String Seminar - April 25, 2003
University of Bonn

The Bosonic String

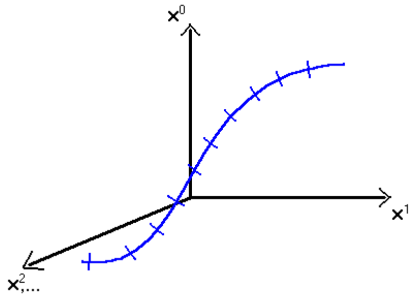
Patrick Vaudrevange

1. Motivation
2. The Polyakov action and its symmetries
3. Oscillator expansions
4. The Quantized Bosonic String
5. Mass spectrum of the bosonic string
6. Summary

1 Motivation

The Relativistic Particle

- Free relativistic particle (mass m) moving in d-dimensional Minkowski space with $\eta_{\mu\nu} = \mathrm{diag}(- + \ldots +)$



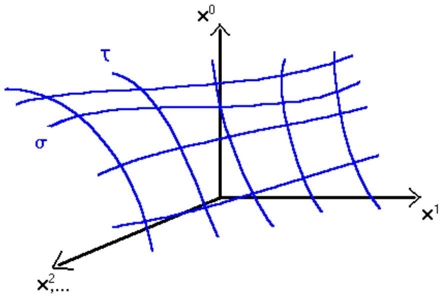
- $x^\mu(\tau)$, $\mu = 1, \ldots, d$ describe the embedding of the point particle in space-time
- Action $\hat{=}$ length of the world-line

$$
\begin{aligned}
S &= -m \int_{s_0}^{s_1} \mathrm{d}s \\
&= -m \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \, \left[-\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \eta_{\mu\nu} \right]^{1/2}
\end{aligned}
$$

2 The Relativistic String

The Nambu-Goto Action

- Free string, described by parameters $\sigma^\alpha = (\sigma, \tau)$
  τ: proper time, σ: $0 \leq \sigma < \bar{\sigma}$
  with: $\bar{\sigma} = \pi$ for open string, $\bar{\sigma} = 2\pi$ for closed string



- $X^\mu(\sigma, \tau)$, $\mu = 1, ..., d$ real functions describe the embedding of the world-sheet in the d-dimensional space
  Notation: $\dot{X} \equiv \frac{\partial X}{\partial \tau}$ and $X' \equiv \frac{\partial X}{\partial \sigma}$
- In analogy: action $\hat{=}$ area of the world-sheet

$$
\begin{aligned}
S_{NG} &= -T \int \mathrm{d}A \\
&= -T \int \mathrm{d}^2\sigma \, \left[-\det \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \right]^{1/2} \\
&= -T \int \mathrm{d}^2\sigma \, \left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{1/2} \\
&= -T \int \mathrm{d}^2\sigma \, \sqrt{-\det \Gamma_{\alpha\beta}}
\end{aligned}
$$

Notation $\Gamma_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

- String tension T = constant, $[T] = \text{mass}^2$
- Disadvantage: square root

The Polyakov Action

- Introduce metric on the world-sheet $h_{\alpha\beta}(\sigma,\tau)$

$$S_P = -\frac{T}{2}\int \mathrm{d}^2\sigma\ \sqrt{h}h^{\alpha\beta}\Gamma_{\alpha\beta}$$

using the definition: $h \equiv -\mathrm{det}\ h_{\alpha\beta}$

- Energy-momentum tensor defined as the response of S to varying $h_{\alpha\beta}$:

$$T_{\alpha\beta} := -\frac{1}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}}$$

it follows

$$T_{\alpha\beta} = \frac{1}{2}\Gamma_{\alpha\beta} - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\Gamma_{\gamma\delta}$$

