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Exercise 2
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Exercises on Elementary Particle Physics

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1. *The Lorentz group*

   The Lorentz group is defined as the set of transformations

   $$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

   which leave the scalar product $\langle x, x\rangle := \eta_{\mu\nu} x^\mu x^\nu$ invariant.

   (a) Show that the Lie algebra of the Lorentz group is the set of all antisymmetric matrices. Hint: Reformulate the statement about the invariance of the scalar product in $\eta_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$ and write an element of the Lorentz group as $\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu - i\lambda^\mu{}_\nu = \delta^\mu_\nu - i\alpha_a(\lambda_a)^\mu{}_\nu$.

   (b) Choose

   $$(M^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$$

   as a basis for the Lie algebra. What do these matrices look like? Describe the form of the matrices in words. Verify the commutation relations

   $$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right). \qquad (1)$$

   (c) We split the generators into 2 groups:

   $$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}, \quad K^i = M^{0i}.$$

   The J's have only spacial indices, the K's have spacial and timelike indices. Verify the commutation relations

   $$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k, \quad \left[J^i, K^j\right] = i\epsilon^{ijk}K^k, \quad \left[K^i, K^j\right] = -i\epsilon^{ijk}J^k,$$

   and describe the meaning of each relation in words. What kind of transformations do the J's and K's correspond to?

   (d) The form of the commutation relations for the Lorentz algebra can still be simplified. Define

   $$T^i_{L/R} = \frac{1}{2}\left(J^i \pm iK^i\right)$$

   and verify the commutation relations

   $$\left[T^i_L, T^j_L\right] = i\epsilon^{ijk}T^k_L, \quad \left[T^i_R, T^j_R\right] = i\epsilon^{ijk}T^k_R, \quad \left[T^i_L, T^j_R\right] = 0.$$

(e) Classify the representations of the Lorentz algebra using what you learned about $su(2)$ on the previous exercise sheet.

Conclusion: Every representation of the Lorentz algebra can be characterized by 2 non-negative integers or half-integers (j_L, j_R).

2. *Weyl spinors*

Summarizing our foregoing considerations, the Lorentz transformation on Minkowski space is given by

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right). \tag{2}$$

Now we take eq. (1) as the definition of the Lorentz algebra and investigate its representations. To make this point clear, we write $D(\Lambda)$ instead of Λ.

(a) Define α, β by the equations $\omega_{ij} = \epsilon_{ijk}\alpha_k$ and $\beta_i = \omega_{0i}$ to show

$$D(\Lambda) = \exp\left(-i\left[\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}\right]\right) = \exp\left(-i(\vec{\alpha} - i\vec{\beta})\cdot\vec{T}_L\right)\exp\left(-i(\vec{\alpha} + i\vec{\beta})\cdot\vec{T}_R\right).$$

Note that T_R^i, T_L^i are still unspecified, we only know their algebra. For a particular representation, we have to make a choice!

(b) Specialize to a representation: Choose the T_R^i, T_L^i to be the Pauli matrices σ_i. The simplest representations of the Lorentz group are $(1/2, 0)$ and $(0, 1/2)$. An object transforming in the $(1/2, 0)$ representation is called a **left-chiral Weyl spinor**. The definition of a right-chiral Weyl spinor is obvious. How many entries does a Weyl spinor have? Write down the transformation laws for the 2 types of Weyl spinors.

Let D_L, D_R denote the transformation matrices for the left- and right-chiral Weyl spinors.

(c) We want to rewrite the transformation laws for Weyl spinor in the standard notation of eq. (2). Therefore, we define the generalized Pauli matrices

$$\sigma^\mu := (1, \sigma^i), \quad \bar{\sigma}^\mu := (1, -\sigma^i).$$

Then, we can define the following quantities:

$$\sigma^{\mu\nu} := \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \qquad \bar{\sigma}^{\mu\nu} := \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu).$$

Show that the Weyl spinors transform as

$$\psi_L \mapsto D_L \psi_L = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\psi_L,$$

$$\psi_R \mapsto D_R \psi_R = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi_R.$$

Hint: You know T_L, T_R explicitly. Using the definitions, express first the K's and J's in terms of T_L, T_R. Second, express the $M^{\mu\nu}$'s in terms of K's and J's. Now identify the components of $\sigma^{\mu\nu}$, $\bar{\sigma}^{\mu\nu}$ with the components of $M^{\mu\nu}$. You will see that they are equal.

(d) Prove that $\sigma_2 = (D_L)^T \sigma_2 D_L$. Hint: First, observe that $D_L^{-1} = D_R^\dagger$. Second, show that $\sigma_2 D_L \sigma_2 = D_R^*$. Third, take the transpose of the last expression.

(e) Let (Φ_L), (Ψ_L) be Weyl spinors. Using the result of the last exercise, show that $(\Phi_L)^T \sigma_2 \Psi_L$ is invariant under Lorentz transformations.

In the following, we will use this result to construct Lorentz invariant combinations of Dirac spinors.

3. *Dirac Spinors*

Remember that the gamma matrices in the Weyl representation are given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

(a) Examine the behaviour of the Weyl spinors under space reflections $\mathbf{P}$ with $\mathbf{P}: (x_0, \vec{x}) \to (x_0, -\vec{x})$ and show that $\mathbf{P}$ exchanges left- and right-chiral spinors.

(b) The simplest representation that closes under $\mathbf{P}$ is therefore given by

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \sim (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}).$$

Ψ is called a *Dirac* spinor. Show that it transforms as

$$\psi \mapsto D\,\psi := \exp\left(\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right)\psi$$

under a Lorentz-transformation, with $\gamma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Use your knowledge on the transformation properties of Weyl spinors.

(c) Derive

$$D^{-1}\gamma^\mu D = \Lambda^\mu{}_\nu \gamma^\nu\,.$$

Hint: Calculate both sides of the equation for infinitesimal transformations and use the identity $[\gamma^\mu, \gamma^{\nu\sigma}] = -(M^{\nu\sigma})^\mu{}_\rho \gamma^\rho$.

(d) Define $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. Calculate γ_5 in the Weyl representation and show that γ_5 anti-commutes with the other gamma matrices: $\{\gamma_5, \gamma^\mu\} = 0$.

(e) In the Weyl representation, we have $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$. