Maldacena Conjecture

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1 The Maldacena Conjecture

The so called Maldacena conjecture was found in 1998 (see [Mal]). It proposes the correspondence of string theory on AdS (Anti-de-Sitter) space and a conformal field theory on the boundary.

Maldacena started with type IIB string theory with fixed coupling constant g. He considered N parallel D3 branes. Now, for low energies the brane decouples from the bulk. That means that the backreaction of the D-brane on the geometry is neglected. Then quantised open string theory is the appropriate description for the D-brane. The open string excitations are constrained to live on the D-branes (by definition). The resulting effective theory is a 3+1 dimensional field theory. More precisely, one obtains $\mathcal{N} = 4$ U(N) SYM (super-Yang-Mills).

Now consider the supergravity solution [1] (by Horowitz and Strominger) carrying D3 brane charge. Performing the low energy limit for the corresponding metric, the resulting metric describes $AdS_5 \times S^5$ with a curvature radius given by $R \propto \sqrt{gN}$. But we can trust supergravity only if the curvature is small that means R is large. Therefore we need large N (number of branes) as g is the fixed string coupling constant.

Now a very important fact is, that the symmetry group of this space is the same as the conformal group in 3+1 dimensions.

This led Maldacena to the following conjecture:

> Type IIB supergravity on $AdS_5 \times S^5$ plus appropriate boundary conditions in the large N limit is dual to $\mathcal{N} = 4$ U(N) super-Yang-Mills

[1] That this solution corresponds to the D3-branes we just discussed in a regime where the back-reaction is taken into account was observed by Polchinski. (This is a very non-trivial observation since in such a regime the (perturbative) open string description is not possible.)

or stronger:

> String theory on $AdS_5 \times S^5$ is equivalent to $N=4$, i.e. stringy corrections to gravity correspond to $1/N$ corrections (or $1/g_{t'Hooft}$) corrections on the SYM side of the conjecture.

This conjecture is called AdS/CFT correspondence.

One justification for example is the fact that calculations for extremal black branes (e.g. cross section of the dilation and the graviton) agree very well with Yang Mills theory.

But of course there are more achievements of this conjecture: If the effective Yang Mills coupling gN is large, then perturbation theory in SYM is impossible. But the calculation may be performed in supergravity on $AdS_5 \times S^5$. Therefore one hopes that the AdS/CFT correspondence may help in a similar way to deal with the problem of large coupling for low energies in QCD. As string theories are the best candidates for quantum gravity, the AdS/CFT correspondence might also be very important in the development of a better theory of gravity. (For short general information on bulk brane etc. see [Ra])

2 Anti de-Sitter Space

Now it should be interesting to see how this AdS/CFT correspondence works in some simple examples. But before we do this, we should know more about AdS space.

It is in general relativity the space time solution with constant negative curvature. AdS_{d+1} can be described with coordinates u, v, x_i by the equation

$$u^2 - v^2 - \sum_{i,j=1}^{d} \eta_{ij} x^i x^j = 1 \tag{1}$$

where η is the Lorentz metric with signature $(-,+,+,\cdots,+)$. The AdS metric is given by

$$ds^2 = du^2 - dv^2 + \sum_{i,j=1}^{d} \eta_{ij} dx^i dx^j \tag{2}$$

Apparently the upper metric has the signature $(+,-,-,+,+,\cdots,+)$. That means, the isometry group of (1) is $SO(2,d)$.

We are now interested in the boundary of AdS_{d+1}. It is useful to replace u, v by the so called light cone coordinates

$$z^+ = u + v \quad , \quad z^- = u - v \tag{3}$$

Therefore the parametrisation (1) of AdS_{d+1} now reads

$$z^+ z^- - \sum_{i,j=1}^{d} \eta_{ij} x^i x^j = 1 \tag{4}$$

and the metric (2) in this coordinates becomes

$$ds^2 = dz^+ dz^- + \sum_{i,j=1}^{d} \eta_{ij} dx^i dx^j \tag{5}$$

The boundary of AdS_{d+1} is now obtained by taking all coordinates to infinity. This has the effect that the 1 at the right hand side of (4) is replaced by zero, i.e. (4) is modified to

