Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises on 'Introduction to Strings and Branes'

Dr. S. Förste

A4. Worldsheet supersymmetry

Supersymmetry is a natural extension of *Poincare* invariance. At the moment we'll restrict ourselves to the twodimensional worldsheet supersymmetry where lots of things simplify. Consider the worldsheet action

$$S = \frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^- \left(\partial_- X^\mu \partial_+ X_\mu + \frac{i}{2} \psi_+^\mu \partial_- \psi_{+\mu} + \frac{i}{2} \psi_-^\mu \partial_+ \psi_{-\mu} \right) . \qquad (1)$$

In addition to the bosonic worldsheet action we have introduced a twodimensional *Majorana-Weyl* spinor ψ which components are labeled by $\psi_\mp$.

Let

$$\rho^\pm \equiv \rho^0 \pm \rho^1 \quad \text{with} \quad \rho^0 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \qquad (2)$$

1) Show that ρ^0 and ρ^1 satisfy a twodimensional *Clifford* algebra.

Majorana-Weyl spinors are *Weyl* spinors, on which a *Majorana* condition can be imposed. This is possible iff $D = 2 \mod 8$. *Majorana* spinors are always real. For details on spinors in various dimensions see e.g. "Polchinski: String theory Vol II, App. B.1".

2) Show for the *Dirac* conjugate

$$\bar{\psi} \equiv \overline{\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}} = i \, (\psi_+, -\psi_-) .$$

3) Use $\not{\partial} = \rho^+ \partial_+ + \rho^- \partial_-$ to prove

$$\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu} = \frac{1}{2} \bar{\psi}^\mu \not{\partial} \psi_\mu . \qquad (3)$$

Therefor it follows from (1) that the *Majorana-Weyl* spinor ψ obeys a massless *Dirac* equation, so we are still dealing with a worldsheet action describing free fields.

Now we are ready to introduce the worldsheet supersymmetry (SUSY) transformations

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon} \psi^\mu & (4) \\ \delta \psi^\mu &= -i \rho^\alpha \partial_\alpha X^\mu \epsilon , & (5) \end{aligned}$$

where ϵ is a constant anticommuting spinor.

4) Discuss the SUSY tranfsformations (4) and (5).
5) Rewrite (4) and (5) to show that

$$\begin{aligned} \delta X^\mu &= i\epsilon_+ \psi_-^\mu - i\epsilon_- \psi_+^\mu & (6) \\ \delta \psi_-^\mu &= -2\epsilon_+ \partial_- X^\mu & (7) \\ \delta \psi_+^\mu &= 2\epsilon_- \partial_+ X^\mu . \end{aligned}$$

6) Show the invariance of (1) under the SUSY transformations (6) and (7). Please write down the steps of your calculation carefully for we'll need them later...
__Hint:__ Remember that e.g. $\epsilon_+ \psi_- = -\psi_- \epsilon_+$.

In analogy to exercise 1 A2 its possible to compute the energy-momentum tensor to be

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + i\psi^\mu_- \partial_+ \psi_{-\mu} \ , \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu + i\psi^\mu_+ \partial_- \psi_{+\mu} \ . \end{aligned}$$

This is of course a consequence of *Poincare* invariance.

Now we want to use the *Noether* method to compute the supercurrent (the conserved current corresponding to SUSY): For constant ϵ the worldsheet action (1) is invariant under the SUSY transformations (6) and (7). If ϵ is not constant then (1) isn't invariant, but its variation takes the general form

$$\delta S = -\frac{1}{2\pi\alpha'} \int d\sigma^+ d\sigma^- (\partial_\alpha \bar{\epsilon}) J^\alpha \ , \tag{8}$$

where J is the supercurrent we are searching for.

7) Go back to your calculations in 6) and write down the additional terms coming from a nonconstant ϵ. Use (8) to derive

$$\begin{aligned} J_- \equiv (J_-)_- &= \psi_{-\mu} \partial_- X^\mu \ , \\ J_+ \equiv (J_+)_+ &= \psi_{+\mu} \partial_+ X^\mu \ , \\ (J_-)_+ &= (J_+)_- = 0 \ , \end{aligned}$$

where the index inside the brackets denote the spinor index and the one outside the vector index.

In analogy to the purely bosonic case its again possible to impose a lightcone gauge. You might want to check your lectures how this works but we won't adress this problem here...

