Seminar Talk on the Maldacena Conjecture

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Overview

- Physical background and Maldacena's Conjecture

- AdS space and it's boundary

- Euclidean Version of AdS space and different representations of AdS_{d+1}

- calculation of correllators for massless and massive scalar fields

The Maldacena Conjecture

Type IIB string theory with fixed coupling constant and N parallel D3 branes

In low energies limit $\longrightarrow$ brane decouples from the bulk

Resulting effective theory is a 3+1 dimensional field theory, namely $\mathcal{N} = 4$ U(N) SYM (super-Yang-Mills)

Corresponding theory for high energies is supergravity carrying D3 brane charge. (Polchinski)

Low energy limit $\longrightarrow$ $AdS_5 \times S^5$ symmetry

$\Rightarrow$ $\boxed{\text{Symmetry group of } AdS_5 \times S^5 \text{ is the same as the conformal group in 3+1 dimensions!}}$

$\Longrightarrow$ Maldacena Conjecture

> Type IIB supergravity on $AdS_5 \times S^5$ plus appropriate boundary conditions in the large *N* limit is dual to $\mathcal{N} = 4$ U(N) super-Yang-Mills

Justification :

calculations for near to extremal black $D3$ branes agree very well with Yang Mills theory (crossection of graviton and dilaton)

Achievement :

effective Young Mills coupling large $\Rightarrow$ perturbation theory in SYM impossible. But possible in supergravity on $AdS_5 \times S^5$!!

Hopes :

- Solving strong coupling problem for low energies in QCD
- Progress in quantum gravity

Anti de-Sitter Space

AdS_{d+1} is given by

$$u^2 - v^2 - \sum_{i,j=1}^{d} \eta_{ij} x^i x^j = 1 \qquad (1)$$

with the metric

$$ds^2 = du^2 - dv^2 + \sum_{i,j=1}^{d} \eta_{ij} dx^i dx^j \qquad (2)$$

AdS metric has signature (+,-,-,+,+, ... ,+).
$\Rightarrow$ isometry group of (1) is $SO(2, d)$.

Look at boundary $\longrightarrow$ light cone coordinates

$$z^+ = u + v \quad , \quad z^- = u - v \qquad (3)$$

AdS_{d+1} now reads

$$z^+ z^- - \sum_{i,j=1}^{d} \eta_{ij} x^i x^j = 1 \qquad (4)$$

and the metric (2) becomes

$$ds^2 = dz^+ dz^- + \sum_{i,j=1}^{d} \eta_{ij} dx^i dx^j \qquad (5)$$

To obtain boundary $(z^+, z^-, x_i) \longrightarrow \infty$
$\Rightarrow$ (4) becomes

$$z^+ z^- - \sum_{i,j=1}^{d} \eta_{ij} x^i x^j = 0 \tag{6}$$

For $z^- \neq 0$ scale $z^- = 1$.

Solve for z^+ and plug into (5). Note $dz^- = 0$
Result :

$$ds^2 = \sum_{i,j=1}^{d} \eta_{ij} dx^i dx^j \tag{7}$$

$\Rightarrow$ boundary of AdS_{d+1} is (compactification) of M_d

Different Representations of Euclidean $\mathbf{AdS}_{\mathrm{d+1}}$

Now we continue with Euclidean version of AdS space

$$u^2 - v^2 - \sum_{i,j=1}^{d} \delta_{ij} x^i x^j = 1 \tag{8}$$

and likewise we now have the metric

$$ds^2 = du^2 - dv^2 + \sum_{i,j=1}^{d} \delta_{ij} dx^i dx^j \tag{9}$$

Metric (9) is d+2 - dim Minkowski metric. Equation (8) is equal to hyperboloid with $\langle\langle x, x \rangle\rangle = -1$ where $\langle\langle \cdot\,, \cdot \rangle\rangle$ is the Minkowski metric.

Now derive 3 different equivalent representations of AdS_{d+1}

Polar coordinates

$$(y, e) \longrightarrow \begin{pmatrix} \sinh(y) \cdot e \\ \cosh(y) \end{pmatrix} \tag{10}$$

where e describes the space like coordinates.