$$z^+z^- - \sum_{i,j=1}^{d} \eta_{ij}x^ix^j = 0 \tag{6}$$

In order to obtain information about the geometry of the boundary we now solve (6) and plug the solution into (5). To this end, we first observe that (6) is invariant under rescaling all coordinates simultaneously by a real number s. As long as $z^- \neq 0$ we can use this scaling relation to set $z^- = 1$. Then equation (6) fixes the value of z^+ uniquely as a function of the remaining coordinates. (The actual form of the solution is obvious but irrelevant since it drops out when we plug $dz^- = d1 = 0$ into (5).) Thus we find for the metric on the boundary of AdS_{d+1}

$$ds^2 = \sum_{i,j=1}^{d} \eta_{ij}dx^idx^j \tag{7}$$

The conclusion that the boundary is the d dimensional Minkowski space would be a bit premature since so far we excluded the point $z^- = 0$. Therefore, in principle, we have to add that point to the Minkowski space (M_d) which results in a compactified version of M_d. (The compactified version of M_d is topologically equivalent to $\left(S^1 \times S^{d-1}\right)/\mathbb{Z}_2$. For details see footnote 1 in [Wit].) Subtilties related to compact versus non compact Minkowski space will not arise in the rest of the talk.

2.1 Different Representations of $\mathbf{AdS_{d+1}}$

So far we have assumed Lorentz signature, but the identification of the boundary of AdS_{d+1} with d-dimensional Minkowski space has it's Euclidean analogue. It will be convenient to continue with the Euclidean version of $AdSd+1$. It is given by

$$u^2 - v^2 - \sum_{i,j=1}^{d} \delta_{ij}x^ix^j = 1 \tag{8}$$

and likewise we now have the metric

$$ds^2 = du^2 - dv^2 + \sum_{i,j=1}^{d} \delta_{ij}dx^idx^j \tag{9}$$

Looking closer at equation (8) we realize that Euclidean version of $AdSd+1$ may be seen as hyperboloid in $\mathbb{R}^{d+2}$ with $\langle\!\langle x,x\rangle\!\rangle = -1$ where $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ is the Minkowski metric. The $AdSd+1$ metric (9) is nothing else but this $d{+}1$-dimensional Minkowski metric.

AdS_{d+1} can be described in different equivalent, geometry preserving ways. That means that we have a coordinate transformation, that maps two elements to the target space in such a way that the metric in the target space gives the same value for the images as the metric in the source space does for the original elements. It will be useful to have some of these descriptions at hand.

Now we can introduce "polar" coordinates :

$$(y,e) \longrightarrow \begin{pmatrix} \sinh(y)\cdot e \\ \cosh(y) \end{pmatrix} \tag{10}$$

where e describes the space like coordinates. We can immediately obtain the corresponding metric. Therefore let

$$c(t) = \begin{pmatrix} \sinh(y(t))\cdot e(t) \\ \cosh(y(t)) \end{pmatrix} \tag{11}$$

denote a curve on the hyperboloid. $\dot{c}(t)$ defines an element of tangent space of this hypersurface with

$$\dot{c}(t) = \begin{pmatrix} \cosh(y(t))\dot{y}(t) \cdot (t) + \sinh(y(t)) \cdot \dot{e}(t) \\ \sinh(y(t))\dot{y}(t) \end{pmatrix} \tag{12}$$

Now we can easily compute the metric in the new coordinates. Let $\langle\langle \cdot , \cdot \rangle\rangle$ again denote the Lorentz metric. Then we have:

$$\langle\langle \dot{c}(t), \dot{c}(t) \rangle\rangle = \dot{y}^2(t) + 0 + \sinh^2(y(t))\langle \dot{e}, \dot{e} \rangle_{\mathbb{R}^d} \tag{13}$$

because of $\langle e(t), \dot{e}(t)\rangle_{\mathbb{R}^d} = 0$. In more familiar notation we have:

$$ds^2 = dy^2 + \sinh^2(y)\, d\Omega^2 \tag{14}$$

where $d\Omega$ is the metric of a $d-$unit sphere. The metric is singular for $y = 0$. This is the case of $v = 0$.

