It is most easily formulated on the general functional of the respective Green functions and relates variations with respect to sources or classical fields, respectively, and external fields to insertions with a well-defined UV and IR-degree. Especially the action principle states that the symmetries of the tree approximation are broken at most by a integrated field polynomial in 1-loop order and proceeds to higher orders by induction:

$$\begin{array}{rcl} (\mathcal{S}(\Gamma))^{(\leq n-1)} & = & 0 \\ (\mathcal{W}_\alpha\Gamma)^{(\leq n-1)} & = & 0 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} (\mathcal{S}(\Gamma))^{(\leq n)} & = & \Delta^{(n)}_{brs} \\ (\mathcal{W}_\alpha\Gamma)^{(\leq n)} & = & \Delta^{(n)}_{\alpha} \end{array} \tag{3.8}$$

The breakings have well-defined properties with respect to the discrete symmetries. For example Δ_{brs} has $\phi\pi$-charge 1, is neutral with respect to electric charge and even under CP, if the classical action is CP-invariant. Furthermore they have a well-defined ultraviolet and infrared degree. Up to this point the analysis has been completely scheme independent just being founded on properties of renormalized perturbation theory, but in classifying the breakings according to their UV- and IR-degree we assume that the renormalized Green functions Γ_{ren} have been constructed within the BPHZL scheme. In the BPHZL scheme the normalization conditions (3.4), which otherwise have to be established by hand, are immediately implemented by the subtraction procedure, guaranteeing, that nowhere infrared divergent contributions are introduced by the subtraction scheme. Infrared divergent contributions are detected by a pure power counting analysis. Especially counterterms of IR dimension less than 4 are forbidden in the BPHZL-scheme because they would destroy the normalization conditions (3.4). Therefore we adopt the UV and IR degrees of fields as given in the appendix. They are uniquely determined by the behaviour of the free-field propagators for $p^2 \to \infty$ and $p^2 \to 0$, respectively. Then by an analysis of the ST identity and the Ward operators it is derived that the breakings of the symmetries have the following UV and IR degrees:

$$\begin{aligned} \dim^{UV} \Delta_{brs} &\leq 4 \qquad\qquad & \dim^{IR} \Delta_{brs} &\geq 3 \\ \dim^{UV} \Delta_{\alpha} &\leq 4 \qquad\qquad & \dim^{IR} \Delta_{\alpha} &\geq 2 \end{aligned}$$

and it is seen, that all symmetries have to be carefully constructed concerning the infrared.

Apart from the IR and UV degree we do not refer to further properties of the BPHZL scheme as the Γ_{eff}, but classify the breakings by the algebra of symmetry transformations, the nilpotency of BRS-transformations and the algebra of $\mathcal{W}_\alpha$. (For details see section 7.) Especially we have to show that all the breakings of the ST identity are variations of the linear ST operator (2.106) and that they can be absorbed into local contributions of Γ_{break} without spoiling the normalization conditions especially (3.4):

$$\mathrm{s}_{\Gamma_{cl}} \Gamma_{break} = -\Delta_{brs} \tag{3.9}$$

Via equs. (3.7) and (3.9) the local contributions are uniquely fixed.

The construction as outlined above is not only interesting from an abstract point of view for having properly defined the standard model but it passes through all the steps, which have been carried out in explicit calculations, too. Especially the construction of the symmetries in 1-loop order including the rigid Ward identity is essential if one wants to proceed to higher orders of perturbation theory. Dimensional regularization makes the analysis of Γ_{break} easier, because it is an invariant scheme for parity conserving gauge theories, but also there it is well-known that the normalization conditions and especially the infrared conditions have to be established by an explicit adjustment of naively non-invariant counterterms, which lead to corrections of the weak mixing angle in the ST identity and the Ward identities.

According to the procedure we have outlined in this section we will proceed for constructing the 1PI Green functions of the standard model as follows:

1. We construct the most general ST operator and Ward operators of rigid symmetry, which are in accordance with the algebraic characterization. (Section 4)
2. We impose normalization conditions according to the on-shell schemes which allow to define proper 2-point functions in the LSZ-limit apart from the problem of unstable particles and derive the most general classical solution, which is in accordance with the normalization conditions and the symmetries. (Section 5)
3. We classify the breakings according to the symmetries and show that they can be absorbed into local contributions to the 1PI Green functions. (Section 7)

4. The algebraic characterization of the symmetry transformations

4.1. The general ansatz and discrete symmetries

In section 2.4 we have derived the symmetries of the standard model for the 1PI-Green functions in the tree approximation: the Slavnov-Taylor identity (2.103), the Ward identities of rigid symmetries (2.112) and the local $U(1)$-Ward identity (2.113). The functional operators depend in lowest order explicitly on the weak mixing angle θ_W, which is in the on-shell schemes defined by the ratio of the W- and Z-mass [26]

$$\cos\theta_W \equiv \frac{M_W}{M_Z} \tag{4.1}$$

It is obvious and can be seen also from explicit 1-loop calculations that the lowest order gets higher order corrections. These higher order corrections depend on the normalization conditions, one has chosen for fixing the 2-point Green functions. Since the standard model includes massless particles, especially the photon and the corresponding $\phi\pi$-ghosts, it is even not possible to define the Green functions of higher orders by the symmetries as given in lowest order. For this reason we construct in a first step towards quantization the symmetry operators in a most general form and characterize them by their algebraic properties. We restrict the analysis to the generating functional of 1PI Green functions as defined in (2.99), which depends on the classical fields as well as on the external fields:

$$\Gamma \equiv \Gamma(V_a, B_a, \Phi, f_i, c_a, \bar{c}_a, \rho_\alpha, Y, \psi_i, \sigma_\alpha, \hat{\mathbf{\Phi}}, \hat{\mathbf{q}}) \tag{4.2}$$

The vectors, $\phi\pi$-ghost and the B-fields have on-shell field indices $a = +, -, Z, A$

$$\begin{array}{rclcrcl} V_a^\mu & = & (W_+^\mu, W_-^\mu, Z^\mu, A^\mu) & & c_a & = & (c_+, c_-, c_Z, c_A) \\ B_a & = & (B_+, B_-, B_Z, B_A) & & \bar{c}_a & = & (\bar{c}_+, \bar{c}_-, \bar{c}_Z, \bar{c}_A) \end{array} \tag{4.3}$$

Since the theory is spontaneously broken, it is more adequate to introduce the scalars as a 4-vector with indices $a = +, -, H, \chi$

$$\begin{array}{rclcrcl} \phi_a & = & (\phi_+, \phi_-, H, \chi) & & \hat{\phi}_a & = & (\hat{\phi}_+, \hat{\phi}_-, \hat{H}, \hat{\chi}) \\ Y_a & = & (Y_+, Y_-, Y_H, Y_\chi) & & q_a & = & (q_+, q_-, q_H, q_\chi) \end{array} \tag{4.4}$$

The external fields ρ_α and σ_α are three component fields

$$\rho_a = (\rho_+, \rho_-, \rho_3) \quad \sigma_a = (\sigma_+, \sigma_-, \sigma_3) \tag{4.5}$$

The charged vectors and scalars are complex fields with

$$\varphi_+^* = \varphi_- \,, \tag{4.6}$$

whereas the neutral vectors and scalars are real fields

$$\varphi_a^* = \varphi_a \qquad a = Z, A, H, \chi, 3\,. \tag{4.7}$$

We group the fermions into a vector according to

$$\begin{array}{rclcrcl} f_i^L & = & (\nu_i^L, e_i^L, u_i^L, d_i^L) & & \psi_{f_i}^R & = & (\psi_{\nu_i}^R, \psi_{e_i}^R, \psi_{u_i}^R, \psi_{d_i}^R) \\ f_i^R & = & (e_i^R, u_i^R, d_i^R) & & \psi_{f_i}^L & = & (\psi_{e_i}^L, \psi_{u_i}^L, \psi_{d_i}^L) \end{array} \tag{4.8}$$

$i = 1, ..., N_F$ denotes the family index. The quantum number of fields are summarized in the appendix.

The algebraic properties of the functional operators acting on Γ are the nilpotency of the Slavnov-Taylor operator (2.105)

$$\begin{array}{rcll} \mathrm{s}_\Gamma\, \mathcal{S}(\Gamma) & = & 0 & \text{for any functional}\, \Gamma \\ \mathrm{s}_\Gamma\, \mathrm{s}_\Gamma & = & 0 & \text{if} \quad \mathcal{S}(\Gamma) = 0 \end{array} \tag{4.9}$$

and the algebra of rigid operators (2.114)

$$[\mathcal{W}_\alpha, \mathcal{W}_\beta] = \hat{\varepsilon}_{\alpha\beta\gamma} \tilde{I}_{\gamma\gamma'} \mathcal{W}_{\gamma'} \tag{4.10}$$

with $\alpha, \beta, \gamma = +, -, 3, 4$ and $\hat{\varepsilon}_{\alpha\beta\gamma}$ denotes the structure constants of $SU(2) \times U(1)$, which are taken as completely antisymmetric in all 3 indices

$$\hat{\varepsilon}_{\alpha\beta\gamma} = \begin{cases} \hat{\varepsilon}_{+-3} &= -i \\ \hat{\varepsilon}_{+-4} &= 0 \end{cases} \tag{4.11}$$

In addition to the algebra the Ward operators are specified by their transformation with respect to complex conjugation:

$$\begin{array}{rclcrcl} \mathcal{W}_+^\dagger &=& \mathcal{W}_- & \quad & \mathcal{W}_3^\dagger &=& \mathcal{W}_3 \\ \mathcal{W}_-^\dagger &=& \mathcal{W}_+ & \quad & \mathcal{W}_4^\dagger &=& \mathcal{W}_4 \end{array} \tag{4.12}$$

which is assigned in agreement with the tree approximation.

Since the functional Γ will be constructed as a simultaneous solution of rigid transformations and the ST identity the respective functional operators have to satisfy the consistency relation

$$\mathcal{W}_\alpha \mathcal{S}(\Gamma) - \mathrm{s}_\Gamma \mathcal{W}_\alpha \Gamma = 0 \quad \text{for any functional}\, \Gamma \tag{4.13}$$

Discrete and global unbroken symmetries, we want to impose on the functional Γ, can be imposed to all orders and are not affected by renormalization. These symmetries are electric and $\phi\pi$-charge neutrality:

$$\begin{aligned} \mathcal{W}_{em} = -i \int dx \Big(& W_+ \frac{\delta}{\delta W_+} - W_- \frac{\delta}{\delta W_-} + B_+ \frac{\delta}{\delta B_+} - B_- \frac{\delta}{\delta B_-} + c_+ \frac{\delta}{\delta c_+} - c_- \frac{\delta}{\delta c_-} \\ & + \bar{c}_+ \frac{\delta}{\delta \bar{c}_+} - \bar{c}_- \frac{\delta}{\delta \bar{c}_-} + \rho_+ \frac{\delta}{\delta \rho_+} - \rho_- \frac{\delta}{\delta \rho_-} + \sigma_+ \frac{\delta}{\delta \sigma_+} - \sigma_- \frac{\delta}{\delta \sigma_-} \\ & + \phi_+ \frac{\delta}{\delta \phi_+} - \phi_- \frac{\delta}{\delta \phi_-} + Y_+ \frac{\delta}{\delta Y_+} - Y_- \frac{\delta}{\delta Y_-} \\ & + \hat{\phi}_+ \frac{\delta}{\delta \hat{\phi}_+} - \hat{\phi}_- \frac{\delta}{\delta \hat{\phi}_-} + q_+ \frac{\delta}{\delta q_+} - q_- \frac{\delta}{\delta q_-} \\ & - \sum_{i=1}^{N_F} \Big(Q_e (\bar{e}_i \frac{\delta}{\delta \bar{e}_i} - \frac{\delta}{\delta e_i} e_i + \bar{\psi}_{e_i} \frac{\delta}{\delta \bar{\psi}_{e_i}} - \frac{\delta}{\delta \psi_{e_i} \psi_{e_i}}) \\ & \quad + Q_u (\bar{u}_i \frac{\delta}{\delta \bar{u}_i} - \frac{\delta}{\delta u_i} u_i + \bar{\psi}_{u_i} \frac{\delta}{\delta \bar{\psi}_{u_i}} - \frac{\delta}{\delta \psi_{u_i}} \psi_{u_i}) \\ & \quad + Q_d (\bar{d}_i \frac{\delta}{\delta \bar{d}_i} - \frac{\delta}{\delta d_i} d_i + \bar{\psi}_{d_i} \frac{\delta}{\delta \bar{\psi}_{d_i}} - \frac{\delta}{\delta \psi_{d_i}} \psi_{d_i}) \Big) \Big) \\ \mathcal{W}_{\phi\pi} = \int dx \Big(& c_a \frac{\delta}{\delta c_a} - \bar{c}_a \frac{\delta}{\delta \bar{c}_a} - \rho_\alpha \frac{\delta}{\delta \rho_\alpha} - 2\sigma_\alpha \frac{\delta}{\delta \sigma_\alpha} - Y_a \frac{\delta}{\delta Y_a} + q_a \frac{\delta}{\delta q_a} \\ & - \sum_{i=1}^{N_F} \Big(\bar{\psi}_{m_i} \frac{\delta}{\delta \bar{\psi}_{m_i}} + \frac{\delta}{\delta \psi_{m_i}} \psi_{m_i} \Big) \Big) \end{aligned} \tag{4.14}$$

Since fermions and quarks cannot be mixed and since we forbid explicitly family mixing we have some further symmetries which correspond to lepton and quark family number conservation:

$$\begin{aligned}
\mathcal{W}_{l_i} = i\int dx\,\Big(&\bar{e}_i\frac{\delta}{\delta\bar{e}_i}-\frac{\delta}{\delta e_i}e_i+\bar{\psi}_{e_i}\frac{\delta}{\delta\bar{\psi}_{e_i}}-\frac{\delta}{\delta\psi_{e_i}}\psi_{e_i}\\
&+\overline{\nu_i^L}\frac{\delta}{\delta\overline{\nu_i^L}}-\frac{\delta}{\delta\nu_i^L}\nu_i^L+\overline{\psi^R_{\nu_i}}\frac{\delta}{\delta\overline{\psi^R_{\nu_i}}}-\frac{\delta}{\delta\psi^R_{\nu_i}}\psi^R_{\nu_i}\Big)
\end{aligned}\tag{4.15}$$

$$\begin{aligned}
\mathcal{W}_{q_i} = i\int dx\,\Big(&\bar{d}_i\frac{\delta}{\delta\bar{d}_i}-\frac{\delta}{\delta d_i}d_i+\bar{\psi}_{d_i}\frac{\delta}{\delta\bar{\psi}_{d_i}}-\frac{\delta}{\delta\psi_{d_i}}\psi_{d_i}\\
&+\bar{u}_i\frac{\delta}{\delta\bar{u}_i}-\frac{\delta}{\delta u_i}u_i+\bar{\psi}_{u_i}\frac{\delta}{\delta\bar{\psi}_{u_i}}-\frac{\delta}{\delta\psi_{u_i}}\psi_{u_i}\Big)
\end{aligned}\tag{4.16}$$

The functional Γ is to all orders invariant under these global symmetries by construction:

$$\begin{array}{rclcrcl}
\mathcal{W}_{em}\Gamma &=& 0 &\qquad& \mathcal{W}_{l_i}\Gamma &=& 0\\
\mathcal{W}_{\phi\pi}\Gamma &=& 0 && \mathcal{W}_{q_i}\Gamma &=& 0
\end{array}\tag{4.17}$$

In addition we have also colour SU(3)-invariance, which we do not consider explicitly. All the functional operators when applied on Γ are seen to be restricted with respect to these global symmetries, especially

$$\begin{array}{rclcrcl}
[\mathcal{W}_{em},\mathcal{W}_+] &=& -i\mathcal{W}_+ &\qquad& [\mathcal{W}_{em},\mathcal{W}_3] &=& 0\\
[\mathcal{W}_{em},\mathcal{W}_-] &=& +i\mathcal{W}_- && [\mathcal{W}_{em},\mathcal{W}_4] &=& 0
\end{array}\tag{4.18}$$

and

$$[\mathcal{W}_{l_i},\mathcal{W}_\alpha]=0 \qquad [\mathcal{W}_{q_i},\mathcal{W}_\alpha]=0 \tag{4.19}$$

The functional Γ we consider in this paper is also invariant with respect to CP-transformations. Therefrom it is derived that the rigid operators as well as the Slavnov-Taylor operator have definite transformation properties with respect to CP. Explicitly it is possible to characterize them by their transformation as given in the tree approximation:

$$\begin{array}{lclclcl}
\mathcal{W}_+ & \xrightarrow{CP} & -\mathcal{W}_- &\qquad& \mathcal{W}_3 & \xrightarrow{CP} & -\mathcal{W}_3\\
\mathcal{W}_- & \xrightarrow{CP} & -\mathcal{W}_+ && \mathcal{W}_4 & \xrightarrow{CP} & -\mathcal{W}_4
\end{array}\tag{4.20}$$

whereas the ST operator is CP-even.

We therefore make for the ST operator the ansatz:

$$\begin{aligned}
\mathcal{S}(\Gamma) \;=\; \int\Big(&Z_4(\sin\Theta_3^g\partial_\mu c_Z+\cos\Theta_3^g\partial_\mu c_A)\Big(\sin\Theta_4^V\frac{\delta\Gamma}{\delta Z_\mu}+\cos\Theta_4^V\frac{\delta\Gamma}{\delta A_\mu}\Big)\\
&+\frac{\delta\Gamma}{\delta\rho_3^\mu}z^\rho\Big(\cos\Theta_3^V\frac{\delta\Gamma}{\delta Z_\mu}-\sin\Theta_3^V\frac{\delta\Gamma}{\delta A_\mu}\Big)+\frac{\delta\Gamma}{\delta\sigma_3}z^\sigma\Big(\cos\Theta_3^g\frac{\delta\Gamma}{\delta c_Z}-\sin\Theta_3^g\frac{\delta\Gamma}{\delta c_A}\Big)
\end{aligned}\tag{4.21}$$

$$
\begin{aligned}
&+\frac{\delta\Gamma}{\delta\rho^\mu_+}\frac{\delta\Gamma}{\delta W_{\mu,-}}+\frac{\delta\Gamma}{\delta\rho^\mu_-}\frac{\delta\Gamma}{\delta W_{\mu,+}}+\frac{\delta\Gamma}{\delta\sigma_+}\frac{\delta\Gamma}{\delta c_-}+\frac{\delta\Gamma}{\delta\sigma_-}\frac{\delta\Gamma}{\delta c_+}+\frac{\delta\Gamma}{\delta Y_a}\tilde{I}_{aa'}\frac{\delta\Gamma}{\delta\phi_{a'}}\\
&+\sum_{i=1}^{N_F}\Big(\frac{\delta\Gamma}{\delta\overline{\psi^L_{f_i}}}\frac{\Gamma\delta}{\delta f^R_i}+\frac{\delta\Gamma}{\delta\overline{\psi^R_i}}\frac{\Gamma\delta}{\delta f^L_i}+\text{h.c.}\Big)\\
&+B_a\hat{g}_{ab}\frac{\delta\Gamma}{\delta\bar{c}_b}+q_a\frac{\delta\Gamma}{\delta\hat{\phi}_a}\Big)=0
\end{aligned}
$$

It is neutral with respect to electric charge, commutes with the operators $\mathcal{W}_{l_i}$ and $\mathcal{W}_{q_i}$ and arises $\phi\pi$ charge by one unit. Defining its linear version s_Γ as given in (2.106) one immediately checks the nilpotency properties (4.9). The three independent angles Θ^V_3, Θ^V_4 and Θ^g_3 describe, how vectors and ghosts are rotated with respect to the abelian subgroup. In the writing of (4.21) we have already anticipated that we are able to absorb constants in external fields at will. The coefficients z^ρ as well as z^σ could be reabsorbed into the external fields ρ_3 and σ_3, but we take them as arbitrary for a proper adjustment later on. For similar reasons we also keep Z_4 in the ansatz. The matrix $\hat{g}_{ab}$ is an arbitrary neutral matrix, it can be introduced into the ST identity without spoiling nilpotency and rigid symmetry (see (4.40)). Its explicit form will be considered, when we give the general classical solution of the gauge-fixing ghost sector.

For the Ward operators of rigid symmetry we take the most general ansatz, which is linear in fields:

$$
\begin{aligned}
\mathcal{W}_\alpha=\tilde{I}_{\alpha\alpha'}\int dx\,\Big(&V^\mu_b\hat{a}^V_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta V^\mu_{c'}}+B_b\hat{a}^B_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta B_{c'}}+c_b\hat{a}^g_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta c_{c'}}+\bar{c}_b\hat{a}^{\bar{g}}_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta\bar{c}_{c'}}\\
&+\phi_b b^\phi_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta\phi_{c'}}+Y_b b^Y_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta Y_{c'}}+\hat{\phi}_b b^{\hat{\phi}}_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta\hat{\phi}_{c'}}+q_b b^q_{bc,\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta q_{c'}}\\
&+v_{c\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta\phi_{c'}}+\hat{v}_{c\alpha'}\tilde{I}_{cc'}\frac{\delta}{\delta\hat{\phi}_{c'}}\\
&+\rho_\beta a^\rho_{\beta\gamma,\alpha'}\tilde{I}_{\gamma\gamma'}\frac{\delta}{\delta\rho_{\gamma'}}+\sigma_\beta a^\sigma_{\beta\gamma,\alpha'}\tilde{I}_{\gamma\gamma'}\frac{\delta}{\delta\sigma_{\gamma'}}\\
&+\sum_{i=1}^{N_F}\Big(\overline{f^L_i}h^{f_i}_{ff',\alpha'}\frac{\delta}{\delta\overline{f'^L_i}}+\frac{\delta}{\delta f'^L_i}h^{f_i\dagger}_{f'f,\beta}f^L_i\tilde{I}_{\beta\alpha'}\\
&\qquad+\overline{\psi^R_{f_i}}h^{\psi_i}_{ff',\alpha'}\frac{\delta}{\delta\overline{\psi^L_{f'_i}}}+\frac{\delta}{\delta\psi^R_{f'_i}}h^{\psi_i\dagger}_{f'f,\beta}\psi^R_{f_i}\tilde{I}_{\beta\alpha'}\\
&\qquad+\overline{f^R_i}\tilde{h}^{f_i}_{ff',\alpha'}\frac{\delta}{\delta\overline{f'^R_i}}+\frac{\delta}{\delta f'^R_i}\tilde{h}^{f_i\dagger}_{f'f,\beta}f^R_i\tilde{I}_{\beta\alpha'}\\
&\qquad+\overline{\psi^L_{f_i}}\tilde{h}^{\psi_i}_{ff',\alpha'}\frac{\delta}{\delta\overline{\psi^L_{f'_i}}}+\frac{\delta}{\delta\psi^L_{f'_i}}\tilde{h}^{\psi_i\dagger}_{f'f,\beta}\psi^L_{f_i}\tilde{I}_{\beta\alpha'}\Big)\Big)
\end{aligned}\tag{4.22}
$$

The coefficients are restricted by the prescription for complex conjugation (4.12) and by electric charge conservation (4.18).

In the notation these properties are taken into account by having neutral index structure throughout and changing + and − by complex conjugation. Well-defined transformation properties under CP (4.20) yields furthermore that $\hat{a}_{bc,\alpha}$ and $a_{\beta\gamma,\alpha}$ as well as $\tilde{h}_{ff',\alpha}$ and $h_{ff',\alpha}$ are imaginary. Similar restrictions are derived for the coefficients $b_{bc,\alpha}$. Family mixing as well as lepton quark mixing are forbidden according to (4.19).

In the following we will solve the algebra (4.10) as well as the consistency equation (4.13) in all generality. Since we construct the Green functions in perturbation theory, it would be also sufficient to start from the tree approximation and consider its possible perturbations. Such a treatment, however, would disguise the simple algebraic structure of the final solution.

4.2. The vector-ghost sector

Evaluating the algebra of rigid operators (4.10) for the vectors, $\phi\pi$-ghosts, the B-fields and the external fields σ_α and ρ_α yields the following representation equations for the coefficients:

$$\begin{aligned} \hat{a}^{\varphi}_{\alpha}\,\tilde{I}\,\hat{a}^{\varphi}_{\beta} - \hat{a}^{\varphi}_{\beta}\,\tilde{I}\,\hat{a}^{\varphi}_{\alpha} &= -\hat{\varepsilon}_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\hat{a}^{\varphi}_{\gamma'} \qquad \varphi \equiv V^{\mu}_{a}, c_a, \bar{c}_a, B_a \\ a^{\Upsilon}_{\alpha}\tilde{I}a^{\Upsilon}_{\beta} - a^{\Upsilon}_{\beta}\tilde{I}a^{\Upsilon}_{\alpha} &= -\varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}a^{\Upsilon}_{\gamma'} \qquad \Upsilon \equiv \rho_\alpha, \sigma_\alpha \end{aligned} \tag{4.23}$$

Here we have introduced a matrix notation: $(\hat{a}_\alpha)_{bc} = \hat{a}_{bc,\alpha}$ denotes 4×4 matrices and $(a_\alpha)_{\beta\gamma} = a_{\beta\gamma,\alpha}$ 3×3 matrices. Due to CP-invariance the non-trivial solutions of (4.23) are uniquely related to the adjoint representation:

$$\hat{a}^{\varphi}_{\alpha} \sim \hat{\varepsilon}_{\alpha} \qquad \hat{a}^{\Upsilon}_{\alpha} \sim \varepsilon_{\alpha} \tag{4.24}$$

with $(\hat{\varepsilon}_\alpha)_{\beta\gamma} = \hat{\varepsilon}_{\beta\gamma\alpha}$ defined by the structure constants of $SU(2)\times U(1)$ (4.11) and $(\varepsilon_\alpha)_{\beta\gamma} = \varepsilon_{\beta\gamma\alpha}$ by the structure constants of $SU(2)$. From this special solution one obtains the general solution by the following equivalence transformations:

$$\begin{aligned} (\hat{a}_\alpha)_{bc} &= \hat{y}_{b\beta}\,\hat{\varepsilon}_{\beta\gamma\alpha}\,\hat{y}^{-1}_{\gamma c} \\ (a_\alpha)_{\beta\gamma} &= y_{\beta\beta'}\,\varepsilon_{\beta'\gamma'\alpha}\,(y^{-1})_{\gamma'\gamma} \end{aligned} \tag{4.25}$$

The matrices y and $\hat{y}$ have to be chosen in accordance with the discrete symmetries. When we consider the general classical solution of the standard model it is seen that the equivalence transformations are related to field redefinitions. Therefore we parameterize them in the following way:

$$(\hat{y}^T)_{\alpha a} \equiv \hat{z}_{\alpha a} = \begin{pmatrix} \hat{z}_W & 0 & 0 & \\ 0 & \hat{z}_W & 0 & 0 \\ 0 & 0 & \hat{z}_Z \cos\theta_Z & -\hat{z}_A \sin\theta_A \\ 0 & 0 & \hat{z}_Z \sin\theta_Z & \hat{z}_A \cos\theta_A \end{pmatrix} \qquad y_{\beta\gamma} \equiv z_{\beta\gamma} = \begin{pmatrix} z_W & 0 & 0 \\ 0 & z_W & 0 \\ 0 & 0 & z_3 \end{pmatrix} \tag{4.26}$$

We have suppressed the field indices, but one has to keep in mind, that the algebra allows independent field redefinitions for each field we consider.

In the tree approximation the representation matrices are given by

$$\hat{a}_\alpha^{(0)} = O^T(\theta_W)\hat{\varepsilon}_\alpha O(\theta_W) \quad \text{and} \quad a_\alpha^{(0)} = \varepsilon_\alpha \tag{4.27}$$

for all fields in question, i.e. one has in perturbation theory for the propagating fields

$$\hat{z}^\varphi_{\alpha b} = O(\theta_W)_{\alpha a}(\mathbf{1} + \delta\hat{z}^\varphi)_{ab} \qquad \text{with} \quad (\delta\hat{z}^\varphi)_{ab} = O(\hbar) \tag{4.28}$$

The matrix $O(\theta_W)$ is the orthogonal matrix, which transforms the $SU(2)\times U(1)$ gauge fields into on-shell fields (2.28). The matrix $\hat{z}$ is however not completely specified by the representation matrices, indeed it is seen that equivalence transformations with diagonal matrices $\hat{z}_{inv}$ leave the adjoint representation invariant:

$$\hat{\varepsilon}_\alpha = \hat{z}_{inv}\hat{\varepsilon}_\alpha \hat{z}_{inv}^{-1} \qquad \varepsilon_\alpha = z_{inv}\varepsilon_\alpha z_{inv}^{-1} \tag{4.29}$$

if

$$\hat{z}_{inv} \equiv \begin{pmatrix} \hat{z}_2 & 0 & 0 & \\ 0 & \hat{z}_2 & 0 & 0 \\ 0 & 0 & \hat{z}_2 & 0 \\ 0 & 0 & 0 & \hat{z}_1 \end{pmatrix} \qquad z_{inv} = \begin{pmatrix} z_2 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{pmatrix} \tag{4.30}$$

There are several possibilities to parameterize the remaining parameters. A symmetric parameterization, which is well adapted to treat higher order corrections of the vectors, is given by

$$r_A = \frac{z_Z}{z_W}\frac{\cos\theta_Z}{\cos\Theta} \qquad r_Z = \frac{z_A}{z_W}\frac{\sin\theta_A}{\sin\Theta} \tag{4.31}$$
$$\cos\Theta = \frac{1}{\sqrt{1+\tan\theta_Z\tan\theta_A}}$$

In this parameterization the general solution of (4.23) reads explicitly

$$\hat{a}_+ \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -ir_Z^{-1}\cos\Theta & ir_A^{-1}\sin\Theta \\ 0 & ir_Z\cos\Theta & 0 & 0 \\ 0 & -ir_A\sin\Theta & 0 & 0 \end{pmatrix} \qquad \hat{a}_3 = \hat{\varepsilon}_3 \tag{4.32}$$

$$\hat{a}_- = \begin{pmatrix} 0 & 0 & ir_Z^{-1}\cos\Theta & -ir_A^{-1}\sin\Theta \\ 0 & 0 & 0 & 0 \\ -ir_Z\cos\Theta & 0 & 0 & 0 \\ ir_A\sin\Theta & 0 & 0 & 0 \end{pmatrix} \qquad \hat{a}_4 = 0 \tag{4.33}$$

One has to determine three independent parameters for each field in the rigid Ward identity. They are fixed by the normalization conditions imposed on the 2-point Green functions. Vice versa it is seen that for the vectors we could also choose $r_A = 1$ and $r_Z = 1$ replacing two normalization conditions by the Ward identities of rigid symmetry. Such a choice corresponds to the minimal on-shell scheme [31].