- Energy-momentum conservation $\nabla^\alpha T_{\alpha\beta} = 0$
- Equation of motion for $h_{\alpha\beta}$:

$$\frac{\delta S}{\delta h^{\alpha\beta}} = 0$$

$$\boxed{T_{\alpha\beta} = 0}$$

$h_{\alpha\beta}$ is a non-propagating field $\longrightarrow$ auxiliary field

- Equation of motion for X^μ:

$$\frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\mu) = 0$$

- If $h_{\alpha\beta}$ fulfills its equation of motion, then S_P and S_{NG} are classically equivalent:

$$\begin{aligned}
T_{\alpha\beta} &= \frac{1}{2}\Gamma_{\alpha\beta} - \frac{1}{4}h_{\alpha\beta}h^{\gamma\delta}\Gamma_{\gamma\delta} = 0 \quad \mid \det, \sqrt{} \\
\frac{1}{2}h^{\gamma\delta}\Gamma_{\gamma\delta} &= \frac{1}{\sqrt{h}}\sqrt{-\det\,\Gamma_{\alpha\beta}} \\
S_P &= -\frac{T}{2}\int \mathrm{d}^2\sigma\ \sqrt{h}h^{\alpha\beta}\Gamma_{\alpha\beta} \\
&= -T\int \mathrm{d}^2\sigma\ \sqrt{-\det\,\Gamma_{\alpha\beta}} = S_{NG}
\end{aligned}$$

- Symmetries of the Polyakov action

  – Poincaré invariance

$$\begin{aligned}
X^\mu &\rightarrow X^\mu + a^\mu{}_\nu X^\nu + b^\mu \\
\delta h_{\alpha\beta} &= 0
\end{aligned}$$

  – reparametrization invariance

$$\begin{aligned}
\sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha(\sigma,\tau) \\
X^\mu(\sigma,\tau) &\rightarrow X^\mu(\sigma,\tau) + \xi^\alpha(\sigma,\tau)\partial_\alpha X^\mu \\
h_{\alpha\beta}(\sigma,\tau) &\rightarrow \frac{\mathrm{D}\sigma^\gamma}{\mathrm{d}\tilde{\sigma}^\alpha}\frac{\mathrm{D}\sigma^\delta}{\mathrm{d}\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma,\tau) \\
\delta h_{\alpha\beta} &= \frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta}
\end{aligned}$$

  – Weyl rescaling

$$\begin{aligned}
h_{\alpha\beta}(\sigma,\tau) &\rightarrow \Omega^2(\sigma,\tau)h_{\alpha\beta}(\sigma,\tau) \\
\Omega^2(\sigma,\tau) &= e^{2\Lambda(\sigma,\tau)} \\
&\approx 1 + 2\Lambda(\sigma,\tau) \\
\delta h_{\alpha\beta} &= 2\Lambda(\sigma,\tau)h_{\alpha\beta} \\
\delta X^\mu &= 0
\end{aligned}$$

- Consequence of Weyl invariance: tracelessness

$$
\begin{aligned}
\delta S &= -T \int \mathrm{d}^2\sigma \, \sqrt{h} T_{\alpha\beta} \delta h^{\alpha\beta} \\
h^{\alpha\beta} T_{\alpha\beta} &= 0
\end{aligned}
$$

(without using the equation of motion $T_{\alpha\beta} = 0$)

- Conformal gauge: use reparametrization invariance to get locally:

$$h_{\alpha\beta} = \Omega^2(\sigma, \tau)\eta_{\alpha\beta}$$

$\Omega^2(\sigma, \tau)$ can be set to 1 by Weyl rescaling:

$$\boxed{h_{\alpha\beta} = \eta_{\alpha\beta}}$$

- Note: Although reparametrization and Weyl invariance have been used, reparametrizations

$$\delta h_{\alpha\beta} = \frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta} \propto h_{\alpha\beta}$$

followed by Weyl rescaling and thus only affect X^μ, not the metric $h_{\alpha\beta} = \eta_{\alpha\beta}$.