Curve on the hyperboloid

$$c(t) = \begin{pmatrix} \sinh(y(t)) \cdot e(t) \\ \cosh(y(t)) \end{pmatrix} \tag{11}$$

with

$$\dot{c}(t) = \begin{pmatrix} \cosh(y(t))\dot{y}(t) \cdot e(t) + \sinh(y(t)) \cdot \dot{e}(t) \\ \sinh(y(t))\dot{y}(t) \end{pmatrix} \tag{12}$$

because of $\langle e(t), \dot{e}(t) \rangle_{\mathbb{R}^d} = 0$ we have

$$\langle\!\langle \dot{c}(t), \dot{c}(t) \rangle\!\rangle = \dot{y}^2(t) + 0 + \sinh^2(y(t)) \langle \dot{e}, \dot{e} \rangle_{\mathbb{R}^d} \tag{13}$$

or

$$ds^2 = dy^2 + \sinh^2(y)\, d\Omega^2 \tag{14}$$

Poincare Disk

AdS_{d+1} is also equivalent to the open d-dim unit ball $B_{d+1} = \left\{ x \in \mathcal{R}^{d+1} \middle| \sum_{i=0}^{d} y_i^2 < 1 \right\}$ with the metric

$$ds^2 = \frac{4 \cdot \sum_{i=0}^{d} dy_i^2}{(1 - |y|^2)^2} \tag{15}$$

Proof by mapping hyperboloid to B_{d+1} with Lorentz inversion $I_L(x) = x / \langle\langle x, x \rangle\rangle$

Metric is singular on the boundary. Take function f with single zero on the boundary. $\longrightarrow$ continue metric with transformation

$$d\tilde{s}^2 = f^2 \cdot ds^2 \tag{16}$$

The transformation $f \longrightarrow f \cdot e^w$, w any real function on $\bar{B}_{d+1}$ induces the conformal transformation

$$d\tilde{s}^2 \longrightarrow e^{2w} \cdot d\tilde{s}^2 \tag{17}$$

$\rightarrow$ AdS_{d+1} has (in Euclidean version) a metric invariant under $SO(1, d+1)$. The boundary S^d has only a conformal structure, which is preserved by the action of $SO(1, d+1)$

Hyperbolic Space

AdS_{d+1} also be represented as $\mathbb{R}^{d+1}$ with $x^0 > 0$ with the metric

$$ds^2 = \frac{1}{x_0^2}\left(\sum_{i=0}^{d}(dx^i)^2\right) \tag{18}$$

Proof with help of Euclidean inversion $I(x) = x/\langle x, x\rangle_{\mathbb{R}^{d+1}}$.

Boundary consists of a copy of $\mathbb{R}^d$ for $x_0 \to 0$ and a single point at $x_0 = \infty$

This representation will be good for calculations

AdS/CFT Correspondence

Concrete realization of AdS/CFT correspondence: Assume equations of motion with solution on $\overline{AdS}_{d+1}$ and given boundary values ϕ_0.

Analysis interactions of fields with branes $\longrightarrow$ coupling of ϕ_0 to a conformal field $\mathcal{O}$ is given by

$$\int_{S_d} \phi_0 \mathcal{O} \tag{19}$$

To find conformal field theory look for correlation functions $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)\rangle$
$\rightarrow$ find generating functional $\langle \exp(\int_{S_d} \phi_0 \mathcal{O})\rangle_{CFT}$

Therefore calculate supergravity (or string) partition function

$$Z_S(\phi_0) = \exp(-I_S(\phi)) \tag{20}$$

Interprete $Z_S(\phi_0)$ as generating functional for CFT.

$$\left\langle \exp \int_{S^d} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0) \qquad (21)$$

Massless Scalar Fields

Conformal dimension of ϕ_0 is zero because ϕ_0 is independent of f (see (16) and (17))

Coupling (19) must be dimensionless $\Rightarrow$ conformal field $\mathcal{O}$ has conformal dimension d

Now compute the two point function for AdS theory that contains massless scalar fields ϕ with boundary value ϕ_0.

Action is given by

$$I(\phi) = \frac{1}{2} \int_{AdS_{d+1}} d^{d+1} y \sqrt{g} |d\phi|^2 \qquad (22)$$

$\Rightarrow$ equations of motion

$$D_i D^i \phi = 0 \qquad (23)$$

To solve, find Green's function K for Laplace equation.

Use AdS representation as upper half space with metric

$$ds^2 = \frac{1}{x_0^2}\left(\sum_{i=0}^{d}(dx^i)^2\right)$$

Covariant Laplace (Beltrami-Laplace) operator is given by

$$\Delta_M = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^\alpha} g^{\alpha\beta}\sqrt{g}\frac{\partial}{\partial \xi^\beta} \tag{24}$$

Metric and boundary conditions are invariant under translations of $x_i, i = 1, \cdots, d \Rightarrow K(x)$ has the same symmetry and depends only on x_0.

(24) simplifies and equations of motion are given by

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0 \tag{25}$$

Solution is $K(x_0) = C_1 \cdot x_0^d + C_2$. A Green's function must vanish everywhere but one point P at the boundary.