Finally the consistency equation between the Ward operators of rigid symmetry and the ST operator (4.13) relates the angles appearing in the ST operator to the parameters of rigid Ward operators. In the parameterization (4.31) one gets

$$\begin{aligned} \tan\Theta_3^V &= \tfrac{r_A^V}{r_Z^V}\tan\Theta^V \qquad \tan\Theta_4^V = \tfrac{r_Z^V}{r_A^V}\tan\Theta^V \\ z^\rho\, r_3^\rho &= \sqrt{\tfrac{1}{(r_Z^V)^2}\cos^2\Theta^V + \tfrac{1}{(r_A^V)^2}\sin^2\Theta^V} \end{aligned} \tag{4.34}$$

and similar equations for the parameters of the ghosts and σ-fields

$$\tan\Theta_3^g = \frac{r_A^g}{r_Z^g}\tan\Theta^g \qquad z^\sigma\, r_3^\sigma = \sqrt{\frac{1}{(r_Z^g)^2}\cos^2\Theta^g + \frac{1}{(r_A^g)^2}\sin^2\Theta^g} \tag{4.35}$$

It has to be proven, that these relations can be consistently maintained to all orders of perturbation theory. In the tree approximation they are obviously fulfilled. Furthermore it is seen, that the normalization constants z^σ and z^ρ can be fixed by the Ward identity of rigid symmetry. Requiring that the external fields transform to all orders just as in the tree approximation

$$a^\rho_{\beta\gamma\alpha} = \varepsilon_{\beta\gamma\alpha} \qquad a^\sigma_{\beta\gamma\alpha} = \varepsilon_{\beta\gamma\alpha} \tag{4.36}$$

the parameters z^σ and z^ρ are uniquely determined.

In order to determine the transformation matrices of the B-fields $\hat{a}^B_\alpha$, it has to be observed, that the gauge fixing is linear in propagating fields. Differentiating the functional of 1PI Green functions with respect to B_a therefore yields a local expression to all orders of perturbation theory, which allows to fix the normalization of the B-fields on the longitudinal parts of the vectors:

$$\frac{\delta\Gamma}{\delta B} = \xi_{ab}B_b + \tilde{I}_{ab}\partial^\mu V_{\mu b} + r_{bc,a}\hat{\phi}_b\phi_c + w_{ca}\phi_c + \hat{w}_{ca}\hat{\phi}_c \tag{4.37}$$

Applying the Ward operators of rigid symmetry on this local expression it is seen, that the transformation of the B_a-fields is completely governed by the vectors:

$$(\hat{a}^B_\alpha) = -(\hat{a}^V_\alpha)^T \tag{4.38}$$

which reads for the parameters introduced above (4.31)

$$r_A^B = \frac{1}{r_A^V} \qquad r_Z^B = \frac{1}{r_Z^V} \qquad \tan\Theta^B = \tan\Theta^V \tag{4.39}$$

One is able to establish rigid symmetry quite trivially on the B-dependent part of the generating functional. Accordance with rigid symmetry directly restricts the independent parameters appearing in (4.37). The explicit form is given in section 5.4.

Finally the consistency condition (4.13) relates the matrix $\hat{g}_{ab}$ to the rigid transformations of anti-ghosts:

$$\hat{a}^{\bar{g}}_{bc,\alpha} = -\hat{g}^T_{bb'} a^V_{b'c',\alpha} \hat{g}^{-1T}_{c'c} = -(\hat{g}z^V)^T_{b\beta} \varepsilon_{\beta\gamma\alpha} (\hat{g}z^V)^{-1T}_{\gamma c} \tag{4.40}$$

From rigid invariance it is therefore allowed to introduce an arbitrary matrix into the BRS-transformation of ghosts. From (4.40) it is obvious that such a general ansatz is related to different field redefinitions of B-fields and anti-ghosts and, finally, vectors and anti-ghosts.

4.3. The scalar sector

The algebra for the coefficients of the scalar fields has the same form as the one for the vectors

$$b^s_\alpha\, \tilde{I}\, b^s_\beta - b^s_\beta\, \tilde{I}\, b^s_\alpha \;=\; -\hat{\varepsilon}_{\alpha\beta\gamma} \tilde{I}_{\gamma\gamma'} b^s_{\gamma'} \qquad s \equiv \phi_a, Y_a, \hat{\phi}_a, q_a \tag{4.41}$$

with $(b_\alpha)_{bc} = b_{bc,\alpha}$. The solution, however, is distinguished from the one of the vector representation equations due to a different transformation behaviour of scalars with respect to CP: The general solution of the scalar representation equations (4.41) is the fundamental representation with its equivalence class:

$$b^s_\alpha \sim \hat{t}_\alpha \tag{4.42}$$

with

$$\hat{t}_+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & -i & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \hat{t}_3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
$$\hat{t}_- = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad G^s\hat{t}_4 = \frac{G^s}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.43}$$

It involves in the abelian component an undetermined parameter for each field. The 4-dimensional representation we have chosen here is equivalent to the complex 2-dimensional representation, which is usually assigned to the scalars in the tree approximation and which we have introduced in section 2. The general solution is obtained from the special solution (4.43) by an equivalence transformation. Because the Ward operators have to be CP-odd, the transformation matrices have to be real and diagonal. This means, that mixing between Higgs- and the neutral would-be Goldstone is forbidden in a CP-invariant theory:

$$(b_\alpha)_{bc} \quad = \quad z_{bb'}\,(\hat{t}_\alpha)_{bc}\,z^{-1}_{c'c} \tag{4.44}$$

with

$$z_{ab} = \begin{pmatrix} z_+ & 0 & 0 & 0 \\ 0 & z_+ & 0 & 0 \\ 0 & 0 & z_H & 0 \\ 0 & 0 & 0 & z_\chi \end{pmatrix} \tag{4.45}$$

We have again suppressed the scalar field indices. The dependence of the representation matrices on these parameters is quite simple, it is seen that the representation matrices only involve the ratios

$$r_+ = \frac{z_+}{z_H} \qquad r_\chi = \frac{z_\chi}{z_H}. \tag{4.46}$$

As in the vector sector, rigid symmetry allows independent field redefinitions for each scalar field. Likewise one can fix the field redefinitions of the charged and CP-odd components by the Ward identity of the tree approximation:

$$r_+ = 1 + \delta r_+ \quad r_\chi = 1 + \delta r_\chi \qquad \text{with} \quad \delta r_a = O(\hbar) \tag{4.47}$$

Finally the consistency condition (4.13) relates the transformation of the external fields Y_a to the transformation of the propagating fields ϕ_a, and the transformation of q_a to the transformation of $\hat{\phi}_a$:

$$b^Y_{bc,\alpha} = b^\phi_{bc,\alpha} \qquad b^q_{bc,\alpha} = b^{\hat{\phi}}_{bc,\alpha} \tag{4.48}$$

which reads for the free parameters involved

$$\begin{aligned} r^Y_a &= r^\phi_a & \qquad G^Y &= G^\phi \\ r^q_a &= r^{\hat{\phi}}_a & \qquad G^q &= G^{\hat{\phi}} \end{aligned}$$

Because we are free to dispose over the external fields at will, as long as we do not find any restrictions in the procedure of quantization, we restrict the transformation of the fields $\hat{\phi}_a$ to be the same as the one of the quantum fields.

$$b^\phi_{bc,\alpha} = b^{\hat{\phi}}_{bc,\alpha} \tag{4.49}$$

The representation equations of the shifts ($\tilde{v}_{b\alpha} \equiv v_{b\alpha}, \hat{v}_{b\alpha}$)

$$\tilde{v}_{b\alpha}\tilde{I}_{bb'}\, b^{\phi}_{b'c\beta} - \tilde{v}_{b\beta}\tilde{I}_{bb'}\, b^{\phi}_{b'c\alpha} = -\hat{\varepsilon}_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\tilde{v}_{c\gamma'} \tag{4.50}$$

are solved by

$$\begin{array}{lll} v_{+-} = -v_{-+} = \frac{i}{2} v r_+^{-1} & v_{\chi 3} = \frac{1}{2} r_\chi^{-1} v & v_{\chi 4} = -\frac{1}{2} G^\phi r_\chi^{-1} v \\ \hat{v}_{+-} = -\hat{v}_{-+} = \frac{i}{2} \zeta v r_+^{-1} & \hat{v}_{\chi 3} = \frac{1}{2} r_\chi^{-1} \zeta v & \hat{v}_{\chi 4} = -\frac{1}{2} G^\phi r_\chi^{-1} \zeta v \end{array} \tag{4.51}$$

All the other components vanish according to charge neutrality and CP-invariance. The free parameters are the shift of the quantum field v and the shift of the external field ζv:

$$v = 2\frac{M_Z}{e}\cos\theta_W \sin\theta_W + O(\hbar) \tag{4.52}$$

4.4. The fermion sector

The algebra for representation matrices of fermions has the following form:

$$\begin{aligned} h^{f_i}_\alpha\, h^{f_i}_\beta - h^{f_i}_\beta\, h^{f_i}_\alpha &= -\hat{\varepsilon}_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\, h^{f_i}_{\gamma'} \\ \tilde{h}^{f_i}_\alpha\, \tilde{h}^{f_i}_\beta - \tilde{h}^{f_i}_\beta\, \tilde{h}^{f_i}_\alpha &= -\hat{\varepsilon}_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\, \tilde{h}^{f_i}_{\gamma'} \end{aligned} \tag{4.53}$$

From the consistency equation with the Slavnov-Taylor operator it is seen, that the transformation of external fields is governed by the transformation of propagating fields:

$$h^{f_i}_\alpha = h^{\psi_i}_\alpha \qquad \tilde{h}^{f_i}_\alpha = \tilde{h}^{\psi_i}_\alpha \tag{4.54}$$

The matrices $(h^{f_i}_\alpha)_{ff'}, f, f' = \nu, e, u, d$, are 4×4 matrices and $(\tilde{h}^{f_i}_\alpha)_{ff'}, f, f' = e, u, d$, are 3×3 matrices. CP-invariance implies, that they are imaginary.

Lepton and quark number conservation enables one to treat quarks and leptons separately and, actually, one only has to consider 2-dimensional representation matrices. The non-trivial solution of the algebra is represented by the Pauli-matrices completed by the unit matrix and its equivalence representations. We know from the tree approximation, that left-handed fermions transform according to doublets, whereas right-handed fields transform trivially under $SU(2)$. This transformation behaviour cannot be spoiled in perturbation theory. Therefore we assign in accordance with the tree approximation

$$h^{\delta_i}_\alpha \sim i\frac{\tau_\alpha}{2} \qquad \tilde{h}^{\delta_i}_\alpha \sim 0 \quad \text{with} \quad \alpha = +,-,3 \tag{4.55}$$

$\delta_i = l_i, q_i$ is the index for quarks and leptons. The abelian component is not well-defined by the algebra, but contains some free parameters. For the nontrivial solution one finds always one undetermined parameter for left-handed leptons and quarks of each family

$$h^{f_i}_4 = \begin{pmatrix} iG^{l_i}\mathbf{1} & 0 \\ 0 & iG^{q_i}\mathbf{1} \end{pmatrix} \tag{4.56}$$

The singlet solution involves undetermined parameters for each right-handed fermion.

$$\tilde{h}_4^{f_i} = \begin{pmatrix} iG^{e_i} & 0 & 0 \\ 0 & iG^{u_i} & 0 \\ 0 & 0 & iG^{d_i} \end{pmatrix} \tag{4.57}$$

Due to charge conservation and CP-invariance the equivalence transformations are carried out by diagonal real matrices, which are related in the course of quantization to the field redefinitions of right- and left-handed fields:

$$h_\alpha^{l_i} = iz^{l_i}\frac{\tau_\alpha}{2}(z^{l_i})^{-1} \qquad h_\alpha^{q_i} = iz^{q_i}\frac{\tau_\alpha}{2}(z^{q_i})^{-1} \tag{4.58}$$

with

$$z^{l_i} = \begin{pmatrix} z^{\nu_i} & 0 \\ 0 & z^{e_i} \end{pmatrix} \qquad z^{q_i} = \begin{pmatrix} z^{u_i} & 0 \\ 0 & z^{d_i} \end{pmatrix} \tag{4.59}$$

The singlet representation is independent from field redefinitions. The charged components of the doublet representation depend on the ratio of field redefinitions carried out for left-handed up-and down type quarks and left-handed neutrinos and charged leptons, respectively.

$$r_{l_i} = \frac{z^{e_i}}{z^{\nu_i}} \qquad r_{q_i} = \frac{z^{d_i}}{z^{u_i}} \tag{4.60}$$

Having analysed the general structure of $SU(2)\times U(1)$ operators it is obvious, that rigid symmetry does not restrict the number of independent field redefinitions. Therefore it is allowed to impose independent normalization conditions for the propagating physical fields as well as for the would-be Goldstones and $\phi\pi$-ghosts without spoiling rigid symmetry.

4.5. The algebraic characterization of an abelian local Ward operator

The algebraic analysis of the last sections has shown that the $SU(2)$-components of the rigid Ward operators are uniquely fixed up to equivalence transformations, which are related to field redefinitions of the different fields in question. The abelian component $\mathcal{W}_4$, however, involves several free parameters, which in higher orders appear as instabilities of the abelian subgroup and have to be determined. If we assume now, that the instabilities of the Ward operator are indeed the only breakings which appear in higher orders, then one is able to fix some of the free parameters to all orders of perturbation theory. But there are left the parameters which correspond to lepton and quark family conservation, and it is obvious that they remain independent parameters of the abelian rigid Ward operator.

When one constructs the electroweak standard model from gauge invariance these parameters are determined on the gauge transformation by the Gell-Mann Nishijima relation, which ensures that the photon couples to the electromagnetic current. In the course of renormalization the gauge symmetries are broken and the role of gauge symmetry is taken over by BRS-symmetry and the Slavnov-Taylor identity. Via the nilpotency properties it contains also the algebraic structure of the group in the external field part. When solving the Slavnov-Taylor identity it is seen that one is lead to representation equations for the BRS-transformations, which have the same form as the ones we have solved for establishing rigid operators. It turns out, that the abelian component is also not uniquely defined in the solution of the Slavnov-Taylor identity. In fact one finds the free parameters which correspond to lepton and quark family conservation to be undetermined as well. Leaving them as free parameters the photon will not couple properly to the electromagnetic current but also on the currents associated with lepton and quark family conservation. For this reason we have to use a local Ward identity in addition to the Slavnov-Taylor identity for defining the gauge transformations of the abelian component in an appropriate way. The local Ward identity of electromagnetic symmetry has non-abelian components and does not exist in renormalizable gauges. Therefore we have to use the abelian Ward operator for fixing the undetermined parameters continuing the Gell-Mann Nishijima relation on a functional level to all orders of perturbation thery.

As we have already mentioned the Ward identities, which correspond to charge conservation and conservation of lepton and quark family number are not affected by renormalization. Therefore the identity

$$\Big(\mathcal{W}_{em} + \sum_{i=1}^{N_F} (g_{l_i}\mathcal{W}_{l_i} + g_{q_i}\mathcal{W}_{q_i})\Big)\Gamma = 0 \tag{4.61}$$

is valid to all orders of perturbation theory with arbitrary parameters g_{l_i} and g_{q_i}. Adding the general Ward operators $\mathcal{W}_3$ and $\mathcal{W}_4$ in a way that for vectors and scalars the electromagnetic Ward operator arises and the shifts vanish

$$\mathcal{W} = \mathcal{W}_3 + \frac{1}{G^\phi}\mathcal{W}_4 \tag{4.62}$$

one gets by using (4.61) the following identity, when acting on the functional Γ:

$$\begin{aligned}\mathcal{W}\Gamma = \int \quad \Big(\quad & \frac{1}{G^\phi}\sum_{i=1}^{N_F}\Big(i(G^{u_i} - G^{q_i} - G^\phi)\overline{u^R_i}\frac{\delta}{\delta\overline{u^R_i}} + i(G^{d_i} - G^{q_i})\overline{d^R_i}\frac{\delta}{\delta\overline{d^R_i}} \\ & + i(G^{e_i} - G^{l_i} - G^\phi)\overline{e^R_i}\frac{\delta}{\delta\overline{e^R_i}} + \text{h.c.}\Big)\Big)\Gamma \end{aligned} \tag{4.63}$$

If one assumes, that these are the only breakings of the rigid Ward operators, which arise in higher orders, then the coefficients appearing therein have to vanish to all orders of perturbation theory, because they can be independently tested on non-vanishing vertices of the classical action, namely on the scalar interaction, the fermion masses and the gauge fixing:

$$G^{u_i} = G^{q_i} + G^{\phi} \qquad G^{d_i} = G^{q_i} \qquad G^{e_i} = G^{l_i} \tag{4.64}$$

These relations just state that the charges of leptons and quarks of each family differ by one unit, which is determined by the charge of the W_+. The final abelian Ward operator acting on Γ takes the form:

$$\mathcal{W}_4\Gamma = \Big(\mathcal{W}_{em} - \mathcal{W}_3 + \sum_{i=1}^{N_F}(g_{l_i}\mathcal{W}_{l_i} + g_{q_i}\mathcal{W}_{q_i})\Big)\Gamma \tag{4.65}$$

with undetermined parameter g_{l_i} and g_{q_i}. Here we have also chosen the overall normalization appropriately, i.e. $G^{\phi} = 1$. The problem of deriving a local Ward identity in connection with the abelian subgroup is therefore not well-posed, but has to be restated by requiring to have a local Ward identity in connection with the electromagnetic current. Defining the local Ward operator connected with electromagnetic current conservation by

$$\mathbf{w}_4^Q \equiv \mathbf{w}_{em} - \mathbf{w}_3 \quad \text{with} \quad \mathcal{W}_3 - \mathcal{W}_{em} = \int(\mathbf{w}_3 - \mathbf{w}_{em}) \tag{4.66}$$

it is seen that it is algebraically unique up to a total derivative acting on the differentiation with respect to the abelian combination of vector fields. The operator

$$\hat{\mathbf{w}}_4^Q = g_1\mathbf{w}_4^Q - \frac{1}{r_Z^V}\partial\frac{\delta}{\delta Z}\sin\Theta^V - \frac{1}{r_A^V}\partial\frac{\delta}{\delta A}\cos\Theta^V \tag{4.67}$$

commutes with the Slavnov-Taylor operator and the Ward operators of rigid symmetry:

$$\Big[\hat{\mathbf{w}}_4^Q, \mathcal{W}_\alpha\Big] = 0 \qquad s_\Gamma\hat{\mathbf{w}}_4\Gamma - \hat{\mathbf{w}}_4^Q\mathcal{S}(\Gamma) = 0 \quad \text{for any } \Gamma \tag{4.68}$$

The final version of the abelian Ward identity we have to prove to all orders of perturbation theory takes the form

$$\left(g_1\mathbf{w}_4^Q - \frac{1}{r_Z^V}\partial\frac{\delta}{\delta Z}\sin\Theta^V - \frac{1}{r_A^V}\partial\frac{\delta}{\delta A}\cos\Theta^V\right)\Gamma = \frac{1}{r_Z^V}\Box B_Z\sin\Theta^V + \frac{1}{r_A^V}\Box B_A\cos\Theta^V \tag{4.69}$$

with r_Z^V, r_A^V and Θ^V determined on the charged rigid $SU(2)$ Ward identities (4.31). It involves an overall normalization parameter, which depends on the parametrization one has chosen and in higher orders on the normalization condition of the coupling. In the QED-like on-shell schemes it is given by

$$g_1 = \frac{e}{\cos\theta_W} + O(\hbar) \tag{4.70}$$

The Ward identity (4.69) has to be established in higher orders of perturbation theory, in order to fix the undetermined parameters appearing in the action as a consequence of the instability of the abelian subgroup. These instabilities are connected with the fact, that it is not possible to algebraically distinguish between gauging the electromagnetic current and the currents associated with lepton and quark familiy conservation. If we were not able to establish the abelian local Ward identity to all orders we had to impose a normalization condition for one charged-fermion photon vertex of each family, but we would loose thereby the control if the gauge symmetry is indeed the electromagnetic symmetry and not the current associated with lepton and quark family conservation, which one gauges in higher orders.

These observations have important consequences for the construction of the gauge fixing and ghost sector: In order to identify the abelian Ward identity according to (4.66) and (4.68) rigid $SU(2) \times U(1)$ Ward identities have to be established. The gauge fixing sector has therefore to be constructed with the help of the external scalar fields as introduced in section 2.2. In this procedure the number of independent gauge parameters is restricted. In order to avoid infrared divergent counterterms for the $\phi\pi$-ghosts one is forced to introduce in higher orders an independent ghost angle, which appears in the Slavnov-Taylor identity and the Ward identities of rigid symmetry via different field redefinitions of vectors and antighosts as derived in (4.40).

5. The general local solution of the Slavnov-Taylor identity and rigid symmetries

5.1. The normalization conditions

As we have outlined in section 2, the construction of higher orders proceeds by proving, that it is possible to adjust local contributions in such a way, that the functional of 1PI Green functions is invariant under the ST identity and the Ward identities of rigid symmetry. Local contributions are algebraically separated into two classes: invariants of the symmetry and non-invariant contributions (cf. (3.5)). The coefficients of the invariants have to be fixed by appropriate normalization conditions and vice versa it has to be shown, that the normalization conditions one wants to impose for a proper particle interpretation only dispose of invariant coefficients.

We impose for all physical fields on-shell conditions as given in the literature (see e.g. [28]). For vectors and scalars they read on the 2-point functions

$$\begin{array}{rclcrcl} \mathrm{Re}\Gamma^T_{+-}(p^2)|_{p^2=M_W^2} & = & 0 & \qquad & \mathrm{Re}\Gamma^T_{ZZ}(p^2)|_{p^2=M_Z^2} & = & 0 \\ \Gamma^T_{AA}(p^2)|_{p^2=0} & = & 0 & \qquad & \mathrm{Re}\Gamma_{HH}(p^2)|_{p^2=m_H^2} & = & 0 \end{array} \tag{5.1}$$

The photon and the Z-boson are not distinguished by quantum numbers and mix from 1-loop order onwards. Therefore they have to be separated on-shell:

$$\Gamma^T_{ZA}(p^2)|_{p^2=0} = 0 \qquad \mathrm{Re}\Gamma^T_{ZA}(p^2)|_{p^2=M_Z^2} = 0 \tag{5.2}$$

The Higgs and the neutral would-be Goldstone are distinguished by their transformation properties under CP. For this reason the respective conditions for scalars are valid by construction in a CP-invariant theory. The transversal part of the vector 2-point functions is defined according to

$$\Gamma_{V_a^\mu V_b^\nu} \equiv -(\eta^{\mu\nu} - \frac{p^\mu p^\mu}{p^2})\Gamma^T_{ab} - \frac{p^\mu p^\mu}{p^2}\Gamma^L_{ab} \tag{5.3}$$

For the unstable particles the counterterms are fixed by the requirement that the real part of the 2-point functions is vanishing. This prescription has to be reanalysed [40], if one constructs the S-matrix and especially wants to prove gauge parameter independence of physical quantities. For the construction of finite Green functions it is certainly a well-defined normalization condition, which continues the tree approximation of the on-shell scheme to higher orders in a proper form.

Because the residua are finally canceled when constructing the S-matrix, there is quite some arbitrariness involved in the respective normalization conditions. In the complete on-shell scheme the residua of all physical particles are fixed at the pole position. In order to avoid on-shell infrared divergencies we modify the complete on-shell scheme by introducing a normalization point κ_a^2 for each vector and impose for the transversal part of the vectors:

$$\mathrm{Re}\partial_{p^2}\Gamma^T_{+-}(p^2)|_{p^2=\kappa_W^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma^T_{ZZ}(p^2)|_{p^2=\kappa_Z^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma^T_{AA}(p^2)|_{p^2=\kappa_A^2} = 1 \tag{5.4}$$

In this form they allow to switch between different normalization conditions by adjusting κ_a^2. As we have already mentioned two of these normalization conditions can be replaced by the Ward identities of rigid symmetry, which corresponds to the minimal on-shell scheme.

Finally one has to specify normalization conditions for the residua of the scalars, which is carried out similarly as above. As it will be seen from the general solution of the ST identity also the residua of unphysical Goldstone bosons are not fixed by the ST identity, but could be fixed on the Ward identities of rigid symmetry. Because they are not considered in external legs in physical scattering processes, the divergencies appearing in dimensional regularization are often subtracted according to the MS scheme. In order to remain quite general in the construction we impose normalization conditions on arbitrary normalization points μ_a^2:

$$\mathrm{Re}\partial_{p^2}\Gamma_{+-}(p^2)|_{p^2=\mu_W^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma_{HH}(p^2)|_{p^2=\mu_H^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma_{\chi\chi}(p^2)|_{p^2=\mu_\chi^2} = 1 \tag{5.5}$$

The normalization conditions for fermions are listed in the literature quite generally also for the case, that there is CP-violation via the CKM matrix [28]. They simplify considerably, if one assumes lepton and quark family conservation. Decomposing the fermion 2-point functions according to

$$\begin{aligned}\Gamma_{\bar f_i f_i} &= \not p\Gamma^L_{f_i}(p^2)\tfrac{1}{2}(1-\gamma_5) + \not p\Gamma^R_{f_i}(p^2)\tfrac{1}{2}(1+\gamma_5) - m_{f_i}\mathbf{1}\Gamma^m_{f_i}(p^2) \\ &\equiv \not p - m_{f_i} + \not p\Sigma^L_{f_i}(p^2)\tfrac{1}{2}(1-\gamma_5) + \not p\Sigma^R_{f_i}(p^2)\tfrac{1}{2}(1+\gamma_5) - m_{f_i}\mathbf{1}\Sigma^m_{f_i}(p^2)\end{aligned} \tag{5.6}$$

the on-shell conditions read

$$\mathrm{Re}(\Gamma^L_{f_i}(p^2) - \Gamma^m_{f_i}(p^2))|_{p^2=m^2_{f_i}} = 0 \qquad \mathrm{Re}(\Gamma^R_{f_i}(p^2) - \Gamma^m_{f_i}(p^2))|_{p^2=m^2_{f_i}} = 0 \tag{5.7}$$

They impose pole conditions on the Dirac spinors and forbid parity violation for the on-shell propagators. On-shell residua are endangered by infrared divergencies as it is the case in QED with a massless photon. We therefore introduce off-shell conditions:

$$\mathrm{Re}\partial_{p^2}(p^2\Gamma^L_{f_i}(p^2) + p^2\Gamma^R_{f_i}(p^2))|_{p^2=\kappa_i^2} = 1 \tag{5.8}$$

Just by construction of vertex functions it is ensured that

$$\Gamma_H = 0 \tag{5.9}$$

which forbids to introduce linear Higgs field terms into the local contributions of higher order corrections.

For proving unitarity of the physical S-matrix we have also to impose normalization conditions on the unphysical fields. The poles of propagators of the longitudinal parts of the vectors, of the unphysical would-be Goldstones and the Faddeev-Popov ghosts are seen to be related by the ST identity. The normalization conditions on the poles of unphysical particles are most easily established on the Faddeev-Popov fields and read:

$$\mathrm{Re}\Gamma_{\bar c_+ c_-}(p^2)|_{p^2=\zeta_W M_W^2} = 0 \quad \mathrm{Re}\Gamma_{\bar c_Z c_Z}(p^2)|_{p^2=\zeta_Z M_Z^2} = 0 \quad \Gamma_{\bar c_A c_A}(p^2)|_{p^2=0} = 0 \tag{5.10}$$

Furthermore one has to require on-shell separation for neutral ghosts

$$\begin{aligned}\mathrm{Re}\Gamma_{\bar c_Z c_A}(p^2)|_{p^2=\zeta_Z M_Z^2} &= 0 \qquad & \Gamma_{\bar c_Z c_A}(p^2)|_{p^2=0} &= 0 \\ \mathrm{Re}\Gamma_{\bar c_A c_Z}(p^2)|_{p^2=\zeta_Z M_Z^2} &= 0 \qquad & \Gamma_{\bar c_A c_Z}(p^2)|_{p^2=0} &= 0\end{aligned} \tag{5.11}$$

Finally we impose also normalization conditions on the residua of the ghost propagators:

$$\mathrm{Re}\partial_{p^2}\Gamma_{\bar{c}_+ c_-}(p^2)|_{p^2=\kappa_W^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma_{\bar{c}_Z c_Z}(p^2)|_{p^2=\kappa_Z^2} = 1 \quad \mathrm{Re}\partial_{p^2}\Gamma_{\bar{c}_A c_A}(p^2)|_{p^2=\kappa_A^2} = 1 \tag{5.12}$$

Solving the ST identity for the most general local action which is compatible with UV dimension 4, the parameters, which are fixed order to order by normalization conditions, have to be free parameters in terms of which all the other couplings are determined. The local contributions which are fixed by the above normalization conditions are given by

$$\begin{aligned}\Gamma^{gen}_{bil} \;=\; \int\Big(& -\tfrac{1}{4}(\partial^\mu V^\nu_a - \partial^\nu V^\mu_a) Z^V_{ab}(\partial_\mu V_{\nu b} - \partial_\nu V_{\mu b}) + \tfrac{1}{2} V^\mu_a \mathcal{M}^V_{ab} V_{\mu b} \\ & + \tfrac{1}{2}\partial^\mu \phi_a Z^S_{ab}\partial_\mu \phi_b - \tfrac{1}{2}M_H^2 H^2(x) \\ & + i Z^R_{f_i} \bar{f}^R_i \not\partial f^R_i + i Z^L_{f_i} \bar{f}^L_i \not\partial f^L_i - M_{f_i}(\bar{f}^R_i f^L_i + \bar{f}^R_i f^L_i) \\ & - \bar{c}_a Z^g_{ab} \Box c_b - \bar{c}_a \mathcal{M}^g_{ab} c_b \Big)\end{aligned} \tag{5.13}$$

The matrices and parameters are chosen in accordance with charge neutrality and CP-invariance, especially Z^S_{ab} is a diagonal matrix. In perturbation theory the parameters are order by order determined by the above normalization conditions:

$$\begin{aligned} Z^V_{ab} &= \tilde{I}_{ab} + \delta Z^V_{ab} \qquad & Z^R_{f_i} &= 1 + \delta Z^R_{f_i} \\ Z^S_{ab} &= \tilde{I}_{ab} + \delta Z^S_{ab} \qquad & Z^L_{f_i} &= 1 + \delta Z^L_{f_i} \end{aligned} \tag{5.14}$$

and respective expressions for the Higgs mass and fermion masses

$$M_H^2 = m_H^2 + \delta m_H^2 \qquad M_{f_i} = m_{f_i} + \delta m_{f_i} \tag{5.15}$$

The vector mass matrix is non-diagonal and can be decomposed into an orthogonal matrix and a diagonal matrix:

$$\mathcal{M}^V_{ab} = O^T(\theta)\begin{pmatrix} 0 & M_{+-} & 0 & 0 \\ M_{+-} & 0 & 0 & 0 \\ 0 & 0 & M_{ZZ} & 0 \\ 0 & 0 & 0 & M_{AA} \end{pmatrix} O(\theta) \quad \text{with} \quad \begin{aligned} M_{+-} &= M_W^2 + \delta M_W^2 \\ M_{ZZ} &= M_Z^2 + \delta M_Z^2 \\ M_{AA} &= 0 \\ \theta &= 0 + \delta\theta \end{aligned} \tag{5.16}$$

M_W^2, M_Z^2, m_H^2 and m_{f_i} are the physical masses of the particles. The explicit form of the local counterterms is of course dependent on the way one has constructed the finite renormalized 1PI Green functions. The objects we are able to talk about in a scheme independent way are the finite Green functions. Constructing them in accordance with the symmetries, they are finally governed by the normalization conditions and are independent of the scheme, one has used for subtracting the divergencies. Especially the conditions for separating massless and massive particle at $p^2 = 0$ (3.4)

$$\begin{aligned} \Gamma_{ZA}(p^2=0) = \Gamma_{AA}(p^2=0) &= 0 \\ \Gamma_{\bar{c}_A c_Z}(p^2=0) = \Gamma_{\bar{c}_Z c_A}(p^2=0) = \Gamma_{\bar{c}_A c_A}(p^2=0) &= 0 \end{aligned} \tag{5.17}$$

have to be established on the finite 2-point functions in order to be able to carry out infrared finite higher order calculations. In the BPHZL scheme, which treats massless particles quite systematically, these normalization conditions are implemented in the scheme. One has therefore $\delta\theta^{BPHZL} = 0$. In dimensional regularization these normalization conditions have to be carefully implemented by adjusting e.g. $\delta\theta^{dim}$.

5.2. The symmetry transformations and the general action

For finding the invariant counterterms, which are added order by order in perturbation theory to the nonlocal contributions, we have to solve the ST identity and the Ward identities of rigid symmetry for the most general local action Γ^{gen}_{cl}, which is compatible with renormalizability by power counting (cf. (3.5) and (3.7)).

$$\mathcal{S}(\Gamma^{gen}_{cl}) = 0 \qquad W_\alpha(\Gamma^{gen}_{cl}) = 0 \tag{5.18}$$

For solving these equations one could proceed in a perturbative expansion, but as for the Ward operators of rigid symmetry such a treatment disguises the simple algebraic structure of the final solution, hence we proceed differently.

For the ST operator and the Ward operators we take the general operators as they are determined by consistency and by the $SU(2)\times U(1)$-algebra in section 3 from the general ansatz (4.21) and (4.22). The ST operator is written in the following form:

$$\begin{aligned} \mathcal{S}(\Gamma) = \int \Big(& (r^g_{4Z}\partial_\mu c_Z + r^g_{4A}\partial_\mu c_A)\Big(\frac{1}{r_Z}\sin\Theta\frac{\delta\Gamma}{\delta Z_\mu} + \frac{1}{r_A}\cos\Theta\frac{\delta\Gamma}{\delta A_\mu}\Big) \\ & + \frac{\delta\Gamma}{\delta\rho^\mu_3}\Big(\frac{1}{r_Z}\cos\Theta\frac{\delta\Gamma}{\delta Z_\mu} - \frac{1}{r_A}\sin\Theta\frac{\delta\Gamma}{\delta A_\mu}\Big) + \frac{\delta\Gamma}{\delta\sigma_3}\frac{1}{\det r^g}\Big(r^g_{4A}\frac{\delta\Gamma}{\delta c_Z} - r^g_{4Z}\frac{\delta\Gamma}{\delta c_A}\Big) \\ & + \frac{\delta\Gamma}{\delta\rho^\mu_+}\frac{\delta\Gamma}{\delta W_{\mu,-}} + \frac{\delta\Gamma}{\delta\rho^\mu_-}\frac{\delta\Gamma}{\delta W_{\mu,+}} + \frac{\delta\Gamma}{\delta\sigma_+}\frac{\delta\Gamma}{\delta c_-} + \frac{\delta\Gamma}{\delta\sigma_-}\frac{\delta\Gamma}{\delta c_+} + \frac{\delta\Gamma}{\delta Y_a}\tilde{I}_{aa'}\frac{\delta\Gamma}{\delta\phi_{a'}} \\ & + \sum_{i=1}^{N_F}\Big(\frac{\delta\Gamma}{\delta\overline{\psi}^L_{f_i}}\frac{\delta\Gamma}{\delta f^R_i} + \frac{\delta\Gamma}{\delta\overline{\psi}^R_{f_i}}\frac{\delta\Gamma}{\delta f^L_i} + \mathrm{h.c.}\Big) \\ & + B_a(r^V)^{-1}_{a\alpha}\delta\hat{g}_{\alpha b}\frac{\delta\Gamma}{\delta\bar{c}_b} + q_a\frac{\delta\Gamma}{\delta\hat{\phi}_a}\Big) \end{aligned} \tag{5.19}$$

The Ward operators involve the representation matrices of the fundamental and adjoint representation with their equivalence classes (cf. (4.24),(4.42) and (4.55)). Because the abelian Ward operator is related to the nonabelian neutral Ward operator $\mathcal{W}_3$ and to the operators of global unbroken symmetries $\mathcal{W}_{em}$ and $\mathcal{W}_{l_i}, \mathcal{W}_{q_i}$ according to equ. (4.65), we only have to consider the non-abelian Ward operators for establishing rigid symmetry ($\alpha = +, -, 3$).

$$
\begin{aligned}
\mathcal{W}_\alpha = \tilde{I}_{\alpha\alpha'} \int dx \Big(& V_b^\mu (r^V)^T_{b\beta} \hat{\varepsilon}_{\beta\gamma\alpha'} (r^V)^{-1T}_{\gamma c} \tilde{I}_{cc'} \frac{\delta}{\delta V^\mu_{c'}} + B_b (r^V)^{-1}_{b\beta} \hat{\varepsilon}_{bc,\alpha'} (r^V)_{\gamma c} \tilde{I}_{cc'} \frac{\delta}{\delta B_{c'}} \\
& + c_b (r^g)^T_{b\beta} \varepsilon_{\beta\gamma\alpha'} (r^g)^{-1T}_{\gamma c} \tilde{I}_{cc'} \frac{\delta}{\delta c_{c'}} + \bar{c}_b (\delta g)^{-1}_{b\beta} \hat{\varepsilon}_{\beta\gamma\alpha'} \delta g_{\gamma c} \tilde{I}_{cc'} \frac{\delta}{\delta \bar{c}_{c'}} \\
& + (r^S_b \phi_b + \delta_{Hb} v) \hat{t}_{bc,\alpha'} r^{S-1}_c \tilde{I}_{cc'} \frac{\delta}{\delta \phi_{c'}} + Y_b r^{S-1}_b \hat{t}_{bc,\alpha'} r^S_c \tilde{I}_{cc'} \frac{\delta}{\delta Y_{c'}} \\
& + (r^S_b \hat{\phi}_b + \hat{\zeta} v \delta_{Hb}) \hat{t}_{bc,\alpha'} r^{S-1}_c \tilde{I}_{cc'} \frac{\delta}{\delta \hat{\phi}_{c'}} + q_b r^S_b \hat{t}_{bc,\alpha'} r^{S-1}_c \tilde{I}_{cc'} \frac{\delta}{\delta q_{c'}} \\
& + \rho_\beta \varepsilon_{\beta\gamma,\alpha'} \tilde{I}_{\gamma\gamma'} \frac{\delta}{\delta \rho_{\gamma'}} + \sigma_\beta \varepsilon_{\beta\gamma,\alpha'} \tilde{I}_{\gamma\gamma'} \frac{\delta}{\delta \sigma_{\gamma'}} \\
& + \sum_{i=1}^{N_F} \sum_{\delta = l,q} \Big(\overline{F^L_{\delta_i}} r^{\delta_i} \frac{i\tau_{\alpha'}}{2} (r^{\delta_i})^{-1} \frac{\delta}{\delta \overline{F^L_{\delta_i}}} - \frac{\delta}{\delta F^L_{\delta_i}} r^{\delta_i} \frac{i\tau_{\alpha'}}{2} (r^{\delta_i})^{-1} F^L_{\delta_i} \\
& \qquad + \overline{\Psi^R_{\delta_i}} (r^{\delta_i})^{-1} \frac{i\tau_{\alpha'}}{2} r^{\delta_i} \frac{\delta}{\delta \overline{\Psi^L_{\delta_i}}} - \frac{\delta}{\delta \Psi^R_{\delta_i}} (r^{\delta_i})^{-1} \frac{i\tau_{\alpha'}}{2} r^{\delta_i} \Psi^R_{\delta_i} \Big) \Big)
\end{aligned} \tag{5.20}
$$

There we have parameterized the equivalence classes of rigid transformations by r^V, r^S, r^g and r^{l_i}, r^{q_i} taking into account, that we are able to determine field redefiniton matrices up to invariant matrices. According to (4.31) and (4.32) we define the matrix r^V by

$$
r^V_{\alpha a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r_Z \cos\Theta & -r_A \sin\Theta \\ 0 & 0 & r_Z \sin\Theta & r_A \cos\Theta \end{pmatrix} \tag{5.21}
$$

The equivalent transformations for scalars and fermions are chosen as in (4.46) and (4.60):

$$
r^S = \begin{pmatrix} r^S_+ & 0 & 0 & 0 \\ 0 & r^S_+ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^S_\chi \end{pmatrix} \qquad r^{\delta_i} = \begin{pmatrix} 1 & 0 \\ 0 & r_{\delta_i} \end{pmatrix} \tag{5.22}
$$

Furthermore the vector transformations in the ST operator (5.19) are parametrized in agreement with the relations gained from the consistency between the general Ward operators and the ST operator (cf. (4.34) and (4.36)). The transformation matrix of ghosts is defined by

$$r^g_{\alpha a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^g_{3Z} & r^g_{3A} \\ 0 & 0 & r^g_{4Z} & r^g_{4A} \end{pmatrix} \tag{5.23}$$

Here we have disposed over the invariant abelian parameter by taking the linear BRS-transformation of vector fields into the transformation matrix. The parameters in the nonlinear transformations of the neutral ghosts are then chosen in accordance with nilpotency and the consistency relation. Finally we have splitted the matrix $\hat{g}_{ab}$ which governs the BRS-transformations of antighosts into the rigid transformation matrices of B-fields and antighosts:

$$\hat{g}_{ab} = (r^V)^{-1}_{a\beta}\delta\hat{g}_{\beta b} \tag{5.24}$$

The rigid transformations of antighosts are determined from the consistency condition as related to $\delta\hat{g}_{\alpha b}$.

With these operators we have to act on Γ^{gen}_{cl}, which consists of all local field polynomials compatible with UV-dimension 4. (See the appendix for quantum numbers.) For finding the invariant counterterms we do not use a specific scheme for treating massless particles. If the parameters of the bilinear action (5.13) are indeed free parameters, it is ensured, that we are able to establish all normalization conditions, especially the ones for separating massless and massive particles at $p^2 = 0$. Further restrictions on Γ^{gen}_{cl} are neutrality with respect to electric and Faddeev-Popov charge (4.14) and lepton and quark family conservation (4.15)

$$\begin{aligned} \mathcal{W}_{em}\Gamma^{gen}_{cl} &= 0 \qquad & \mathcal{W}_{l_i}\Gamma^{gen}_{cl} &= 0 \\ \mathcal{W}_{\phi\pi}\Gamma^{gen}_{cl} &= 0 \qquad & \mathcal{W}_{q_i}\Gamma^{gen}_{cl} &= 0 \end{aligned} \tag{5.25}$$

According to lepton and quark family conservation one is able to restrict the analysis to CP-invariant field polynomials. From formal unitarity it is required, that local contributions are hermitean:

$$\Gamma^{gen}_{cl} = \Gamma^{gen\,\dagger}_{cl} \tag{5.26}$$

The general ansatz for Γ^{gen}_{cl} is quite lengthy. For the purpose of the present paper we explicitly give only the most general external field part. When solving (5.18) it is seen that the solution is traced back to representation equations for the general couplings of the external field part. Finally it has to be shown that these couplings as well as the ST identity and the Ward operators of rigid symmetry are uniquely determined as functions of those parameters of which one disposes by the normalization conditions (5.13). Vice versa the bilinear part of the action cannot possibly be restricted by the ST identity and rigid symmetry, because otherwise unitarity and particle interpretation of the field theory is endangered.

We start the presentation of the general classical solution in the vector, scalar and fermion part of the action. Having established there the ST identity, rigid invariance follows as a consequence with well determined coefficients of rigid transformations for vectors, scalars and fermions. The gauge fixing sector can then be established in accordance with rigid invariance and allows to compute the ghost sector and at the same time the transformation parameters of ghosts and antighosts. The procedure for adjusting symmetric local contributions as outlined here in the abstract approach has to be done exactly the same way in practice when calculating order by order in perturbation theory.

5.3. The vector-scalar and fermion part of the action

In this section we present the general solution of the ST identity in the vector, scalar and fermion part of the action, which can be solved in combination with the external field part self-consistently. The bilinear part of the most general local action Γ_{cl}^{gen} is given in (5.13). Because the parameters appearing therein are fixed by normalization conditions, they should not be determined in the course of the calculations. For this reason we redefine the vector and scalar fields in such a way, that the bilinear part takes a simple form:

$$\begin{array}{rclcl} V^o_{\mu a} & = & z^V_{ab} V_{\mu a} & \qquad & f_i^{oR} = \tilde{z}_{f_i} f_i^R \\ \phi^o_a & = & z^S_a \phi_a & & f_i^{oL} = z_{f_i} f_i^L \end{array} \tag{5.27}$$

In a first step we have to show that these parameters are uniquely determined from the parameters appearing in the bilinear part of the action Γ_{bil}^{gen}, we fix by the normalization conditions. Due to CP-invariance all field redefinitions can be chosen real. Fermion und scalar field redefinitions are determined on the kinetic parts up to a sign, which is irrelevant in perturbation theory and can finally be adjusted in the tree approximation:

$$Z^S_{ab} = \begin{pmatrix} 0 & z_+^2 & 0 & 0 \\ z_+^2 & 0 & 0 & 0 \\ 0 & 0 & z_H^2 & 0 \\ 0 & 0 & 0 & z_\chi^2 \end{pmatrix} \qquad \begin{array}{rcl} Z^R_{f_i} & = & \tilde{z}^2_{f_i} \\ Z^L_{f_i} & = & z^2_{f_i} \end{array} \tag{5.28}$$

Because photon and Z-boson are not distinguished by any quantum numbers, the vector redefinition matrix nondiagonal in the neutral sector. We parameterize this matrix as in (4.26):

$$z^V_{ab} = \begin{pmatrix} z_W & 0 & 0 & \\ 0 & z_W & 0 & 0 \\ 0 & 0 & z_Z \cos\theta_Z & -z_A \sin\theta_A \\ 0 & 0 & z_Z \sin\theta_Z & z_A \cos\theta_A \end{pmatrix} \tag{5.29}$$

On the kinetic parts z^V_{ab} is determined up to an orthogonal matrix:

$$Z^V_{ab} = {z^V_{aa'}}^T \tilde{I}_{a'b'} z^V_{b'b} = (O(\theta)z^V)^T_{aa'} \tilde{I}_{a'b'} (O(\theta)z^V)_{b'b} \tag{5.30}$$

This remaining orthogonal matrix can be fixed on the vector mass matrix. Requiring $\mathcal{M}^o_{ab}$ to be diagonal

$$\mathcal{M}^o_{ab} = (z^V)^{-1T}_{aa'} \mathcal{M}_{a'b'} (z^V)^{-1}_{b'b} \quad \text{with} \quad \mathcal{M}^o_{ab} = \begin{pmatrix} 0 & {M^o_W}^2 & 0 & 0 \\ {M^o_W}^2 & 0 & 0 & 0 \\ 0 & 0 & {M^o_Z}^2 & 0 \\ 0 & 0 & 0 & {M^o_A}^2 \end{pmatrix} \tag{5.31}$$

finally determines z^V_{ab} uniquely up to signs. Transforming likewise the masses of the fermions and the Higgs into bare masses:

$$m^o_{f_i} = \tilde{z}^{-1}_{f_i} z^{-1}_{f_i} M_{f_i} \qquad {m^o_H}^2 = M^2_H z^{-2}_H \tag{5.32}$$

the bilinear part of the action is transformed into the standard form expressed in terms of bare quantities, which depend by definition on the arbitrary field redefinitions z^V_{ab}, z^S_a, z_{f_i} and $\tilde{z}_{f_i}$.

$$\begin{aligned}
\Gamma^{gen}_{bil} &= \int \Big(-\tfrac{1}{4}(\partial^\mu V^\nu_a - \partial^\nu V^\mu_a) Z^V_{ab} (\partial^\mu V_{\nu b} - \partial^\nu V_{\mu b}) + \tfrac{1}{2} V^\mu_a \mathcal{M}^V_{ab} V_{\mu b} \\
&\quad + \tfrac{1}{2}\partial^\mu \phi_a Z^S_{ab} \partial_\mu \phi_b - \tfrac{1}{2} M^2_H H^2(x) \\
&\quad + i Z^R_{f_i} \bar{f}^R_i \partial\!\!\!/ f^R_i + i Z^L_{f_i} \bar{f}^L_i \partial\!\!\!/ f^L_i - M_{f_i}(\bar{f}^R_i f^L_i + \bar{f}^R_i f^L_i) \Big) \\
&= \int \Big(-\tfrac{1}{4}(\partial^\mu V^{o\nu}_a - \partial^\nu V^{o\mu}_a) \tilde{I}_{ab} (\partial^\mu V^o_{\nu b} - \partial^\nu V^o_{\mu b}) + \tfrac{1}{2} V^{o\mu}_a \mathcal{M}^o_{ab} V^o_{\mu b} \\
&\quad + \tfrac{1}{2}\partial^\mu \phi^o_a \tilde{I}_{ab} \partial_\mu \phi^o_b - \tfrac{1}{2} m^{o2}_H H^{o2}(x) \\
&\quad + i \bar{f}^{oR}_i \partial\!\!\!/ f^{oR}_i + i \bar{f}^{oL}_i \partial\!\!\!/ f^{oL}_i - m^o_{f_i}(\bar{f}^{oR}_i f^{oL}_i + \bar{f}^{oR}_i f^{oL}_i) \Big)
\end{aligned} \tag{5.33}$$

These redefinitions are carried out throughout in Γ^{gen}_{cl} by redefining also all the arbitrary couplings appearing therein as we did it for the masses. At the same time we have to transform the original fields into bare fields in the ST identity and the Ward identities of rigid symmetry. The arbitrary field redefinitions appearing thereby are absorbed into a redefinition of parameters and external fields:

$$\begin{aligned}
\rho^o_{\mu\alpha} &= z^{-1}_W \rho_{\mu\alpha} & \psi^{oR}_{f_i} &= z^{-1}_{f_i} \psi^R_{f_i} \\
Y^o_a &= z^{S-1}_a Y_a & \psi^{oL}_{f_i} &= \tilde{z}^{-1}_{f_i} \psi^L_{f_i}
\end{aligned} \tag{5.34}$$

The bare parameters are defined via the representation matrices of rigid invariance:

$$\begin{aligned}
(r^{oV})^{-1} \hat{\varepsilon}_\alpha (r^{oV}) &= z^V (r^V)^{-1} \hat{\varepsilon}_\alpha r^V (z^V)^{-1} \\
(r^{oS})^{-1} \hat{t}_\alpha (r^{oS}) &= z^S (r^S)^{-1} \hat{t}_\alpha r^S (z^S)^{-1} \\
(r^{o\delta_i})^{-1} \tau_\alpha (r^{o\delta_i}) &= z^\delta_i (r^{\delta_i})^{-1} \tau_\alpha r^{\delta_i} (z^\delta_i)^{-1}
\end{aligned} \tag{5.35}$$

We are now ready to apply the ST operator on Γ^{gen}_{cl}. The computation is quite lengthy, therefore we only quote the final result and the crucial equations. Most important for the solution is the external field part of the general action:

$$
\begin{aligned}
\Gamma^{gen}_{ext.f.} = \int \Big(& -\frac{1}{2}\sigma_\alpha f_{\alpha,bc} c_b c_c \\
& +\rho^o_{\mu\alpha}(a'^g_{\alpha b}\partial^\mu c_b + \hat{a}'_{a,bc} V^{o\mu}_b c_c) + Y^o_a (t'_{ab,c}\phi^o_b c_c + v'_{ab} c_b) \\
& +\sum_{i=1}^{N_F} (\bar{\psi}^{oR}_{f'_i} f^{oL}_i h'^i_{f'f,a} c_a - \bar{f}^{oL}_i \psi^{oR}_{f'_i} h'^i_{ff',a} c_a \\
& \qquad + \bar{\psi}^{oL}_{f'_i} f^{oR}_i \tilde{h}'^i_{f'f,a} c_a - \bar{f}^{oR}_i \psi^{oL}_{f'_i} h'^i_{ff',a} c_a)\Big)
\end{aligned}
\tag{5.36}
$$

For simplicity we have suppressed the interaction polynomials of external scalars $\hat{\phi}_a$. These polynomials are considered in the context of the gauge fixing and ghost sector.

The arbitrary coupling matrices are restricted by the global symmetries (5.25), complex conjugation (5.26) and CP-invariance. We have already carried out the transformation into bare fields for vectors, scalars and fermions and the respective external fields and have transformed at the same time the original couplings into primed couplings (see (5.37)).

Via the ST identity the couplings of the vector, scalar and fermion part of the general action are determined as functions of the coupling matrices appearing in the external field part and of the parameters of the ST identity. Explicitly they depend on the following combinations:

$$
\begin{aligned}
\hat{a}^o_{a,bc} &= (r^{oV})^{-1}_{a\alpha}\hat{a}'_{\alpha bc'}\tilde{z}^{g-1}_{cc'} = z_W (r^{oV})^{-1}_{a\alpha}\hat{a}_{\alpha bc'} z^{V-1}_{b'b}\tilde{z}^{g-1}_{cc'} \\
t^o_{ab,c} &= t'_{ab,c'}\tilde{z}^{g-1}_{cc'} = z^S_{aa'} t_{a'b',c'} z^{S-1}_{b'b}\tilde{z}^{g-1}_{cc'} \\
v^o_{ac} &= v'_{ac'}\tilde{z}^{g-1}_{c'c} = z^S_{aa'} v_{a'c'}\tilde{z}^{g-1}_{c'c} \\
h^{oi}_{ff'c} &= h'^i_{ff'c'}\tilde{z}^{g-1}_{c'c} = z_f z^{-1}_{f'} h^i_{ff'c'}\tilde{z}^{g-1}_{c'c} \\
\tilde{h}^{oi}_{ff'c} &= \tilde{h}'^i_{ff'c'}\tilde{z}^{g-1}_{c'c} = \tilde{z}_f \tilde{z}^{-1}_{f'}\tilde{h}^i_{ff'c'}\tilde{z}^{g-1}_{c'c}
\end{aligned}
\tag{5.37}
$$

$\tilde{z}^g_{ab}$ denotes a ghost transformation matrix which arises from the linear part of vector transformations and the matrix r^{oV}: Vector transformations consist of the linear part appearing in the ST identity r^g_{4b} and the linear part of the nonlinear vector transformations, $a'^g_{\alpha b} = z_W a^g_{\alpha b}, \alpha = +,-,3$:

$$
\tilde{z}^g_{ab} = \sum_{\substack{\alpha,\alpha'= \\ +,-,3}} (r^{oV})^{-1}_{a\alpha}\tilde{I}_{\alpha'\alpha} a'_{\alpha b} + (r^{oV})^{-1}_{a4} r^g_{4b} \tag{5.38}
$$

Evaluating the ST identity one finds, that the couplings defined in (5.37) have to satisfy the following equations:

- On the part containing the 4-dimensional vector polynomials $\hat{a}^o_{abc}$ is determined to be completely antisymmetric and is seen to be the solution of the Jacobi identity:

$$\hat{a}^o_{abc} = -\hat{a}^o_{bac} = \hat{a}^o_{bca} \tag{5.39}$$

$$\hat{a}^o_{abc}\tilde{I}_{aa'}\hat{a}^o_{b'a'c'} + \hat{a}^o_{acc'}\tilde{I}_{aa'}\hat{a}^o_{b'a'b} + \hat{a}^o_{ac'b}\tilde{I}_{aa'}\hat{a}^o_{b'a'c} = 0$$

- On the scalar-vector part, which contains the bare mass matrix as defined in (5.31), the matrix v^o_{ab} is determined in terms of the bare masses:

$$\mathcal{M}^o_{ab} = \tilde{I}_{a'b'}v^o_{a'a}v^o_{b'b} \Longrightarrow \begin{cases} |v^o_{+-}|^2 = M^o_W{}^2 \\ (v^o_{\chi Z})^2 = M^o_Z{}^2 \\ (v^o_{\chi A})^2 = 0 \end{cases} \tag{5.40}$$

Therefrom it follows that the mass of the photon has to vanish and is not an independent parameter of the theory:

$$M^o_A{}^2 = 0 \tag{5.41}$$

The matrices $t^o_{ab,c}$ have to be antisymmetric in the scalar indices

$$t^o_{ab,c} = -t^o_{ba,c} \tag{5.42}$$

and satisfy the following representation equations:

$$t^o_{ba,b'}\tilde{I}_{aa'}t^o_{a'c,c'} - t^o_{ba,c'}\tilde{I}_{aa'}t^o_{a'c,b'} = -\hat{a}^o_{b'c'a'}\tilde{I}_{a'a}t^o_{bc,a} \tag{5.43}$$

and

$$t^o_{ba,b'}\tilde{I}_{aa'}v^o_{a'c'} - t^o_{ba,c'}\tilde{I}_{aa'}v^o_{a'b'} = -\hat{a}^o_{b'c'a'}\tilde{I}_{a'a}v^o_{ba} \tag{5.44}$$

- In the fermion part of the action we find on the bare mass terms

$$\tilde{h}^{oi}_{ffA} = h^{oi}_{ffA} \tag{5.45}$$

for all massive fermions $f = e, u, d$. Because the kinetic terms of bare fields are normalized, one has furthermore

$$h^{oi}_{ff'+} = -h^{oi}_{f'f-} \qquad \begin{array}{rcll} f & = & \nu, & u \\ f' & = & e, & d \end{array} \tag{5.46}$$

In addition one gets the following representation equations for each family:

$$\begin{aligned} h^{oi}_{ff',b}h^{oi}_{f'f'',c} - h^{oi}_{ff',c}h^{oi}_{f'f'',b} &= -\hat{a}^o_{bca'}\tilde{I}_{aa'}h^{oi}_{ff'',a} \\ \tilde{h}^{oi}_{ff',b}\tilde{h}^{oi}_{f'f'',c} - \tilde{h}^{oi}_{ff',c}\tilde{h}^{oi}_{f'f'',b} &= -\hat{a}^o_{bca'}\tilde{I}_{a'a}\tilde{h}^{oi}_{ff'',a} \end{aligned} \tag{5.47}$$

It is straightforward to solve these equations: Due to (5.39) $\hat{a}^o_{abc}$ are qualified as structure constants of $SU(2) \times U(1)$ and are related to the structure constants $\hat{\varepsilon}_{\alpha\beta\gamma}$ by:

$$\hat{a}^o_{abc} = g^o_2 \hat{\varepsilon}_{\alpha\beta\gamma} O_{\alpha a}(\theta^o_W) O_{\beta b}(\theta^o_W) O_{\gamma c}(\theta^o_W) \tag{5.48}$$

θ^o_W and g^o_2 are at this stage two arbitrary parameters, which parameterize the two remaining parameters of the coupling matrix $\hat{a}^o_{abc}$. Therefore the representation equations (5.43) and (5.44) are equivalent to equs. (4.41) and (4.50) and their solutions can be read off from the solutions (4.51) and (4.44):

$$t^o_{bc,a} = g^o_2 \hat{t}_{bc,\alpha} O_{\alpha a}(\theta^o_W) \qquad v^o_{ba} = v^o_{b\alpha} O_{\alpha a}(\theta^o_W) \tag{5.49}$$

Here we have already used that antisymmetry (5.42) singles out from the equivalence class the antisymmetric solution. Most important are the solutions of the shift equations, which relate the remaining undetermined parameters to the masses of Z-boson and W-boson. Inserting the relations (5.40) into (4.51) determines G^ϕ, θ^o_W and v^o:

$$\cos\theta^o_W = \frac{M^o_W}{M^o_Z} \quad G^\phi = -\frac{\sin\theta^o_W}{\cos\theta^o_W} \quad v^o = \frac{2}{g^o_2} M^o_Z cos\theta^o_W \tag{5.50}$$

(The signs of $\cos\theta^o_W$ and v^o are chosen in accordance with the tree approximation.) Only the nonabelian coupling g^o_2 remains undetermined.

In the same way the representation equations of the fermion matrices (5.47) are equivalent to equs. (4.53) we have solved in equs. (4.55), (4.56) and (4.57). The relation (5.46) singles out the antisymmetric solution:

$$\begin{aligned} h^{o\ \delta_i}_+ &= ig^o_2 \frac{\tau_+}{2} \qquad h^{o\ \delta_i}_- = ig^o_2 \frac{\tau_-}{2} \\ h^{o\ \delta_i}_Z &= ig^o_2 (\cos\theta^o_W \frac{\tau_3}{2} + \sin\theta^o_W G^{\delta_i} \frac{\mathbf{1}}{2}) \\ h^{o\ \delta_i}_A &= ig^o_2 (-\sin\theta^o_W \frac{\tau_3}{2} + \cos\theta^o_W G^{\delta_i} \frac{\mathbf{1}}{2}) \end{aligned} \tag{5.51}$$

is the solution of the doublet representation and

$$\begin{aligned} \tilde{h}^{oi}_Z &= ig^o_2 \sin\theta^o_W \begin{pmatrix} iG^{e_i} & 0 & 0 \\ 0 & iG^{u_i} & 0 \\ 0 & 0 & iG^{d_i} \end{pmatrix} \\ \tilde{h}^{oi}_A &= ig^o_2 \cos\theta^o_W \begin{pmatrix} iG^{e_i} & 0 & 0 \\ 0 & iG^{u_i} & 0 \\ 0 & 0 & iG^{d_i} \end{pmatrix} \end{aligned} \tag{5.52}$$

is the solution of the singlet representation. There we have introduced a matrix notation as in section 3.4 ($\delta_i = l_i, q_i$). Although it is not relevant for perturbation theory, we want to mention, that on the mass terms doublet and singlet representations are distinguished. The ST identity is only solved, if we assign to one chirality the doublet representation and to the second the singlet representation. Inserting furthermore the relation (5.45), which relates the algebraically undetermined parameters of the abelian subgroup, yields immediately the relations (4.64). With (5.50) they read

$$\frac{1}{2}(-\tan\theta^o_W + G^{q_i}) = G^{u_i} \qquad \frac{1}{2}(\tan\theta^o_W + G^{q_i}) = G^{d_i} \qquad \frac{1}{2}(\tan\theta^o_W + G^{l_i}) = G^{e_i} \tag{5.53}$$

As expected we remain with one undetermined parameter for each lepton and quark family. Parameterizing G^{δ_i} by the electric charge, $Q_e = -1, Q_d = -\frac{1}{3}$, and an remainder g_{δ_i},

$$G^{q_i} = -\tan\theta^o_W(2Q_d + 1 + 2g_{q_i}) \qquad G^{l_i} = -\tan\theta^o_W(2Q_e + 1 + 2g_{l_i}) \tag{5.54}$$

it is seen that the free parameters correspond to coupling the Noether currents of lepton and quark family conservation to the photon.