AdS_{d+1} may also be identified with $B_{d+1} = \{y \in \mathbb{R}^{d+1} \,|\, \sum_{i=0}^{d} y_i^2 < 1\}$, the open unit ball, with the metric

$$ds^2 = \frac{4 \cdot \sum_{i=0}^{d} dy_i^2}{(1 - |y|^2)^2} \tag{15}$$

This can be proven with the help of the Lorentz inversion $I_L(p) := p/\langle\langle p, p \rangle\rangle$. The details can be taken from [Ha, p. 30, 31]. The boundary of B_{d+1} is the unit sphere S^d. The fact, that S^d is the boundary of $\bar{B}_{d+1}$ is the Euclidean version of the statement that Minkowski space is the boundary of AdS_{d+1}. The metric (15) on B_{d+1} does not extend over $\bar{B}_{d+1}$ because it is singular at $|y| = 1$. This can be fixed by taking a function f on $\bar{B}_{d+1}$ which is positive on B_{d+1} and has a first order zero on the boundary (e.g. $f = 1 - |y|^2$) and replacing ds by

$$d\tilde{s}^2 = f^2 \cdot ds^2 \tag{16}$$

As there is no natural choice of f, this metric is only well defined up to conformal transformations. The transformation

$$f \longrightarrow f \cdot e^w \tag{17}$$

with w any real function on $\bar{B}_{d+1}$ would induce the conformal transformation

$$d\tilde{s}^2 \longrightarrow e^{2w} \cdot d\tilde{s}^2 \tag{18}$$

Thus, while AdS_{d+1} has (in its Euclidean version) a metric invariant under $SO(1, d+1)$, the boundary S^d has only a conformal structure, which is preserved by the action of $SO(1, d+1)$

Finally, one can regard AdS_{d+1} as the upper half space of $\mathbb{R}^{d+1}$ with $x^0 > 0$ with the metric

$$ds^2 = \frac{1}{x_0^2} \left(\sum_{i=0}^{d} (dx^i)^2 \right) \tag{19}$$

This description can be equivalent to the previous one. Again the proof can be found in [Ha, p. 28, 29]. In this representation the boundary consists of a copy of $\mathbb{R}^d$ at $x^0 = 0$ together with a single point P at $x^0 = \infty$ since the metric vanishes for $x^0 \to \infty$. This description of AdS_{d+1} will be very useful in the following calculations.

3 AdS/CFT Correspondence

As we have seen explicitly in the previous section the isometric group on AdS_d acts as the conformal group on the boundary. This is the key for Witten's realization of the Maldacena conjecture (see [Wit]) :,

> Boundary values of fields that have dynamics defined inside the AdS space act as sources of correlation functions of the boundary conformal field theory

This should be understood in the following way : Suppose we have some equations of motion with the existence of a unique solution on $\bar{B}_{d+1}$ ($\stackrel{\triangle}{=} \overline{AdS}_{d+1}$) with any given boundary values ϕ_0. This, for instance, is the case for massless scaler fields obeying the Laplace equation $D_i D^i \phi = 0$, as we will see in the next subsection. It shall be assumed that in the correspondence between AdS_{d+1} and conformal field theory on the boundary, ϕ_0 should be considered to couple to a conformal field $\mathcal{O}$, via the coupling

$$\int_{S_d} \phi_0 \mathcal{O} \tag{20}$$

This assumption is natural given by some analysis of interactions of fields with branes. ϕ is, in the massless case, conformally invariant - it has conformal weight zero - since the use of a function f as in (16) to define a metric in on S^d does not enter at all in the definition of ϕ_0. So conformal invariance dictates that $\mathcal{O}$ must have conformal dimension (or scaling dimension) d, because the coupling (20) must be dimensionless. (For dilations it can easily be seen, that the spacial dimension of the fields and the scaling dimension are the same.) This is different for the massive case which is discussed in section (3.2).

