Physikalisches Institut
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Theoretische Physik

Exercise 1
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Exercises on 'Introduction to Strings and Branes'

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A1. Warmup: Some metric calculus

Let $g_{\alpha\beta}$ be a metric in D-dimensional space, and $g^{\delta\gamma}$ the inverse metric.

1) Compute
$$\frac{\delta g_{\alpha\beta}}{\delta g^{\delta\gamma}} \; .$$

2) Let g be the determinant of $g_{\alpha\beta}$. Compute
$$\frac{\delta g}{\delta g^{\alpha\beta}} \; .$$

<u>*Hint:*</u> It might be useful to remember that $\det \exp(A) = \exp(trA)$.

A2. *Polyakov*- vs. *Nambu-Goto*-action

The *Polyakov*-action is given by

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \; , \tag{1}$$

where the index μ is raised and lowered with the *Minkowski*-metric $\eta_{\mu\nu}$ and $\alpha, \beta \in \{\sigma, \tau\}$.

1) Compute the energy-momentum tensor via the functional variation

$$T_{\alpha\beta} = -\frac{4\pi\alpha'}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} \; . \tag{2}$$

2) The equation of motion with respect to $g_{\alpha\beta}$ forces the energy-momentum tensor to vanish identically. Solve $T_{\alpha\beta} = 0$ and plug the solution back into (1). The resulting action is the Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \; .$$

3) Compute the trace of the energy-momentum tensor

$$g^{\alpha\beta} T_{\alpha\beta} \; .$$

A3. Gauge fixing of the *Polyakov*-action

The *Polyakov*-action (1) respects the following symmetries:

(i) global symmetries:

$*$ *Poincare* invariance

$$\begin{aligned} \delta X^\mu &= a^\mu{}_\nu X^\nu + b^\mu \quad (a_{\mu\nu} = -a_{\nu\mu}) \\ \delta g_{\alpha\beta} &= 0 \end{aligned}$$

(ii) local symmetries:

* reparametrization invariance

$$\begin{aligned}\delta X^\mu &= \zeta^\alpha \partial_\alpha X^\mu \\ \delta g_{\alpha\beta} &= \zeta^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \zeta^\gamma g_{\gamma\beta} + \partial_\beta \zeta^\gamma g_{\alpha\gamma} \\ &= \nabla_\alpha \zeta_\beta + \nabla_\beta \zeta_\alpha \\ \delta\sqrt{-g} &= \partial_\alpha(\zeta^\alpha \sqrt{-g})\end{aligned}$$

* *Weyl* rescaling

$$\begin{aligned}\delta g_{\alpha\beta} &= 2\Lambda g_{\alpha\beta} \\ \delta X^\mu &= 0\end{aligned}$$

where ζ and Λ are infinitesimal functions of τ and σ. With help of the local symmetries it's always possible to choose the worldsheet metric to be *Minkowskian*, i.e. $g_{\alpha\beta} = \eta_{\alpha\beta}$.

Take $g_{\alpha\beta} = \eta_{\alpha\beta}$ and perform the coordinate transformation

$$\sigma^\pm = \tau \pm \sigma \ .$$

This coordinates are called conformal coordinates.

1) Rewrite the *Polyakov*-action (1) and the energy-momentum tensor (2) in these coordinates, i.e. show that

$$S_P^{\text{gauged}} = \frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^- \partial_- X^\mu \partial_+ X^\mu \ . \tag{3}$$

2) How does the *Minkowski*-metric look like in conformal coordinates?
3) Compute how the *Minkowskian* worldsheet metric changes under the so called conformal coordinate transformations

$$\sigma^+ \to \tilde{\sigma}^+\left(\sigma^+\right) \quad , \quad \sigma^- \to \tilde{\sigma}^-\left(\sigma^-\right) \ .$$

From this it's clear, that the gauge fixed *Polyakov*-action (3) still is invariant under conformal coordinate transformations. One can show that conformal symmetry is equivalent (here!) to the fact, that $T_{+-} = 0$. But one still has to satisfy the constraints coming from $T_{++} = T_{--} = 0$.

4) Choose the so called light cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^1\right) .$$

and write down the constraints $T_{++} = T_{--} = 0$ in these coordinates.
5) Compute the equations of motion from (3).
6) Take
$$X^+ = x^+ + p^+\tau \qquad \text{with } x^+, p^+ = const$$
(why is this always possible?) and solve the constraints $T_{++} = T_{--} = 0$ for X^-.
7) Interpret the results.