Finally the angle Θ^o, which appears as a free parameter in the ST identity is determined as function of the bare vector mass ratio by inverting the relation between $\hat{a}^o_{abc}$ and $\hat{a}'_{\alpha bc}, \alpha = +, -, 3$:

$$\hat{a}^o_{Z+-} = -ig^o_2 \cos\theta^o_W = \frac{1}{r^o_Z} cos\Theta^o \hat{a}'_{3+-}\frac{1}{z^g_{++}} \tag{5.55}$$

$$\hat{a}_{A+-} = ig^o_2 \sin\theta^o_W = -\frac{1}{r^o_A} \sin\Theta^o \hat{a}'_{3+-}\frac{1}{z^g_{++}} \tag{5.56}$$

i.e.

$$\tan\Theta^o = \frac{r^o_A}{r^o_Z}\tan\theta^o_W \tag{5.57}$$

On the ST identity it is not possible to fix r^{oV}_A and r^{oV}_Z, but they are determined by using rigid symmetry.

Having solved the above equations the action of vectors, scalars and fermions and the external field part is determined from the ST identity

$$\mathcal{S}(\Gamma^{gen}_{GSW} + \Gamma^{gen}_{ext.f}) = 0 \tag{5.58}$$

without having specified the ghost redefinition matrices and without using rigid or local gauge symmetry. Ingredients are only nilpotency of the ST identity, the bare form of the bilinear action, which states especially that there are massive vector bosons, and the global symmetries as charge neutrality and lepton and quark family number conservation. Explicitly we find as solution of the ST identity in the vector, fermion and scalar sector the following general action expressed in terms of bare fields:

$$\Gamma^{gen}_{GSW}(V^o_a, \phi^o_a, f^o_i) = \Gamma_{YM}(V^o_a) + \Gamma_{scalar}(\phi^o_a, V^o_a) + \Gamma_{Yuk}(\phi^o_a, f^o_i) + \Gamma_{matter}(V^o_a, f^o_i) \tag{5.59}$$

with

$$
\begin{aligned}
\Gamma_{YM} &= -\frac{1}{4}\int G^{o\mu\nu}_{\alpha}\tilde{I}_{\alpha\alpha'}G^{o}_{\mu\nu\alpha'} \qquad (5.60)\\
\Gamma_{scalar} &= \int\Big((D^\mu\Phi^o)^\dagger D_\mu\Phi^o - g_2^o\frac{1}{4}\frac{m_H^{o2}}{M_W^{o2}}(\Phi^{o\dagger}\Phi^o + 4\frac{M_W^o}{g_2^o}H^o)^2\Big)\\
\Gamma_{matter} &= \sum_{i=1}^{N_F}\int\Big(\overline{F^{oL}_{l_i}}i\not{D}F^{oL}_{l_i} + \overline{F^{oL}_{q_i}}i\not{D}F^{oL}_{q_i} + \overline{f^{oR}_i}i\not{D}f^{oR}_i\Big)\\
\Gamma_{Yuk} &= -\sum_{i}^{N_F}\int\Big(m^o_{f_i}\bar{f}^o_i f^o_i + \frac{g^o_2}{\sqrt{2}M^o_W}(m^o_{e_i}\overline{F^{oL}_{l_i}}\Phi^o e^{oR}_i\\
&\qquad + m^o_{u_i}\overline{F^{oL}_{q_i}}\Phi^o u^{oR}_i + m^o_{d_i}\overline{F^{oL}_{q_i}}\tilde{\Phi}^o d^{oR}_i + \text{h.c.})\Big)
\end{aligned}
$$

For notational convenience we have rewritten the 4-vector of scalars ϕ^o_a into the complex scalar doublet Φ^o and $\tilde{\Phi}^o$. The structure of the individual ST-invariant terms is the same as in the tree approximation. Therefore it is seen that the information on the $SU(2)\times U(1)$ algebra is completely transfered to the ST identity. Because the bare form of the action has been fixed, the covariant derivatives are immediately computed as functions of M^o_W and M^o_Z. The weak mixing angle in its bare form θ^o_W is defined by the bare vector mass ratio

$$\cos\theta^o_W = \frac{M^o_W}{M^o_Z} \qquad (5.61)$$

and is not an independent parameter of the theory (cf. (5.50)). From the above construction it is obvious, that the broken theory is considered and characterized in its own right and one never refers to the underlying symmetric theory.

$$
\begin{aligned}
G^{o\mu\nu}_{\alpha} &= O_{\alpha a}(\theta^o_W)(\partial^\mu V^{o\nu}_a - \partial^\nu V^{o\mu}_a) + g^o_2\tilde{I}_{\alpha\alpha'}\hat{\varepsilon}_{\alpha\beta\gamma}O_{\beta b}(\theta^o_W)O_{\gamma c}(\theta^o_W)V^{o\mu}_b V^{o\nu}_c \qquad (5.62)\\
D_\mu\Phi^o &= \partial_\mu\Phi^o - ig^o_2\Big(\frac{\tau_\alpha}{2}O_{\alpha a}(\theta^o_W) - \tan\theta^o_W O_{4a}(\theta^o_W)\Big)V^o_{\mu a}(\Phi^o + \frac{\sqrt{2}}{g^o_2}\begin{pmatrix}0\\ M^o_W\end{pmatrix})\\
D^\mu F^{oL}_{\delta_i} &= \Big(\partial^\mu - ig^o_2(\frac{\tau_\alpha}{2}O_{\alpha a}(\theta^o_W)V^{o\mu}_a + \frac{G^{\delta_i}}{2}O_{4a}(\theta^o_W)V^{o\mu}_a)\Big)F^{oL}_{\delta_i} \qquad \delta = l,q\\
D^\mu f^{oR}_i &= (\partial^\mu + ig^o_2\frac{1}{2}(\tan\theta^o_W + G^{\delta_i}))f^R_i O_{4a}(\theta^o_W)V^{o\mu}_a \qquad f_i = e_i, d_i\\
D^\mu f^{oR}_i &= (\partial^\mu + ig^o_2\frac{1}{2}(-\tan\theta^o_W + G^{q_i}))f^{oR}_i O_{4a}(\theta^o_W)V^{o\mu}_a \qquad f_i = u_i
\end{aligned}
$$

The external field part depends on the ghost redefinition matrices $a'^g_{\alpha b}$ and r^g_{4b}. In accordance with rigid symmetry (5.23) we introduce the following notation

$$a'^g_{\alpha b} = \hat{z}^g_W\tilde{I}_{\alpha\beta}r^g_{\beta b} \qquad (5.63)$$

These parameters will be finally fixed in the ghost sector on the bilinear parts of the ghosts:

$$\Gamma^{gen}_{ext.f.} = \int\Big(-\hat{z}^g_W\frac{g^o_2}{2}\sigma_\alpha\hat{\varepsilon}_{\alpha\beta\gamma}r^g_{\beta b}c_b r^g_{\gamma c}c_c$$

$$
\begin{aligned}
&+\rho^o_{\mu\alpha}\hat{z}^g_W(\tilde{I}_{\alpha\beta}\partial^\mu r^g_{\beta b}c_b + g^o_2\hat{\varepsilon}_{\alpha\beta\gamma}O_{\beta b}(\theta^o_W)V^{o\mu}_b r^g_{\gamma c}c_c)\\
&+g^o_2\Big(Y^{o\dagger}(i\frac{\tau_\alpha}{2}\hat{z}^g_W r^g_{\alpha a} - i\frac{\mathbf{1}}{2}\tan\theta^o_W r^g_{4a}))c_a(\Phi^o + \frac{\sqrt{2}}{g^o_2}\begin{pmatrix}0\\ M^o_W\end{pmatrix}) + \text{h.c.}\Big) \qquad (5.64)\\
&+\sum_{i=1}^{N_F}\Big(\sum_{\delta=l,q}\bar{\Psi}^{oR}_{\delta_i} ig^o_2(\frac{\tau_\alpha}{2}\hat{z}^g_W r^g_{\alpha a} + G^{\delta_i}\frac{\mathbf{1}}{2}r^g_{4a})c_a F^{oL}_{\delta_i}\\
&\qquad + \bar{\psi}^{oL}_{e_i} ig^o_2\frac{1}{2}(\tan\theta^o_W + G^{l_i})e^{oR}_i r^g_{4a}c_a\\
&\qquad + \bar{\psi}^{oL}_{d_i} ig^o_2\frac{1}{2}(\tan\theta^o_W + G^{q_i})d^{oR}_i r^g_{4a}c_a\\
&\qquad + \bar{\psi}^{oL}_{u_i} ig^o_2\frac{1}{2}(-\tan\theta^o_W + G^{q_i})u^{oR}_i r^g_{4a}c_a + \text{h.c.}\Big)\Big)
\end{aligned}
$$

The general form of the ST-invariant action is obtained by transforming the bare field into the original fields according to (5.27). The parameters of the bilinear action remain arbitrary as we did not to have to dispose on them when solving the ST identity. Only the bare mass of the photon is determined as zero from the ST identity and is not an independent parameter of the theory.

Besides the ghost redefinition matrix $r^g_{\alpha b}$ (5.38) there remain undetermined the non-abelian coupling g^o_2 and the fermion couplings G^{l_i} and G^{q_i}. In order to embed the structure of quantum electrodynamics into the standard model they have to be fixed on the local abelian Ward identity as given in (4.66) and (4.67), remaining with one free parameter g^o_2, which can be finally adjusted to the fine structure constant in the Thompson limit. For this reason the Ward identities of rigid invariance have to be established.

The solution of the ST identity Γ^{gen}_{GSW} in the vector, scalar and fermion sector as given above is immediately seen to be invariant under rigid symmetry. Applying the Ward operators $\mathcal{W}_\alpha$ (5.20) on Γ^{gen}_{GSW} determines uniquely the matrices of equivalence transformations:

$$
r^{oV} = O(\theta^o_W) \qquad r^{oS} = \mathbf{1} \qquad r^{o\delta_i} = \mathbf{1} \tag{5.65}
$$

For the parameter v^o, which appears as a free parameter in the Ward operators, one gets

$$
v^o = \frac{2M^o_W}{g^o_2} \tag{5.66}
$$

Inverting the relations (5.35) finally yields r^S, r^V, r^{δ_i} as functions of masses and field redefinitions:

$$
\begin{aligned}
r_Z &= \frac{z_Z}{z_W}\cos(\theta^o_W + \theta_Z)\sqrt{1 + \tan(\theta^o_W + \theta_Z)\tan(\theta^o_W + \theta_A)}\\
r_A &= \frac{z_A}{z_W}\sin(\theta^o_W + \theta_A)\sqrt{1 + \cot(\theta^o_W + \theta_Z)\cot(\theta^o_W + \theta_A)}
\end{aligned}
$$

$$\tan\Theta = \sqrt{\tan(\theta_W^o+\theta_Z)\tan(\theta_W^o+\theta_A)} \tag{5.67}$$

and

$$r_a^S = \frac{1}{z_H} z_a^S \qquad r_{l_i} = \frac{z^{e_i}}{z_{\nu_i}} \qquad r_{q_i} = \frac{z^{d_i}}{z_{u_i}} \tag{5.68}$$

The shift parameter v as defined in (4.51) is determined on the general classical action to

$$v = \frac{2M_W^o}{g_2^o z_H} \tag{5.69}$$

The Ward identities of rigid symmetry hold then without further restrictions on Γ_{GSW}^{gen} and $\Gamma_{ext.f}^{gen}$.

One is now ready to apply the local Ward operator $\mathbf{w}^Q$ (4.67). Requiring the Ward identity (4.69) to be valid on the $\Gamma_{GSW}^{gen} + \Gamma_{ext.f}^{gen}$

$$\left(g_1 \mathbf{w}_4^Q - \frac{1}{r_Z^V}\partial\frac{\delta}{\delta Z}\sin\Theta^V - \frac{1}{r_A^V}\partial\frac{\delta}{\delta A}\cos\Theta^V\right)(\Gamma_{GSW}^{gen} + \Gamma_{ext.f}^{gen}) = 0 \tag{5.70}$$

determines G^{l_i} and G^{q_i} as functions of the electric charge of the charged leptons ($Q_e = -1$) and down-type quarks ($Q_d = -\frac{1}{3}$):

$$G^{q_i} = -\tan\theta_W^o(2Q_d+1) \qquad G^{l_i} = -\tan\theta_W^o(2Q_e+1) \tag{5.71}$$

The overall normalization constant g_1 is determined on the general classical action as function of the nonabelian coupling g_2^o, the wave function renormalization and the bare masses:

$$g_1 = z_W g_2^o \tan\theta_W^o \sqrt{\frac{\tan(\theta_W^o+\theta_A)}{\tan(\theta_W^o+\theta_Z)}} \tag{5.72}$$

After having applied the local Ward identity there remains only one coupling g_2^o, which is not fixed on the mass terms and by symmetries. In a QED-like parametrization this coupling is determined by the fine structure constant, which measures the interaction strength of the photon to the electromagnetic current in the Thompson limit:

$$\bar u(p)\Gamma_{eeA_\mu}(p,p,0)u(p)\big|_{p^2=m_e^2} = ie\bar u(p)\gamma_\mu u(p) \tag{5.73}$$

In the tree approximation this relation yields

$$g_2^o = \frac{e}{\sin\theta_W} + O(\hbar) \tag{5.74}$$

Respectively one can parameterize the bare coupling g_2^o by the electromagnetic bare coupling e^o and the bare mass ratio of W- and Z-boson

$$g_2^o = \frac{e^o}{\sin\theta_W^o} \qquad \text{with} \qquad \text{with} \quad e^o = e + \delta e \tag{5.75}$$

Transforming the bare fields of the general action back to original fields and expanding the free parameters in perturbation theory determines the symmetric local contributions $\Gamma^{(n)}_{inv}$ (3.5), which are in agreement with the ST identity, rigid symmetry and the local Ward identity, as specified by algebra and nilpotency. Since some combinations of parameters (5.67) explicitly enter as parameters the symmetry operators, also the explicit form of the ST identity and the Ward identities is modified in higher orders of perturbation theory.

In concrete calculation it is wideley used that dimensional regularization is an invariant scheme if parity is conserved. Under such circumstances only the symmetric counterterms appearing in the general ST-invariant action would appear as counterterms in the Γ_{eff}, which governs the calculation of Green functions in the Gell-Mann Low formula. The Feynman rules of the vector, scalar and fermion part derived from such a symmetric Γ_{eff} are listed for example in [37] and have exploited in constructing ST-invariant 1-loop Green functions in the physical sector. Furthermore, if one had an invariant scheme, the parameters of the ST identity and the Ward identity would be related to the respective counterterms in renormalized perturbation theory as derived in (5.67), (5.68) and (5.72). Of course a necessary prerequisite of exploiting such relations is the complete construction of the ST identity, the rigid symmetries and the local abelian gauge Ward identity order by order in perturbation theory for all Green functions involved, especially also for the ones of the ghost sector. In the next sections we outline the construction of the gauge fixing and ghost sector in the classical approximation and in higher orders, taking care in preserving the Ward identities of rigid and local gauge invariance.

5.4. The gauge fixing and ghost sector

Classically the ghost sector has been completely determined as BRS-variation of the gauge fixing function (cf. (2.69)). The respective relation is immediately derived for the generating functional of 1PI Green functions by differentiating the ST identity (5.19) with respect to the B-fields:

$$s_\Gamma\left(\frac{\delta\Gamma}{\delta B_a}\right) = -(r^V)^{-1}_{a\alpha}\delta\hat{g}_{\alpha b}\frac{\delta\Gamma}{\delta\bar{c}_b} \qquad (5.76)$$

In the linear gauges, moreover, $\frac{\delta\Gamma}{\delta B_a}$ is local (4.37), because there are no vertices which could constitute loop diagrams with external B-legs. Then eq. (5.76) yields the linear ghost equations, which have to be established to all orders of perturbation theory.

In the tree approximation we have constructed the gauge fixing sector to be invariant under Ward identities of rigid symmetry by introducing the external scalar fields $\hat{\phi}_a$ (see (2.51) – (2.57)). As the abelian fermion couplings are not well determined from the ST identity, Ward identities of rigid symmetries have to be maintained for the generating functional of Green functions. For this reason we have to choose the gauge fixing sector as being invariant under rigid transformations as specified in the vector and scalar part of the action. To the external scalars we assign the transformation behaviour of the propagating scalars and derive for the most general rigid invariant gauge-fixing sector the following expression:

$$\begin{aligned}\Gamma^B &= \int\Big(\frac{1}{2}\xi B_a (r^V)^{-1}_{a\alpha}\tilde{I}_{\alpha\beta}(r^V)^{-1T}_{\beta b}B_b + \frac{1}{2}\hat{\xi}B_a(r^V)^{-1}_{a4}(r^V)^{-1T}_{4b}B_b + B_a\tilde{I}_{ab}\partial^\mu V_{\mu b} \\ &+ g_\xi\Big(\sum_{\substack{\alpha=\\+,-,3}} B_a(r^V)^{-1}_{a\alpha}(r^S_b\phi_b + \delta_{Hb}v)\hat{t}_{bc,\alpha}(r^S_c\hat{\phi}_c + \delta_{Hc}\hat{\zeta}v) \\ &+ \hat{G}B_a(r^V)^{-1}_{a4}(r^S_b\phi_b + \delta_{Hb}v)\hat{t}_{bc,4}(r^S_c\hat{\phi}_c + \delta_{Hc}\hat{\zeta}v)\Big)\Big)\end{aligned} \tag{5.77}$$

The matrix $r^V_{\alpha a}$ is the three parameter matrix, which parameterizes the rigid transformations of vectors (5.21) and r^S_b are the three parameters of the scalar rigid transformations (5.22), v denotes the shift parameter of the scalar field as it appears as parameter in the Ward operators. These parameters are determined by the normalization conditions on the vector and scalar 2-point functions and one cannot dispose of them in the gauge fixing sector anymore. Γ^B as given in (5.77) holds for the B-dependent part of the generating functional of 1PI Green functions in linear gauges to all orders. The free parameters of the gauge fixing sector are

$$\xi,\ \hat{\xi},\ \hat{\zeta},\ \hat{G},\ g_\xi \tag{5.78}$$

Finally a further free parameter is the overall normalization of the external scalar field $\hat{\phi}_a$, which can be used to fix the parameter g_ξ at will. In QED-like parameterizations it is convenient to adjust

$$g_\xi = \frac{e}{\sin\theta_W} \tag{5.79}$$

(5.77) yields the linear ghost equations, which are valid in this form to all orders (see [3] for details) ($\alpha = +,-,3$):

$$\begin{aligned}\partial^\mu\frac{\delta\Gamma}{\delta\rho^\mu_\alpha} + g_\xi\frac{\delta\Gamma}{\delta Y'_b}r^S_{b'}\tilde{I}_{b'b}\hat{t}_{bc,\alpha}(r^S_c\hat{\phi}_c + \delta_{Hc}\hat{\zeta}v)& \\ +(r^S_b\phi_b + \delta_{Hb}v)\hat{t}_{bc,\alpha}r^S_c\hat{q}_c &= -\delta\hat{g}_{\alpha b}\frac{\delta}{\delta\bar{c}_b} \\ \Box(r^g_{4Z}c_Z + r^g_{4A}c_A) + g_\xi\hat{G}\frac{\delta\Gamma}{\delta Y'_b}r^S_{b'}\tilde{I}_{b'b}\hat{t}_{bc,4}(r^S_c\hat{\phi}_c + \delta_{Hc}\hat{\zeta}v)& \\ +(r^S_b\phi_b + \delta_{Hb}v)\hat{t}_{bc,4}r^S_c\hat{q}_c &= -\delta\hat{g}_{4b}\frac{\delta}{\delta\bar{c}_b}\end{aligned} \tag{5.80}$$

Using rigid symmetery the ghost equations are immediately integrated yielding that the generating functional of 1PI Green functions depends on specific combinations between the external fields ρ^μ_α and Y_a and antighosts, whereas the remaining contributions are local. Splitting off also the local B-dependent part the generating functional of 1PI Green functions can be decomposed in the following way:

$$\begin{aligned}
&\Gamma(V^\mu_a, B_a, c_a, \phi_a, \hat{\phi}_a, \sigma_\alpha, \rho^\mu_\alpha, Y_a, \bar{c}_a, f_i, \psi_{f_i}) \\
&= \Gamma^{nl}(V^\mu_a, c_a, \phi_a, \hat{\phi}_a, \sigma_\alpha, \rho'^\mu_\alpha, Y'_a, f_i, \psi_{f_i}) + \Gamma^B(B_a, V^\mu_a, \phi_a, \hat{\phi}_a) \\
&-\bar{c}_a(\delta\hat{g})^{-1}_{a4} r^g_{4b} \Box c_b - g_\xi \bar{c}_a \Big(\sum_{\substack{\alpha= \\ +,-,3}} (\delta\hat{g})^{-1}_{a\alpha} \hat{t}_{bc,\alpha} + (\delta\hat{g})^{-1}_{a4} \hat{G} \hat{t}_{bc,4} \Big)(\phi_b + \delta_{Hb} v)\hat{q}_c
\end{aligned} \tag{5.81}$$

with

$$\begin{aligned}
\rho'^\mu_\alpha &= \rho^\mu_\alpha + \partial^\mu \bar{c}_a (\delta\hat{g})^{-1}_{a\alpha} \\
Y'_b &= Y_b - \bar{c}_a g_\xi \tilde{I}_{bb'} \Big(\sum_{\substack{\alpha= \\ +,-,3}} (\delta\hat{g})^{-1}_{a\alpha} \hat{t}_{b'c,\alpha} + \hat{G}(\delta\hat{g})^{-1}_{a4} \hat{t}_{b'c,4} \Big)(\hat{\phi}_c + \delta_{Hc} \hat{\zeta} v)
\end{aligned} \tag{5.82}$$

For simplification we have absorbed the irrelevant scalar redefinitions, i.e. $r^S_b = 1$. All non-local contributions are contained in the part Γ^{nl}. The proof that all breakings of the ST identity can be absorbed by adjusting local contributions, can be finally restricted to this functional (see section 7). The solution of the ghost equations is quite trivial in a massless symmetric gauge. This is not the case for the standard model where the ghosts are massive. There one not only has to solve the ghost equation, but also one has to show, that the on-shell normalization conditions for the ghost 2-point functions and in particular the infrared conditions

$$\Gamma_{\bar{c}_A c_Z}(p^2 = 0) = \Gamma_{\bar{c}_Z c_A}(p^2 = 0) = \Gamma_{\bar{c}_A c_A}(p^2 = 0) = 0 \tag{5.83}$$

can be fulfilled by adjusting the parameters $\hat{\zeta}, \hat{G}$ and the BRS-transformation matrix $\delta\hat{g}_{\alpha b}$

5.4.1. The classical approximation

First we solve the ghost equations in the classical approximation, taking into account that we impose normalization conditions on the ghost 2-point functions as specified in (5.10), (5.11) and (5.12). The bilinear part of the ghost action is therefore fixed (see (5.13)) and all parameters have to be determined as functions of Z^g_{ab} and $\mathcal{M}^g_{ab}$, and of the free parameters of the vector and scalar part of the action. We proceed therefore as in the vector sector and define on the bilinear part bare ghosts by the following field redefinitions

$$c^o_a = z^g_{ab} c_b \qquad \bar{c}^o_a = \bar{z}^g_{ab} \bar{c}_b \tag{5.84}$$

Inserting these bare fields in the bilinear part of the ghost action and requiring the bare action to have a standard form determines z^g and $\bar{z}^g$ up to a diagonal matrix:

$$\Gamma^{gen}_{bil,ghost} = -\int (\bar{c}_a Z^g_{ab} \Box c_b + \bar{c}_a \mathcal{M}^g_{ab} c_b)$$

$$
\begin{aligned}
&= -\int (\bar{c}^o_a (\bar{z}^g)^{-1T}_{aa'} Z^g_{a'b'} \Box (z^g)^{-1}_{b'b} c^o_b + \bar{c}^o_a (\bar{z}^g)^{-1T}_{aa'} \mathcal{M}^g_{a'b'} (z^g)^{-1}_{b'b} c^o_b) \\
&\stackrel{!}{=} -\int (\bar{c}^o_a \tilde{I}_{ab} \Box c^o_b + \bar{c}^o_a \mathcal{M}^{og}_{ab} c^o_b)
\end{aligned} \tag{5.85}
$$

Explicitly one gets a relation between the field redefinitions of ghosts and antighosts on the kinetic part:

$$
\tilde{I}_{ab'} (z^g)_{b'b} = (\bar{z}^g)^{-1T}_{aa'} Z^g_{a'b} \tag{5.86}
$$

The remaining undetermined matrix can be used to fix the ghost mass matrix as being diagonal:

$$
\mathcal{M}^{og}_{ab} = \begin{pmatrix} 0 & \zeta^o_W M^{o2}_W & 0 & 0 \\ \zeta^o_W M^{o2}_W & 0 & 0 & 0 \\ 0 & 0 & \zeta^o_Z M^{o2}_Z & 0 \\ 0 & 0 & 0 & M^{o2}_{gA} \end{pmatrix} \tag{5.87}
$$

and

$$
\mathcal{M}^{og}_{ab} = (\bar{z}^g)^{-1T}_{aa'} \mathcal{M}^g_{a'c} \tilde{I}_{cc'} (Z^g)^{-1}_{c'b'} (\bar{z}^g)^T_{b'b} \tag{5.88}
$$

We have parameterized the W and Z-ghost mass by the masses of the W- and Z-boson and independent parameters ζ^o_W and ζ^o_Z. In contrast to the vector mass matrix the ghost mass matrix is not required to be symmetric in the neutral components. Indeed considering the 1-loop diagrams it is seen, that the 2-point function $\Gamma_{\bar{c}_Z c_A}$ and $\Gamma_{\bar{c}_A c_Z}$ get different loop corrections, because interactions of scalars and ghosts are unsymmetric in Z-ghosts and A-ghosts in the tree approximation (cf. (2.75)). For diagonalizing an arbitrary matrix the equivalence transformation has to be carried out by an invertible matrix. Therefore diagonalization of the ghost mass matrix determines the wave functions renormalization of anti-ghosts $\bar{z}^g_{ab}$ up to a diagonal matrix. Taking for the BRS-transformation matrix $\delta\hat{g}_{\alpha b}$ the ansatz

$$
\delta\hat{g}_{\alpha b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\gamma_Z & -\sin\gamma_A \\ 0 & 0 & \sin\gamma_Z & \cos\gamma_A \end{pmatrix} \tag{5.89}
$$

it is possible to diagonalize the ghost matrix by adjusting the arbitrary angles γ_Z and γ_A, and at the same time the three undetermined parameters of the diagonal matrix are fixed. We want to mention that in massless nonabelian gauge theories the antighost field redefinitions are not determined on the bilinear part of the action and can therefore be completely fixed by the ST identity.

In the classical approximation the external field vertices $\delta\Gamma^{gen}_{cl}/\delta\rho^\mu_\alpha$ and $\delta\Gamma^{gen}_{cl}/\delta Y_a$ are local field polynomials. They have been determined in the last section as functions of vector field redefinitions z^V_{ab}, of scalar field redefinitions z^S_a and of the matrix $r^g_{\alpha b}$ and $\hat{z}^g_W$

(5.63). One can either take the explicit form of $\Gamma^{gen}_{ext.f.}$ or better one goes back to the bare form of the action as given in (5.64). For proceeding with the bare form, we have also to transform the scalars and vectors in the gauge fixing part to bare fields. The parameters, which appear by carrying out this transformation, are absorbed into a redefinition of B-fields, into the overall redefinition of the external scalar fields and into a redefinition of the arbitrary parameters into a bare form. One has to note that in the classical approximation the matrices $r^V_{\alpha b}$ and r^S_a are determined as functions of field redefinitions and bare masses (cf. (5.67) and (5.68)), especially one has in the classical approximation

$$
\begin{aligned}
(r^V)^{-1}_{a\alpha} O_{\alpha b'}(\theta^o_W)(z^V)_{b'b} &= z_W \delta_{ab} + z_W \Big(\sqrt{\tfrac{\tan(\theta^o_W+\theta_Z)}{\tan(\theta^o_W+\theta_A)}} - 1 \Big) \delta_{a4}\delta_{b4} && (5.90)\\
(r^S)^{-1}_{a\alpha}(z^S)_{\alpha b} &= z_H \delta_{ab} && (5.91)
\end{aligned}
$$

The bare form of the gauge fixing is then given by

$$
\begin{aligned}
\Gamma^{gen}_{g.f.} = \int \Big(& \frac{1}{2}\xi^o B^o_a \tilde{I}_{ab} B^o_b + \frac{1}{2}\hat{\xi}^o (\sin\theta^o_W B^o_Z + \cos\theta^o_W B^o_A)^2 + B^o_a \tilde{I}_{ab} \partial^\mu V^o_{\mu b} \qquad (5.92)\\
& + g_\xi \Big(\sum_{\substack{\alpha= \\ +,-,3}} B^o_a O^T_{a\alpha}(\theta^o_W)(\phi^o_b + \delta_{Hb} v^o) \hat{t}_{bc,\alpha} (\hat{\phi}_c + \delta_{Hc} \hat{\zeta}^o v^o) \\
& + \hat{G}^o B^o_a O^T_{a4}(\theta^o_W)(\phi^o_b + \delta_{Hb} v^o) \hat{t}_{bc,4} (\hat{\phi}^o_c + \delta_{Hc} \hat{\zeta}^o v^o) \Big) \Big)
\end{aligned}
$$

with the bare fields

$$
B^o_a = (z^V)^{-1T}_{ab} B_b \qquad \hat{\phi}^o_a = r^S_a \frac{z_W}{z_H} \hat{\phi}_a \qquad (5.93)
$$

and bare parameters

$$
\begin{aligned}
\xi^o &= z_W^2 \xi && (5.94)\\
\hat{\xi}^o &= z_W^2 \Big(\frac{\tan(\theta^o_W + \theta_Z)}{\tan(\theta^o_W + \theta_A)} - 1 \Big) \xi + z_W^2 \frac{\tan(\theta^o_W + \theta_Z)}{\tan(\theta^o_W + \theta_A)} \hat{\xi} \\
\hat{\zeta}^o &= \frac{z_W}{z_H^2} \hat{\zeta} \\
\hat{G}^o &= \sqrt{\frac{\tan(\theta^o_W + \theta_Z)}{\tan(\theta^o_W + \theta_A)}} \hat{G}
\end{aligned}
$$

One has finally to transform the ghost fields and B-fields in the ST identity and in the external field part (5.64) to bare fields, absorbing the field redefinition parameters into a redefinition of the by now undetermined parameters $r^g_{\alpha b}$ and $\hat{z}^g_W$ (5.63) and γ_Z, γ_A. The bare transformation matrix $\delta\hat{g}_{\alpha b}$, which depends on the bare angles γ^o_Z and γ^o_A as defined in (5.89), is computed via

$$
O^T_{a\alpha}(\theta^o_W) \delta\hat{g}^o_{\alpha b} = \Big((z^V)\, O(\theta^o_W)\, (r^V)^{-1} \Big)_{a\beta} \delta\hat{g}_{\beta b'} (\bar{z}^g)^T_{b'b} \qquad (5.95)
$$

This equation determines also those three parameters of the antighost field redefinition matrix $(\bar{z}^g)_{ab}$, which are not specified on the bilinear ghost part of the action.