- World-sheet light-cone coordinates $\sigma^\pm = \tau \pm \sigma$: (notation $\sigma^x = (\sigma^+, \sigma^-)$)

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$$

and the metric transforms as

$$\tilde{h}_{xy}(\sigma^+, \sigma^-) = \frac{\partial\sigma^\gamma}{\partial\sigma^x}\frac{\partial\sigma^\delta}{\partial\sigma^y} h_{\gamma\delta}(\sigma, \tau) = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}$$

- Polyakov action in conformal gauge: the world-sheet metric is $\eta_{\alpha\beta} \Rightarrow \sqrt{\eta} = 1$

$$S_P = \frac{T}{2} \int \mathrm{d}^2\sigma \; (\dot{X}^2 - X'^2)$$

the conjugate momentum is: $\Pi^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = T\dot{X}^\mu$
and the Poisson brackets are:

$$\begin{aligned} \{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} = 0 \\ \{X^\mu(\sigma,\tau), T\dot{X}^\nu(\sigma',\tau)\}_{\text{P.B.}} &= \eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned}$$

Vanishing of energy-momentum tensor is in conformal gauge equivalent to:

$$\frac{1}{2}(\dot{X} \pm X')^2 = 0$$

- Equation of motion: varying with respect to X^μ:

$$\begin{aligned} \delta S &= T \int \mathrm{d}^2\sigma \; (\partial_\sigma^2 - \partial_\tau^2) X^\mu \delta X_\mu - T \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \; X'_\mu \delta X^\mu \Big|_0^{\bar{\sigma}} \stackrel{!}{=} 0 \\ \Rightarrow \quad & (\partial_\sigma^2 - \partial_\tau^2) X^\mu = 4\partial_+\partial_- X^\mu = 0 \end{aligned}$$

with the conditions:

$$\begin{aligned} X^\mu(\sigma + 2\pi, \tau) &= X^\mu(\sigma,\tau) \quad \text{(closed string)} \\ X'_\mu\Big|_0^{\bar{\sigma}} &= 0 \qquad\qquad\quad \text{(open string)} \end{aligned}$$

general solution: $X^\mu(\sigma^+,\sigma^-) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)$

$X^\mu_{R,L}$ describe "right"- (respectively "left"-) moving modes of the string.

- Oscillator expansion

closed string

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}$$

X^μ are real functions, i.e. $(X^\mu)^\dagger = X^\mu \Rightarrow$
- x^μ and p^μ are real
- $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$ and $\bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^\dagger$

Compute the center of mass momentum:

$$P^\mu = \int_0^{2\pi} \mathrm{d}\sigma\ T\dot{X}^\mu(\sigma,\tau) = p^\mu$$

and the center of mass position:

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\sigma\ X^\mu(\sigma,\tau=0) = x^\mu$$

The Hamiltonian is defined as:

$$\begin{aligned} H &= \int_0^{\bar{\sigma}} \mathrm{d}\sigma\ (\dot{X}\cdot\Pi - L) \\ &= T\int_0^{\bar{\sigma}} \mathrm{d}\sigma\ ((\partial_+ X)^2 + (\partial_- X)^2) \\ &= \frac{1}{2}\sum_{n=-\infty}^{+\infty}(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) \end{aligned}$$

Notation $\alpha_0^\mu = \bar{\alpha}_0^\mu \equiv \frac{1}{\sqrt{4\pi T}}p^\mu$

open string

$$X_R^\mu(\sigma,\tau) = x^\mu + \frac{1}{\pi T} p^\mu \tau + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)$$

as in the closed string case:

- $\alpha_0^\mu \equiv \frac{1}{\sqrt{\pi T}} p^\mu$
- x^μ, p^μ are real
- $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$

and by inserting X^μ:

$$H = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n$$

- Constraint $T_{\alpha\beta} \stackrel{!}{=} 0$ in light-cone coordinates:

$$\begin{aligned}
\tilde{T}_{xy}(\sigma^+, \sigma^-) &= \frac{\partial \sigma^\gamma}{\partial \sigma^x} \frac{\partial \sigma^\delta}{\partial \sigma^y} T_{\gamma\delta}(\sigma,\tau) \\
T_{++} &= \frac{1}{2} \partial_+ X \cdot \partial_+ X \stackrel{!}{=} 0 \\
T_{--} &= \frac{1}{2} \partial_- X \cdot \partial_- X \stackrel{!}{=} 0 \\
T_{+-} &\equiv T_{-+} \equiv 0
\end{aligned}$$

