Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 3
27. Apr. 2006
SS 06

Exercises on Elementary Particle Physics II

Prof. Dr. H.-P. Nilles

1. *The SUSY Algebra and the Chiral Representation*

The SUSY algebra is an extension of the Poincaré algebra. It contains new fermionic generators Q_α. The (anti-) commutation relations involving the Q_αs read:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu \qquad \text{with } \bar{Q}_{\dot\alpha} := (Q_\alpha)^\dagger \\ [Q_\alpha, P_\mu] &= 0 \end{aligned}$$

(a) Show that

$$[(\theta Q), (\bar{Q}\bar{\theta})] = 2(\theta\sigma^\mu\bar{\theta})P_\mu$$

(b) An element of the SUSY group reads:

$$S(a^\mu, \alpha, \bar{\alpha}) := \exp[\alpha Q + \bar{Q}\bar{\alpha} - i a^\mu P_\mu]$$

Show that $S(a^\mu, \alpha, \bar{\alpha})S(b^\mu, \beta, \bar{\beta})$ is again a group element.

(c) Next, we define a vector space on which the SUSY transformation $S(a^\mu, \alpha, \bar{\alpha})$ acts. The elements of this space are called superfields $\Phi(x^\mu, \theta, \bar{\theta})$. They are defined by their transformation properties:

$$S(a^\mu, \alpha, \bar{\alpha})[\Phi(x^\mu, \theta, \bar{\theta})] = \Phi(x^\mu + a^\mu - i\alpha\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\alpha}, \theta + \alpha, \bar{\theta} + \bar{\alpha}).$$

By using an infinitesimal transformation, show that the representation of the SUSY algebra on the space of superfields $\Phi(x^\mu, \theta, \bar{\theta})$ reads:

$$\begin{aligned} P_\mu &= i\partial_\mu \\ Q_\alpha &= \partial_\alpha - i(\sigma^\mu)_{\alpha\dot\beta}\bar{\theta}^{\dot\beta}\partial_\mu \\ \bar{Q}_{\dot\beta} &= -\bar{\partial}_{\dot\beta} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu \end{aligned}$$

(d) Additionally, check that these operators form a representation of the SUSY algebra by explicitly verifying the (anti-) commutation relations.

(e) The (SUSY) covariant derivative D_α is defined by the commutator

$$D_\alpha(\delta_S\Phi) = \delta_S(D_\alpha\Phi).$$

Show that the following derivatives fulfill this condition and are therefore covariant:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar D_{\dot\beta} &= -\bar\partial_{\dot\beta} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu \end{aligned}$$

(f) Next, we define different representations of the SUSY group, i.e. the left and right chiral representations:

$$\begin{aligned} S_L(a^\mu,\alpha,\bar\alpha) &:= \exp[\alpha Q - ia^\mu P_\mu]\exp[\bar Q\bar\alpha] \\ S_R(a^\mu,\alpha,\bar\alpha) &:= \exp[\bar\alpha\bar Q - ia^\mu P_\mu]\exp[\alpha Q] \end{aligned}$$

In the following, we will concentrate on the left chiral representation. How does it relate to the representation $S(a^\mu,\alpha,\bar\alpha)$ discussed in part (b).

(g) Show that $S_L(a^\mu,\alpha,\bar\alpha)S_L(b^\mu,\beta,\bar\beta)$ is again a group element.

(h) Again, a superfield is defined by its transformation properties:

$$S_L(a^\mu,\alpha,\bar\alpha)[\phi_L(x^\mu,\theta,\bar\theta)] = \phi_L(x^\mu + a^\mu + 2i\theta\sigma^\mu\bar\alpha, \theta+\alpha, \bar\theta+\bar\alpha).$$

Find the representations of the SUSY generators Q_L and $\bar Q_L$.

(i) In the left chiral representation, the covariant derivatives are

$$\begin{aligned} D_{L\alpha} &= \partial_\alpha + 2i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar D_{L\dot\beta} &= -\bar\partial_{\dot\beta}. \end{aligned}$$

You might check the commutator with the SUSY transformation S_L.

(j) Next, we define chiral superfields by the conditions:

$$\begin{aligned} &\bar D\Phi(x,\theta,\bar\theta) = 0 && \text{for left chiral sf} \\ &D\Phi(x,\theta,\bar\theta) = 0 && \text{for right chiral sf} \end{aligned}$$

The definition is independent of the representation. But choosing a specific representation (the left chiral representation $\bar D\Phi = \bar D_L\phi_L$, for example) gives us some insight. So, what can you say about a left chiral superfield ϕ_L? What is the general form of a left chiral superfield?

Hint: Make a Taylor expansion in θ.

(k) Consider the infinitesimal SUSY transformation $S_L(0,\delta\theta,\delta\bar\theta)$ of a left chiral superfield ϕ_L. How do the component fields of ϕ_L transform?

Hint: Use the left chiral representation of the SUSY generators Q_L and $\bar Q_L$ and assume that the transformation is small: $\delta\theta\sigma^\mu\delta\bar\theta \approx 0$.