Having transformed the general bilinear ghost action into its standard form the ghost equations are solved quite simply. On the kinetic parts the matrix $a^{og}_{\alpha b}$ is related to the angles of antighost transformations:

$$\hat{z}^{og}_W = 1$$

$$\begin{array}{rclcrcl} r^{og}_{3Z} & = & \cos\gamma^o_Z & \qquad & r^{og}_{3A} & = & -\sin\gamma^o_A \\ r^{og}_{4Z} & = & \sin\gamma^o_Z & & r^{og}_{4A} & = & \cos\gamma^o_Z \end{array} \tag{5.96}$$

The parameter $\hat{\zeta}^o$ of the gauge fixing part is determined from the mass ratio of W-boson and W-ghost:

$$\frac{g_\xi}{g^o_2}\hat{\zeta}^o = \zeta^o_W \tag{5.97}$$

Inserting this result yields on the neutral part the following equations:

$$\begin{aligned} \zeta^o_W M^{o2}_W \cos(\gamma^o_Z - \theta^0_W) &= \cos\theta^o_W \cos\gamma^o_Z \zeta^o_Z M^{o2}_Z \\ \zeta^o_W M^{o2}_W \sin(\gamma^o_A - \theta^0_W) &= \cos\theta^o_W \sin\gamma^o_A M^{o2}_{gA} \\ -\hat{G}^o \zeta^o_W M^{o2}_W \cos(\gamma^o_Z - \theta^0_W) &= \cos\theta^o_W \sin\gamma^o_Z \zeta^o_Z M^{o2}_Z \\ \hat{G}^o \zeta^o_W M^{o2}_W \sin(\gamma^o_A - \theta^0_W) &= \cos\theta^o_W \cos\gamma^o_A M^{o2}_{gA} \end{aligned} \tag{5.98}$$

They determine the BRS-transformations of antighosts, i.e. γ^o_Z, γ^o_A, and the abelian parameter of the gauge fixing part $\hat{G}^o$ as functions of the vector boson mass ratio and the ghost mass ratio. The mass of the photon ghost is seen to be not a free parameter of the model but has to vanish ($\cos\theta^o_W \equiv M^o_W/M^o_Z$):

$$\frac{\zeta^o_W M^{o2}_W}{\zeta^o_Z M^{o2}_Z} = \frac{\cos\gamma^o_Z \cos\theta^o_W}{\cos(\gamma^o_Z - \theta^o_W)} \qquad M^o_{gA} = 0 \tag{5.99}$$

$$\hat{G}^o = -\tan\gamma^o_Z \qquad \tan\gamma^o_A = \tan\theta^o_W$$

The whole point in this calculation is the adjustment of the abelian coupling $\hat{G}$ via the mass of the Z-ghost. For arbitrary $\hat{G}$ indeed one has to introduce the angle γ_Z into the BRS-transformations of ghosts, as otherwise one is not able to keep the normalization condition

$$\Gamma_{\bar{c}_A c_Z}\Big|_{p^2=0} = 0 \tag{5.100}$$

which is crucial for infrared finite computations for off-shell Green functions. In the tree approximation, of course it is possible to fix the ghost mass ratio equal to the vector mass ratio by the normalization condition:

$$\mathrm{Re}\Gamma_{\bar{c}_+ c_-}(p^2)|_{p^2=\zeta M^2_W} = 0 \quad \mathrm{Re}\Gamma_{\bar{c}_Z c_Z}(p^2)|_{p^2=\zeta M^2_Z} = 0 \tag{5.101}$$

Then the expressions (5.99) simplify to the ansatz we have taken in the classical approximation:

$$\cos\gamma_Z = \frac{M_W}{M_Z} + O(\hbar) \tag{5.102}$$

and the BRS-transformation of antighosts is diagonal. For higher orders, however, the normalization conditions (5.101) together with the infrared condition (5.100) does not imply a diagonal transformation matrix for antighosts. Conversely requiring (5.101) and a diagonal ghost transformation one has to introduce then counterterms $\bar{c}_A c_Z$ in order to fulfil the ST identity. These counterterms produce in the next order off-shell infrared divergencies. That the ghost mass ratio and the parameter $\hat{G}$ are indeed independent parameters of the standard model, is already indicated by the computations we have carried out in the classical approximation: Starting with the general bilinear ghost action it is seen, that there is no parameter left, which could adjust the bare ghost mass ratio to the bare vector mass ratio. Likewise when we transformed the gauge fixing part to the bare form, we had to treat the parameter $\hat{G}$ as an independent parameter of the theory. The Callan-Symanzik equation, we derive in the section 4, unambiguously allows to determine the independent parameters of the model. There it is finally proven, that the ghost mass ratio is a further independent parameter of the theory. The coupling $\hat{G}$ and γ_Z are then determined order by order by normalization conditions, which fix the mass of the Z-ghost and diagonalize the neutral ghost mass matrix at $p^2 = 0$. Taking therefore the ghost mass ratio as arbitrary also in the tree approximation as specified in (5.10) we find

$$\begin{aligned}
\hat{G} &= \tan\theta_W \frac{1 - \frac{\zeta_W}{\zeta_Z}\cos^2\theta_W}{\frac{\zeta_W}{\zeta_Z}(1-\cos^2\theta_W)} + O(\hbar) \\
&= \frac{\zeta_Z M_Z^2 - \zeta_W M_W^2}{\zeta_W M_W \sqrt{M_Z^2 - M_W^2}} + O(\hbar)
\end{aligned} \tag{5.103}$$

This independent parameter does not only enter the gauge fixing part but also enters the ST identity and the ghost interactions, and has, wherever it appears, to be differently treated from the vector mass ratio, even if we fix the ghost mass ratio to be the same as the vector mass ratio by the normalization conditions (5.101). The ghost interactions in the classical approximation are immediately read off from (5.81):

$$\begin{aligned}
&\Gamma^{gen}_{ext.f.}(\rho^o_\alpha, Y^o_a) + \Gamma^{gen}_{ghost}(\bar{c}^o_a) = \Gamma^{gen}_{ext.f}(\rho'^o_\alpha, Y'^o_a) - \bar{c}^o_a(\delta\hat{g}^o_{a4})^{-1}(\delta\hat{g}^o)_{4b}\Box c^o_b \\
&-g_\xi \bar{c}^o_a\Big(\sum_{\substack{\alpha=\\+,-,3}} (\delta\hat{g}^o)^{-1}_{a\alpha}\hat{t}_{bc,\alpha} - \tan\theta^o_G(\delta\hat{g}^o)^{-1}_{a4}\hat{t}_{bc,4}\Big)(\phi^o_b + \delta_{Hb}2\tfrac{M^o_W}{g^o_2})\hat{q}^o_c
\end{aligned} \tag{5.104}$$

with

$$\rho'^o_\alpha = \rho^o_\alpha + \partial\bar{c}^o_a(\delta\hat{g}^o)^{-1}_{a\alpha} \tag{5.105}$$

$$Y_b'^o = Y_b^o - \bar{c}_a^o g_\xi \tilde{I}_{bb'}\Big(\sum_{\substack{\alpha=\\+,-,3}} (\delta\hat{g}^o)^{-1}_{a\alpha}\hat{t}_{b'c,\alpha} - \tan\theta_G^o(\delta\hat{g}^o)^{-1}_{a4}\hat{t}_{b'c,4}\Big)(\hat{\phi}_c^o + \delta_{Hc}\zeta 2\frac{M_W^o}{g_2^o})$$

The interactions of the external fields with propagating fields are summarized in $\Gamma^{gen}_{ext.f}$ (5.64). It is not modified by solving the ghost equations, but for bare ghosts the arbitrary parameters therein are now specified to be related to $\delta\hat{g}^o$:

$$\delta\hat{g}^o_{\alpha b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_G^o & -\sin\theta_W^o \\ 0 & 0 & \sin\theta_G^o & \cos\theta_W^o \end{pmatrix} \tag{5.106}$$

For having simple notations we have introduced the ghost angle θ_G^o in analogy to the weak mixing angle θ_W^o, which both are defined in the on-shell scheme by the vector mass and the ghost mass ratio, respectively:

$$\begin{aligned} \cos\theta_W^o &\equiv \frac{M_W^o}{M_Z^o} \\ \cos\theta_G^o &\equiv \frac{\zeta_W^o M_W^o}{\zeta_Z^o M_Z^o}\frac{\sqrt{1-\frac{M_W^{o2}}{M_Z^{o2}}}}{\sqrt{1-\frac{\zeta_W^o M_W^{o2}}{\zeta_Z^o M_Z^{o2}}}} \end{aligned} \tag{5.107}$$

Respective expressions are defined for the physical on-shell masses. It is quite instructive to consider the ghost transformation matrix in the context of the classical field transformations: When we introduced the ghosts in the classical approach by changing the infinitesimal parameters of gauge transformations $\epsilon_\alpha(x)$ to anticommuting parameters c_a (2.59), we have already mentioned that there is an arbitrariness in defining them. This arbitrariness has now been exploited to construct a diagonal ghost mass matrix by the transformation

$$\epsilon_\alpha \longrightarrow c_\alpha = \delta\hat{g}_{\alpha b}c_b \tag{5.108}$$

Finally it remains to solve the ST identity for the interactions, which depend on the external scalar field we have suppressed up to now. Quite generally the dependence of the generating functional of 1PI Green functions on the external scalars is governed by the following equation, one derives from the ST identity:

$$s_\Gamma\left(\frac{\delta\Gamma}{\delta q_a}\right) = -\frac{\delta\Gamma}{\delta\hat{\phi}_a} \tag{5.109}$$

Solving it in the classical approximation it turns out that one further parameter appears, due to a field redefinition of the propagating scalars into propagating and external scalars. The transformation

$$\begin{pmatrix}\phi_a^o \\ \hat{\phi}_a^o\end{pmatrix} \longrightarrow \begin{pmatrix}\phi_a^o - x^o\hat{\phi}_a^o \\ \hat{\phi}_a^o\end{pmatrix} \tag{5.110}$$

is compatible with the ST identity and rigid symmetry if we redefine the external field part by

$$Y^o_a \hat{t}_{abc}(\phi^o_b + v^o\delta_{bH})c^o_c \longrightarrow Y^o_a \hat{t}_{abc}(\phi^o_b - x^o\hat{\phi}^o_b + v^o\delta_{bH})c^o_c - x^o Y^o_a \tilde{I}_{ab} q^o_b \tag{5.111}$$

As long as we do not want to interpret $\hat{\phi}_a$ as a background field, the normalization of x is irrelevant, because finally Green functions are considered at $\hat{\phi}_a = 0$. One can fix x by the following normalization condition,

$$\partial_{p^2}\Gamma_{H\hat{H}}(p^2)\Big|_{p^2=\mu^2_H} = x \tag{5.112}$$

For the purpose of this paper we choose $x = 0$, but nevertheless one gets nonlocal higher order contributions between propagating and external scalars.

5.4.2. The solution of the ghost equations in higher orders

The purpose of this section is to prove, that the ghost equations can be indeed established to all orders in accordance with the normalization conditions on the ghost 2-point functions (5.10), (5.11) and (5.12). If one is able to implement the ghost normalization conditions by adjusting the free parameters in the external field part, the gauge fixing and the ST identity, then in the construction of higher orders one has only to consider the non-local functional Γ^{nl} as defined in (5.81).

First we give the ghost equations in momentum space and test them with respect to the mass normalization conditions. Introducing

$$\Gamma_{\rho^\mu_\alpha c_b}(p,-p) = -ip^\mu \Gamma_{\rho_\alpha c_b}(p^2) = -ip^\mu + O(\hbar) \tag{5.113}$$

the ghost equation of the charged ghost tested at $p^2 = \zeta_W M^2_W$ reads:

$$\zeta_W M^2_W \mathrm{Re}\Gamma_{\rho_+ c_-}(\zeta_W M^2_W) + i\hat{\zeta} M_W \mathrm{Re}\Gamma_{Y_+ c_-}(\zeta_W M^2_W) = 0 \tag{5.114}$$

The $SU(2)$-components of the ghost equation are tested at $p^2 = \zeta_Z M^2_Z$ and $p^2 = 0$:

$$\begin{aligned}
\mathrm{Re}(\zeta_Z M^2_Z \Gamma_{\rho_3 c_Z}(\zeta_Z M^2_Z) - \hat{\zeta} M_W \Gamma_{Y_\chi c_Z}(\zeta_Z M^2_Z)) &= 0 \\
\mathrm{Re}(\zeta_Z M^2_Z \Gamma_{\rho_3 c_A}(\zeta_Z M^2_Z) - \hat{\zeta} M_W \Gamma_{Y_\chi c_A}(\zeta_Z M^2_Z)) &= -\sin\gamma_A \mathrm{Re}\Gamma_{\bar{c}_A c_A}(\zeta_Z M^2_Z) \\
-\hat{\zeta} M_W \Gamma_{Y_\chi c_Z}(0) &= \cos\gamma_Z \Gamma_{\bar{c}_Z c_Z}(0) \\
\hat{\zeta} M_W \Gamma_{Y_\chi c_A}(0)) &= 0
\end{aligned} \tag{5.115}$$

The same test is carried out on the abelian component of the ghost equations:

$$\begin{aligned}
\zeta_Z M^2_Z r^g_{4Z} + \hat{G}\hat{\zeta} M_W \mathrm{Re}\Gamma_{Y_\chi c_Z}(\zeta_Z M^2_Z)) &= 0 \\
\zeta_Z M^2_Z r^g_{4A} + \hat{G}\hat{\zeta} M_W \mathrm{Re}\Gamma_{Y_\chi c_A}(\zeta_Z M^2_Z) &= \cos\gamma_A \mathrm{Re}\Gamma_{\bar{c}_A c_A}(\zeta_Z M^2_Z) \\
\hat{G}\hat{\zeta} M_W \Gamma_{Y_\chi c_Z}(0) &= \sin\gamma_Z \Gamma_{\bar{c}_Z c_Z}(0) \\
\hat{G}\hat{\zeta} M_W \Gamma_{Y_\chi c_A}(0)) &= 0
\end{aligned} \tag{5.116}$$

In order to evaluate these equations one has to take into account that the vertex functions $\Gamma_{\rho_\alpha c_b}(p^2)$ and $\Gamma_{Y_a c_b}(p^2)$ are not independent from each other but are related by the ST identity. We assume now, that we had already established the ST identity and the Ward identities of rigid symmetry to order $n-1$ for all Green functions and to order n for all tests with respect to vectors and scalars. Having also applied the normalization conditions on the 2-point functions and the one for fixing the nonabelian coupling the external field part is determined up to local contributions of order n. These local contributions are read off from the classical approximation (5.64). If the vertex functions $\Gamma_{\rho_\alpha c_b}(p^2)$ and $\Gamma_{Y_a c_b}(p^2)$ solve the ST identity then also the vertex functions $\Gamma'_{\rho_\alpha c_b}(p^2)$ and $\Gamma'_{Y_a c_b}(p^2)$ solve the ST identity and they are related by

$$\begin{aligned}
\Gamma'^{(n)}_{\rho_\alpha c_b}(p^2) &= \Gamma^{(n)}_{\rho_\alpha c_b}(p^2) + a^{(n)}_{\alpha b} \\
\Gamma'^{(n)}_{Y_\pm c_\mp}(p^2) &= \Gamma^{(n)}_{Y_\pm c_\mp}(p^2) \pm iM_W a^{(n)}_{++} \\
\Gamma'^{(n)}_{Y_\chi c_a}(p^2) &= \Gamma^{(n)}_{Y_\chi c_a}(p^2) + M_Z(\cos\theta_W a^{(n)}_{3a} + \sin\theta_W a^{(n)}_{4a})c_a
\end{aligned} \tag{5.117}$$

and $a^{(n)}_{++} = a^{(n)}_{--}$ due to CP-invariance. Inserting in the ghost equations and taking advantage of the quantum action principle, which restricts the breakings of order n to local expressions, it is seen that the ghost equations can be fulfilled, if we adjust the arbitrary parameters $a^{(n)}_{\alpha b}$, the parameters of the gauge fixing part $\hat\zeta$ and $\hat G$ and the linear transformation parameters of the vectors $r^{g(n)}_{4a}$. In fact it is the same calculation as in the classical approximation. In particular the diagonalization of the ghost mass matrix at $p^2 = 0$ (5.83) implies

$$\Gamma_{Y_\chi c_A}(0) = 0 \qquad \text{and} \qquad \Gamma_{Y_\chi c_Z}(0) = \cos\gamma_Z \frac{\Gamma_{\bar c_Z c_Z}(0)}{\hat\zeta M_W} \tag{5.118}$$

and

$$\hat G = -\tan\gamma_Z \tag{5.119}$$

Inserting the last relation into the solution of the ghost equation (5.81) we find, that the condition

$$\Gamma_{\bar c_A c_Z}(p^2 = 0) = 0 \tag{5.120}$$

is now fulfilled by construction, whereas

$$\Gamma_{\bar c_Z c_A}(p^2 = 0) = \Gamma_{\bar c_A c_A}(p^2 = 0) = 0 \tag{5.121}$$

has to be established by requiring

$$\Gamma_{Y_\chi c_A}(p^2 = 0) = 0 \tag{5.122}$$

We want to mention, that in the BPHZL-scheme this condition is implemented due to the infrared degrees of c_A and Y_χ. Otherwise this condition has to be introduced in addition to the usual ones in order to be able to protect internal ghost loops from off-shell infrared divergencies.

Explicitly it is seen that the normalization conditions at $p^2 = 0$ fix the counterterms

$$\begin{aligned} \cos\theta_W a_{3Z}^{(n)} &+ \sin\theta_W r_{4Z}^{g(n)} \\ \cos\theta_W a_{3_A}^{(n)} &+ \sin\theta_W r_{4A}^{g(n)} \end{aligned} \tag{5.123}$$

On-shell conditions for the charged ghosts and Z-ghost and the separation of massive and massless ghosts at $p^2 = \zeta_Z M_Z^2$ determine finally the parameters $\hat{\zeta}, \gamma_Z, \gamma_A$ whereas a_{3Z}, a_{3A} and a_{++} are fixed by the normalization conditions on the residua of ghosts.

The construction of higher orders can be therefore indeed restricted to Γ^{nl} as defined in (5.81), but we have to take into account, that we are not able to dispose of the counterterms $Y_\chi c_Z$ and $Y_\chi c_A$, because these counterterms have to be adjusted for establishing the ghost equations without introducing infrared divergencies. Thanks to the fact, that the antighost transformations can be modified by introducing the angles γ_Z and γ_A, the coefficients appearing in the ST identity r^g_{4Z} and r^g_{4A} are at our disposal for absorbing local breakings of the ST identity into corrections of the ST operator, even if the ghost 2-point functions are constructed with on-shell conditions.

5.5. Summary of the classical approximation

Because the approach we have chosen here for determining the invariant local counterterms is somewhat unconventional, we want to summarize the results of the last sections.

The general invariant action has been determined by requiring invariance with respect to the ST identity and with respect to rigid $SU(2)$-symmetry,

$$\mathcal{S}(\Gamma_{cl}^{gen}) = 0 \quad \mathcal{W}_\alpha \Gamma_{cl}^{gen} = 0 \tag{5.124}$$

The analysis is unconventional in so far as we did not prescribe the symmetry operators explicitly as e.g. in the tree form, but we specified them by field content and algebraic properties, i.e. nilpotency of the ST operator, algebra of rigid operators and the consistency relation. This is the only form appropriate for the treatment of the standard model, because the parameters of the tree approximation get higher order corrections as indicated by the classical approximation, if one separates the massless and massive particles at $p^2 = 0$. This result can be read off from the explicit expressions for the vector 2-point functions, which have been calculated in the on-shell scheme in the literature (see e.g. [37] for a complete list). Actually, in the abstract approach one notices that one has to modify the symmetry operators of the tree approximation, only if one classifies the higher order breakings of the ST identity according to their infrared and ultraviolet degree, as we do it in section 5. Indeed it will be seen there, that we have already solved the infrared part of the problem by solving the classical approximation with the general symmetry operators.

A further important point in the treatment is the observation, that the ST identity does not uniquely determine all parameters of the standard model. For gauging the electromagnetic current rather than the currents of lepton and quark number conservation the Gell-Mann Nishijima relation has to be extended for off-shell Green functions to higher orders. For doing this one has to derive a $U(1)$ Ward identity and specify therein the free parameters as the ones of electromagnetic current conservation:

$$\mathbf{w}^Q = \mathbf{w}_{em} - \mathbf{w}_3 \tag{5.125}$$

The general invariant action as solution (5.124) can be decomposed as in the tree approximation into

$$\Gamma^{gen}_{cl} = \Gamma^{gen}_{GSW} + \Gamma^{gen}_{ext.f.} + \Gamma^{gen}_{ghost} + \Gamma^{gen}_{g.f.} \tag{5.126}$$

Apart from the mass of the photon and the photon ghosts, the parameters of the bilinear action, i.e. masses, residua and nondiagonal mass matrix elements, are free parameters of the theory, and one can dispose of them by the normalization conditions we have specified in section 4.1. Masslessness of the photon and the photon ghost has to be proven to be in agreement with the ST identity in higher orders. If we furthermore use the local $U(1)$-Ward identity for determining the lepton and quark family couplings, then we remain with one free parameter, the nonabelian coupling, which can be chosen to the electromagnetic fine structure constant in the Thompson limit by the normalization condition on the electron-photon vertex (5.73).

The general action can be written in the bare form by eliminating the non-diagonal mass matrix elements of the general bilinear part and the arbitrary residua into a field redefinition, which transforms the original fields to bare fields. The general expression is obtained by undoing this transformations. Expressed in bare fields the general action depends on the bare vector boson masses, the bare ghost masses, the bare fermion masses and the bare Higgs mass and the the nonabelian coupling g_2^o or likewise e^o.

$$\begin{array}{rclcrcl} M^{o2}_W &=& M^2_W + \delta M^2_W & \qquad & m^{o2}_H &=& m^2_H + \delta m^2_H \\ M^{o2}_Z &=& M^2_Z + \delta M^2_Z & \qquad & m^{o2}_{f_i} &=& m^2_{f_i} + \delta m^2_{f_i} \end{array} \tag{5.127}$$

$$\begin{array}{rcl} \zeta^o_W M^{o2}_W &=& \zeta_W M^2_W + \delta\zeta_W M^2_W + \zeta_W \delta M^2_W \\ \zeta^o_Z M^{o2}_Z &=& \zeta_Z M^2_Z + \delta\zeta_Z M^2_Z + \zeta_Z \delta M^2_Z \end{array} \tag{5.128}$$

We want to point out that $\delta\zeta_Z$ and $\delta\zeta_W$ are independent higher order corrections even if we choose $\zeta_W = \zeta_Z$ in the tree approximation. In a QED-like parameterization one has

$$g_2^o = e\frac{M_Z}{\sqrt{M_Z^2 - M_W^2}} + \delta g_2 \tag{5.129}$$

Γ^{gen}_{GSW} has been given in (5.59), where we have to replace the couplings of lepton and quark currents according to (5.71) in order ot fulfil the abelian Ward identity of electromagnetic and weak current conservation. Taking for the gauge fixing function the most general ansatz, which is compatible with rigid symmetry and is linear in propagating fields (cf. (5.77) and (5.92)), the ghost equations relate Γ^{gen}_{ghost} to the external field part according to (5.104). Using the notations for the ghost angle and weak mixing angle as introduced in (5.106) and (5.107) the external field part as function of the bare vector masses and bare ghost masses is given by

$$\begin{aligned}
\Gamma^{gen}_{ext.f.} = \int \Big(& -\frac{1}{2}\sigma^o_a \hat{\varepsilon}_{\alpha\beta\gamma}\delta\hat{g}^o_{\beta b}c^o_b\delta\hat{g}^o_{\gamma c}c^o_c \\
& +\rho^o_{\mu\alpha}(\partial^\mu \tilde{I}_{\alpha\beta}\delta\hat{g}^o_{\beta b}c^o_b + \hat{\varepsilon}_{\alpha\beta\gamma}O_{\beta b}(\theta^o_W)V^{o\mu}_b\delta\hat{g}^o_{\gamma c}c^o_c) \\
& +\Big(Y^{o\dagger}\Big(ig^o_2(\frac{\tau_\alpha}{2}\delta\hat{g}^o_{\alpha a} - \frac{\mathbf{1}}{2}\tan\theta^o_W\delta\hat{g}^o_{4a})(\Phi^o + \frac{\sqrt{2}}{g^2_o}\begin{pmatrix}0\\ M^o_W\end{pmatrix})c^o_a - x\hat{\mathbf{q}}^o\Big) + \text{h.c.}\Big) \\
& +\sum_{i=1}^{N_F}\Big(\sum_{\delta=l,q}(\bar{\Psi}^{oR}_{\delta_i}ig^o_2(\frac{\tau_\alpha}{2}\delta g^o_{\alpha a} + \frac{G^{\delta_i}}{2}\delta\hat{g}^o_{4a})c^o_a F^{oL}_{\delta_i} \\
& \qquad + \sum_{f=e,d}\bar{\psi}^{oL}_{f_i}ig^o_2\frac{1}{2}(\tan\theta^o_W + G^{\delta_i}))f^{oR}_i\delta\hat{g}^o_{4a}c^o_a \\
& \qquad + \bar{\psi}^{oL}_{u_i}ig^o_2\frac{1}{2}(-\tan\theta^o_W + G^{q_i})u^{oR}_i\delta\hat{g}^o_{4a}c^o_a + \text{h.c.}\Big)\Big)
\end{aligned} \tag{5.130}$$

Here we have again rewritten the scalars into complex doublets.

The ST operator and the Ward operators depend in the bare form also on the vector mass ratio and ghost mass ratio:

$$\begin{aligned}
\mathcal{S}(\Gamma^{gen}_{cl}) = \int \Big(& (\sin\theta^o_G\partial_\mu c_Z + \cos\theta^o_W\partial_\mu c^o_A)\Big(\sin\theta^o_W\frac{\delta}{\delta Z^o_\mu} + \cos\Theta^o_W\frac{\delta}{\delta A^o_\mu}\Big) \\
& +\frac{\delta\Gamma^{gen}_{cl}}{\delta\rho^{o\mu}_3}\Big(\cos\theta^o_W\frac{\delta}{\delta Z^o_\mu} - \sin\theta^o_W\frac{\delta}{\delta A^o_\mu}\Big) + \frac{\delta\Gamma^{gen}_{cl}}{\delta\sigma^o_3}\frac{1}{\cos(\theta^o_W-\theta^o_G)}\Big(\cos\theta^o_W\frac{\delta}{\delta c^o_Z} - \sin\theta^o_G\frac{\delta}{\delta c^o_A}\Big) \\
& +\frac{\delta\Gamma^{gen}_{cl}}{\delta\rho^{o\mu}_+}\frac{\delta}{\delta W^o_{\mu-}} + \frac{\delta\Gamma^{gen}_{cl}}{\delta\rho^{o\mu}_-}\frac{\delta}{\delta W^o_{\mu+}} + \frac{\delta\Gamma^{gen}_{cl}}{\delta\sigma^o_+}\frac{\delta}{\delta c^o_-} + \frac{\delta\Gamma^{gen}_{cl}}{\delta\sigma^o_-}\frac{\delta}{\delta c^o_+} + \frac{\delta\Gamma^{gen}_{cl}}{\delta Y^o_a}\tilde{I}_{aa'}\frac{\delta}{\delta\phi^o_{a'}} \\
& +\sum_{i=1}^{N_F}\Big(\frac{\delta\Gamma^{gen}_{cl}}{\delta\overline{\psi^{oL}_{f_i}}}\frac{\delta}{\delta f^{oR}_i} + \frac{\delta\Gamma^{gen}_{cl}}{\delta\overline{\psi^{oR}_{f_i}}}\frac{\delta}{\delta f^{oL}_i} + \text{h.c.}\Big)
\end{aligned} \tag{5.131}$$

$$+B^o_a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta^o_G - \theta^o_W) & 0 \\ 0 & 0 & \sin(\theta^o_G - \theta^o_W) & 1 \end{pmatrix}_{ab} \frac{\delta}{\delta \bar c_b} + q^o_a \frac{\delta}{\delta \hat\phi^o_a}\Big)\Gamma^{gen}_{cl} = 0$$

The Ward operators of rigid symmetry are determined to ($\alpha = +, -, 3$)

$$\begin{aligned} \mathcal{W}_\alpha = \tilde I_{\alpha\alpha'} \int dx \,\Big(& V^{o\mu}_b O^T_{b\beta}(\theta^o_W)\hat\varepsilon_{\beta\gamma\alpha'} O_{\gamma c}(\theta^o_W)\tilde I_{cc'}\frac{\delta}{\delta V^{o\mu}_{c'}} + \{B^o_a\} \\ & + c^o_b (\delta\hat g^o)^T_{b\beta}\varepsilon_{\beta\gamma\alpha'}(\delta \hat g^o)^{-1T}_{\gamma c}\tilde I_{cc'}\frac{\delta}{\delta c^o_{c'}} + \bar c^o_b (\delta \hat g^o)^{-1}_{b\beta}\hat\varepsilon_{\beta\gamma\alpha'}(\delta\hat g^o)_{\gamma c}\tilde I_{cc'}\frac{\delta}{\delta \bar c^o_{c'}} \\ & + (\phi^o_b + \delta_{Hb} 2\frac{M^o_W}{g^o_2})\hat t_{bc,\alpha'}\tilde I_{cc'}\frac{\delta}{\delta\phi^o_{c'}} + \{Y^o_b, \hat\phi^o_b + \hat\zeta^o 2\frac{M^o_W}{g^o_2}, \hat q^o_a\} \\ & + \rho^o_\beta \varepsilon_{\beta\gamma,\alpha'}\tilde I_{\gamma\gamma'}\frac{\delta}{\delta\rho^o_{\gamma'}} + \sigma^o_\beta\varepsilon_{\beta\gamma,\alpha'}\tilde I_{\gamma\gamma'}\frac{\delta}{\delta\sigma^o_{\gamma'}} \\ & + \sum_{i=1}^{N_F}\sum_{\delta=l,q}\Big(\overline{F^{oL}_{\delta_i}} i\frac{\tau_{\alpha'}}{2}\frac{\delta}{\delta \overline{F^{oL}_{\delta_i}}} - \frac{\delta}{\delta F^{oL}_{\delta_i}}\frac{\tau_{\alpha'}}{2}F^{oL}_{\delta_i} \\ & \qquad + \overline{\Psi^{oR}_{f_i}}\frac{i\tau_{\alpha'}}{2}\frac{\delta}{\delta\overline{\Psi^{oR}_{f_i}}} - \frac{\delta}{\delta\Psi^{oR}_{\delta_i}}\frac{i\tau_{\alpha'}}{2}\Psi^R_{\delta_i}\Big)\Big) \end{aligned} \tag{5.132}$$

Comparing the ST identity and the Ward operators of rigid symmetries expressed in terms of bare fields, with the ones of the tree approximation, it is seen that they differ due to the appearance of the ghost angle, which signals – apart from field redefinitions – different corrections of the vector mass ratio and the ghost mass ratio in higher orders.