Energy-momentum conservation $\nabla^\alpha T_{\alpha\beta} = 0$ in light-cone coordinates:

$$\begin{aligned}
0 &= \partial_- T_{++} + \partial_+ T_{-+} = \partial_- T_{++} \\
0 &= \partial_+ T_{--} + \partial_- T_{+-} = \partial_+ T_{--}
\end{aligned}$$

This gives:

$$\begin{aligned}
T_{++} &= T_{++}(\sigma^+) \\
T_{--} &= T_{--}(\sigma^-)
\end{aligned}$$

Implies set of infinitely many conserved charges L_f:

$$\partial_-(f(\sigma^+)T_{++}) = 0$$
$$L_f = 2T\int_0^{\bar\sigma} \mathrm{d}\sigma\ f(\sigma^+)T_{++}(\sigma^+)$$

and analogously for T_{--}.

- Virasoro constraints and Virasoro algebra

closed string

Choose $f_m(\sigma^\pm) = \exp(im\sigma^\pm) \Rightarrow$ charges $\hat{=}$ Fourier coefficients of T_{xx} at $\tau = 0$:

$$\begin{aligned} L_m &= 2T\int_0^{2\pi} \mathrm{d}\sigma\ e^{-im\sigma}T_{--} \\ &= T\int_0^{2\pi} \mathrm{d}\sigma\ e^{-im\sigma}(\partial_- X)^2 \\ &= \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n \end{aligned}$$

$$\begin{aligned} \bar{L}_m &= 2T\int_0^{\bar\sigma} \mathrm{d}\sigma\ e^{+im\sigma}T_{++} \\ &= \frac{1}{2}\sum_n \bar\alpha_{m-n}\cdot\bar\alpha_n \end{aligned}$$

The charges satisfy the Virasoro algebra:

$$\begin{aligned} \{L_m, L_n\}_{\text{P.B.}} &= -i(m-n)L_{m+n} \\ \{\bar{L}_m, \bar{L}_n\}_{\text{P.B.}} &= -i(m-n)\bar{L}_{m+n} \\ \{\bar{L}_m, L_n\}_{\text{P.B.}} &= 0 \end{aligned}$$

Compare to Hamiltonian:

$$H = L_0 + \bar{L}_0$$

open string

Because left- and right-movers are not independent, define:

$$
\begin{aligned}
L_m &= 2T \int_0^\pi \mathrm{d}\sigma \; (e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}) \\
&= \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n
\end{aligned}
$$

again they satisfy:

$$\{L_m, L_n\}_{\text{P.B.}} = -i(m-n)L_{m+n}$$

Compare to Hamiltonian:

$$H = L_0$$

3 The Quantized Bosonic String

- Dirac's correspondence principle

$$\begin{aligned} X^\mu &\rightarrow \hat{X}^\mu \\ \alpha^\mu_m &\rightarrow \hat{\alpha}^\mu_m \\ \{\ ,\ \}_{\text{P.B.}} &\rightarrow \frac{1}{i}[\ ,\ ] \end{aligned}$$

- By analogy to Poisson brackets: commutator relations

$$\begin{aligned} [X^\mu(\sigma,\tau), T\dot{X}^\nu(\sigma',\tau)] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') \\ [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= [\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)] = 0 \end{aligned}$$

by inserting the oscillator expansions

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha^\mu_m, \alpha^\nu_n] &= [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\delta_{(m+n),0}\,\eta^{\mu\nu} \\ [\bar{\alpha}^\mu_m, \alpha^\nu_n] &= 0 \end{aligned}$$

- Compare to harmonic oscillator
  - the hermiticity condition still holds
  - rescale α^μ_m for $m > 0$:

$$a^\mu_m \equiv \frac{1}{\sqrt{m}}\alpha^\mu_m \quad (a^\mu_m)^\dagger \equiv \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$$

    they fulfill harmonic oscillator commutation relations

$$[a^\mu_m, (a^\nu_n)^\dagger] = \delta_{m,n}\eta^{\mu\nu}$$

    So α_m are lowering operators for $m > 0$ and raising operators for $m < 0$.

