Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 5
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General Relativity

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1. Bianchi Identities

(a) Verify the *Bianchi identities*

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0\,, \tag{1}$$

where $X_{;\nu}$ denotes the covariant derivative

$$X^{\mu\cdots}{}_{\nu\cdots;\rho} = \frac{\partial}{\partial x^\rho} X^{\mu\cdots}{}_{\nu\cdots} + \Gamma^\mu{}_{\rho\sigma} X^{\sigma\cdots}{}_{\nu\cdots} + \ldots - \Gamma^\sigma{}_{\nu\rho} X^{\mu\cdots}{}_{\sigma\cdots} - \ldots\,. \tag{2}$$

Use the fact that (1) is explicitly covariant and work in a locally inertial system where the Γ*s (but not their derivatives) vanish.*

(b) Why is $g_{\mu\nu;\rho}$ always zero?

(c) Use (b) to contract the indices in (1) multiple times to arrive at

$$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{;\mu} = 0\,. \tag{3}$$

What does this imply for energy-momentum conservation in General Relativity?

2. *Energy-Momentum in Hydrodynamics*

A comoving observer in a *perfect fluid* will by definition see his surroundings as isotropic. In this frame, the energy-momentum tensor will be

$$\tilde{T}^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \tag{4}$$

where ρ is the density and p the pressure of the liquid.

(a) Calculate the energy-momentum tensor $T^{\mu\nu}$ for the rest frame. Assume the comoving observer's velocity to be $\vec{v}$.

(b) Show that $T^{\mu\nu}$ can also be written as

$$T^{\mu\nu} = (p + \rho)\, U^\mu U^\nu + p\, \eta^{\mu\nu} \tag{5}$$

where U^μ is the four-velocity of the fluid.

(c) Consider an ideal gas (point particles that only interact in local collisions). Its energy-momentum tensor is

$$T^{\mu\nu} = \sum_N \frac{p_N^\mu\, p_N^\nu}{E_N}\, \delta^3(\vec{x} - \vec{x}_N)\, . \tag{6}$$

Calculate the density ρ and pressure p for a comoving observer.

(d) What is the relation between ρ and p for a non-relativistic gas?
What is the relation for a highly relativistic gas?
(What relation exists between E and $\vec{p}$ in those limits?
Use the particle number density $n \equiv \sum_N \delta^3(\vec{x} - \vec{x}_N)$.)