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Exercise 1
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Exercises on 'Elementary Particle Physics'

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1. *The Dirac equation*

   The *Dirac* equation was found in an attempt to write down a relativistic equivalent of the non-relativistic *Schroedinger* equation, i.e. being linear in $\partial/\partial t$. It therefore corresponds to 'taking the square root' of the relativistic energy-momentum relation.

   (a) The ansatz for the *Dirac* equation is given by

   $$H\psi = (\alpha_i P_i + \beta m)\psi \ , \tag{1}$$

   where i runs from 1 to 3. Why?

   (b) Use the relativistic energy-momentum relation to prove that

   i. the α_i and β all anticommute with each other.

   ii. $\alpha_i^2 = \beta^2 = \mathbb{1}$ (no sum over i).

   (c) Define the *Dirac* matrices $\gamma^\mu \equiv (\beta, \beta\alpha)$ and show that

   i. the *Dirac* equation can be written in the covariant form

   $$(i\gamma^\mu \partial_\mu - m)\psi = 0 \ .$$

   ii. they fulfill the *Clifford* algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ with g the (mostly minus) *Minkowski* metric.

   iii. it is $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

   The lowest dimensionality matrices satisfying the above constraints are 4×4. There is however a choice in defining these matrices, leaving the physics unchanged of course. We will for now use the so called *Dirac-Pauli* representation

   $$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \ , \qquad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \ ,$$

   with the *Pauli* matrices

   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

2. *Free solutions of the Dirac equation*

We will label the indices of the *Dirac* matrices by greek letters from the beginning of the alphabet. The ψ's are four-component column 'vectors' (spinors), labeled correspondingly.

(a) Use the covariant form of the *Dirac* equation to show that for every α:

$$(\Box + m^2)\psi_\alpha = 0 \,.$$

For free particles we can therfore make the following ansatz:

$$\psi = u(\vec{p})e^{-ip\cdot x} \,.$$

(b) Plug this ansatz into (1) and use the *Pauli-Dirac* representation to show that

$$Hu = \begin{pmatrix} m\mathbb{1}_2 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = E \begin{pmatrix} u_A \\ u_B \end{pmatrix} ,$$

with u splitted into two two-component spinors u_A and u_B.

(c) What are the energy eigenvalues for a particle at rest? Interpret the result.

(d) Now take $\vec{p} \neq 0$. We will label the solutions by an index (s). You can find two solutions by choosing $u_A^{(s)} = \chi^{(s)}$ with $\chi^{(1)} = (1,0)^T$ and $\chi^{(2)} = (0,1)^T$. What are the corresponding u_B? What can you say about the energy eigenvalues of this solutions? Proceed analogously for the remaining two solutions. Don't bother about normalizations for now.

(d) It's convenient to choose the so called covariant normalization

$$\int \psi^\dagger \psi dV = 2E \,.$$

Use this to derive the normalizations of the $u^{(s)}$s.

From the solutions we see, that there are always two solutions per eigenvalue and we therfore got a degeneracy. Such degeneracies are always due to additional symmetries (c.f. *Runge-Lenz* vector in H-atom).

(e) Show that the operator

$$\Sigma\cdot\hat{p} \equiv \begin{pmatrix} \vec{\sigma}\cdot\hat{\vec{p}} & 0 \\ 0 & \vec{\sigma}\cdot\hat{\vec{p}} \end{pmatrix} \qquad \text{with} \quad \hat{\vec{p}} \equiv \vec{p}/|\vec{p}|$$

corresponds to an observable, i.e. that it commutes with H and P. The associated quantum number $\frac{1}{2}\vec{\sigma}\cdot\hat{\vec{p}}$ is called *helicity*. Choose $\vec{p}$ along the z-axis. What are the helicities of the $\chi^{(s)}$?