    The corresponding number operator (for $m > 0$) is: $N_m = \alpha_{-m}\alpha_m$.

- Ghosts (negative norm states)
  Denote ground state by: $|0, p^\mu\rangle$ consider $(m > 0)$:

$$\begin{aligned}
[\alpha_m^0, \alpha_{-m}^0] &= -m \\
\langle 0|[\alpha_m^0, \alpha_{-m}^0]|0\rangle &= \langle 0|\alpha_m^0 \alpha_{-m}^0|0\rangle - \underbrace{\langle 0|\alpha_{-m}^0 \alpha_m^0|0\rangle}_{=0} \\
&= -m\langle 0|0\rangle
\end{aligned}$$

  so we have a negative norm state:

$$\begin{aligned}
\langle 0|\alpha_m^0 \alpha_{-m}^0|0\rangle &= \langle 0|(\alpha_{-m}^0)^\dagger \alpha_{-m}^0|0\rangle \\
&= \| \, \alpha_{-m}^0|0\rangle \, \|^2 \\
&= -m\langle 0|0\rangle \; < 0
\end{aligned}$$

  However: one can prove that in $d = 26$ these states decouple from the physical Hilbert space.

- The Virasoro algebra and normal ordering
  Constraint $T_{\alpha\beta} = 0$ corresponds to vanishing of L_m:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n$$

  but α_m now operators, so order matters. Define:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\alpha_{m-n} \cdot \alpha_n:$$

  Normal ordering: raising operators to the left and lowering operators to the right. Problem only arises for L_0. Define L_0:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$$

Introduce normal ordering constant a in all formulas by replacing L_0 by $(L_0 - a)$.

Now: determine the algebra of the L_m's

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$$

c appears due to quantum effects and is called central charge. Here: $c = \eta^\mu_{\ \mu} = d$.

Classically all L_m must vanish, but:

$$\langle\phi|[L_m, L_{-m}]|\phi\rangle = \langle\phi|2mL_0|\phi\rangle + \frac{d}{12}m(m^2-1)\langle\phi|\phi\rangle$$

so we can not require $L_m|\phi\rangle = 0 \ \ \forall m$, but only

$$\begin{aligned} L_m|\text{phys}\rangle &= 0 \qquad m > 0 \\ (L_0 - a)|\text{phys}\rangle &= 0 \end{aligned}$$

This defines the physical states $|\text{phys}\rangle$.

Indeed L_m form closed subalgebra for $m > 0$.

One can show that $(L_0 - \bar{L}_0)$ generates σ translations, which do not affect the string, so:

$$(L_0 - \bar{L}_0)|\text{phys}\rangle = 0$$

- Mass-shell condition

open string

Using the definitions $\frac{1}{2}\alpha_0^2 = \frac{1}{2\pi T}p^\mu p_\mu = \alpha' p^\mu p_\mu$ and $N = \sum_{n=1}^{\infty} N_m$ yields:

$$
\begin{aligned}
L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \\
&= \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty} N_m \\
&= -\alpha' m^2 + N
\end{aligned}
$$

Applied on a physical state:

$$
\begin{aligned}
(L_0 - a)|\mathrm{phys}\rangle &= 0 \\
\alpha' m^2 &= N - a
\end{aligned}
$$

closed string

$$
\begin{aligned}
L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{\alpha'}{4} p^\mu p_\mu + N \\
\bar{L}_0 &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n = \frac{\alpha'}{4} p^\mu p_\mu + \bar{N}
\end{aligned}
$$

the mass m is now given as:

$$
m^2 = -p^\mu p_\mu = m_L^2 + m_R^2
$$

L_0 and $\bar{L}_0$ applied on a physical state:

$$
\begin{aligned}
\alpha' m_L^2 &= 2(\bar{N} - a) \\
\alpha' m_R^2 &= 2(N - a)
\end{aligned}
$$

with $m_L^2 = m_R^2$ as a consequence of $L_0 - \bar{L}_0 = 0$.

