Universität Bonn
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Theoretische Physik

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Example sheet 9
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General Relativity

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1. *Gravitational waves*

In order to describe gravitational waves, we decompose the metric into Minkowski metric η and a perturbation h,

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} + h_{\mu\nu}\ . \tag{1}$$

h is small, i.e. $h_{\mu\nu} \ll 1$, so that we can work in linear order in h throughout this problem.

(a) Consider a coordinate change

$$x^\mu \;\to\; x'^\mu = x^\mu + \varepsilon^\mu(x) \tag{2}$$

where $\partial\varepsilon^\mu/\partial x^\mu$ is at most of the same order of magnitude as $h_{\mu\nu}$. Calculate the metric in the new coordinate system (described by x').

(b) Make an ansatz for the solution of the field equations for gravitational waves:

$$h_{\mu\nu}(x) \;=\; e_{\mu\nu}\,\exp(\mathrm{i}\,k_\lambda x^\lambda) + e^*_{\mu\nu}\,\exp(-\mathrm{i}\,k_\lambda x^\lambda)\ . \tag{3}$$

Show that h solves the field equations (see Eq. (8) below) if

$$k^\mu k_\mu \;=\; 0 \tag{4}$$

and that the choice of a harmonic coordinate system (cf. Eq. (9)) corresponds to

$$k_\mu e^\mu{}_\nu \;=\; \frac{1}{2} k_\nu e^\mu{}_\mu\ . \tag{5}$$

Why is the matrix $e_{\mu\nu}$ symmetric?

(c) Consider a wave traveling in z-direction, i.e.

$$k^1 \;=\; k^2 \;=\; 0 \quad\text{and}\quad k^3 \;=\; k^0 \;=:\; k > 0\ . \tag{6}$$

Express e_{i0} ($1 \leq i \leq 3$) and e_{22} in terms of the other $e_{\mu\nu}$s.

(d) Perform a coordinate transformation (2) with

$$\varepsilon^\mu(x) \;=\; \mathrm{i}\,\epsilon^\mu\,\exp(\mathrm{i}\,k_\lambda x^\lambda) - \mathrm{i}\,\epsilon^{\mu *}\,\exp(-\mathrm{i}\,k_\lambda x^\lambda)\;. \tag{7}$$

How does h change?

(e) Invent a coordinate transformation that brings all $e_{\mu\nu}$ to 0 except for e_{11}, e_{12} and e_{22}. How many physical components does h have?

(f) How does h (and $e_{\mu\nu}$) change when we subject the coordinate system to a rotation about the z-axis? Discuss the result! How does the situation compare to the case of electromagnetic waves?

Hint: The field equations for free gravitational waves read:

$$\Box\, h_{\mu\nu} \;=\; 0\;. \tag{8}$$

One can further simplify the calculation by working with harmonic coordinates where

$$\frac{\partial}{\partial x^\mu} h^\mu{}_\nu \;=\; \frac{1}{2}\frac{\partial}{\partial x^\nu} h^\mu{}_\mu\;. \tag{9}$$

2. *Robertson-Walker metric*

The Robertson-Walker metric reads

$$\mathrm{d}s^2 \;=\; \mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-\alpha\,r^2} + r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\varphi^2\right\} \tag{10}$$

with $\alpha = 0, \pm 1$. Calculate

(a) the Christoffel symbols,

(b) the spatial Riemann tensor, the spatial Ricci tensor and the spatial curvature scalar, and

(c) the (4-dimensional) Riemann and Ricci tensors as well as the curvature scalar.