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Theoretische Physik

Exercise 1
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Exercises on Elementary Particle Physics

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1. *The Dirac Equation*

   If we use the correspondence

   $$\vec{p} \to -i\nabla, \quad E \to i\partial_t,$$

   the relativistic energy-momentum relation

   $$E^2 = \vec{p}^{\,2} + m^2$$

   gives the Klein-Gordon equation:

   $$(\Box + m^2)\psi = 0.$$

   Dirac's basic idea was to "factorize" the above relation to obtain an equation which is first-order in the derivatives.

   (a) Make the ansatz

   $$H\psi = (\alpha_i p_i + \beta m)\psi. \tag{1}$$

   Squaring eq. (1) should give the Klein-Gordon equation. Show that from this requirement, it follows:

   $$\beta^2 = \alpha_i^2 = 1, \quad \{\beta, \alpha_i\} = \{\alpha_i, \alpha_j\} = 0, \quad i \neq j$$

   (b) Define the Dirac matrices γ^μ, $\mu = 0, \ldots, 3$ by

   $$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i, \quad i = 1, 2, 3.$$

   Show that the Dirac equation can be written in the covariant form

   $$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

   (c) Show that the gamma matrices fulfill the *Clifford algebra*

   $$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4, \tag{2}$$

   where $\eta^{\mu\nu} =$ diag(1,-1,-1,-1) .

(d) The lowest dimensional matrices satisfying the Clifford algebra eq. (2) are 4×4 matrices. The choice of the matrices is not unique. One convenient choice is the Weyl or chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Verify that this set of matrices fulfills the Clifford algebra eq. (2).

(e) One can prove that the matrices α_i, β must have a minimum of 4 dimensions. The proof has 4 steps:

i. Show that the matrices α_i, β are traceless. Hint: Calculate $\beta\alpha_i\beta$ and take the trace.

ii. Show that the eigenvalues of α_i, β are ± 1.

iii. Show that the dimension of the matrices must be even. Hint: Combine the results of (i) and (ii).

iv. Show that the dimension must be greater than 2. Hint: How many traceless Hermitean matrices are there in n dimensions?

2. *Representations of* $su(2)$

A Lie algebra $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacoby identity.

A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}(V)$$

which is an algebra homomorphism. The dimension of V is called the dimension of the representation.

If there is a vector space $W \subset V$ so that $\rho(\mathfrak{g})W \subset W$ (invariant subspace), then the representation is called *reducible*, otherwise *irreducible*.

(a) The group SU(2) is the set of all 2-dimensional unitary matrices with determinant 1. Show that the corresponding Lie algebra $su(2)$ is the set of traceless Hermitean matrices. Hint: $\det A = \exp \mathrm{Tr} \log A$.

(b) Choose the basis

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the traceless Hermitean matrices. Define

$$J_3 = \frac{1}{2}\sigma_3, \quad J_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad J_- = \frac{1}{2}(\sigma_1 - i\sigma_2),$$

and verify the commutation relations

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3.$$

(c) J_3 is diagonalizable, so the matrix $\rho(J_3)$ on V is diagonalizable (preservation of Jordan decomposition). Therefore V can be decomposed into eigenspaces:

$$V = \bigoplus V_\alpha$$

For $v \in V_\alpha$, the "action of J_3" yields a scalar multiple of v:

$$J_3(v) := \rho(J_3)\, v = \alpha v, \quad \alpha \in \mathbb{C}$$

Show that $J_+(v) \in V_{\alpha+1}$ and $J_-(v) \in V_{\alpha-1}$.

(d) **From now on we assume the representation to be irreducible.** Prove that all complex numbers α which appear in the above decomposition differ from one another by 1. Hint: Choose an arbitrary $\alpha_0 \in \mathbb{C}$ from the decomposition and prove that

$$\bigoplus_{k\in\mathbb{Z}} V_{\alpha_0+k} \subset V$$

is indeed equal to V using the irreducibility of the representation.

(e) Argue that there is $k \in \mathbb{N}$ for which $V_{\alpha_0+k} \neq 0$ and $V_{\alpha_0+k+1} = 0$. Define $n := \alpha_0 + k$. Note that up to now, we only know that $n \in \mathbb{C}$.

Draw a diagram. Write the vector spaces V_{n-2}, V_{n-1}, V_n in a row and indicate the action of J_3, J_+, J_- on these vector spaces by arrows.

The eigenvalue n is called **highest weight** and a vector $v \in V_n$ is called **highest weight vector**. Is it clear why?

(f) Choose an arbitrary vector $v \in V_n$ (highest weight vector). Prove that the vectors $\{v, J_-v, J_-^2v, \ldots\}$ span V. Hint: Show that the vector space spanned by these vectors is invariant under the action of J_3, J_+, J_- and use the irreducibility of the representation.

(g) Argue that all the eigenspaces V_α are 1-dimensional.

(h) Prove that n is a non-negative integer or half-integer and that $V = V_{-n} \oplus \ldots \oplus V_n$. Complement your diagram drawn in part (e). Hint: The representation is finite dimensional, so there exists $m \in \mathbb{Z}$ (!) for which $J_-^{m-1}v \neq 0$ and $J_-^m v = 0$. Evaluate the product $J_+J_-^m v$.

We have learned so far:

- Every irreducible representation is characterized by a non-negative integer or half-integer n which is called the highest weight.
- The eigenvalues range from $-n$ to n and differ by integers. The dimension of the representation is $2n+1$.
- The eigenspaces are 1-dimensional.
- Given any non-negative integer or half-integer, there is a corresponding irreducible representation. (This can be proven, we have not shown it in the exercises.)

(i) *Tensor Products of irreps*
Consider the tensor product of a 2-dimensional and a 3-dimensional irreducible representation of $su(2)$:

$$V = V^{(2)} \otimes V^{(3)}$$

Is the resulting representation V irreducible? If not, decompose V into its irreducible representations. Hint: The first thing to note is that the action of a Lie algebra on the tensor product of 2 representations is given by $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, i.e. the eigenvalues of J_3 on V is the sum of the eigenvalues of J_3 on $V^{(2)}$ and $V^{(3)}$. Draw the x-axis and mark the eigenvalues (with multiplicities) by circles. Then use the fact that the irreducible representations are 1-dimensional.

Web page for exercises and other information
http://www.th.physik.uni-bonn.de/nilles/exercises.html