A5. Lightcone gauge quantisation of the closed bosonic string

From the bosonic worldsheet action (compare A2) one finds the equation of motion

$$\partial_+ \partial_- X^\mu = 0 \ . \tag{9}$$

We parametrize the closed string by $\sigma \in [0, \pi]$. Of course the general solution for (9) is given by

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \ ,$$

where the indices L and R denote left- and rightmoving waves in correspondence to the classical picture. Because of the periodicity of the closed string one can expand $X_{L/R}$ in a *Fourier* series yielding

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu \sigma^+ + \frac{i}{2}\sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-2in\sigma^+} \ , \tag{10}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu \sigma^- + \frac{i}{2}\sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-2in\sigma^-} \ . \tag{11}$$

Classically, x^μ corresponds to the center of mass position, p^μ to the center of mass momentum and α^μ_n ($\tilde{\alpha}^\mu_n$) to the amplitude of the n'th left moving (right moving) vibration mode.

Reality of X^μ requires $\alpha^{\mu\dagger}_n = \alpha^\mu_{-n}$ and $\tilde{\alpha}^{\mu\dagger}_n = \tilde{\alpha}^\mu_{-n}$.

Following canonical quantization the equal time (τ) commutators are given by

$$\begin{aligned} [X^\mu(\sigma), X^\nu(\sigma')] = \left[\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\right] &= 0 \\ \left[\dot{X}^\mu(\sigma), X^\nu(\sigma')\right] &= -i\pi\delta(\sigma - \sigma')\eta^{\mu\nu} \ . \end{aligned} \tag{12}$$

Formally one can assign a *Fourier* series to the delta function

$$\delta(\sigma) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} e^{2ik\sigma} \ .$$

1) Use the canonical commutators (12) to derive

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha_n^\mu, \alpha_k^\nu] &= n\delta_{n,-k}\eta^{\mu\nu} \\ [\tilde{\alpha}_n^\mu, \tilde{\alpha}_k^\nu] &= n\delta_{n,-k}\eta^{\mu\nu} \ . \end{aligned}$$

2) Now we choose the lightcone gauge. What follows for the oscillators α_n^+ and $\tilde{\alpha}_n^+$ ($n \neq 0$)?
3) Use the constraints $T_{++} = T_{--} = 0$ to show that

$$p^+\alpha_n^- = \sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i \alpha_m^i: -2a\delta_{n,0} \tag{13}$$

$$p^+\tilde{\alpha}_n^- = \sum_{m=-\infty}^{\infty} :\tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i: -2a\delta_{n,0} \ , \tag{14}$$

where a is a normal ordering constant, $\alpha_0^\mu \equiv \tilde{\alpha}_0^\mu \equiv p^\mu/2$ and i runs over the transverse oscillators ($i \in [2, D-1]$). (13) and (14) serve as conditions on physical states.

Choose the vacuum as an eigenstate of the p^μ, i.e. $p^\mu|k\rangle = k^\mu|k\rangle$. Define the α_n's with $n > 0$ to be annihilators, and those with $n < 0$ to be creators. Now the normal ordering correspondes to creation/annihilation normal ordering.

4) Rewrite (13) and (14) for $n = 0$ and make the normal ordering explicit. Use this to derive the spectrum

$$M^2 \equiv 2p^+p^- - p^ip^i = 8(N-a) = 8(\tilde{N}-a) \ , \tag{15}$$

where the number operators are given by

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \ , \tag{16}$$

and $\tilde{N}$ analogously.

5) What is the first excited state consistent with (15)? By decomposing this state in a symmetric traceless part, an antisymmetric part and the trace itself, you'll find irreducible representations of SO$(D-2)$. Comparing with (15), what can you say about M and a?

Now we want to derive a in another way. Naturally the normal ordering would be symmetric, i.e.

$$N - a = \frac{1}{2} \sum_{n \neq 0} \alpha_{-n}^i \alpha_n^i \ . \tag{17}$$

6) Use (17), (16) and the commutators for the α's to show that

$$a = -\frac{D-2}{2} \sum_{n=1}^{\infty} n \ .$$

7) Use the 'zeta function regularisation', i.e. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\zeta(-1) = -1/12$ to derive the critical dimension for the closed bosonic string.