$\Rightarrow K(x_0) = c \cdot x_0^d$ with P the point at $x_0 = \infty$.

To show $K(x_0)$ is Green's function map P to origin with $SO(1,d)$ isometry transformation

$$x_i \longrightarrow \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2}, \quad i = 0, \ldots, d \qquad (26)$$

Trafo preserves Laplace operator. K transforms to

$$K(x) = c \cdot \frac{x_0^d}{(x_0^2 + \sum_{j=1}^d x_j^2)^d} \qquad (27)$$

For $x_0 \to 0$ $K(x)$ vanishes except for $x_1 = \ldots = x_n = 0$ and $K(x) > 0$.

$K(x)$ is Green's function if $\int dx_1 \ldots dx_d \, K(x)$ is independent of x_0.

$$\int dx_1 \dots dx_d \frac{\frac{1}{x_0^d}}{\left(1 + \sum_{j=1}^d \frac{x_j^2}{x_0^2}\right)^d} \tag{28}$$

Transformation $x_j \to x_0 \cdot x_j$ eliminates x_0

Using Green's function

$$\phi(x_0, x_i) = c \cdot \int d\mathbf{x}' \frac{x_0^d}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d} \phi_0(x_i') \tag{29}$$

(abbreviation $\mathbf{x} = (x_1, \dots, x_d)$, $|\mathbf{x}| = \sum_{j=1}^d x_j^2$ and $d\mathbf{x} = dx_1 dx_2 \cdots dx_d$)

For small x_0 we have

$$\frac{\partial \phi}{\partial x_0} \approx d c x_0^{d-1} \int d\mathbf{x}' \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2d}} + O(x_0^{d+1}) \tag{30}$$

Rewriting the term $|d\phi|^2 = D_i \phi D^i \phi$ in the action we obtain

$$I(\phi) = \frac{1}{2} \int_{\substack{\mathbb{R}^{d+1},\\ x_0 > 0}} d\mathbf{x} \sqrt{g} \left(D_i(\phi D^i \phi) - \phi D_i D^i \phi\right) \tag{31}$$

Using Gauss lemma on non-vanishing term

$$I(\phi) = \lim_{\epsilon\to 0} \frac{1}{2}\int_{T_\epsilon} d\mathbf{x}\,\sqrt{h}\,\phi(\vec{n}\cdot\nabla\phi) \tag{32}$$

T_ϵ is the surface with $x_0 = \epsilon$

h the induced metric on $T_\epsilon \Rightarrow \sqrt{h} = x_0^{-d}$

n is the unit normal vector to T_ϵ
$\Rightarrow \quad n = (x_0, 0, \ldots, 0)$

T_ϵ is flat space $\Rightarrow \nabla$ is normal gradient

$\longrightarrow \vec{n}\cdot\nabla\phi = x_0(\partial\phi/\partial x_0)$ (see (30))

plug in everything we have

$$I(\phi) = \frac{c\cdot d}{2}\int d\mathbf{x}\,d\mathbf{x}'\frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{2d}} \tag{33}$$

We now have the two point function

$$\langle\mathcal{O}(\mathbf{u})\mathcal{O}(\mathbf{v})\rangle = \frac{\delta}{\delta\phi_0(u)}\frac{\delta}{\delta\phi_0(v)}\exp\left(-\frac{cd}{2}\int d\mathbf{x}d\mathbf{x}'\frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{2d}}\right)\Bigg|_{\phi_0=0} \tag{34}$$

A short calculation gives

$$\langle\mathcal{O}(\mathbf{u})\mathcal{O}(\mathbf{v})\rangle = -\frac{c\cdot d}{|\mathbf{w}-\mathbf{v}|^{2d}} \tag{35}$$

$\longrightarrow \langle \mathcal{O}(\mathbf{u})\mathcal{O}(\mathbf{v})\rangle \propto |\mathbf{x}-\mathbf{x}'|^{-2d}$ as it must be for every field $\mathcal{O}$ of conformal dimension d.