The local $U(1)$ Ward identity

$$\Big(g^o_2 \tan\theta^o_W \mathbf{w}^Q_4 - \partial\frac{\delta}{\delta Z^o}\sin\theta^o_W - \partial\frac{\delta}{\delta A^o}\cos\theta^o_W\Big)\Gamma^{gen}_{cl} = \sin\theta^o_W \Box B^o_Z + \cos\theta^o_W\Box B^o_A \tag{5.133}$$

has been derived in the matter part of the action. It is broken only by gauge fixing the longitudinal parts of the abelian vector field combination. It is valid in the ghost part of the action as it is, if we take care to maintain rigid symmetry.

Transforming the bare fields to original fields yields the general form of the action and the general form of the ST identity and Ward identities. It allows order by order to determine the invariant local contributions by expanding the parameters in perturbation theory. There it has to be noted that some of the parameters appear explicitly in the symmetry operators. According to our conventions (cf. (5.21),5.22 and (5.23)) these parameters are

$$r^V_Z, r^V_A, \Theta^V, r^S_a, r_{l_i}, r_{q_i}, r^g_{\alpha a}, \gamma_Z, \gamma_A \tag{5.134}$$

As long as they are not needed for removing infrared divergent contributions, they could be fixed in the symmetry transformations as it is e.g. for r^S and r_{q_i}, r_{l_i}. Furthermore one has explicit dependence on the shift of the Higgs field v and the shift of the external Higgs $\hat{\zeta} v$ in the Ward operators of rigid symmetry. These parameters are fixed by on-shell conditions on the mass of the W-boson and on the mass of charged ghosts,

$$M_W, \zeta_W M_W \tag{5.135}$$

The remaining parameters which are only specified on the general classical action are the field redefinitions

$$z_W, \frac{\tan(\theta_W^o + \theta_A)}{\tan(\theta_W^o + \theta_A)}, Z_{+-}^g, z_H, z_{\nu_i}, z_{u_i}, \tilde{z}_{f_i} \text{ and } x^o \tag{5.136}$$

the nonabelian coupling, which is fixed to the fine structure constant in QED-like schemes, and the remaining masses of the standard model:

$$e^o \quad \text{and} \quad M_Z^o, m_H^o, \zeta_Z^o M_Z^o, m_{f_i}^o \tag{5.137}$$

These parameters, in any case, have to be fixed by normalization conditions on the finite 1PI Green functions.

The Callan-Symanzik equation, we derive in the next section, enables one to calculate the asymptotic logarithmic corrections to the invariants. There it is seen, that the invariants, which are connected with the parameters (5.136) and (5.137) get independent logarithmic corrections to the next order.

6. The Callan-Symanzik equation

The Callan-Symanzik (CS) equation describes the response of the Green functions to the scaling of all momenta by an infinitesimal factor. The dilatational operator

$$\mathcal{W}^D = \sum_{\text{all fields}} \int (d_{\varphi_k} + x\partial_x)\varphi_k(x) \frac{\delta}{\delta\varphi_k(x)} \qquad d_{\varphi_k} = \dim^{UV} \varphi_k \tag{6.1}$$

acts on the 1PI-Green functions in the same way as the differentiation with respect to all the mass parameters of the theory:

$$\begin{aligned} &\mathcal{W}^D\Gamma = -m\partial_m\Gamma \quad \text{with} \\ &m\partial_m \equiv M_W\partial_{M_W} + M_Z\partial_{M_Z} + m_H\partial_{m_H} + \sum_{i=1}^{N_F}\sum_f m_{f_i}\partial_{m_{f_i}} + \sum_{\kappa_a}\kappa_a\partial_{\kappa_a} \end{aligned} \tag{6.2}$$

κ_a are the normalization points, which we have introduced to fix the residua of fields. In concrete calculations one will introduce only one κ for fixing all the infrared divergent residua of charged particles off-shell.

The CS equation [41] is of utmost interest in the abstract approach as well as for concrete calculations. In the abstract approach the β-functions and anomalous dimensions, which describe the breaking of dilatations by anomalies in higher orders, allow to determine the independent parameters of the theory in a scheme independent way. When we solved the ST identity and the Ward identities of rigid symmetries for the local contributions, we gave a list of free parameters, which were not fixed by the symmetries and can be adjusted by normalization conditions. The CS equation singles out from these parameters the ones which get independent logarithmic corrections in the asymptotic region, where all momenta are much larger than the masses of the theory. To be specific we consider as an example the ghost mass ratio: It can, of course, be fixed by a normalization condition to the vector boson mass ratio. However, from the CS equation it is immediately derived, that the vertices, which depend on the ghost mass ratio, get different logarithmic corrections in higher orders from the ones, which depend on the vector mass ratio. Similar statements are true for other mass ratios, one would like to relate to the vector boson mass ratio in lowest order, as it is required for example in the context of noncommutative geometry [42]. With the help of the CS equation it is at least in massless theories possible to find relations between independent parameters, which are compatible with renormalizability, using the principle of reduction of couplings [43, 44]. It is, however, not obvious, how reduction has to be applied to spontaneously broken theories, due to the fact that the β-functions do not only depend on the perturbative expansion parameters but also on the mass ratios. This property has also the consequence, that the β-functions of the CS equation depend on mass logarithms from 2-loop order onwards and that they differ from the ones of the symmetric theory [45].

For the purpose of the present paper concerning the renormalization of the standard model, the CS equation, especially the dilatational operator, is also of special interest, because it allows to simplify the algebraic cohomology as carried out in the next section considerably. One can derive that all algebraic anomalies of the ST identity and the Ward identities are restricted to 4-dimensional field polynomials, 3-dimensional breakings are immediately seen to be variations of terms with quantum numbers of the action [21]. In this context we also want to mention that the CS equation plays an important role if one wants to prove the nonrenormalization theorems for the Adler-Bardeen anomaly as it appears in the ST identity (see [46] and references therein).

For concrete calculations the most important outcome of the CS equation is the determination of higher orders leading logarithms. If there are large asymptotic logarithms in 1-loop order, the CS equation allows consistently to determine those large leading logarithms of higher orders, which are induced by the 1-loop logarithms of the lowest order.

In this section we construct the symmetric operators of the CS equation, which define the β-functions and anomalous dimensions of 1-loop order. It is important to note, that we are able to classify the dilatational anomalies also if we have not proved that the ST identity and Ward identities of rigid symmetries are established in 1-loop. The only ingredients are the symmetries of lowest order and the linear gauge fixing. The soft breakings, however, can be only consistently classified when we have established rigid symmetry of 1-loop order. Then the classification works as we present here in the tree approximation.

6.1. The soft breaking of dilatations

In the standard model dilatations are already broken in the tree approximation by all terms with mass dimensions less than 4, especially by the mass terms of the fields. Due to the spontaneous symmetry breaking all the masses of the physical fields are generated by the shift of the Higgs field. According to the construction of the gauge fixing sector using rigid symmetry the ghost masses and the masses of would-be-Goldstones are generated by the shift of the external Higgs (cf. (2.51),(2.52) and (2.75)). In the tree approximation one reads off the breakings of dilatations by applying $m\partial_m$ on the classical action. For proceeding to higher orders it is unavoidable to characterize also the soft breakings by their symmetries. For this reason we do the same in the tree approximation and solve thereby also the problem of constructing the soft breakings of higher orders, if the ST identity and rigid symmetry are established. We have in the tree approximation

$$m\partial_m\Gamma_{cl} = \Delta_m \tag{6.3}$$

where Δ_m is an integrated field polynomial with mass dimension less than 4, CP-even and neutral with respect to electric and $\phi\pi$ charge. Commuting the operator $m\partial_m$ with the ST operator it is seen that Δ_m is $\mathrm{s}_{\Gamma_{cl}}$-invariant.

$$\mathrm{s}_\Gamma m\partial_m\Gamma - m\partial_m\mathcal{S}(\Gamma) = 0 \Longrightarrow \mathrm{s}_{\Gamma_{cl}}\Delta_m = 0 \tag{6.4}$$

The dilatational operator does not commute with the Ward identities of rigid symmety:

$$\big[\mathcal{W}_\alpha, m\partial_m\big] = \big[\mathcal{W}_\alpha, \int v\big(\frac{\delta}{\delta H} + \hat\zeta\frac{\delta}{\delta \hat H}\big)\big] \tag{6.5}$$

This implies

$$\mathcal{W}_\alpha\Delta_m = \mathcal{W}_\alpha\int v\big(\frac{\delta}{\delta H} + \hat\zeta\frac{\delta}{\delta \hat H}\big)\Gamma_{cl} \tag{6.6}$$

Noting that the differentiation with respect to the Higgs and the external Higgs is a BRS-variation quite generally (5.109)

$$\mathrm{s}_\Gamma Y_H = \frac{\delta\Gamma}{\delta H} \qquad \text{and} \qquad \mathrm{s}_\Gamma\left(\frac{\delta\Gamma}{\delta\hat q_H}\right) = \frac{\delta\Gamma}{\delta\hat H} \tag{6.7}$$

the polynomial Δ_m can be decomposed into

$$\Delta_m = \int \Big(v \frac{\delta \Gamma_{cl}}{\delta H} + \hat{\zeta} v \frac{\delta \Gamma_{cl}}{\hat{\delta} H} \Big) + \Delta_{inv} \tag{6.8}$$

and

$$\mathrm{s}_{\Gamma_{cl}} \Delta_{inv} = 0 \qquad \mathcal{W}_\alpha \Delta_{inv} = 0 \tag{6.9}$$

Inspecting all 2- and 3-dimensional field polynomial it is seen that there is only one rigid and $\mathrm{s}_{\Gamma_{cl}}$- invariant polynomial:

$$\Delta_{inv} \equiv \int (2\phi_+\phi_- + \chi^2 + H^2 + 2vH) \tag{6.10}$$

This invariant field polynomial we couple to an external scalar field $\hat{\varphi}_o$ with UV dimension 2 and IR-dimension 2 and add it to the classical action

$$\Gamma_{cl} \to \Gamma_{cl} + \int \hat{\varphi}_o (2\phi_+\phi_- + \chi^2 + H^2 + 2vH) \,. \tag{6.11}$$

The enlarged classical action is ST-invariant and rigid invariant if we assign:

$$\mathrm{s}\hat{\varphi}_o = 0 \qquad \delta_\alpha \hat{\varphi}_o = 0 \tag{6.12}$$

Finally we are able to write Δ_m as a field operator acting on the classical action:

$$m\partial_m \Gamma_{cl} = \int v \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} + \frac{m_H^2}{2v} \frac{\delta}{\delta \hat{\varphi}_o} \Big) \Gamma_{cl} \tag{6.13}$$

and v is determined in the QED-like parameterization to

$$v = 2 \frac{M_Z}{e} \sin\theta_W \cos\theta_W + O(\hbar) \qquad\qquad \cos\theta_W \equiv \frac{M_W}{M_Z} \tag{6.14}$$

The soft breaking of dilatations, and in particular the breaking of the tree approximation, is therefore completely characterized by its properties under the symmetry transformations and one can proceed in higher orders as in the tree approximation, if the ST identity and rigid symmetries are established. We want to mention, that also at this stage rigid symmetry plays an important role for classifying the soft breakings of dilatations. If one breaks rigid symmetry in the gauge fixing and ghost sector it is not possible to derive an unambiguous expression for the soft breakings in higher orders.

Because we have rewritten the dilatational breaking of lowest order into a sum of field differentiations we are able to proceed immediately to the next order. When one applies the symmetric operator

$$\mathcal{W}^D_{sym} \equiv \mathcal{W}^D - \int v \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} + \frac{m_H^2}{2v} \frac{\delta}{\delta \hat{\varphi}_o} \Big) \tag{6.15}$$

on the functional of 1PI Green functions, all breakings of 1-loop order are now known to be local due to the action principle. But in the construction of the ST identity and rigid Ward identities we have also to consider Green functions which include the invariant external field $\hat{\varphi}_o$. For the invariant counterterms as derived in the last sections the only linear dependence on $\hat{\varphi}_o$ enters via the interaction with the 2-dimensional scalar invariant (6.11), which reads for the bare scalar fields $\tilde{\phi}^o_a = \phi^o_a + x^o \hat{\phi}^o_a$ (cf. (5.110)):

$$\Gamma^{gen}_{\hat{\varphi}_o} = \int \hat{\varphi}^o_o (2\tilde{\phi}^o_+ \tilde{\phi}^o_- + \tilde{\chi}^{o2} + \tilde{H}^{o2} + 2v^o \tilde{H}^o) \tag{6.16}$$

The field renormalization of the external scalar field $\hat{\varphi}_o$ can be fixed by setting the coefficient of the field differentiation with respect to $\hat{\varphi}_o$ in the CS equation.

6.2. The dilatational anomalies – hard breakings

In higher orders the dilatations are not only broken by the soft mass terms but also by hard terms, the dilatational anomalies. The importance of the CS equation is founded in the fact, that these anomalies can be absorbed into differential operators, which are connected with the anomalous dimensions and β-functions.

The dilatational anomalies of one-loop order are normalization point independent, but the differential operators introduced depend on the special parameterization and the specific form of the breaking mechanism. They are essentially characterized by the symmetries of the tree approximation. Because we want finally to use the CS operator for classifying the breakings of the ST identity we only assume that the ST identity is established in lowest order and so for the rigid symmetries. However, for deriving the CS equation we cannot completely stick to the tree approximation as given in the section 2, but we have to treat the ghost mass ratio as an independent parameter. The ST identity and Ward identities of the tree approximation have then the form as given in (5.131) and (5.132), replacing all fields and parameters by ordinary fields and parameters. Therefore the symmetry operators depend in the tree approximation on the vector mass ratio, i.e. $\cos\theta_W$, and ghost mass ration, i.e. $\cos\theta_G$ (5.107).

Using the action principle in its quantized version one derives that all symmetries of the tree approximation are broken in 1-loop order by integrated field polynomials

$$\Big(\mathcal{S}(\Gamma)\Big)^{(\leq 1)} = \Delta^{(1)}_{brs} \qquad \Big(\mathcal{W}_\alpha \Gamma\Big)^{(\leq 1)} = \Delta^{(1)}_\alpha \tag{6.17}$$

$$\Big(\mathcal{W}^D_{sym}\Gamma\Big)^{(\leq 1)} = \Delta^{(1)}_m$$

These field polynomials are restricted according to the UV-degree and IR-degree of the classical operators:

$$\begin{array}{rclcrcl} \dim^{UV}\Delta^{(1)}_{brs} & \leq & 4 & \qquad & \dim^{IR}\Delta^{(1)}_{brs} & \geq & 3 \\ \dim^{UV}\Delta^{(1)}_{\alpha} & \leq & 4 & & \dim^{IR}\Delta^{(1)}_{\alpha} & \geq & 2 \\ \dim^{UV}\Delta^{(1)}_{m} & \leq & 4 & & \dim^{IR}\Delta^{(1)}_{m} & \geq & 2 \end{array} \tag{6.18}$$

We want to mention already here that breakings with infrared dimension 2 are alarming, because they do in general not exist as integrated insertions in higher orders. They potentially contain mass insertions for massless particles, which are seen to be infrared divergent off-shell. In particular, absence of mass insertions for massless particles, i.e. $A^\mu A_\mu$ and $\bar{c}_A c_A$, has to be verified in the CS equation by a test with respect to the respective normalization conditions to all orders.

Because the functional of 1PI Green functions is invariant under the global charge symmetries (4.14) and (4.15) and CP-transformations, all breakings have a well-defined transformation behaviour under these symmetries: Δ_{brs} is CP-even, has $\phi\pi$ charge 1 and is neutral with respect to electric charge, whereas Δ_α is CP-odd, neutral with respect to $\phi\pi$ charge and has electric charge ± 1 for $\alpha = +,-$ and is neutral for $\alpha = 3$. The important point is that Δ_m in fact has the same quantum numbers as the general classical action: It is neutral with respect to electric and $\phi\pi$-charge and CP-even. Up to linear field polynomials the most general basis for the integrated insertion Δ_m is equivalent to the general renormalizable action, from which one has constructed the general invariant solution in section 5. For this reason we are able to make use of the symmetric dilatational operator, when we consider the algebraic cohomology problem, for classifying the breakings of the ST identity.

For proceeding we take advantage from the property that the operator $\mathcal{W}^D_{sym}$ (6.15) commutes with the ST operator and the Ward operators of rigid symmetry. Therefrom one derives the following consistency relations:

$$\mathcal{W}^D_{sym}\Delta^{(1)}_\alpha = -\mathcal{W}_\alpha \Delta^{(1)}_m \qquad \mathcal{W}^D_{sym}\Delta^{(1)}_{brs} = -\mathrm{s}_\Gamma \Delta^{(1)}_m \tag{6.19}$$

We decompose now the local insertions Δ_α, Δ_m and Δ_{brs} according to their transformation under $\mathcal{W}^D_{sym}$

$$\Delta^{(1)}_{op} \;=\; \sum_{k=1}^{4} \Delta^k_{op} \qquad \text{with} \qquad \mathcal{W}^D_{sym}\Delta^k_{op} = (k-4)\Delta^k_{op} \tag{6.20}$$

It is proven immediately that this decomposition is unique, because it is nothing else but classifying field polynomials according to their mass dimension and taking the Higgs, the external Higgs and the neutral scalar $\hat{\varphi}_o$ in a shifted version

$$H + v\,, \quad \hat{H} + \hat{\zeta} v\,, \quad \hat{\varphi}_o + \frac{m_H^2}{4} \tag{6.21}$$

The consistency relations split up into equations for any of the Δ^k_{op} and give different informations as long as we did not establish the ST operator and rigid symmetry in 1-loop order. The 4-dimensional breakings Δ^4_m are seen to be $\mathrm{s}_{\Gamma_{cl}}$-symmetric and rigid symmetric

$$0 = -\mathcal{W}_\alpha \Delta^4_m \qquad 0 = -\mathrm{s}_{\Gamma_{cl}} \Delta^4_m \tag{6.22}$$

These equations are used to construct the hard anomalies of the CS equation, which are related to β-functions and anomalous dimensions. The further equations for lower dimensional field polynomials state that the breakings $\Delta^{\leq 3}_{brs}$ and $\Delta^{\leq 3}_{\alpha}$ can be immediately written as $\mathrm{s}_{\Gamma_{cl}}$ and $\mathcal{W}_\alpha$ variations, respectively, of integrated field polynomials with quantum numbers of the action.

For constructing the hard breakings of the CS equation we have therefore the task to find all independent field polynomials satisfying the above constraints and to express them in form of symmetric differential operators. The first problem has been already solved in the last section, because Δ_m has the same quantum numbers as the general renormalizable action. We have only to fix the parameters, which appear explicitly in the symmetry operators (5.134) to their tree values and have to expand the remaining ones (5.136) and (5.137) to the next order, singling out the polynomials which contain only lower dimensional field polynomials.

$$\begin{aligned} \mathcal{S}(\Gamma^{gen}_{cl})(\theta_W, \theta_G) &= \mathcal{S}(\Gamma_{cl} + \sum_{n=1}^{\infty} \Gamma^{(n)}_{inv})(\theta_W, \theta_G) \\ &= \Big(\mathcal{S}(\Gamma_{cl}) + \mathrm{s}_{\Gamma_{cl}} \sum_{n=1}^{\infty} \Gamma^{(n)}_{inv}\Big)(\theta_W, \theta_G) = 0 \end{aligned} \tag{6.23}$$

For finding the invariant operators corresponding to the invariant field polynomials we construct in the usual way symmetric operators, which commute with the lowest order rigid Ward operators and the lowest order ST operator and this all done we identify the operators with invariant field polynomials. It is well-known that the invariant field polynomials are separated into two classes: The first one contains all invariant field polynomials, which are $\mathrm{s}_{\Gamma_{cl}}$-variations. These invariants are generated by acting with symmetric field differentiation operators on the classical action and are connected with field redefinitions and anomalous dimensions. The second class comprises the invariants which are generated by differentiation with respect to independent parameters of the theory. These field polynomials are $\mathrm{s}_{\Gamma_{cl}}$-invariants without being variations.

First we give a list of all symmetric field differentiation operators. Because the vectors, B-fields and ghosts are rotated from the $SU(2)$ and $U(1)$-fields to mass eigenstates by the matrix $O_{\alpha a}(\theta_W)$ and $\delta g_{\alpha b}$, the field differentiation operators are not purely leg counting operators, but mix massless and massive neutral fields. In the vector ghost sector we find the following invariant field differentiation operators

$$\begin{aligned}
\mathcal{N}_V &= \int \Big(V_a \frac{\delta}{\delta V_a} - \rho_\alpha \frac{\delta}{\delta \rho_\alpha} + \tfrac{1}{\cos(\theta_W - \theta_G)} (\sin\theta_G c_Z + \cos\theta_W c_A)(\sin\theta_W \frac{\delta}{\delta c_Z} + \cos\theta_G \frac{\delta}{\delta c_A}) \Big) \\
\hat{\mathcal{N}}_V &= \int \Big((\sin\theta_W Z + \cos\theta_W A)(\sin\theta_W \frac{\delta}{\delta Z} + \cos\theta_W \frac{\delta}{\delta A}) \\
&\quad + \tfrac{1}{\cos(\theta_W - \theta_G)} (\sin\theta_G c_Z + \cos\theta_W c_A)(\sin\theta_W \frac{\delta}{\delta c_Z} + \cos\theta_G \frac{\delta}{\delta c_A}) \Big) \\
\mathcal{N}_B &= \int \Big(B_a \frac{\delta}{\delta B_a} + \bar{c}_a \frac{\delta}{\delta \bar{c}_a} \Big) \qquad (6.24) \\
\hat{\mathcal{N}}_B &= \int \Big((\sin\theta_W B_Z + \cos\theta_W B_A)(\sin\theta_W \frac{\delta}{\delta B_Z} + \cos\theta_W \frac{\delta}{\delta B_A}) \\
&\quad + \tfrac{1}{\cos(\theta_W - \theta_G)} (\sin\theta_W \bar{c}_Z + \cos\theta_G \bar{c}_A)(\sin\theta_G \frac{\delta}{\delta \bar{c}_Z} + \cos\theta_W \frac{\delta}{\delta \bar{c}_A}) \Big) \\
\mathcal{N}_c &= \int \Big(c_+ \frac{\delta}{\delta c_+} + c_- \frac{\delta}{\delta c_-} + \tfrac{1}{\cos(\theta_W - \theta_G)} (\cos\theta_G c_Z - \sin\theta_W c_A)(\cos\theta_W \frac{\delta}{\delta c_Z} - \sin\theta_G \frac{\delta}{\delta c_A}) \\
&\quad - \sigma_+ \frac{\delta}{\delta \sigma_+} - \sigma_- \frac{\delta}{\delta \sigma_-} - \sigma_3 \frac{\delta}{\delta \sigma_3} \Big)
\end{aligned}$$

The symmetric field differentiation operators in the fermion sector have to commute also with the operators of lepton and quark family conservation. They are also not leg-counting operators for massive fermions, but involve the γ_5. It is convenient to split these operators into the ones for left-handed and right handed fields, which are both invariant operators:

$$\begin{aligned}
\mathcal{N}^L_{F_{\delta_i}} &= \int \Big(\overline{F^L_{\delta_i}} \frac{\delta}{\delta \overline{F^L_{\delta_i}}} - \overline{\Psi^R_{\delta_i}} \frac{\delta}{\delta \overline{\Psi^R_{d,i}}} + \frac{\delta}{\delta F^L_{\delta_i}} F^L_{\delta_i} - \frac{\delta}{\delta \Psi^R_{\delta_i}} \Psi^R_{\delta_i} \Big) \quad \delta = l, q \qquad (6.25) \\
\mathcal{N}^R_{f_i} &= \int \Big(\overline{f^R_i} \frac{\delta}{\delta \overline{f^R_i}} - \overline{\psi^L_i} \frac{\delta}{\delta \overline{\psi^L_i}} + \frac{\delta}{\delta f^R_i} f^R_i - \frac{\delta}{\delta \psi^L_i} \psi^L_i \Big) \qquad f_i = e_i, d_i, u_i
\end{aligned}$$

The invariant scalar field differentiation operators comprise the ones of the propagating scalars and of external scalars. They are symmetric with respect to the rigid operators, if one includes the shift of the Higgs and external Higgs in the transformation.

$$\begin{aligned}
\mathcal{N}_S + v \int \frac{\delta}{\delta H} &= \int \Big(\phi_a \frac{\delta}{\delta \phi_a} + v \frac{\delta}{\delta H} - Y_a \frac{\delta}{\delta Y_a} \Big) \qquad (6.26) \\
\mathcal{N}_{\hat{S}} + \hat{\zeta} v \int \frac{\delta}{\delta \hat{H}} &= \int \Big(\hat{\phi}_a \frac{\delta}{\delta \hat{\phi}_A} + \hat{\zeta} v \int \frac{\delta}{\delta \hat{H}} + \hat{q}_a \frac{\delta}{\delta q_a} \Big)
\end{aligned}$$

When acting on Γ_{cl} the invariant differentiation operators summarized in (6.24), (6.25) and (6.26) are in one to one correspondence with the field redefinition parameters listed in (5.137):

$$
\begin{array}{rcl}
(\mathcal{N}_V - \mathcal{N}_B)\ \Gamma_{cl} & \longleftrightarrow & z_W \\
(\hat{\mathcal{N}}_V - \hat{\mathcal{N}}_B)\ \Gamma_{cl} & \longleftrightarrow & \frac{\tan\theta_W + \theta_Z}{\tan(\theta_W + \theta_A)} \\
\mathcal{N}_c\ \Gamma_{cl} & \longleftrightarrow & Z^g_{+-}
\end{array}
\qquad
\begin{array}{rcl}
\mathcal{N}_S\ \Gamma_{cl} & \longleftrightarrow & z_H \\
\mathcal{N}^L_{F_{l_i}}\ \Gamma_{cl} & \longleftrightarrow & z_{\nu_i} \\
\mathcal{N}^L_{F_{q_i}}\ \Gamma_{cl} & \longleftrightarrow & z_{u_i} \\
\mathcal{N}^R_{f_i}\ \Gamma_{cl} & \longleftrightarrow & \tilde{z}_{f_i}
\end{array}
\tag{6.27}
$$

The operators $\mathcal{N}_B, \hat{\mathcal{N}}_B$ and $\mathcal{N}_{\hat{S}}$ correspond to field redefinitions of B-fields and external scalars, which are, however, fixed in the gauge fixing part to the ones of vectors, propagating scalars and coupling redefinitions (cf. (5.93)). Taking them as independent operators in the CS equation their coefficients are determined quite simply by a test on the local gauge fixing polynomial.

The invariant field polynomial, which corresponds to the field redefinition of the propagating into the external scalar, i.e. to the parameter x^o (5.110), is generated by the mixed field differentiation operator:

$$
\tilde{\mathcal{N}}_S + \hat{\zeta} v \int \frac{\delta}{\delta H} \;=\; \int \Big(\hat{\phi}_a \frac{\delta}{\delta \phi_a} + \hat{\zeta} v \int \frac{\delta}{\delta H} \Big) \tag{6.28}
$$

It is symmetric with respect to rigid symmetry but not with respect to the ST operator. The $s_{\Gamma_{cl}}$-invariant insertion is then given by

$$
(\tilde{\mathcal{N}}_S + \hat{\zeta} v \int \frac{\delta}{\delta H})\Gamma_{cl} + \int q_a \tilde{I}_{ab} Y_b \tag{6.29}
$$

Now we turn to the non-variations among invariant field polynomials. From the general classical symmetric solution they are read off by expanding the independent parameters (5.137) in perturbation theory. Equivalently they are generated by differentiating the classical action with respect to the independent parameters: These are the coupling e, which is the perturbative expansion parameter, and furthermore the mass ratios, $\frac{M_W}{M_Z}$, for the weak interactions, $\frac{m_H}{M_Z}$ for the scalar interaction and $\frac{m_{f_i}}{M_Z}$ for the Yukawa interactions. At this stage it is unavoidable to treat θ_G i.e. the ghost mass ratio as an independent parameter, because its differentiation corresponds to an independent insertion in the gauge fixing and ghost sector. Similarly it turns out that also the differentiation with respect to the both gauge parameters, ξ and $\hat{\xi}$, has to be included. The differentiation with respect to parameters, which do not appear in the ST identity and the rigid Ward operators of the tree approximation immediately correspond to respective invariant field polynomials:

$$
m_H \partial_{m_H},\ m_{f_i} \partial_{m_{f_i}},\ \xi \partial_\xi,\ \hat{\xi} \partial_{\hat{\xi}} \tag{6.30}
$$

The differentiation with respect to the coupling e is not a rigid invariant, but has to be symmetrized by including the shift: the operator

$$
e\partial_e - e\partial_e v \int \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} \Big) = e\partial_e + \frac{2}{e} M_Z \sin\theta_W \cos\theta_W \int \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} \Big) \tag{6.31}
$$

is $s_{\Gamma_{cl}}$- and rigid symmetric. Without using the local $U(1)$-Ward identity there would be seemingly invariant field polynomials corresponding to the lepton and quark family coupling G_{δ_i}. They are however singled out by deriving quite in analogy to (6.19) and (6.22), that the field polynomials Δ^4_m are $U(1)$-gauge invariant. This result finally relates also the β-functions of the coupling e and the mass ratio $\frac{M_W}{M_Z}$ to the anomalous dimensions (see (6.39)).