- Light cone gauge

Define: light cone coordinates in space-time

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^{d-1}) \qquad X^- = \frac{1}{\sqrt{2}}(X^0 - X^{d-1})$$

New coordinates: $X^+, X^-, X^i,\ i = 1, \ldots, d-2.$

Remaining reparametrization invariance:

$$\frac{\mathrm{D}\xi^\beta}{\mathrm{d}\sigma^\alpha} + \frac{\mathrm{D}\xi^\alpha}{\mathrm{d}\sigma^\beta} \propto h_{\alpha\beta}$$

in world-sheet light-cone coordinates:

$$\partial_+\xi^- = \partial_-\xi^+ = 0 \ \text{ i.e. } \ \xi^\pm = \xi^\pm(\sigma^\pm)$$

$\tilde{\sigma}^\pm = \sigma^\pm + \xi^\pm(\sigma^\pm)$ is the infinitesimal form of the reparametrization $\sigma^\pm \to \tilde{\sigma}^\pm = \tilde{\sigma}^\pm(\sigma^\pm)$.

Therefore

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$$

transforms into

$$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-))$$

so $\tilde{\tau}$ arbitrary solution of wave equation:

$$\partial_+\partial_-\tilde{\tau} = 0$$

Choose $\tilde{\tau} \propto X^+$ and insert this into constraint

$$\frac{1}{2}(\dot{X} \pm X')^2 = 0$$

solve result with respect to $X^- = X^-(X^i)$ and thus X^+ and X^- are eliminated. One can show that

$$\alpha_{-n} \cdot \alpha_n \Rightarrow \sum_{i=1}^{d-2} \alpha^i_{-n}\alpha^i_n$$

- Spectrum of the bosonic string
  States are generated by transverse oscillators acting on the ground state.

  open string spectrum

  Mass operator:

$$\alpha' m^2 = (N - a)$$

  acting on ground state gives

$$\alpha' m^2 |0, p^i\rangle = -a|0, p^i\rangle$$

  acting on first excited state gives

$$\alpha' m^2 (\alpha^i_{-1}|0, p^j\rangle) = (1 - a)\alpha^i_{-1}|0, p^j\rangle$$

  $\alpha^i_{-1}|0, p^j\rangle$ is a vector of SO$(d-2)$. Lorentz group is SO(d).
  Lorentz invariance: little group is

  - SO$(d-2)$ for massless particles
  - SO$(d-1)$ for massive particles

  can not combine the fundamental SO$(d-2)$ representation to SO$(d-1)$. So first excited state must be massless.

$$\text{Lorentz invariance} \;\Rightarrow\; a = 1$$

Normal ordering constant is fixed: $a = 1$. Consider

$$\begin{aligned}
\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n &= \frac{1}{2}\sum_{n\neq 0}:\alpha^i_{-n}\alpha^i_n: + \underbrace{\frac{d-2}{2}\sum_{n=1}^{\infty} n}_{=-a} \\
&= \sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \frac{d-2}{2}\sum_{n=1}^{\infty} n
\end{aligned}$$

Consider: $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$ using the Riemann zeta function. Converges for $s > 1$, unique analytic continuation: $\zeta(-1) = -1/12$. It follows

$$\begin{aligned}
-1 &= -\frac{d-2}{2}\frac{1}{12} \\
d &= 26
\end{aligned}$$

Due to Lorentz invariance $a = 1$ and $d = 26$

closed string spectrum

Excitation level for left- and right movers equal
Mass operator:

