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Example sheet 1
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Elementary Particle Physics II

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1. ***Calculation tools for Weyl spinors***

- θ_α ($\alpha = 1, 2$) and $\bar{\theta}^{\dot{\alpha}}$ ($\dot{\alpha} = 1, 2$) are anticommuting *Grassmann* variables:

$$\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta_\alpha, \bar{\theta}^{\dot{\beta}}\} = 0\,. \tag{1}$$

  θ transforms as a *left-handed* ($\frac{1}{2}$,0) Weyl spinor under the Lorentz group, $\bar{\theta}$ as a *right-handed* (0,$\frac{1}{2}$) one. (*For more background about Weyl spinors, see last term's example sheet 2.*)

- $\epsilon_{\alpha\beta} \equiv \epsilon^{\alpha\beta} \equiv \epsilon_{\dot{\alpha}\dot{\beta}} \equiv \epsilon^{\dot{\alpha}\dot{\beta}}$ are totally antisymmetric tensors, defined through $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{12} = 1$.

(a) Verify that $\epsilon_{\alpha\beta}\,\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$.

(b) The ϵ are Lorentz invariant and can be used to raise and lower spinor indices. We define

$$\theta_\alpha \equiv \epsilon_{\alpha\beta}\,\theta^\beta\,, \qquad \bar{\theta}_{\dot{\alpha}} \equiv \epsilon_{\dot{\alpha}\dot{\beta}}\,\bar{\theta}^{\dot{\beta}}\,. \tag{2}$$

What is the inverse of these relations?

(c) Verify the following identities (*note the conventions for index contraction*):

$$\xi\psi(\equiv \xi^\alpha\psi_\alpha) = \psi\xi\,, \quad \bar{\xi}\bar{\psi}(\equiv \bar{\xi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) = \bar{\psi}\bar{\xi}\,, \tag{3}$$

$$\xi_\alpha\xi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\,\xi\xi\,, \qquad \xi^\alpha\xi^\beta = ?\,, \qquad \bar{\xi}^{\dot{\alpha}}\bar{\xi}^{\dot{\beta}} = -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\xi}\bar{\xi}\,, \qquad \bar{\xi}_{\dot{\alpha}}\bar{\xi}_{\dot{\beta}} = ?\,. \tag{4}$$

(d) We will write the derivative with respect to a Grassmann variable as $\frac{\partial\theta^\beta}{\partial\theta^\alpha} \equiv \partial_\alpha\theta^\beta = \delta_\alpha^\beta = \partial^\beta\theta_\alpha$ and $\frac{\partial\bar{\theta}_{\dot{\beta}}}{\partial\bar{\theta}_{\dot{\alpha}}} \equiv \bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}} = \bar{\partial}_{\dot{\beta}}\bar{\theta}^{\dot{\alpha}}$. Note that the product rule must include a negative sign: $\partial_\alpha(\theta^\beta\theta^\gamma) = \delta_\alpha^\beta\theta^\gamma - \theta^\beta\delta_\alpha^\gamma$. Why is this neccessary? Check that $\partial^\alpha = \epsilon^{\alpha\beta}\partial_\beta$ (*with the opposite sign to (b)!*). Now check that $\partial^\alpha\partial_\alpha(\theta\theta) = \bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}(\bar{\theta}\bar{\theta}) = 4\,.$

2. ***Pauli matrices and Weyl spinors***

The Pauli matrices can be used to link the spinorial indices $\alpha, \dot{\alpha}$ to the spacetime index μ (a Lorentz vector x can be written either as x^μ or $x_{\alpha\dot{\beta}} \equiv x_\mu \sigma^\mu_{\alpha\dot{\beta}}$). We will use these conventions:

$$\sigma^\mu_{\alpha\dot{\beta}} = (\mathbf{1}, \vec{\sigma})_{\alpha\dot{\beta}}\,, \qquad \bar{\sigma}^{\dot{\beta}\alpha}_\mu = (\mathbf{1}, \vec{\sigma})^{\dot{\beta}\alpha}\,, \qquad \eta^{\mu\nu} = diag(1, -1, -1, -1)\,. \tag{5}$$

(a) Check that the definitions in (5) are consistent with the use of ϵ to raise and lower indices. (*Hint from last term:* $-\epsilon\sigma^\mu\epsilon = (\bar{\sigma}^\mu)^T$)

(b) Check

$$\sigma^\mu_{\alpha\dot{\beta}}\, \bar{\sigma}^{\dot{\gamma}\delta}_\mu = 2\, \delta^\delta_\alpha\, \delta^{\dot{\gamma}}_{\dot{\beta}}\,, \qquad \mathrm{Tr}(\sigma^\mu \bar{\sigma}^\nu) = 2\, \eta^{\mu\nu}\,. \tag{6}$$

(c) Show

$$V^\mu = \frac{1}{2}\, (\bar{\sigma}^\mu)^{\dot{\beta}\alpha}\, V_{\alpha\dot{\beta}}\,. \tag{7}$$

(d) Verify the following identities:

$$(\bar{\xi}\bar{\sigma}^\mu\psi) = -(\psi\sigma^\mu\bar{\xi})\,, \tag{8}$$

$$\psi_\alpha \bar{\xi}_{\dot{\beta}} = \frac{1}{2}\, \sigma^\mu_{\alpha\dot{\beta}}\, (\psi\sigma_\mu\bar{\xi})\,, \qquad (\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta})\,. \tag{9}$$

3. ***SUSY generators***

The SUSY algebra is defined through the following relations:

$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = 2\, \sigma^\mu_{\alpha\dot{\beta}}\, P_\mu\,, \quad [Q_\alpha, P_\mu] = \left[\bar{Q}_{\dot{\alpha}}, P_\mu\right] = \{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0 \tag{10}$$

(a) Show that $\left[\theta Q, \bar{Q}\bar{\theta}\right] = 2\, \theta\sigma^\mu\bar{\theta}\, P_\mu$.

(b) Check that

$$\begin{aligned} P_\mu &= i\frac{\partial}{\partial x_\mu} \equiv i\partial_\mu\,, & (11)\\ Q_\alpha &= \partial_\alpha - i\, \sigma^\mu_{\alpha\dot{\beta}}\, \bar{\theta}^{\dot{\beta}}\, \partial_\mu\,, & (12)\\ \bar{Q}_{\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}} + i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}}\, \partial_\mu\,. & (13) \end{aligned}$$

form a representation of the SUSY algebra by explicitly verifying that they satisfy the (anti-)commutators in (10).