To establish the conformal field theory on the boundary of AdS_{d+1} we would like to compute the n-point functions $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)\rangle$. The most elegant way to do this is to find the generating functional $\langle \exp(\int_{S_d} \phi_0 \mathcal{O})\rangle_{CFT}$ for a given ϕ_0. At least in the case of massless fields this function is convergent in some neighbourhood of $\phi_0 = 0$ as we will see in the next subsection. But in general in field theory there will arise difficult questions of convergence and renormalisation at this point.

But how do we obtain the CFT partition function? Let $Z(\phi_0)$ be the supergravity (or string) partition function on B_{d+1} computed with the boundary condition that at infinity ϕ approaches a given function ϕ_0. For example, in the approximation of classical supergravity, one computes $Z_S(\phi_0)$ by simply extending ϕ_0 over B_{d+1} as a solution ϕ of the classical supergravity equations, and then writing

$$Z_S(\phi_0) = \exp(-I_S(\phi)) \tag{21}$$

where I_S is the classical supergravity action. Now our ansatz for the precise relation of conformal field theory on the boundary to AdS space is that

$$\left\langle \exp \int_{S^d} \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_S(\phi_0) \tag{22}$$

3.1 Massless Scalar Fields

Now we will do a sample calculation for massless fields in the approximation of classical supergravity. First we consider an AdS theory that contains a massless scalar field ϕ with boundary value ϕ_0 and the action

$$I(\phi) = \frac{1}{2} \int_{AdS_{d+1}} d^{d+1}y \sqrt{g} |d\phi|^2 \tag{23}$$

where g is the AdS metric. Following the above ideas, we now want to compute this action in order to obtain the AdS partition function (21) and therefore the conformal generating function of the n-point functions.

To do this, we must solve for ϕ in terms of its boundary value ϕ_0. For this we first look for a "Green's function", a solution K of the Laplace equation on AdS_{d+1} whose boundary value is a delta function at a point P on the boundary. To find this function, it is convenient to use the representation of AdS_{d+1} as the upper half space with metric (19). Take P to be the point at $x_0 = \infty$. The metric is obviously invariant under translations of the x_i, $i = 1, \dots, d$ and so are the boundary conditions, because for fixed x_0 translations of the x_i of course preserve the boundary. That means K will have this symmetry and is a function only of x_0.

We now want to compute the Laplace equation in the coordinates we are using now. This can be done with help of the Beltrami-Laplace operator

$$\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial \xi^\beta} \tag{24}$$

where g is the metric in the new coordinates. From (19) we read off $\sqrt{g} = x_0^{-(d+1)}$ and $g^{\alpha\beta} = x_0^2 \delta_{\alpha\beta}$. The Laplace equation with the transformed Operator (24) simplifies considerably as ϕ is a function of x_0 only. $\Delta = 0$ becomes now:

$$\frac{d}{dx_0} x_0^{-d+1} \frac{d}{dx_0} K(x_0) = 0 \tag{25}$$

This is an ordinary differential equation with the unique solution $K(x_0) = C_1 \cdot x_0^d + C_2$. As K is meant to be a Green's function, it should vanish everywhere but for $x_0 = P$. So we have

$$K(x_0) = c \cdot x_0^d \tag{26}$$

Since this grows at infinity, there is some sort of singularity at the boundary point P. To show that this singularity is a δ-function, it helps to map P to a finite point. This is done by the $SO(1,d)$ transformation

$$x_i \longrightarrow \frac{x_i}{x_0^2 + \sum_{j=1}^d x_j^2}, \quad i = 0, \dots, d \tag{27}$$

which maps P to the origin and vice versa. Note that this transformation is an isometry of our vector space and maps boundary points to boundary points. Under this mapping K transforms to

$$K(x) = c \cdot \frac{x_0^d}{(x_0^2 + \sum_{j=1}^d x_j^2)^d} \tag{28}$$

As the transformation is an isometry that preserves the metric and the boundary, K remains a solution of the Laplace equation on the upper half space with metric (19).

