Universität Bonn
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Theoretische Physik

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Example sheet 3
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General Relativity

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1. *Non-orthogonal coordinates*

Let us take a Cartesian coordinate system spanned by the unit vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$, and a non-orthogonal system spanned by $\vec{e}_1$ and $\vec{e}_2$. The tilted system's basis vectors are given by

$$\vec{e}_1 = \vec{\epsilon}_1 , \qquad \vec{e}_2 = \vec{\epsilon}_1 + 2\vec{\epsilon}_2 . \tag{1}$$

We can write any point X in the Cartesian system as

$$X = \xi^1 \cdot \vec{\epsilon}_1 + \xi^2 \cdot \vec{\epsilon}_2 \equiv \xi^a \vec{\epsilon}_a \equiv \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}_\epsilon . \tag{2}$$

The coefficients ξ^a $(a = 1, 2)$ are called *contravariant* coordinates.

(a) Rewrite X in the tilted system to obtain the contravariant coordinates x^i of $\vec{\xi}$ in terms of the ξ^a.

(b) Show that the distance s from point X to the origin can be written in the Cartesian system as $s^2 = \eta_{ab}\, \xi^a \xi^b$. What does the *metric* η_{ab} of the Cartesian system look like?

(c) If we require that distances should not depend on the choice of coordinate system, we can write s^2 as

$$s^2 = \eta_{ab}\, \xi^a \xi^b = g_{ij}\, x^i x^j , \tag{3}$$

where we now denote the metric of the tilted system by g_{ij}.
Write g_{ij} in terms of η_{ab} for our example and generally ($dx^i = (\partial x^i / \partial \xi^a) d\xi^a$).

(d) The contraction $x_i = g_{ij}\, x^j$ is called *covariant* coordinate.
Draw a sketch of the two coordinate systems and choose a point X arbitrarily. Identify the coordinates ξ^a, ξ_a, x^i and x_i of your chosen point on the axes.

2. Locally varying coordinates

One familiar example of location-dependent coordinates are spherical coordinates

$$\xi^1 = r\cos\theta\,, \quad \xi^2 = r\sin\theta\cos\phi\,, \quad \xi^3 = r\sin\theta\sin\phi\,. \tag{4}$$

Let us write $(r, \theta, \phi) \equiv (q^1, q^2, q^3)$. Since the coordinate systems change from point to point, the invariant length in equation (3) has to be rewritten infinitesimally as

$$ds^2 = \eta_{ab}\, d\xi^a d\xi^b = g_{ij}(q)\, dq^i dq^j \tag{5}$$

(a) Determine $g_{ij}(q)$ from ds^2 for the spherical coordinates.

(b) What are the covariant coordinates q_i?

3. Lagrange formalism with general coordinates

(a) Obtain the equations of motion for the general coordinates q^k with metric $g_{ij}(q)$ from the Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \frac{\partial \mathcal{L}}{\partial q^k} = 0\,, \tag{6}$$

where $\mathcal{L} = T(\dot{q}, q) - V(q)$. Note the dependence of the kinetic energy T on the coordinates q which originates in

$$T = \frac{m}{2}v^2 = \frac{m}{2} g_{ij}(q)\, \dot{q}^i \dot{q}^j\,. \tag{7}$$

(b) Verify that the equations of motion in (a) have the form

$$\ddot{q}^\ell + \Gamma^\ell_{ij}\, \dot{q}^i \dot{q}^j = -\frac{1}{m} g^{\ell k} \frac{\partial V}{\partial q^k}\,. \tag{8}$$

The Γ^ℓ_{ij} are called *Christoffel symbols.* Note that they parameterize pseudo-forces coming from the local change of the metric tensor g_{ij}:

(c) Show that

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{\ell k} \left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right)\,. \tag{9}$$

4. Preparation for next week: Spherical coordinates again

(a) Calculate all Christoffel symbols for the spherical coordinates given in exercise 2.

(b) Use equation (8) to calculate the equations of motion for r, θ and ϕ.