Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 4
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General Relativity

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1. *Free movement on geodesics*

(a) Use the action principle $\delta \int \mathcal{L}\, dt = 0$ and conservation of energy to show that the movement of a free particle is always given by $\delta \int ds = 0$ (*i.e.* free particles move on a path of extremal length, the *geodesic*).

(b) Write the equations of motion in last week's question 4 for a free particle on a sphere of fixed radius R.

(c) Using the conserved conjugate momentum to ϕ, show that the remaining equation of motion can be written as

$$\ddot{\theta} - B^2 \frac{\cos\theta}{\sin^3\theta} = 0\,, \tag{1}$$

where $B = \mathit{const} = \sin^2\theta\,\dot{\phi}$. In the following, we will try to determine which geometrical form the motion will take.

(d) Show that energy conservation implies

$$\frac{\dot{\theta}^2}{\sin^4\theta\,\dot{\phi}^2} = \frac{A}{B^2} - \frac{1}{\sin^2\theta}\,; \qquad A = \mathit{const} = \frac{2E}{mR^2}\,. \tag{2}$$

(e) Calculate $\frac{d}{d\phi}\cot\theta$ and substitute $u \equiv \cot\theta$ to rewrite (2) as

$$\left(\frac{du}{d\phi}\right)^2 = \frac{A}{B^2} - 1 - u^2\,. \tag{3}$$

(f) Show that the solution of (3) can be rewritten in cartesian coordinates as

$$z = \alpha\, x + \beta\, y\,, \qquad x^2 + y^2 + z^2 = R^2\,, \tag{4}$$

where α and β are suitably chosen constants.
What form do the trajectories of a free particle on a sphere take?

2. Riemann Tensor

The Christoffel symbols are *not* tensors, and thus are not suitable to describe a curved geometry in a coordinate-invariant way. The only tensor that can be constructed from the metric and its first and second derivatives is the *Riemann tensor*

$$R^\lambda{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^\lambda{}_{\mu\nu}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda{}_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta{}_{\mu\nu}\Gamma^\lambda{}_{\kappa\eta} - \Gamma^\eta{}_{\mu\kappa}\Gamma^\lambda{}_{\nu\eta}\,. \tag{5}$$

Through self-contractions we get the *Ricci tensor* $R_{\mu\kappa} \equiv R^\lambda{}_{\mu\lambda\kappa}$ and the *curvature scalar* $R \equiv g^{\mu\kappa}R_{\mu\kappa}$.

(a) Using the metric, the Riemann tensor can be made fully covariant:

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= g_{\lambda\eta}R^\eta{}_{\mu\nu\kappa} \\ &= \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right) \\ &\quad + g_{\eta\sigma}\left(\Gamma^\eta{}_{\nu\lambda}\Gamma^\sigma{}_{\mu\kappa} - \Gamma^\eta{}_{\kappa\lambda}\Gamma^\sigma{}_{\mu\nu}\right)\,. \end{aligned} \tag{6}$$

Check the symmetry properties $R_{\lambda\mu\,\nu\kappa} = -R_{\lambda\mu\,\kappa\nu}$ and $R_{\lambda\mu\,\nu\kappa} = +R_{\nu\kappa\,\lambda\mu}$.

(b) Calculate the components of $R^\ell{}_{mnk}$, R_{mk} and the curvature scalar R for a space with coordinates (θ, ϕ) and metric $g_{mn} = diag(a^2, a^2 \sin^2\theta)$.
(Use last week's Christoffel symbols again!)

(c) Do the same for a space with coordinates (x^1, x^2) and metric $g_{mn} = diag(1, (x^1)^2)$. What is the geometry?

(d) What is the curvature of a circle with fixed radius R?