Now, for $x_0 \to 0$ $K(x)$ vanishes except for $x_1 = x_2 = \dots = s_d = 0$. Furthermore K is always positive. So for K to be proportional to a δ-function we only need to show, that the normalisation integral $\int dx_1 \dots dx_d \, K(x)$ is independent of x_0. This can be seen if we put the integral in the following form:

$$\int dx_1 \dots dx_d \, \frac{\frac{1}{x_0^d}}{\left(1 + \sum_{j=1}^d \frac{x_j^2}{x_0^2}\right)^d}$$

Now the transformation $x_j \to x_0 \cdot x_j$ eliminates x_0. So $K(x)$ is the desired Green's function.

Henceforth we write $\mathbf{x}$ for the d-tuple $(x_1, \ldots, s_d)$ and $|\mathbf{x}|$ for $\sum_{j=1}^d x_j^2$. Furthermore let $d\mathbf{x}'$ be the abbreviation for $dx_1' dx_2' \ldots dx_d'$

Using this Green's function, the solution of the Laplace equation on the upper half space with boundary value ϕ_0 is

$$\phi(x_0, x_i) = c \cdot \int d\mathbf{x}' \, \frac{x_0^d}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^d} \, \phi_0(x_i') \tag{29}$$

For small x_0 we have

$$\frac{\partial \phi}{\partial x_0} \approx d c x_0^{d-1} \int d\mathbf{x}' \, \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2d}} + O(x_0^{d+1}) \tag{30}$$

Rewriting the term $|d\phi|^2 = D_i \phi D^i \phi$ in (23) we obtain

$$I(\phi) = \frac{1}{2} \int_{\substack{\mathbb{R}^{d+1}, \\ x_0 > 0}} d\mathbf{x} \sqrt{g} \, \left(D_i(\phi D^i \phi) - \phi D_i D^i \phi \right) \tag{31}$$

The second term vanishes because ϕ fulfils the covariant Laplace equation which is the equation of motion for this action. Now using Gauss lemma on the remaining term we obtain

$$I(\phi) = \lim_{\epsilon \to 0} \frac{1}{2} \int_{T_\epsilon} d\mathbf{x} \sqrt{h} \, \phi(\vec{n} \cdot \nabla \phi) \tag{32}$$

where T_ϵ is the surface with $x_0 = \epsilon$ and n is the unit normal vector to T_ϵ and h the induced metric on the surface, i.e. h is given by $ds^2 = \frac{1}{x_0^2} \sum_{i=1}^d dx_i^2$. Therefore we have $\sqrt{h} = x_0^{-d}$. ∇ is the covariant gradient, but in this coordinates it is equal to the normal Euclidean gradient, because for fixed x_0 the Christoffel symbols of the given metric vanish. We now need to derive n but this is obviously given by $(x_0, 0, \ldots, 0)$. Of course we then have $\vec{n} \cdot \nabla \phi = x_0 (\partial \phi / \partial x_0)$. Since $\phi \to \phi_0$ for $x \to x_0$ and $(\partial \phi / \partial x_0)$ was calculated in (30), (32) gives

$$I(\phi) = \frac{c \cdot d}{2} \int d\mathbf{x} \, d\mathbf{x}' \, \frac{\phi_0(\mathbf{x}) \phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2d}} \tag{33}$$

Now we can calculate the two point function. The conjecture (22) tells us to interpret the partition function (21) with the just calculated action as generating function of the conformal field theory. So we have

$$\langle \mathcal{O}(\mathbf{u}) \mathcal{O}(\mathbf{v}) \rangle = \frac{\delta}{\delta \phi_0(u)} \frac{\delta}{\delta \phi_0(v)} \exp \left(-\frac{cd}{2} \int d\mathbf{x} d\mathbf{x}' \frac{\phi_0(\mathbf{x}) \phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2d}} \right) \Bigg|_{\phi_0 = 0} \tag{34}$$

A short calculation gives

$$\langle \mathcal{O}(\mathbf{u}) \mathcal{O}(\mathbf{v}) \rangle = -\frac{c \cdot d}{|\mathbf{w} - \mathbf{v}|^{2d}} \tag{35}$$

So the two point function of the operator $\mathcal{O}$ is proportional to $|\mathbf{x} - \mathbf{x}'|^{-2d}$ as it must be for every field $\mathcal{O}$ of conformal dimension d. This may be seen as a justification of the conjecture (22).

