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Exercise 6
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Exercises on 'Elementary Particle Physics'

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1. *C, P & T for spinors*

   Under parity, charge conjugation and time reversal a Dirac field ψ transforms as

   $$\begin{aligned}\hat{P}\psi(x)\hat{P}^{-1} &= \eta_P\gamma^0\psi(x_P) \qquad x_P^\mu = (x_0, -\mathbf{x})\\ \hat{C}\psi(x)\hat{C}^{-1} &= \eta_C C\bar{\psi}(x)^T\\ \hat{T}\psi(x)\hat{T}^{-1} &= \eta_T B\psi(x_T) \qquad x_T^\mu = (-x_0, \mathbf{x})\,,\end{aligned}$$

   with $\hat{P}$, $\hat{C}$, $\hat{T}$ the linear, linear and antilinear operators (respectively) implementing these operations (note that antilinear means $\hat{T} \mid \lambda\phi\,\rangle = \lambda^*\hat{T} \mid \phi\,\rangle$ and that if $\hat{T} \mid \phi\,\rangle = \mid \phi_T\,\rangle$ then $\langle\,\phi' \mid \phi\,\rangle = \langle\,\phi_T \mid \phi'_T\,\rangle$). $\eta_{P/C/T}$ are the intrinsic parity, charge conjugation parity etc. (remember: $\mid \eta_{P/C/T} \mid^2 = 1$). According to *Wigner*'s theorem any linear operator must be unitary and any antilinear operator must be antiunitary. The matrix C is defined via $C\gamma^{\mu\,T}C^{-1} = -\gamma^\mu$ (where we assume that $\gamma^{\mu\,\dagger} = (\gamma^0, -\gamma)$). One can show that $C^t = -C$ and $C^\dagger C = \mathbb{1}$. The matrix B is defined by $B\gamma^{\mu\,*}B^{-1} = (\gamma^0, -\gamma)$ and one can show that $B^\dagger B = \mathbb{1}$. If you need a specific representation for your calculations, take the *Dirac-Pauli* one (of course any would do).

   (a) Show that $B\gamma_5^* B^{-1} = \gamma_5$ and verify that with the assumed form of $\gamma^{\mu\,\dagger}$ we may take $B = \pm\gamma_5 C$.

   (b) Compute $\hat{P}\bar{\psi}(x)\hat{P}^{-1}$, $\hat{C}\bar{\psi}(x)\hat{C}^{-1}$ and $\hat{T}\bar{\psi}(x)\hat{T}^{-1}$.

   (c) Show that, if X is a matrix acting on Dirac spinors,

   $$\begin{aligned}\hat{C}\bar{\psi}(x)X\psi(x)\hat{C}^{-1} &= \bar{\psi}(x)X_C\psi(x)\\ \hat{T}\bar{\psi}(x)X\psi(x)\hat{T}^{-1} &= \bar{\psi}(x_T)X_T\psi(x_T)\,,\end{aligned}$$

   where $X_C = CX^TC^{-1}$ (ψ and $\bar{\psi}$ anti-commute (!)) and $X_T = BX^*B^{-1}$. What is the expression for $\hat{P}\bar{\psi}(x)X\psi(x)\hat{P}^{-1}$.

   (d) Hence determine the transformation properties of the fermionic bilinears $\mathbb{1}, i\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5$ and $i[\gamma^\mu, \gamma^\nu]$ under parity, charge conjugation and time reversal .

   (e) If $|\,p\,\rangle$ is a boson with momentum p and $\langle\,0\,|\,\bar{\psi}(0)\,i\gamma_5\psi(0)\,|\,p\,\rangle \neq 0$, show that in a theory where P and C are conserved the boson must have negative intrinsic parity and also positive charge conjugation parity.

Homeworks

1. Lecture clean-up

   (a) Let $\psi = (\xi^\beta, \bar{\chi}_{\dot{\beta}})^T$ be a *Dirac* spinor in two-component notation. Show that then $\psi^C = (\chi^\alpha, \bar{\xi}_{\dot{\alpha}})^T$ in the *Weyl* representation.

   (b) For the proof that $j_C^\mu = -j^\mu$ we needed that for a *Dirac* spinor $\bar{\psi}_C = -\psi^T C^{-1}$. Show this.

   (c) Consider the kinetic term of a *Majorana* field theory, i.e.

   $$\mathcal{L} = i\bar{\chi}\bar{\sigma}^\mu \partial_\mu \chi \ .$$

   Show that $\mathcal{L}$ is real and *Lorentz* invariant.

   (d) Combine two independent *Weyl* spinors χ_1 and χ_2 into a *Dirac* spinor $\psi_D = (\chi_1, \bar{\chi}_2)^T$ and show, using the *Weyl* representation, that then the Lagrangian

   $$\mathcal{L} = i\bar{\chi}_1\bar{\sigma}^\mu \partial_\mu \chi_1 + i\bar{\chi}_2\bar{\sigma}^\mu \partial_\mu \chi_2$$

   can be rewritten in the form

   $$\mathcal{L}_D = i\bar{\psi}_D \gamma^\mu \partial_\mu \psi_D \ .$$