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Exercise 5
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Exercises on 'Introduction to Strings and Branes'

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A8. The *Buscher* formulae

Last time we saw, that compactifying one dimension changes the left and right moving momenta (neglecting oscillators, $B = 0$ and G diagonal for simplicity) to

$$p_L = \frac{\alpha'}{R}n + \frac{wR}{\alpha'} \qquad \text{and} \qquad p_R = \frac{\alpha'}{R}n - \frac{wR}{\alpha'} \ .$$

Using the transformation

$$R \leftrightarrow \frac{\alpha'}{R} \qquad \text{and} \qquad n \leftrightarrow w$$

leads to $p_L \to p_L$ and $p_R \to -p_R$. The spectrum is unchanged because it is quadratic in the momenta. What we found is a discrete symmetry of the theory, called T-duality.

Now we want to show with help of the nonlinear σ-model, that T-duality is a symmetry even for nontrivial backgrounds. This calculation was first done by *Buscher*.

The (gauge fixed) worldsheet action is (up to the Dilaton term) given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(G_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right) \ . \tag{1}$$

Compactifying one target space dimension is possible if (1) is invariant under constant shifts in this direction. Define the translations in the $d-1$ direction

$$X^{d-1} \to X^{d-1} + c \ . \tag{2}$$

Call the other coordinates X^i. One can show that for consistency with compactification it is sufficient that B doesn't depend on X^{d-1} anymore.

We now want to gauge the symmetry (2) and to 'undo' this by constraining the gauge fields to be of pure gauge. Therefore we introduce twodimensional gauge fields A_α changing under (2) as

$$A_\alpha \to A_\alpha - \partial_\alpha c$$

and replace

$$\partial_\alpha X^{d-1} \to D_\alpha X^{d-1} + A_\alpha \tag{3}$$

in the $d-1$ direction.

1) Plug (3) into (1) and write S as a sum $S + S_A$ of old terms and terms including the gauge field.
2) How can you implement the constraint that the field strength of A has to vanish? How could you get back the original σ model?

One can show that if X^{d-1} repsects a periodicity with period 2π (compactification radius absorbed in the metric), than a related periodicity has to hold for the *Lagrange* multiplier λ.

3) Because A does not propagate you can integrate out A by using it's equation of motion. Plug the result back into S_A (you have to be very careful about what you are doing!!!). Summing up S and S_A again you should find the dual model

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left(\tilde{G}_{\mu\nu} \partial_\alpha \tilde{X}^\mu \partial^\alpha \tilde{X}^\nu + \epsilon^{\alpha\beta} \tilde{B}_{\mu\nu} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \right)$$

with $\left\{ \tilde{X}^\mu \right\} = \{\lambda, X^i\}$ and the dual backgrounds given by the *Buscher* formulae

$$\begin{gathered}
\tilde{G}_{d-1,d-1} = \frac{1}{G_{d-1,d-1}} \,, \quad \tilde{G}_{d-1,i} = \frac{B_{d-1,i}}{G_{d-1,d-1}} \,, \\
\tilde{G}_{i,j} = G_{ij} - \frac{G_{i,d-1}G_{d-1,j} + B_{i,d-1}B_{d-1,j}}{G_{d-1,d-1}} \,, \\
\tilde{B}_{d-1,i} = \frac{G_{d-1,i}}{G_{d-1,d-1}} \,, \quad \tilde{B}_{i,j} = B_{ij} - \frac{G_{i,d-1}B_{d-1,j} + B_{i,d-1}G_{d-1,j}}{G_{d-1,d-1}} \,.
\end{gathered}$$