3.2 The Massive Case

Now we consider a scalar with mass m in AdS_{d+1}. The action is given by

$$I(\phi) = \frac{1}{2} \cdot \int dy^{d+1} \sqrt{g} (|d\phi|^2 + m^2 \phi^2) \tag{36}$$

with g the the metric on $AdS_{d+1=}$. This gives the equations of motion

$$(-D_i D^i + m^2)\phi = 0 \tag{37}$$

We will now calculate the Laplace operator in "polar coordinates" as they are given by (14). We obviously have $\sqrt{g} = \sinh(y)^d$ and $g^{\alpha\beta} \propto \delta_{\alpha\beta}$ with $g^{00} = 1$ and $g^{jj} = \sinh(y)^{-2}$ for $j = 1, \ldots, d$. So we have for the equations of motion

$$\left(-\frac{1}{\sinh(y)^d}\frac{d}{dy}\sinh(y)^d\frac{d}{dy} + \frac{L^2}{\sinh(y)^2} + m^2\right)\phi = 0 \tag{38}$$

were L^2 is independent of y and therefore irrelevant in the case of large y we are looking at now. For large y (38) simplifies to

$$-\phi'' - d \cdot \phi' + m^2\phi = 0 \tag{39}$$

This ordinary differential equation has the general solution (as can be calculated fast with Maple or Mathematica)

$$\phi = A \cdot e^{\frac{1}{2}(-d+\sqrt{d^2+4m^2})y} + B \cdot e^{\frac{1}{2}(-d-\sqrt{d^2+4m^2})y} \tag{40}$$

Denote the larger exponent by λ_+ and the smaller one by λ_-. Then this solution behaves at infinity as $e^{\lambda_+ y}$. That means in we cannot have a solution that approaches a constant at infinity. The closest we can do is pick any positive function f on B_{d+1} that has a simple zero on the boundary. For instance take $f = e^{-y}$. Then one can look for a solution of the equations of motion that behaves as

$$\phi \sim f^{-\lambda_+}\phi_0 \tag{41}$$

with an arbitrary "function" ϕ_0 on the boundary. As e^y has a simple pole on the boundary, a function ϕ obeying (41) has the right asymptotic behaviour.

The definition of ϕ as a function depends on the choice of a particular f, which is the same used in (16) to define a metric on the boundary of AdS_{d+1}. If we transform $f \to e^w f$ the metric will transform by $d\tilde{s}^2 \to e^{2w}d\tilde{s}^2$. At the same time, (41) shows that ϕ_0 will transform by $\phi_0 \to e^{w\lambda_+}\phi_0$ in order to keep ϕ fixed. This transformation under conformal rescaling of the metric shows that ϕ_0 has scaling dimension $\Delta = -\lambda_+$. That can be seen from the transformation relation for conformal fields: $\varphi \to |\partial x'/\partial x|^{-\Delta/d}\varphi = (\Lambda^{-d/2})^{\Delta/d}\varphi = \Lambda^{-\Delta/2}\varphi$ were $g' = \Lambda g$ when g is the metric an Δ the scaling dimension.

If in a conformal field theory on the boundary of AdS_{d+1} there is a coupling $\int \phi_0 \mathcal{O}$ for some operator $\mathcal{O}$, then $\mathcal{O}$ must have conformal dimension

$$\Delta = d + \lambda_+ = \frac{1}{2}\left(d + \sqrt{d^2 + 4m^2}\right) \tag{42}$$

(because mass dimension and scaling dimension have a one to one correspondence, as can easily be seen for dilations). This is a relation between scaling dimension and mass of the scalar field in AdS_{d+1} space!!