This may be seen as a justification of the conjecture

$$\left\langle \exp \int_{S^d} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0)$$

The Massive Case

Consider scalar with mass m in AdS_{d+1}. The action is given by

$$I(\phi) = \frac{1}{2} \cdot \int dy^{d+1} \sqrt{g}(|d\phi|^2 + m^2\phi^2) \quad (36)$$

with g the AdS metric.
$\Rightarrow$ equations of motion

$$(-D_i D^i + m^2)\phi = 0 \quad (37)$$

Consider polar coordinates with metric

$$ds^2 = dy^2 + \sinh^2(y)\, d\Omega^2$$

calculate Beltrami Laplace operator

$$\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial \xi^\beta}$$

we have in polar coordinates

$\sqrt{g} = \sinh(y)^d$

$g^{\alpha\beta} \propto \delta_{\alpha\beta}$ with $g^{00} = 1$ and

$g^{jj} = \sinh(y)^{-2}$ for $j = 1, \ldots, d$

$\longrightarrow$ equations of motion

$$\left(-\frac{1}{\sinh(y)^d} \frac{d}{dy} \sinh(y)^d \frac{d}{dy} + \frac{L^2}{\sinh(y)^2} + m^2 \right) \phi = 0 \tag{38}$$

were L^2 is independent of y (angular part)

For large y we get

$$-\phi'' - d \cdot \phi' + m^2 \phi = 0 \tag{39}$$

with solution

$$\phi = A\cdot\exp(\underbrace{\frac{1}{2}(-d+\sqrt{d^2+4m^2})}_{\lambda_+}\,y)+B\cdot\exp(\underbrace{\frac{1}{2}(-d-\sqrt{d^2+4m^2})}_{\lambda_-}\,y) \tag{40}$$

$\Rightarrow$ solution has first order pole at the boundary

Take function f with first order zero on the boundary. Boundary value or ϕ can be described by

$$\phi \sim f^{-\lambda_+}\phi_0 \tag{41}$$

with arbitrary ϕ_0 on the boundary.

Trafo $f \to e^w f$

$\Rightarrow d\tilde{s}^2 \to e^{2w} d\tilde{s}^2$ and $\phi_0 \to e^{w\lambda_+}\phi_0$

$\Rightarrow \phi_0$ has scaling dim $\Delta = -\lambda_+$

(first talk: $\varphi \to |\partial x'/\partial x|^{-\Delta/d}\varphi = \Lambda^{-\Delta/2}\varphi$ were $g' = \Lambda g$ when g is the metric)

coupling term $\int \phi_0 \mathcal{O}$ must be dim-less

$\longrightarrow$ conformal field $\mathcal{O}$ has scaling dimension

$$\Delta = d + \lambda_+ = \frac{1}{2}\left(d + \sqrt{d^2+4m^2}\right) \tag{42}$$

Computation of the Two Point Function

Equations of motion were given by

$$(-D_i D^i + m^2)\phi = 0$$

Represent AdS as upper half space with metric

$$ds^2 = \frac{1}{x_0^2}\left(\sum_{i=0}^{d}(dx^i)^2\right)$$

$K(x)$ only depends on x_0 (same reasons as massless case)

Equations of motion in this coordinates (use Beltrami Laplace operator)

$$\left(-x_0^{d+1}\frac{d}{dx_0}x_0^{-d+1}\frac{d}{dx_0} + m^2\right)K(x_0) = 0 \tag{43}$$

The solution is

$$K = A \cdot x_0^{\overbrace{\frac{1}{2}(d+\sqrt{d^2+4m^2})}^{=d+\lambda_+}} + B \cdot x_0^{\overbrace{\frac{1}{2}(d-\sqrt{d^2+4m^2})}^{=d+\lambda_-}} \tag{44}$$

Solution vanishing for $x_0 = 0$ is

$$K(x_0) = c' \cdot x_0^{d+\lambda_+} \tag{45}$$

The $SO(1,d)$ inversion gives

$$K = c' \cdot \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{46}$$

$\Rightarrow$ behaves as $x_0^{d+\lambda_+}$ for $x_0 \to 0$ except for a singularity at $\mathbf{x} = 0$.

Function is supported near $x = 0$.

$$\lim_{x_0 \to 0} \frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{47}$$

is a multiple of $\delta(\mathbf{x})$ (similar argument to massless case) :

$$\int d\mathbf{x} \, \frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{48}$$

(reduce the fraction and make transformation $x_i \to x_0 \cdot x_i$)

$\Rightarrow$ Green's function

If K used as Green's function to construct solution

$$\phi(x) = c' \int d\mathbf{x} \, \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d+\lambda_+}} \, \phi_0(\mathbf{x}) \tag{49}$$

ϕ behaves for $x_0 \to 0$ like $x_0^{-\lambda_+} \phi(x)_0$ as expected (compare (47) to K)

Action evaluated analogue to massless case

(Integration by parts, reduction to surface term)

Result :

$$I(\phi) = \frac{c'(d + \lambda_+)}{2} \int d\mathbf{x} d\mathbf{x}' \, \frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(\lambda_+ + d)}} \tag{50}$$

Two point function obtained analogue to massless case

Read off

$$\boxed{\Delta = d + \lambda_+ = \tfrac{1}{2}(d + \sqrt{d^2 + 4m^2})}$$