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Exercise 2
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Exercises on 'Elementary Particle Physics'

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1. *The Lorentz group*

   (a) The *Lorentz* transformations Λ are defined by $\Lambda^{\mu}{}_{\nu}\Lambda^{\rho}{}_{\sigma}\eta_{\mu\rho} = \eta_{\nu\sigma}$ (we use $\eta_{\mu\nu} = diag(+1,-1,-1,-1)$). Show that the

   $$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

   generate the group.

   (b) Compute the *Poincaré* algebra, i.e. $[P,P]$, $[M,M]$ and $[P,M]$, where $P_\mu = -i\partial_\mu$ are the generators of translations.

   (c) Decompose the *Lorentz* algebra into rotations $J_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}$ and boosts $K_i = M_{0i}$. Compute the algebra in this basis.

   (d) Show that the generators $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$ fullfill the commutation relations of $SU(2)_A \times SU(2)_B$. Because of this the representations of the *Lorentz* group can be characterized by two spins (j_A, j_B).

2. *Representations of the Lorentz group*

   (a) From exercise 1d one sees, that the two smallest (non trivial) representations of the *Lorentz* group are given by $\psi_L \sim (\frac{1}{2}, 0)$ und $\psi_R \sim (0, \frac{1}{2})$ (so called *Weyl* spinors). Give the transformation matrices $D_{L,R}(\Lambda)$ explicitly.

   *(b) Prove that $D_R^* = \sigma_2 D_L \sigma_2$ and $(D_{L,R})^{-1^T} = \sigma_2 D_{L,R}\sigma_2$. Use the results to construct a bilinear *Lorentz* invariant in $\psi_{L,R}$

Homework

3. *Spinors*

   (a) Examine the behaviour under space reflections $\mathbf{P}$ with $\mathbf{P} : (x_0, \vec{x}) \to (x_0, -\vec{x})$ and show that $\mathbf{P}$ exchanges left- and righthanded spinors. The simplest representation that closes under $\mathbf{P}$ is therefore given by

   $$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \sim (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}).$$

   ψ is called a *Dirac* spinor.

   (b) The *Dirac* matrices γ^μ can be given in terms of the *Pauli* matrices by

   $$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} .$$

   Show that these matrices fullfill the *Clifford* algebra $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_4$.

   (c) Show that a Dirac spinor transforms as

   $$\psi \mapsto D(\Lambda)\psi := \exp\left(\frac{i}{2}\Lambda_{\mu\nu}\gamma^{\mu\nu}\right) \psi$$

   under a Lorentz-transformation, with $\gamma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Use the results of the exercises! (i.e. 2a)

   (d) Calculate the (fundamental) matrix representation of the Lorentz Algebra $M_{\rho\sigma}$ on fourdimensional Minkowski space, called $m_{\rho\sigma}$. (*Hint: Apply the generator to the four vector* x^ν*...*) Hence, a Lorentz transformation in Minkowski space is given by $L(\Lambda) = \exp(-\frac{i}{2}\Lambda_{\rho\sigma}m^{\rho\sigma})$. Derive

   $$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = L(\Lambda)^\mu{}_\nu\gamma^\nu .$$

   (*Hint: Calculate for infinitesimals (it's a Lie algebra!). First derive the identity* $[\gamma^\mu, \gamma^{\nu\sigma}] = -(m^{\nu\sigma})^\mu{}_\rho\gamma^\rho$.)

   (e) We can assume the hermiticity property $(\gamma^\mu)^\dagger = (\gamma^0, -\gamma)$. Show that the fermionic bilinears ($\bar{\psi} = \psi^\dagger\gamma_0$)

   $$(i)\ \bar{\psi}\psi \quad (ii)\ \bar{\psi}\, i\gamma_5\psi \quad (iii)\ \bar{\psi}\gamma^\mu\psi$$
   $$(iv)\ \bar{\psi}\gamma^\mu\gamma_5\psi \quad (v)\ \bar{\psi}\, i[\gamma^\mu, \gamma^\nu]\psi \qquad (*)$$

   transform as scalar, pseudoscalar, vector, pseudovector, (2,0)-tensor respectively under a Lorentz transformation. (What's the difference..?)