$$\alpha' m^2 = 4(N - a)$$

again $a = 1$ and $d = 26$

Table 3.1: The five lowest mass levels of the oriented open bosonic string

level	$\alpha'(\text{mass})^2$	states and their $SO(24)$ representation contents	little group	representation contents with respect to the little group
0	-1	$\vert 0\rangle$ $\bullet$ (1)	$SO(25)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\vert 0\rangle$ $\square$ (24)	$SO(24)$	$\square$ (24)
2	$+1$	$\alpha^i_{-2}\vert 0\rangle$: $\square$, (24) $\alpha^i_{-1}\alpha^j_{-1}\vert 0\rangle$: $\square\square+\bullet$, $(299)+(1)$	$SO(25)$	$\square\square$ (324)
3	$+2$	$\alpha^i_{-3}\vert 0\rangle$: $\square$, (24) $\alpha^i_{-2}\alpha^j_{-1}\vert 0\rangle$: $\begin{smallmatrix}\square\\\square\end{smallmatrix}+\square\square+\bullet$, $(276)+(299)+(1)$ $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\vert 0\rangle$: $\square\square\square+\square$, $(2576)+(24)$	$SO(25)$	$\underset{(2900)}{\square\square\square}+\underset{(300)}{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$
4	$+3$	$\alpha^i_{-4}\vert 0\rangle$: $\square$, (24) $\alpha^i_{-3}\alpha^j_{-1}\vert 0\rangle$: $\square\square+\begin{smallmatrix}\square\\\square\end{smallmatrix}+\bullet$, $(299)+(276)+(1)$ $\alpha^i_{-2}\alpha^j_{-2}\vert 0\rangle$: $\square\square+\bullet$, $(299)+(1)$ $\alpha^i_{-2}\alpha^j_{-1}\alpha^k_{-1}\vert 0\rangle$: $2\times\square+\square\square\square+\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$, $2\times(24)+(2576)+(4576)$ $\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}\alpha^l_{-1}\vert 0\rangle$: $\square\square\square\square+\square\square+\bullet$, $(17250)+(299)+(1)$	$SO(25)$	$\underset{(20150)}{\square\square\square\square}+\underset{(5175)}{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}+\underset{(324)}{\square\square}+\underset{(1)}{\bullet}$

Table 3.2: The three lowest mass levels of the oriented closed bosonic string

level	$\alpha'(\text{mass})^2$	states and their $SO(24)$ representation contents	little group	representation contents with respect to the little group
0	-4	$\vert 0\rangle$ $\bullet$ (1)	$SO(25)$	$\bullet$ (1)
1	0	$\alpha^i_{-1}\bar{\alpha}^j_{-1}\vert 0\rangle$ $\square \times \square$ $(24)\ \ (24)$	$SO(24)$	$\underset{(299)}{\ydiagram{2}} \times \underset{(276)}{\ydiagram{1,1}} \times \underset{(1)}{\bullet}$
2	$+4$	$\alpha^i_{-2}\bar{\alpha}^j_{-2}\vert 0\rangle$: $\square \times \square$, $(24)\ \ (24)$ $\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-1}\bar{\alpha}^l_{-1}\vert 0\rangle$: $(\ydiagram{2} + \bullet) \times (\ydiagram{2} + \bullet)$, $(299)+(1)\ \ (299)+(1)$ $\alpha^i_{-2}\bar{\alpha}^j_{-1}\bar{\alpha}^k_{-1}\vert 0\rangle$: $\square \times (\ydiagram{2} + \bullet)$, $(24)\ \ (299)+(1)$ $\alpha^i_{-1}\alpha^j_{-1}\bar{\alpha}^k_{-2}\vert 0\rangle$: $(\ydiagram{2} + \bullet) \times \square$, $(299)+(1)\ \ (24)$	$SO(25)$	$\underset{(324)}{\ydiagram{2}} \times \underset{(324)}{\ydiagram{2}} = \underset{(20150)}{\ydiagram{4}} + \underset{(32175)}{\ydiagram{2,2}}$ $+ \underset{(52026)}{\ydiagram{5,1}} + \underset{(324)}{\ydiagram{2}} + \underset{(300)}{\ydiagram{1,1}} + \underset{(1)}{\bullet}$

4 Summary

- Starting point: relativistic particle
- In analogy: string-action $\hat{=}$ area of world-sheet
- Equivalent: Polyakov action + constraints
- Quantization
- Constraints correspond to Virasoro algebra
- Zero mode Virasoro operator leads to mass^2
- Physical states organize into representations of little group
- Spin 2 (graviton), "spin 1" (antisymmetric tensor field) and spin 0 (dilaton) in spectrum of the <u>closed</u> string
- Problem: Tachyon with negative mass^2 $\Rightarrow$ Supersymmetric string theory