3.2.1 Computation of the Two Point Function

To start with, we need to find the explicate form of a function ϕ that obeys the massive wave equations and behaves as $f^{-\lambda_+}\phi_0$ at infinity. The calculation is in principle completely analogue to the massless case. Again we represent AdS space as the upper half space with the metric (19). As before the first step is to find a Green's function K for the equations of motion (37) that vanishes at the boundary except at one point. Again taking this point to be at $x_0 = \infty$, K should be a function of x_0 only. Plugging the already known transformed Laplace operator (24) (with $\sqrt{g} = x_0^{-(d+1)}$ and $g^{\alpha\beta} = x_0^2\delta_{\alpha\beta}$) in the equations of motion (37) we obtain

$$\left(-x_0^{d+1}\frac{d}{dx_0}x_0^{-d+1}\frac{d}{dx_0} + m^2\right)K(x_0) = 0 \tag{43}$$

Maple immediately gives the solution

$$\begin{aligned} K &= A \cdot x_0^{\frac{1}{2}(d+\sqrt{d^2+4m^2})} + B \cdot x_0^{\frac{1}{2}(d-\sqrt{d^2+4m^2})} \\ &= A \cdot x_0^{(d+\lambda_+)} + B \cdot x_0^{(d+\lambda_-)} \end{aligned} \tag{44}$$

The first term vanishes for $x_0 = 0$, the second not. So we have $K(x_0) = c' \cdot x_0^{d+\lambda_+}$. The already known inversion (27) gives

$$K = c' \cdot \frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{45}$$

which behaves as $x_0^{d+\lambda_+}$ for $x_0 \to 0$ except for a singularity at $\mathbf{x} = 0$. Now

$$\lim_{x_0\to 0}\frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{46}$$

is a multiple of $\delta(\mathbf{x})$ as one can prove by using a similar scaling argument as in the massless case to show that

$$\int d\mathbf{x}\,\frac{x_0^{d+2\lambda_+}}{(x_0^2 + |\mathbf{x}|^2)^{d+\lambda_+}} \tag{47}$$

is independent of x_0 (reduce the fraction in (47) by $x_0^{2d+2\lambda_+}$ and make the transformation $x_i \to x_0 \cdot x_i$) and observing that for $x_0 \to 0$ the function in (46) is supported near $x = 0$. So if we use K as the Green's function (instead of (46)) to construct a solution

$$\phi(x) = c' \int d\mathbf{x}\,\frac{x_0^{d+\lambda_+}}{(x_0^2 + |\mathbf{x} - \mathbf{x}'|^2)^{d+\lambda_+}}\,\phi_0(\mathbf{x}) \tag{48}$$

of the massive wave equation, then ϕ behaves for $x_0 \to 0$ like $x_0^{-\lambda_+}\phi(x)$, as expected from the argumentation at the beginning subsection (3.2).

The action (36) can be evaluated exactly by the same arguments that we have already see in the massless case. Again rewrite $|d\phi|^2 = D_i\phi D^i\phi$ by partial integration. We then obtain

$$I(\phi) = \frac{1}{2}\int d^{d+1}y\sqrt{g}\left(D_i(\phi D^i\phi) - \phi D_i D^i\phi + m^2\phi^2\right) \tag{49}$$

Because ϕ fulfils the equations of motion the last two terms vanish and we are left with the first term, that has the same form as what we obtain in equation (31) except that d in the exponent of the denominator (49) is replaced by $d + \lambda_+$ in comparison to the expression for ϕ in the massless case.

>From there we obtain the action in complete analogy to the massless case. The result is:

$$I(\phi) = \frac{c'(d+\lambda_+)}{2}\int d\mathbf{x}d\mathbf{x}'\,\frac{\phi_0(\mathbf{x})\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(\lambda_+ + d)}} \tag{50}$$

Like in the massless case this result gives the scaling dimension of a conformal field $\mathcal{O}$ via the two point function. The dimension is $\Delta = \lambda_+ + d$ as we have already expected. This again justifies the conjecture (22).

In sum, the scaling dimension Δ of conformal field on the boundary that is related to a field of mass m on AdS_{d+1} space is

$$\Delta = d + \lambda_+ = \frac{1}{2}(d + \sqrt{d^2 + 4m^2}) \tag{51}$$

References

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