In order to find the $s_{\Gamma_{cl}}$-invariants of the mass ratios $\frac{M_W}{M_Z}$ and $\frac{\zeta_W M_W}{\zeta_Z M_Z}$ it is not sufficient to expand the mass ratios only in the general symmetric classical action, but one has to take into account, that such a mass expansion concerns also the ST operator and Ward operators of rigid symmetry. This subtlety comes in, because these mass ratios take a twofold role: They appear in the field transformation matrices, which are introduced for constructing mass eigenstates (cf. (2.28) and (5.106)) , and they take at the same time the role of the abelian gauge coupling and the abelian coupling $\hat{G}$, respectively (cf. (2.29) and (5.103)). Expanding the bare mass ratios in perturbation theory

$$\cos\theta^o_W = \cos(\theta_W + \delta\theta_W) \qquad \cos\theta^o_G = \cos(\theta_G + \delta\theta_G) \tag{6.32}$$

one finds from the general symmetric solution:

$$s^{(0)}_{\Gamma_{cl}}\Gamma^{(1)}_{inv}(\delta\theta_W, \delta\theta_G) + (\delta S)^{(1)}(\Gamma_{cl})(\delta\theta_W, \delta\theta_G) = 0 + O(\hbar^2) \tag{6.33}$$

with

$$\mathcal{S}(\Gamma) = (\mathcal{S}^{(0)} + \delta S^{(1)})(\Gamma) + O(\hbar^2) \tag{6.34}$$

Corresponding to these expressions differentiation with respect to θ_W as well as θ_G are not $s_{\Gamma_{cl}}$-invariant operators, because their action on Γ_{cl} produces only $\Gamma^{(1)}_{inv}$. In addition one has to enlarge them both with mixed massless – massive field differentiation operators for being $s_{\Gamma_{cl}}$-invariant operators:

$$\begin{aligned}
\tilde{\partial}_{\theta_W} \equiv\ & \partial_{\theta_W} + \int\Big(A\frac{\delta}{\delta Z} - Z\frac{\delta}{\delta Z} + B_A\frac{\delta}{\delta B_Z} - B_Z\frac{\delta}{\delta B_A}\Big) \\
& + \tfrac{1}{\cos(\theta_W - \theta_G)}\int c_A\Big(\frac{\delta}{\delta c_Z} + \sin(\theta_W - \theta_G)\frac{\delta}{\delta c_Z}\Big) \\
& - \tfrac{1}{\cos(\theta_W - \theta_G)}\int\Big(\bar{c}_Z + \sin(\theta_W - \theta_G)\bar{c}_A\Big)\frac{\delta}{\delta\bar{c}_A}
\end{aligned} \tag{6.35}$$

$$\begin{aligned}
\tilde{\partial}_{\theta_G} \equiv\ & \partial_{\theta_G} - \tfrac{1}{\cos(\theta_W - \theta_G)}\int c_Z\Big(\frac{\delta}{\delta c_A} + \sin(\theta_W - \theta_G)\frac{\delta}{\delta c_Z}\Big) \\
& + \tfrac{1}{\cos(\theta_W - \theta_G)}\int\Big(\sin(\theta_W - \theta_G)\bar{c}_Z + \bar{c}_A\Big)\frac{\delta}{\delta\bar{c}_Z}
\end{aligned} \tag{6.36}$$

The operator $\tilde{\partial}_{\theta_G}$ is a symmetric operator with respect to rigid symmetry, whereas $\tilde{\partial}_{\theta_W}$ has to be enlarged by the contributions from the shift, because it acts on v already in the lowest order. The operator

$$\tilde{\partial}_{\theta_W} - \partial_{\theta_W} v \int \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} \Big) = \tilde{\partial}_{\theta_W} - \frac{2}{e} M_Z cos 2\theta_W \int \Big(\frac{\delta}{\delta H} + \hat{\zeta} \frac{\delta}{\delta \hat{H}} \Big) \tag{6.37}$$

is then also rigid symmetric.

Acting with the symmetric operators (6.24), (6.25), (6.26), (6.30), (6.31), (6.36) and (6.37) on the classical action one produces together with the polynomial (6.29) a complete basis for the hard breakings of the symmetric dilatational operator (6.15) in 1-loop order. Therefore it is possible to give the dilatational anomalies in the form of a CS equation, i.e. as a linear combination of differential operators. Writing all the soft breakings produced by symmetrization with respect to the shift on the r.h.s we get the CS equation of the standard model in 1-loop order:

$$\begin{aligned}
\Big(\quad & m\partial_m + \beta_e e \partial_e - \beta_{M_W} \tilde{\partial}_{\theta_W} + \beta_{m_H} m_H \partial_{m_H} + \sum_{i=1}^{N_F} \sum_f \beta_{m_{f_i}} m_{f_i} \partial_{m_{f_i}} \\
- \quad & \gamma_V (\mathcal{N}_V - \mathcal{N}_B + 2\xi\partial_\xi + 2\hat{\xi}\partial_{\hat{\xi}} + \sin\theta_G \cos\theta_G \tilde{\partial}_{\theta_G}) - \gamma_c \mathcal{N}_c \\
- \quad & \hat{\gamma}_V (\hat{\mathcal{N}}_V - \hat{\gamma}_B \hat{\mathcal{N}}_B + 2(\xi + \hat{\xi})\partial_{\hat{\xi}}) - \gamma_S \mathcal{N}_S - \gamma_{\hat{S}} \mathcal{N}_{\hat{S}} - \tilde{\gamma}_S \tilde{\mathcal{N}}_S \\
- \quad & \sum_{i=1}^{N_F} (\gamma_{F_{l_i}} \mathcal{N}^L_{F_{l_i}} + \gamma_{F_{q_i}} \mathcal{N}^L_{F_{q_i}} + \gamma_{e_i} \mathcal{N}^R_{e_i} + \gamma_{u_i} \mathcal{N}^R_{u_i} + \gamma_{d_i} \mathcal{N}^R_{d_i}) \Big) \Gamma^{(\leq 1)} \Big|_{\hat{\varphi}_o = 0} \\
= \quad & \int \Big((1 + \beta_e e \partial_e - \beta_{M_W} \partial_{\theta_W}) v \Big(\frac{\delta \Gamma}{\delta H} + \hat{\zeta} \frac{\delta \Gamma}{\delta \hat{H}} \Big) + v(\gamma_S + \tilde{\gamma}_S) \frac{\delta \Gamma}{\delta H} + \hat{\zeta} v \gamma_{\hat{S}} \frac{\delta \Gamma}{\delta \hat{H}} + \frac{m_H^2}{2} \frac{\delta \Gamma}{\delta \hat{\varphi}_o} \Big) \\
+ \quad & \int \tilde{\gamma}_S \hat{q}_a \tilde{I}_{ab} Y_b + \Delta_m^{\leq 3}
\end{aligned} \tag{6.38}$$

On the right hand side we have also collected all local lower dimensional 1-loop breakings, which are not classified by the lowest order symmetries, into $\Delta_m^{\leq 3}$. When finally the ST identity and Ward identities are established, we are able to prove that $\Delta_m^{\leq 3}$ is vanishing. The CS equation takes then essentially this form to all orders, in particular the number of independent operators we have introduced is exhausted by the list giving above. It is only the explicit form of higher order operators, which is changed due to the symmetrization with respect to the general Ward operators and the generalized ST identity.

Calculating the commutator of the CS operator and the local $U(1)$-Ward operator yields the abelian relation between the β-functions and anomalous dimensions.

$$\beta_e = \frac{\sin\theta_W}{\cos\theta_W} \beta_{M_W} + \gamma_V + \hat{\gamma}_V \tag{6.39}$$

In the CS equation we have already inserted the result, which comes out from the test on the local B-dependent part of the action (5.77) and (5.81). In addition one derives for the anomalous dimension of the external scalars $\hat{\gamma}_S$ in the QED-like parameterization (5.79)

$$\gamma_{\hat{S}} = \beta_e + \frac{\cos\theta_W}{\sin\theta_W}\beta_{M_W} + \gamma_V - \gamma_S \tag{6.40}$$

whereas $\tilde{\gamma}_S$ is an independent anomalous dimension and can be determined on the mixed external – propagating scalar 2-point functions. This function is only important, when one wants to interpret $\hat{\phi}_a$ as a background field.

We want to give here the results for the β-functions and anomalous dimensions of the vector-scalar sector. A complete list of the β-functions can be found in [47]. The β-functions of the electromagnetic coupling and the vector mass ratio are determined to

$$\begin{aligned}
\beta_e &= -\frac{e^2}{24\cdot 4\pi^2}\Big(42 - \tfrac{64}{3}N_F\Big)\\
\beta_{M_W} &= -\frac{e^2}{4\cdot 24\pi^2\sin\theta_W\cos\theta_W}\Big((43-8N_F) - (42-\tfrac{64}{3}N_F)\sin^2\theta_W\Big).
\end{aligned}\tag{6.41}$$

The remarkable point is, that the β-function of the electromagnetic coupling is indeed QED-like in the sense, that it does not involve mass ratios, as does β_{M_W}. In contrast to QED it involves the nonabelian contributions of the charged vectors. This has as a consequence that the sign of β_e is negative, if one considers the standard model without fermions or includes only one family. The anomalous dimensions of vectors in general gauges compatible with rigid symmetry are given by

$$\gamma_V = \frac{e^2}{4\pi^2\sin^2\theta_W}\Big(\frac{6\xi-25}{24} + \frac{1}{3}N_F\Big) \tag{6.42}$$

$$\hat{\gamma}_V = \frac{e^2}{4\pi^2}\Big(-\frac{6\xi-25}{24\sin^2\theta_W} + \frac{1}{24\cos^2\theta_W} + \frac{-3+8\sin^2\theta_W}{9\sin^2\theta_W\cos^2\theta_W}N_F\Big), \tag{6.43}$$

The CS-functions (6.41) and (6.42) are seen to fulfil the abelian relation (6.39).

Because the anomalous dimension γ_V is nonvanishing, the coefficient of the ghost mass differentiation ∂_{θ_G} is also nonvanishing and the ghost angle has therefore to be treated as an independent parameter. This proves finally that the ghost mass ratio gets independent higher order corrections. A choice compatible with renormalizability is to set all ghost masses equal, i.e. $\theta_G = 0$. Such a choice, however, is connected with nondiagonal vector-scalar propagators and not adequate for concrete calculations. Similarly it is seen that also the abelian gauge parameter $\hat{\xi}$ is an independent parameter of the theory.

Before we turn to the higher order breakings of the ST identity and Ward identities we want to consider the off-shell infrared problem as it appears in the CS equation. Because Δ_m has infrared dimension 2, it has to be proven explicitly that the insertions

$$\int A_\mu A^\mu\,, \bar{c}_A c_A,\ H \tag{6.44}$$

do not appear on the r.h.s. The proof is carried out best by using the Zimmermann algebraic identities, which relate insertions of infrared dimension 2 to insertions with infrared dimension 3. (The technique has been presented and applied for deriving the CS equation in the spontaneously broken Higgs-Yukawa model [49] and works here in the same way.) Then the r.h.s. contains all the 2-dimensional field polynomials explicitly and one is able to test the CS equation with respect to these field polynomials at $p^2 = 0$. Due to the existence of the field mixing operators appearing e.g. in $\hat{\mathcal{N}}_V$, and in the operator $\tilde{\partial}_{\theta_W}$ the l.h.s. will only vanish, if the mixed 2-point functions of massless and massive fields vanish at $p^2 = 0$:

$$\Gamma_{ZA}\Big|_{p^2=0} = 0 \qquad \Gamma_{\bar{c}_Z c_A}\Big|_{p^2=0} = \Gamma_{\bar{c}_A c_Z}\Big|_{p^2=0} = 0 \tag{6.45}$$

Otherwise, nonintegrable infrared divergencies appear to the next order and make it impossible to derive the CS equation of higher orders.

7. Higher orders

7.1. The quantum numbers of higher orders breakings

We complete now the analysis of the renormalization of the standard model by proving, that the ST identity (5.19), Ward identities of rigid symmetry (5.20) and local $U(1)$-symmetry (4.69) can be established in the general form as given in section 5.2 to all orders and lead to unique expressions for finite renormalized Green functions. The basic ingredient of this proof is the action principle in its quantized version, as it is valid in presence of massless particles in the framework of the BPHZL scheme [35, 39]. If the symmetries are established to a definite order n in perturbation theory, the breakings of the next order are restricted to be local field polynomials with definite ultraviolet and infrared degree:

$$\begin{array}{rclcrcl} (\mathcal{S}(\Gamma))^{(\leq n-1)} & = & 0 & \Longrightarrow & (\mathcal{S}(\Gamma))^{(\leq n)} & = & \Delta^{(n)}_{brs} \\ (\mathcal{W}_\alpha(\Gamma))^{(\leq n-1)} & = & 0 & & (\mathcal{W}_\alpha(\Gamma))^{(\leq n)} & = & \Delta^{(n)}_{\alpha} \end{array} \tag{7.1}$$

with

$$\begin{array}{rclcrcl} \dim^{UV} \Delta^{(n)}_{brs} & \leq & 4 & \qquad & \dim^{IR} \Delta^{(n)}_{brs} & \geq & 3 \\ \dim^{UV} \Delta^{(n)}_{\alpha} & \leq & 4 & & \dim^{IR} \Delta^{(n)}_{\alpha} & \geq & 2 \end{array} \tag{7.2}$$

The ultraviolet degree of the breakings is deduced from a pure power counting analysis of renormalizable quantum field theory, but the infrared degree is assigned due to the

BPHZL scheme. The BPHZL scheme implements those normalization conditions in the scheme, which have to be fulfilled for being able to carry out infrared finite computations for off-shell Green functions in presence of massless particles. Having the Green functions constructed in a different scheme as dimensional regularization these normalization conditions have to be established finally by adjusting local counterterms. The conditions, which ensure infrared finitiness for off-shell Green functions are read off form the BPHZL-scheme:

$$\begin{array}{rclcrcl}\Gamma_{Z^\mu A^\nu}(p^2=0) & = & 0 & \qquad & \Gamma_{A^\mu A^\nu}(p^2=0) & = & 0 \\ \Gamma_{Y_\chi c_A}(p^2=0) & = & 0 & & \Gamma_{\bar{c}_Z c_A}(p^2=0) & = & 0 \\ \Gamma_{\bar{c}_A c_Z}(p^2=0) & = & 0 & & \Gamma_{\bar{c}_A c_A}(p^2=0) & = & 0\end{array} \tag{7.3}$$

The breakings are furthermore restricted by the global symmetries, i.e. by electromagnetic charge conservation (4.14)

$$\mathcal{W}_{em}\Delta^{(n)}_{brs} = 0 \qquad \begin{array}{rcl}\mathcal{W}_{em}\Delta^{(n)}_{\pm} & = & \mp i\Delta^{(n)}_{\pm} \\ \mathcal{W}_{em}\Delta^{(n)}_{3} & = & 0\end{array} \tag{7.4}$$

by Faddeev-Popov charge conservation

$$\mathcal{W}_{\phi\pi}\Delta^{(n)}_{brs} = \Delta^{(n)}_{brs} \qquad \mathcal{W}_{\phi\pi}\Delta^{(n)}_{\alpha} = 0$$

and by conservation of lepton and quark family number (4.15)

$$\mathcal{W}_{\delta_i}\Delta^{(n)}_{brs} = 0 \qquad \mathcal{W}_{\delta_i}\Delta^{(n)}_{\alpha} = 0$$

The algebraic restrictions on the breakings derived from nilpotency of the ST operator (4.9), algebra of Ward operators (4.10) and the consistency equation (4.13) read:

$$\mathrm{s}_{\Gamma_{cl}}\Delta^{(n)}_{brs} = 0 + O(\hbar^{n+1}) \tag{7.5}$$

$$\mathcal{W}_\alpha\Delta^{(n)}_\beta - \mathcal{W}_\beta\Delta^{(n)}_\alpha = \varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}\Delta^{(n)}_{\gamma'} + O(\hbar^{n+1}) \tag{7.6}$$

$$\mathrm{s}_{\Gamma_{cl}}\Delta^{(n)}_\alpha - \mathcal{W}_\alpha\Delta^{(n)}_{brs} = 0 + O(\hbar^{n+1}) \tag{7.7}$$

The symmetry operators involved are the ones of the tree approximation, because higher order contributions are not effective when acting in perturbation theory on a polynomial of order $\hbar^n$.

In the higher order analysis one has to find all the breakings compatible with quantum numbers and symmetries, and one has to prove, that they can be absorbed into a redefinition of the ST operator, the Ward operators of rigid symmetry and by adjusting finite counterterms, without destroying the on-shell normalization conditions on the 2-point functions. This computation is well-defined and can be carried out straightforwardly. However, there are a lot of terms which have to be considered in the standard model, even if CP-invariance is assumed. The analysis is simplified enormously, when we take advantage of the fact, that the breakings are also classified under the symmetric dilatational operator $\mathcal{W}^D_{sym}$ (6.15), and when we finally use the knowledge about local symmetry invariants. These local invariants have been completely characterized, when we solved the ST identiy and Ward identities in the classical approximation in generality without using a perturbative expansion and explicit form of the operators.

We proceed now as follows: First we consider the breakings of the ST identity and classify them in variations and non-variations, the anomalies (section 7.2):

$$\Delta^{(n)}_{brs} = s_{\Gamma_{cl}} \Gamma^{(n)}_{gen} + r^{(n)} \Delta^{anom}_{brs} \tag{7.8}$$

Here $\Gamma^{(n)}_{gen}$ is a general local field polynomial with quantum numbers of the classical action and UV dimension less than four. One is not able to exclude at this stage field polynomials of infrared dimension three. It is well-known, that there are anomalies in the standard model, which are given in the next section. The anomalies have to be shown to vanish – in the 1-loop order by inspection of diagrams, in higher orders by applying the non-renormalization theorems. These theorems state, that the anomalies of the ST identity vanish in higher orders, if they vanish in lowest order ([46] and references therein). Application of the non-renormalization theorems to the standard model will be considered elsewhere and we take its validity as granted for the purpose of the paper. The variations we absorb as far as possible into finite counterterms to the action. In particular, breakings of IR dimension three cannot be absorbed into counterterms for reasons of infrared definiteness, although they are variations. Then we establish Ward identities of rigid symmetry and are finally able to define a unique ST identity (section 7.3). In this analysis we do not have to consider tests with respect to B-field and with respect to antighosts, because this part has been already constructed in agreement with the symmetries by solving the ghost equation (see section 5.4). In particular we can carry out the variable transformation $\rho_\alpha \to \rho'_\alpha$ and $Y_a \to Y'_a$ as given in (5.81) and establish the symmetries on $\Gamma^{nl}(\rho', Y'_a)$, as usually done (cf. e.g [3]). When we have established both, the rigid Ward identities and the ST identity, the abelian local Ward identity is identified. Its breakings are known to be total divergencies. The variations can be absorbed into a redefinition of the lepton and quark couplings of the abelian subgroup and the anomalous currents vanish, if the anomalies in the ST identity vanish. At the very end one has one single unspecified parameter, which is not fixed on the 2-point functions. This parameter can be finally adjusted to be the electromagnetic fine structure constant in the Thompson limit (5.73), as it is done in the QED-like on shell schemes (see for a review [22] and references therein).

7.2. The cohomolgy and the Adler-Bardeen anomaly

In the first step we concentrate completely in finding the non-variations of the breakings under the ST identity, the Adler-Bardeen anomalies [17, 18, 19]. In the construction of the Callan-Symanzik equation we have shown that anomalies of the ST identity can appear only in the 4-dimensional field polynomials (6.19). This analysis is valid to all orders, once the CS equation is established to all orders (section 7.4).

$$\mathcal{W}^D_{sym}\Delta^{anom}_{brs} = 0 \tag{7.9}$$

all lower dimensional polynomials have been already seen to be variations. The consistency equation with the rigid transformation operators furthermore tells that only rigid invariant field polynomials can contribute to the anomaly:

$$\mathcal{W}_\alpha\Delta^{anom}_{brs} = 0 \tag{7.10}$$

Therefore the algebraic problem can be indeed formulated in symmetric variables $\alpha = +, -, 3, 4$:

$$\begin{array}{rclcrcl} c_\alpha & = & \delta\hat{g}_{\alpha b}c_b & \qquad & H' & = & H + v \\ V_\alpha & = & O_{\alpha a}(\theta_W)V_b & & \hat{H}' & = & \hat{H} + \hat{\zeta}v \end{array} \tag{7.11}$$

and the analysis is the same as one has to carry out in the symmetric theory. Here we see in the abstract approach that ultraviolet divergencies of the spontaneously broken theories are not worse than the ones of the symmetric theory [7, 8].

Since we have split off the transformation of the abelian component in the ST identity, the external field part is essentially treated as in a $SU(2)$ gauge theory. Therefore we remain finally with polynomials depending on vectors, scalars and fermions and arrive at the well-known Wess-Zumino consistency condition [51]:

$$\mathbf{w}_\alpha P_\beta - \mathbf{w}_\beta P_\alpha = \varepsilon_{\alpha\beta\gamma}\tilde{I}_{\gamma\gamma'}P_{\gamma'} \tag{7.12}$$

where $\mathbf{w}_\alpha$ are the gauge transformations of the tree approximation given in (2.35) and P_α is a 4-dimensional polynomial depending only on the propagating fields of the standard model:

$$\Delta^{anom}_{brs} = \int c_a O^T_{a\alpha}(\theta_W)P_\alpha(V_a, \phi_a, f^L_i, f^R_i) \tag{7.13}$$

The solution of the consistency equation has been analyzed quite generally in [21] and can be evaluated in the standard model without further complications. Having CP-invariance there are even no abelian contributions, which escape the algebraic treatment of consistency, because those terms are CP odd. We end up with the following explicit expression for the anomalies ($a = +, -, Z, A$ are physical field indices, $O(\theta_W)$ is defined in (2.28):

$$\begin{aligned}\Delta^{anom}_{brs} &= r_1 \int \varepsilon_{\mu\nu\rho\sigma} O_{4a}(\theta) c_a \partial^\mu \Big(O_{4b}(\theta_W) V^\nu_b \partial^\rho O_{4c}(\theta_W) V^\sigma_c \Big) \qquad (7.14)\\ &+ r_2 \int \varepsilon_{\mu\nu\rho\sigma} O_{4a}(\theta_W) c_a \partial^\mu \Big(V^\nu_b \tilde{I}_{bc} \partial^\rho V^\sigma_c - \frac{1}{3} \hat{\varepsilon}_{bcd}(\theta_W) V^\nu_b V^\rho_c V^\sigma_d \Big)\end{aligned}$$

with

$$\hat{\varepsilon}_{abc} = \varepsilon_{\alpha\beta\gamma} O_{\alpha a}(\theta_W) O_{\beta b}(\theta_W) O_{\gamma c}(\theta_W) \qquad (7.15)$$

The form of the anomaly is unique up to the addition of BRS-variations. The general classical action contains two rigid invariant field polynomials in vectors which are odd under parity transformations:

$$\Gamma^P_{cl}(V_a) = \int \varepsilon_{\mu\nu\rho\sigma} O_{4a}(\theta_W) V^\mu_a (k_1 V^\nu_b \tilde{I}_{bc} \partial^\rho V^\sigma_c + k_2 \hat{\varepsilon}_{abc} V^\mu_a V^\nu_b V^\rho_c V^\sigma_4) \qquad (7.16)$$

We have used the $s_{\Gamma_{cl}}$-variations of these field polynomials to bring the anomaly in the form given above, where it only depends on the abelian ghost combination.

In 1-loop order the coefficients of the anomaly vanish. One has to note that the purely abelian part vanishes due to electromagnetic current conservation and therefore depends crucially on establishment of a local Ward identity in connection with electromagnetic current conservation.

In the following, we assume that the non-renormalization theorems on the Adler-Bardeen is valid, if we are able to prove a local abelian Ward identity and establish the Callan-Symanzik equation order by order in perturbation theory. These two equations are the necessary prerequisite for proving the non-renormalization theorems in higher orders [46].

7.3. The establishment of symmetries

We start the consideration in 1-loop order and take for the lowest order the usual standard model Lagrangian as given in section 2. If one calculates the finite Green functions Γ^{ren} with the Feynman rules of the tree approximation in a specific scheme to 1-loop order, the ST identity is in general broken by the local field polynomial Δ_{brs}:

$$(\mathcal{S}\Gamma^{ren})^{(\leq 1)} = \Delta_{brs} \qquad (7.17)$$

Because the coefficient of the anomaly vanishes in 1-loop order, the breaking can be rewritten as a variation of integrated field polynomials:

$$\Delta_{brs} = s_{\Gamma_{cl}} \Gamma^{(1)}_{gen} \qquad (7.18)$$

and

$$\dim^{UV}\Gamma^{(1)}_{gen}\leq 4 \qquad \dim^{IR}\Gamma^{(1)}_{gen}\geq 3 \tag{7.19}$$

In the BPHZL scheme it is obvious that we do not have to introduce counterterms with respect to a photon mass term, because the variation of the photon mass term has infrared dimension 2.

$$\mathrm{s}_{\Gamma_{cl}}\int A^\mu A_\mu = \int 2\partial^\mu c_A A_\mu + ... \tag{7.20}$$

But all further field polynomials appear in principle in $\Gamma^{(1)}_{gen}$. In order to be able to establish the on-shell conditions and conditions on the residua of all propagating fields, we have to show that we have not to dispose of those field polynomials which are fixed by the normalization conditions. These field polynomials are listed in (5.13). Due to the fact that we have eliminated the antighost contributions by using the ghost equations (5.81), the normalization conditions specified on the ghost 2-point functions, are now translated into normalization conditions on the external field part. Explicitly we are not able to dispose of the terms $Y_\chi c_Z$ and $Y_\chi c_A$ for establishing on-shell conditions without introducing infrared divergencies (cf. eqs. (5.115), (5.116) and also (7.4)). The terms $\rho_\alpha a^g_{\alpha b}\partial c_b$ are kept arbitrary and are finally adjusted on the residua of ghost propagators. Therefore we find the following list of field polynomials, which are not available for adjusting finite counterterm contributions

$$\begin{aligned}\Gamma^{gen}_{bil} &= \int\Big(-\tfrac{1}{4}(\partial^\mu V^\nu_a-\partial^\nu V^\mu_a)Z^V_{ab}(\partial_\mu V_{\nu b}-\partial_\nu V_{\mu b})+\tfrac{1}{2}V^\mu_a\mathcal{M}^V_{ab}V_{\mu b}\\ &\quad+\tfrac{1}{2}\partial^\mu\phi_a Z^S_{ab}\partial_\mu\phi_b-\tfrac{1}{2}M_H^2H^2(x)\\ &\quad+iZ^R_{f_i}\bar f^R_i\partial\!\!\!/ f^R_i+iZ^L_{f_i}\bar f^L_i\partial\!\!\!/ f^L_i-M_{f_i}(\bar f^R_i f^L_i+\bar f^R_i f^L_i)\\ &\quad+\rho^\mu_\alpha a^g_{\alpha b}\partial_\mu c_b+Y_\chi m^g_{\chi b}c_b\Big)\end{aligned} \tag{7.21}$$

These terms can be eliminated if they are in one to one correspondence with $\mathrm{s}_{\Gamma_{cl}}$-invariants. From the detailed considerations of the classical approximation it is seen, that there are left three polynomials, namely $\int Z^\mu A_\mu$, $\int Y_\chi c_Z$ and $\int Y_\chi c_A$ which do not correspond to $\mathrm{s}_{\Gamma_{cl}}$-invariants. Therefore we are able to write

$$\begin{aligned}\Delta_{brs} &= \mathrm{s}_{\Gamma_{cl}}\Gamma'_{break}+u_1\mathrm{s}_{\Gamma_{cl}}\int M_Z^2Z^\mu A_\mu+u'_2M_Z\mathrm{s}_{\Gamma_{cl}}\int Y_\chi c_Z+u'_3M_Z\mathrm{s}_{\Gamma_{cl}}\int Y_\chi c_A\\ &= \mathrm{s}_{\Gamma_{cl}}\Gamma_{break}+u_1\mathrm{s}_{\Gamma_{cl}}\int\Big(A^\mu\frac{\delta}{\delta Z^\mu}-Z^\mu\frac{\delta}{\delta A^\mu}\Big)\Gamma_{cl}\\ &\quad+u_2\mathrm{s}_{\Gamma_{cl}}\int(\sin\theta_W c_Z+\cos\theta_W c_A)\Big(\sin\theta_W\frac{\delta}{\delta c_Z}+\cos\theta_W\frac{\delta}{\delta c_A}\Big)\Gamma_{cl}\\ &\quad+u_3\mathrm{s}_{\Gamma_{cl}}\int(\cos\theta_W c_Z-\sin\theta_W c_A)\Big(\sin\theta_W\frac{\delta}{\delta c_Z}+\cos\theta_W\frac{\delta}{\delta c_A}\Big)\Gamma_{cl}\end{aligned} \tag{7.22}$$

There we have rewritten the field polynomials with infrared dimension 3 into field operators acting on the classical action, and have the remaining terms shifted into Γ_{break}. Γ_{break} consists of all integrated CP-even field polynomials except the ones listed in (7.21), i.e. it has especially infrared dimension 4. Applying the consistency equation between the Ward operators of rigid symmetry and the ST identity (7.5) we find that the breakings of the Ward operators take the following form:

$$\begin{aligned}\Big(\mathcal{W}_\alpha\Gamma^{ren}\Big)^{(\leq 1)} &= \Delta_\alpha^{inv} + \mathcal{W}_\alpha\Gamma_{break} + u_1\mathcal{W}_\alpha\int\Big(A^\mu\frac{\delta}{\delta Z^\mu} - Z^\mu\frac{\delta}{\delta A^\mu}\Big)\Gamma_{cl} \\ &\quad + u_3\mathcal{W}_\alpha\int(\cos\theta_W c_Z - \sin\theta_W c_A)\Big(\sin\theta_W\frac{\delta}{\delta c_Z} + \cos\theta_W\frac{\delta}{\delta c_A}\Big)\Gamma_{cl}\end{aligned} \tag{7.23}$$

Δ_α^{inv} comprises all field polynomials, which are $\mathrm{s}_{\Gamma_{cl}}$-invariants.