(f) Now define $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Calculate γ_5 in the representation of (b) and hence argue that it is sensible to define a spinor left- or right-handed if it is an eigenvector of γ_5 with eigenvalue $1/-1$ resp. To show that this is well-defined show for a general (!) Clifford algebra that any 4-spinor can be decomposed uniquely as $\psi = \psi_L + \psi_R$, where $\psi_{L/R}$ are left/right-handed spinors. (*Hint: Find the suitable projectors..*) Calculate the identity $\{\gamma^\mu, \gamma_5\} = 0$ and use this to show that γ^μ acting on a right-handed spinor gives a left handed spinor and vice versa.

(g) Show that the Dirac equation is equivalent to the pair of equations

$$i\gamma^\mu \partial_\mu \psi_R - m\psi_L = 0 \quad i\gamma^\mu \partial_\mu \psi_L - m\psi_R = 0 \,.$$

What is the role of the mass term?

4. *C, P & T for spinors*

Under parity, charge conjugation and time reversal a Dirac field ψ transforms as

$$\begin{aligned}
\hat{P}\psi(x)\hat{P}^{-1} &= \eta_P \gamma^0 \psi(x_P) \qquad x_P^\mu = (x_0, -\mathbf{x}) \\
\hat{C}\psi(x)\hat{C}^{-1} &= \eta_C C \bar{\psi}(x)^t \\
\hat{T}\psi(x)\hat{T}^{-1} &= \eta_T B \psi(x_T) \qquad x_T^\mu = (-x_0, \mathbf{x}) \,,
\end{aligned}$$

with $\hat{P}$, $\hat{C}$, $\hat{T}$ the linear, linear and antilinear operators (respectively) implementing these operations (note that antilinear means $\hat{T} \mid \lambda\phi \rangle = \lambda^* \hat{T} \mid \phi \rangle$ and that if $\hat{T} \mid \phi \rangle = \mid \phi_T \rangle$ then $\langle \phi' \mid \phi \rangle = \langle \phi_T \mid \phi'_T \rangle$). $\eta_{P/C/T}$ are the intrinsic parity, charge conjugation parity etc. (remember: $\mid \eta_{P/C/T} \mid^2 = 1$). The matrix C is defined via $C\gamma^{\mu\, t} C^{-1} = -\gamma^\mu$ (where we assume that $\gamma^{\mu\,\dagger} = (\gamma^0, -\gamma)$). One can show that $C^t = -C$ and $C^\dagger C = \mathbb{1}$. The matrix B is defined by $B\gamma^{\mu\,*} B^{-1} = (\gamma^0, -\gamma)$ and one can show that $B^\dagger B = \mathbb{1}$.

(a) Give an explicit construction for C. (*Hint: γ-matrices..*) Show that $B\gamma_5^* B^{-1} = \gamma_5$ and verify that with the assumed form of $\gamma^{\mu\,\dagger}$ we may take $B = \pm\gamma_5 C$.

(b) Show that, if X is a matrix acting on Dirac spinors,

$$\begin{aligned}
\hat{C}\bar{\psi}(x) X \psi(x) \hat{C}^{-1} &= \bar{\psi}(x) X_C \psi(x) \\
\hat{T}\bar{\psi}(x) X \psi(x) \hat{T}^{-1} &= \bar{\psi}(x_T) X_T \psi(x_T) \,,
\end{aligned}$$

where $X_C = CX^t C^{-1}$ (ψ and $\bar{\psi}$ anti-commute (!)) and $X_T = BX^* B^{-1}$. What is the expression for $\hat{P}\bar{\psi}(x) X \psi(x) \hat{P}^{-1}$.

(c) Hence determine the transformation properties of the fermionic bilinears $(*)$ (above) under parity, charge conjugation and time reversal .

(d) If $|p\rangle$ is a boson with momentum p and $\langle 0 | \bar{\psi}(0)\, i\gamma_5 \psi(0) \,|p\rangle \neq 0$, show that in a theory where P and C are conserved the boson must have negative intrinsic parity and also positive charge conjugation parity.