$$\mathrm{s}_{\Gamma_{cl}}\Delta_\alpha^{inv} = 0 \tag{7.24}$$

Considering the list of all possible breakings compatible with the algebra of rigid symmetry and discrete and global symmetries it is seen, that Δ_α^{inv} itself can be written as a $\mathrm{s}_{\Gamma_{cl}}$ and $\mathcal{W}_\alpha$-variation. Explicitly we find the following list of contributions:

$$\Delta_\alpha = \mathcal{W}_\alpha\Big(\sum_k u_k\mathcal{N}_k\Gamma_{cl} + \int v^{(1)}\frac{\delta}{\delta H}\Gamma_{cl}\Big) \tag{7.25}$$

and $\mathcal{N}_k$ comprises the following field operators

$$\begin{aligned}
\mathcal{N}_{ZA} &= \int\Big(Z\frac{\delta}{\delta Z} + A\frac{\delta}{\delta A} + c_Z\frac{\delta}{\delta c_Z} + c_A\frac{\delta}{\delta c_A} - \rho_3\frac{\delta}{\delta\rho_3} - \sigma_3\frac{\delta}{\delta\sigma_3}\Big) \\
\mathcal{N}_{\widetilde{ZA}} &= \int\Big(\Big(\sin\theta_W Z + \cos\theta_W A\Big)\Big(\cos\theta_W\frac{\delta}{\delta Z} - \sin\theta_W\frac{\delta}{\delta A}\Big) - \rho_3\partial(\sin\theta_W c_Z + \cos\theta_W c_A) \\
&\qquad + \Big(\sin\theta_W c_Z + \cos\theta_W c_A\Big)\Big(\cos\theta_W\frac{\delta}{\delta c_Z} - \sin\theta_W\frac{\delta}{\delta c_A}\Big)\Big) \\
\mathcal{N}_{c_Z c_Z} &= \int\Big(\Big(\cos\theta_W c_Z - \sin\theta_W c_A\Big)\Big(\cos\theta_W\frac{\delta}{\delta c_Z} - \sin\theta_W\frac{\delta}{\delta c_A}\Big) - \sigma_3\frac{\delta}{\delta\sigma_3}\Big) \\
\mathcal{N}_{\widetilde{c_Z c_A}} &= \int\Big(\sin\theta_W c_Z + \cos\theta_W c_A\Big)\Big(\cos\theta_W\frac{\delta}{\delta c_Z} - \sin\theta_W\frac{\delta}{\delta c_A}\Big) \\
\mathcal{N}_{\phi_+} &= \int\phi_+\frac{\delta}{\delta\phi_+} + \phi_-\frac{\delta}{\delta\phi_-} - Y_+\frac{\delta}{\delta Y_+} - Y_-\frac{\delta}{\delta Y_-} \\
\mathcal{N}_\chi &= \int\Big(\chi\frac{\delta}{\delta\chi} - Y_\chi\frac{\delta}{\delta Y_\chi}\Big) \\
\mathcal{N}_{e_i} &= \int\Big(\overline{e_i^L}\frac{\delta}{\delta\overline{e_i^L}} + \frac{\delta}{\delta e_i^L}e_i^L - \overline{\psi_{e_i}^R}\frac{\delta}{\delta\overline{\psi_{e_i}^R}} - \frac{\delta}{\delta\psi_{e_i}^R}\psi_{e_i}^R\Big) \\
\mathcal{N}_{d_i} &= \int\Big(\overline{d_i^L}\frac{\delta}{\delta\overline{d_i^L}} + \frac{\delta}{\delta d_i^L}d_i^L - \overline{\psi_{d_i}^R}\frac{\delta}{\delta\overline{\psi_{d_i}^R}} - \frac{\delta}{\delta\psi_{d_i}^R}\psi_{d_i}^R\Big)
\end{aligned} \tag{7.26}$$

For absorbing these polynomials into the Ward operators (5.20) we have finally to note that the expansion of the coefficients to 1-loop order can be also rewritten into a field differentiation acting on Γ_{cl}. Denoting with $\mathcal{W}_\alpha^{(0)}$ the Ward operator of the tree approximation (2.112) then we write

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha{}^{(0)} + \delta\mathcal{W}_\alpha^{(1)} + O(\hbar^2) \tag{7.27}$$

with

$$\begin{aligned}
\delta\mathcal{W}_\alpha^{(1)}\Gamma_{cl} \quad = \quad & \mathcal{W}_\alpha^{(0)} \int \Big(\delta r_Z Z \frac{\delta}{\delta Z} + \delta r_A A \frac{\delta}{\delta A} + \delta\theta^V (Z\frac{\delta}{\delta A} - A\frac{\delta}{\delta Z}) \\
& + \delta r_{33}^g \Big(\cos\theta_W c_Z - \sin\theta_W c_A\Big)\Big(\cos\theta_W \frac{\delta}{\delta c_Z} - \sin\theta_W \frac{\delta}{\delta c_A}\Big) \\
& + \delta r_{34}^g \Big(\cos\theta_W c_Z - \sin\theta_W c_A\Big)\Big(\sin\theta_W \frac{\delta}{\delta c_Z} + \cos\theta_W \frac{\delta}{\delta c_A}\Big) \\
& + \delta r_{43}^g \Big(\sin\theta_W c_Z + \cos\theta_W c_A\Big)\Big(\cos\theta_W \frac{\delta}{\delta c_Z} - \sin\theta_W \frac{\delta}{\delta c_A}\Big) \\
& + \delta r_+^S \mathcal{N}_{\phi_+} + \delta r_\chi^S \mathcal{N}_\chi + \sum_{i=1}^{N_F} (\delta r_{l_i} \mathcal{N}_{e_i} + \delta r_{q_i} \mathcal{N}_{d_i}) + \delta v \frac{\delta}{\delta H}\Big)\Gamma_{cl}
\end{aligned} \tag{7.28}$$

Therefrom it is seen that the scalar and fermion contributions are immediately absorbed into a redefinition of the tree Ward operators compatible with the algebra. For the ghosts and vectors only parts of the invariants are absorbed, but a straigthforward calculation shows that all the remaining contributions can be shifted into a $\hat{\Gamma}_{break}$, which again includes only interaction terms. Therefore we remain with

$$\Big(\mathcal{W}\Gamma^{ren}\Big)^{(\leq 1)} = -\delta\mathcal{W}_\alpha^{(1)}\Gamma_{cl} + \mathcal{W}_\alpha(\Gamma_{break} + \hat{\Gamma}_{break}) \tag{7.29}$$

If one goes back with these expressions to the ST identity we have now for reasons of consistency to split off therein the corresponding contributions $\delta\mathcal{S}^{(1)}$, because otherwise the consistency relations are not valid to the next order.

$$\mathcal{S}(\Gamma^{ren})^{(\leq 1)} = -\delta S^{(1)}\Gamma_{cl} + s_{\Gamma_{cl}}(\Gamma_{break} + \hat{\Gamma}_{break}) \tag{7.30}$$

where

$$\begin{aligned}
\delta S^{(1)}\Gamma_{cl} \quad = \quad & s_{\Gamma_{cl}} \int \Big(\delta r_Z Z \frac{\delta}{\delta Z} + \delta r_A A \frac{\delta}{\delta A} + \delta\theta^V (Z\frac{\delta}{\delta A} - A\frac{\delta}{\delta Z}) \\
& + \delta r_{33}^g \Big(\cos\theta_W c_Z - \sin\theta_W c_A\Big)\Big(\cos\theta_W \frac{\delta}{\delta c_Z} - \sin\theta_W \frac{\delta}{\delta c_A}\Big) \\
& + \delta r_{34}^g \Big(\cos\theta_W c_Z - \sin\theta_W c_A\Big)\Big(\sin\theta_W \frac{\delta}{\delta c_Z} + \cos\theta_W \frac{\delta}{\delta c_A}\Big) \\
& + \delta r_{44}^g \Big(\sin\theta_W c_Z + \cos\theta_W c_A\Big)\Big(\sin\theta_W \frac{\delta}{\delta c_Z} + \cos\theta_W \frac{\delta}{\delta c_A}\Big)\Big)\Gamma_{cl}
\end{aligned} \tag{7.31}$$

Defining the generating functional of Green functions of the standard model by

$$\Gamma = \Gamma_{cl} + \Gamma^{ren} - \Gamma_{break} - \hat{\Gamma}_{break} + O(\hbar^2) \tag{7.32}$$

and the Ward operators and ST operator to 1-loop order by

$$\mathcal{W}_\alpha = \mathcal{W}_\alpha^{(o)} + \delta\mathcal{W}_\alpha^{(1)} \qquad \mathcal{S}(\Gamma) = \mathcal{S}^{(0)}(\Gamma) + \delta\mathcal{S}^{(1)}(\Gamma) \tag{7.33}$$

we have proceeded to absorb all breakings into counterterms of the action and a redefinition of the symmetry operators compatible with the algebra.

$$\mathcal{S}(\Gamma) = 0 + O(\hbar^2) \qquad \mathcal{W}_\alpha\Gamma = 0 + O(\hbar^2) \tag{7.34}$$

The field polynomials, on which the normalization conditions on the 2-point functions are established, are not touched in the construction. It is worth to note that at higher orders as it was in the classical approximation the ST identity is only completely specified, if we construct simultanouesly the Ward identities of rigid symmetry. By now we have suppressed those contributions which depend on the external field $\hat{\phi}_a$. They do not contribute to anomalies and the absorption of their breakings proceeds as in [50], where we have carried out the same analysis in the abelian Higgs model.

Since we have only determined the normalization conditions on the 2-point functions in the above construction, the finite Green functions are not unique by now. First we have to fix the remaining coupling constant by a normalization condition on an interaction vertex as given e.g in (5.73). From the construction of the general classical invariant action it is seen, that furthermore the abelian couplings of fermions are not specified. The contributions which remain arbitrary can be read off from the fermionic part Γ^{gen}_{matter} (5.60) and the external field part $\Gamma^{gen}_{ext.f.}$ (5.64) of the general invariant classical action. They are obtained in their explicit form by expanding G_{δ_i} to 1-loop order and setting all other coefficients to their tree value. We denote with Γ_{δ_i} the corresponding $s_{\Gamma_{cl}}$-invariant field polynomials.

$$\Gamma' = \Gamma + \sum_{i=1}^{N_F} \sum_{\delta=l,q} G^{(1)}_{\delta_i}\Gamma_{\delta_i} \tag{7.35}$$

and Γ and Γ' satisfy both the ST identity and rigid Ward identities in the same form.

For fixing these undetermined parameters one has to use the local $U(1)$ Ward identity (4.69). Having constructed Γ in accordance with ST identity and rigid symmetry, it is obvious that the abelian Ward identity is only broken by total divergencies in 1-loop order. Consistency (4.68) furthermore restricts the breakings to be again $s_{\Gamma_{cl}}$-invariants and rigid invariants. One has

$$\begin{aligned}&\Big(\frac{e}{\cos\theta_W}\mathbf{w}^Q - \partial\Big(\frac{1}{r_Z}\sin\Theta\frac{\delta}{\delta Z} + \frac{1}{r_A}\cos\Theta\frac{\delta}{\delta A}\Big)\Big)\Gamma \\ &= \Box\Big(\frac{1}{r_Z}\sin\Theta B_Z + \frac{1}{r_A}\cos\Theta B_A\Big) \\ &+\delta g_1\partial_\mu j^{matter}_\mu + \delta g_{\delta_i}\partial_\mu j^{\delta_i} + r_i\partial_\mu J^{anom}_i + O(\hbar^2)\end{aligned} \tag{7.36}$$

The breakings are given by

$$\partial^\mu j_\mu^{matter} = \mathbf{w}^Q \Gamma_{cl} \qquad \partial^\mu j_\mu^{\delta_i} = \mathbf{w}_{\delta_i}\Gamma_{cl} \tag{7.37}$$

$\mathbf{w}_{\delta_i}$ denotes the non-integrated version of the operators of lepton and quark familiy conservation (4.15). The anomalous contribution is determined to

$$\begin{aligned} \partial^\mu J_\mu^{anom} &= r_1 \int \varepsilon_{\mu\nu\rho\sigma}\partial^\mu\Big(O_{4b}(\theta_W)V_b^\nu \partial^\rho O_{4c}(\theta_W)V_c^\sigma\Big) \\ &+ r_2 \int \varepsilon_{\mu\nu\rho\sigma}\partial^\mu\Big(V_b^\nu \tilde{I}_{bc}\partial^\rho V_c^\sigma - \frac{1}{3}\hat{\varepsilon}_{bcd}(\theta_W)V_b^\nu V_c^\rho V_d^\sigma\Big) \end{aligned} \tag{7.38}$$

The coefficients of the anomalous currents in the Ward identity are related to the ones of the ST identity. (This can be seen by establishing the abelian Ward identity as a $s_{\Gamma_{cl}}$-variation of a ghost equation.) In particular, they vanish in 1-loop order, and they vanish to all orders, if the non-renormalization theorems are valid in the standard model. Vanishing of the purely abelian current anomaly to all orders can be proved only by means of the local abelian Ward identity [46].

The absorption of non-anomalous currents proceeds as in the classical approximation (cf. (5.70) – (5.73)): The lepton and quark family currents are absorbed by fixing the by now undetermined constants $G_{\delta_i}^{(1)}$ in (7.35) and the matter current ∂j^{matter} is absorbed into the overall normalization of the Ward identity.

The finite renormalized Green functions are constructed in 1-loop order uniquely: They satisfy the ST identity, the Ward identities of rigid symmetry and the local abelian Ward identity. The 2-point functions of physical fields and of Faddeev-Popov ghosts have one particle properties, and especially the mass matrices of massiv massless particles are diagonalized at $p^2 = 0$. This property ensures that the Green functions of the next order exist in renormalized perturbation theory.

7.4. Induction to all orders

Having constructed the Green functions of 1-loop order in accordance with the symmetries and in accordance with off-shell infrared existence (7.3), the action principle can be applied to the renormalized Green functions of the next order in the same way, as it applied, when we proceeded from lowest order to 1-loop (7.17) and (7.18). For this reason we are able to carry out the proof to all orders by induction (7.1). Assuming that the ST identity (5.19), the Ward identities of rigid symmetry (5.20) and the local $U(1)$ Ward identity (4.69) are established for the Green functions to order $n-1$, then one can make the induction step to order n. The important point is the fact that the UV dimension and IR dimension of the breakings is not changed, because the counterterms, we had to add for establishing the symmetries, are compatible with UV and IR dimension 4. In particular, the Γ_{eff}, which governs the perturbative expansion of Green functions in the BPHZL scheme, is a 4-4 insertion (see [3] for details).

Because the one-loop breakings have been absorbed in accordance with the algebraic properties and the consistency equation

$$\begin{aligned} \mathrm{s}_\Gamma \mathcal{S}(\Gamma) &= 0 \quad \text{for any} \quad \Gamma \\ \mathrm{s}_\Gamma \mathrm{s}_\Gamma &= 0 \quad \text{if} \quad \mathcal{S}(\Gamma) = 0 \\ \mathcal{W}_\alpha \mathcal{S}(\Gamma) - \mathrm{s}_\Gamma \mathcal{W}_\alpha \Gamma &= 0 \quad \text{for any} \quad \Gamma \\ \left[\mathcal{W}_\alpha, \mathcal{W}_\beta\right] &= \varepsilon_{\alpha\beta\gamma} \tilde{I}_{\gamma\gamma'} \mathcal{W}_{\gamma'} \end{aligned} \tag{7.39}$$

the breakings of order n are algebraically characterized by (7.5). Therefore we are able to proceed from order $n-1$ to order n in the same way as from lowest order to 1-loop order, since we did not use explicit expressions of 1-loop order, but only algebraic and power counting properties.

The only ingredient of 1-loop order has been the characterization of the anomaly candidates by the CS equation (7.9). In order to close the arguments we have finally to derive the Callan-Symanzik to order $n-1$. The CS equation of 1-loop is given in (6.38). Since the symmetries are established to order n the the unsymmetric soft field polynomial $\Delta_{\overline{m}}^{\leq 3}$ vanishes. The construction of the higher order CS equation proceeds for the hard breakings as given in section 6.2, especially there are the same number of independent parameters and $\mathrm{s}_{\Gamma_{cl}}$-invariants. All differentiations with respect to couplings and mass parameters act on the parameters, which appear in higher orders as corrections in the ST identity and the Ward identities of rigid symmetry. As it is for the differentiation with respect to M_W (6.35) in 1-loop order, they have all supplemented by field operators in order to commute with the ST operator and Ward operators of rigid symmetry. The explicit expressions can be read off from eqs. (7.28) and (7.30) and will be given in detail elsewhere. We denote with $\tilde{\partial}_\lambda, \lambda = e, m_H, m_{f_i}, \theta_W, \theta_G$ the rigid and $\mathrm{s}_{\Gamma_{cl}}$ symmetric operators of higher orders. (The weak mixing angle and the ghost angle are given by the on-shell definition (5.107)). Also the higher order field differentiation operators (6.24), (6.25) and (6.26), which correspond to the anomalous dimensions, are modified in an obvious way. The soft breakings of the CS equation are constructed as in the tree approximation, because they are completely characterized by their algebraic properties. The CS equation is then finally given by

$$\Big(\; m\partial_m + \beta_e e \tilde{\partial}_e - \beta_{M_W} \tilde{\partial}_{\theta_W} + \beta_{m_H} m_H \tilde{\partial}_{m_H} + \sum_{i=1}^{N_F} \sum_f \beta_{m_{f_i}} m_{f_i} \tilde{\partial}_{m_{f_i}} \tag{7.40}$$

$$
\begin{aligned}
&- \ \gamma_V(\mathcal{N}_V - \mathcal{N}_B + 2\xi\partial_\xi + 2\hat{\xi}\partial_{\hat{\xi}}) - \beta_{\theta_G}\tilde{\partial}_{\theta_G} - \gamma_c\mathcal{N}_c \\
&- \ \hat{\gamma}_V(\hat{\mathcal{N}}_V - \hat{\gamma}_B\hat{\mathcal{N}}_B + 2(\xi+\hat{\xi})\partial_{\hat{\xi}}) - \gamma_S\mathcal{N}_S - \gamma_{\hat{S}}\mathcal{N}_{\hat{S}} - \tilde{\gamma}_S\tilde{\mathcal{N}}_S \\
&- \ \sum_{i=1}^{N_F}(\gamma_{F_{l_i}}\mathcal{N}^L_{F_{l_i}} + \gamma_{F_{q_i}}\mathcal{N}^L_{F_{q_i}} + \gamma_{e_i}\mathcal{N}^R_{e_i} + \gamma_{u_i}\mathcal{N}^R_{u_i} + \gamma_{d_i}\mathcal{N}^R_{d_i})\Big)\,\Gamma\Big|_{\hat{\varphi}_o=0} \\
&= \ \int\Big(\Big((1+\sum_\lambda\beta_\lambda\partial_\lambda)v\Big)\frac{\delta\Gamma}{\delta H} + \Big((1+\sum_\lambda\beta_\lambda\partial_\lambda)\hat{\zeta}v\Big)\frac{\delta\Gamma}{\delta\hat{H}} \\
&\quad + v(\gamma_S+\tilde{\gamma}_S)\frac{\delta\Gamma}{\delta H} + \hat{\zeta}v\gamma_{\hat{S}}\frac{\delta\Gamma}{\delta\hat{H}} + \frac{m_H^2}{2}\frac{\delta\Gamma}{\delta\hat{\varphi}_o}\Big) + \int\tilde{\gamma}_S\hat{q}_a\tilde{I}_{ab}Y_b
\end{aligned}
$$

Infrared existence of the CS equation can be proved as in 1-loop order, since the conditions for off-shell infrared existence (7.3) have been maintained in the construction of symmetry operators.

With the establishment of the CS equation the construction of standard model Green functions to all orders is completed. The ST identity and Ward identities of rigid symmetry

$$
\mathcal{S}(\Gamma) = 0 \qquad \mathcal{W}_\alpha\Gamma = 0 \tag{7.41}
$$

the local abelian Ward identity

$$
\Big(g_1\mathbf{w}^Q - \partial\Big(\frac{1}{r_Z}\sin\Theta\frac{\delta}{\delta Z} + \frac{1}{r_A}\cos\Theta\frac{\delta}{\delta A}\Big)\Big)\Gamma = \Box\Big(\frac{1}{r_Z}\sin\Theta B_Z + \frac{1}{r_A}\cos\Theta B_A\Big) \tag{7.42}
$$

with

$$
\mathbf{w}^Q \equiv \mathbf{w}_{em} - \mathbf{w}_3 \tag{7.43}
$$

define uniquely the Green functions of the standard model of electroweak interactions to all orders of perturbation theory in the on-shell scheme.

8. Conclusions and outlook

In this article we have constructed the finite renormalized Green functions of the standard model of electroweak interactions to all orders of perturbation theory. Special attention has been paid to the construction of 2-point functions in the on-shell scheme. Only if the Green functions have one-particle properties in the LSZ-limit (apart from the problem of unstable particles), can one proceed to construct the S-matrix and finally prove unitarity of the physical S-matrix. These properties are the main requirements for being able to interpret a quantum field theory as a physical theory of fundamental interactions. Since the standard model contains massless particles, mass diagonalization of massless and massive fields is connected with off-shell infrared existence of finite renormalized Green functions.

The analysis has been carried out using the method of algebraic renormalization, which until now was applied mainly to theories with semisimple gauge groups. In order to apply the method to the standard model with the non-semisimple $SU(2) \times U(1)$ group we had to generalize the method of algebraic renormalization at some points. In particular, we had to obtain the symmetry operators by means of their algebraic properties instead of postulating them in an explicit form a priori. The parameters, which appear in the general solution of the algebra, correspond to field redefinitions of individual fields and, in particular, to non-diagonal field redefinitions of neutral massive/massless fields. The adjustment of the latter parameters is essential for diagonalizing the mass matrix of neutral vectors at $p^2 = 0$. Due to the non-semisimple group structure, the abelian component of the action contains additional free parameters, which are not specified by the Slavnov-Taylor identity. These are interpreted as the couplings of the currents of lepton and quark family conservation. Classically, these currents are conserved in the standard model, if one neglects mixing of quarks due to the CKM matrix. In the general case there are classically the conserved currents of fermion family number conservation and of baryon number conservation. These currents are not gauged, but are not distinguished from the electromagnetic current in the theoretical prescription, since they have the same quantum numbers. In order to characterize the interactions prescribed by the standard model as the ones of weak and electromagnetic interaction, the electromagnetic and the lepton and quark number currents have to be identified and fixed by a Ward identity. Because the electromagnetic Ward identity of current conservation cannot be derived for off-shell Green functions, we have to use a specific form of the abelian local Ward identity. This identity is the functional generalization of the Gell-Mann Nishijima relation. We want to point out that in the general case, with quark family mixing, the local Ward identity becomes even more important for correct adjustment of the electromagnetic current.

An abelian Ward identity cannot be derived a priori, but has to be characterized in the group structure as being abelian. For this reason, we have to require invariance under the nonabelian rigid symmetries. The construction of Green functions in agreement with rigid symmetry restricts the gauge fixing sector and the number of independent parameters appearing therein. In order to be able to diagonalize the mass matrix of neutral ghosts at $p^2 = 0$, one has to introduce an additional ghost angle into the BRS-transformations of antighosts. In the on-shell scheme, this angle is related to the ghost mass ratio in a way similar to that in which the vector mass ratio is related to the weak mixing angle. From the Callan-Symanzik equation, one can see that the ghost mass ratio indeed has to be introduced as an independent parameter of the theory, since it has independent higher order corrections.

The most remarkable consequence of the higher order construction is the observation, that the standard model provides exactly the right number of parameters to bring the propagators to a form in which they have one-particle properties in the LSZ-limit. As we have pointed out, one has to adjust all of these parameters and one has to take into account *all* deformations allowed by the algebra. If we had not succeeded with the analysis as prescribed in the paper, then we would have had to prove that one-particle properties are the consequence of a symmetry. Such a procedure has to be carried out finally in the unphysical sector proving mass degeneracy for all unphysical fields by means of the Slavnov-Taylor identity [14, 20]. From this point of view the vector and the unphysical sector will be analysed carefully, when the renormalization is extended to CP-violating interactions.

Having constructed the symmetry operators, we can apply them immediately to explicit one-loop and higher loop calculations. The parameters appearing therein are mainly determined on 2-point functions, which are listed for one-loop order in the literature. One is then able to prove, if finite Green functions satisfy the Slavnov-Taylor identity. In dimensional regularization one has to pay most attention to such breakings, which are absorbed into parity violating counterterms of the effective action. Due to parity non-conservation, it is not evident that the finite Green functions satisfy the Slavnov-Taylor identity, if poles are subtracted by means of d-dimensional symmetric counterterms.

A first insight into higher order non-local contributions can be gained by considering the Callan-Symanzik equation. The Callan-Symanzik equation of the standard model of electroweak interaction has a completely different form from that of the corresponding symmetric theory. It contains mixed field operators between massless/massive neutral fields and, in particular, β-functions with respect to the independent mass parameters of the theory. It is, however, not a matter of taste, whether one wants to derive the

Callan-Symanzik equation in terms of physical fields or symmetric fields, since it does not even exist, if one does not include the mass diagonalization conditions of massless/massive particles at $p^2 = 0$ in the construction. For this reason, the considerations which concern the renormalization group analysis of the unbroken $SU(2)\times U(1)$-theory are not applicable to the standard model. It was one of the main intentions of the present work, to make the differences between unbroken and broken theories apparent. In particular, what one has to consider in the spontaneously broken case, are the large mass logarithms, which are induced from the lowest order β-functions to higher orders. These large-mass logarithms are specific for the model in its spontaneously broken form and have been analysed in a much simpler broken theory in [45]. The corresponding systematic investigation is now also feasible in the electroweak standard model.

Acknowledgements I want to thank K. Sibold for initial common work, many helpful discussions and a critical reading of the manuscript. I am grateful to G. Weiglein, B.A. Kniehl and A. Denner for comments on explicit one-loop renormalization.

A The quantum numbers of fields

In this appendix we list the quantum numbers of fields. We give the electromagnetic charge Q_{em}, the Faddeev-Popov charge $Q_{\phi\pi}$ and the properties under charge conjugation C and parity transformation P. Parity transformation to massive fermions is assigned in accordance with parity conservation in electromagnetic interactions. The infrared ($\dim^{IR}$) and ultraviolet ($\dim^{UV}$) dimensions of fields is adjusted in agreement with the BPHZL-scheme [34, 35].

	dim^{UV}	dim^{IR}	Q_{em}	$Q_{\phi\pi}$	C	$P(x^\mu \to x_\mu)$
e^L	$\frac{3}{2}$	2	-1	0	$-i\gamma^2 e^{R*}$	$\gamma^0 e^R$
e^R	$\frac{3}{2}$	2	-1	0	$-i\gamma^2 e^{L*}$	$\gamma^0 e^L$
u^L	$\frac{3}{2}$	2	$+\frac{2}{3}$	0	$-i\gamma^2 u^{L*}$	$\gamma^0 u^R$
u^R	$\frac{3}{2}$	2	$+\frac{2}{3}$	0	$-i\gamma^2 u^{R*}$	$\gamma^0 u^L$
d^L	$\frac{3}{2}$	2	$-\frac{1}{3}$	0	$-i\gamma^2 d^{L*}$	$\gamma^0 d^R$
d^R	$\frac{3}{2}$	2	$-\frac{1}{3}$	0	$-i\gamma^2 d^{R*}$	$\gamma^0 d^L$
ν^L	$\frac{3}{2}$	$\frac{3}{2}$	0	0	$CP : -i\gamma^2\gamma^0\nu^{L*}$	
ψ_e^R	$\frac{5}{2}$	2	-1	1	$-i\gamma^2\psi_e^{L*}$	$\gamma^0\psi_e^L$
ψ_e^L	$\frac{5}{2}$	2	-1	1	$-i\gamma^2\psi_e^{R*}$	$\gamma^0\psi_e^R$
ψ_u^R	$\frac{5}{2}$	2	$+\frac{2}{3}$	1	$-i\gamma^2\psi_u^{L*}$	$\gamma^0\psi_u^L$
ψ_u^L	$\frac{5}{2}$	2	$+\frac{2}{3}$	1	$-i\gamma^2\psi_u^{R*}$	$\gamma^0\psi_u^R$
ψ_d^R	$\frac{5}{2}$	2	$-\frac{1}{3}$	1	$-i\gamma^2\psi_d^{L*}$	$\gamma^0\psi_d^L$
ψ_d^L	$\frac{5}{2}$	2	$-\frac{1}{3}$	1	$-i\gamma^2\psi_d^{R*}$	$\gamma^0\psi_d^R$
ψ_ν^R	$\frac{5}{2}$	$\frac{5}{2}$	0	1	$CP : -i\gamma^2\gamma^0\psi_\nu^{R*}$	

Table 1: Quantum numbers of the fermion fields

	dim^{UV}	dim^{IR}	Q_{em}	$Q_{\phi\pi}$	C	$P(x^\mu \to x_\mu)$
$W^\mu_\pm$	1	2	± 1	0	$-W^\mu_\mp$	$W_{\mu\pm}$
Z^μ	1	2	0	0	$-Z^\mu$	Z_μ
A^μ	1	1	0	0	$-A^\mu$	A_μ
$\rho^\mu_\pm$	3	3	± 1	-1	$-\rho^\mu_\mp$	$\rho_{\mu\pm}$
ρ^μ_3	3	3	0	-1	$-\rho^\mu_3$	$\rho_{\mu 3}$
$c_\pm$	0	1	± 1	1	$-c_\mp$	$c_\pm$
c_Z	0	1	0	1	$-c_Z$	c_Z
c_A	0	0	0	1	$-c_A$	c_A
$\sigma_\pm$	4	4	± 1	-2	$-\sigma_\mp$	$\sigma_\pm$
σ_3	4	4	0	-2	$-\sigma_3$	σ_3
$\bar{c}_\pm$	2	3	± 1	-1	$-\bar{c}_\mp$	$\bar{c}_\pm$
$\bar{c}_Z$	2	3	0	-1	$-\bar{c}_Z$	$\bar{c}_Z$
$\bar{c}_A$	2	1	0	-1	$-\bar{c}_A$	$\bar{c}_A$
$B_\pm$	2	3	± 1	0	$-B_\mp$	$B_\pm$
B_Z	2	3	0	0	$-B_Z$	B_Z
B_A	2	2	0	0	$-B_A$	B_A
$\phi_\pm$	1	1	± 1	0	$\phi_\mp$	$\phi_\pm$
H	1	2	0	0	H	H
χ	1	2	0	0	$-\chi$	χ
$Y_\pm$	3	3	± 1	-1	$Y_\mp$	$Y_\pm$
Y_H	3	3	0	-1	Y_H	Y_H
Y_χ	3	3	0	-1	$-Y_\chi$	Y_χ
$\hat{\phi}_\pm$	1	1	± 1	0	$\hat{\phi}_\mp$	$\hat{\phi}_\pm$
$\hat{H}$	1	2	0	0	$\hat{H}$	$\hat{H}$
$\hat{\chi}$	1	2	0	0	$-\hat{\chi}$	$\hat{\chi}$
$\hat{q}_\pm$	1	1	± 1	1	$+\hat{q}_\mp$	$\hat{q}_\pm$
$\hat{q}_H$	1	2	0	1	$\hat{q}_H$	$\hat{q}_H$
$\hat{q}_\chi$	1	2	0	1	$-\hat{q}_\chi$	$\hat{q}_\chi$
$\hat{\varphi}_0$	2	2	0	0	$\hat{\varphi}_0$	$\hat{\varphi}_0$

Table 2: Quantum numbers of boson fields

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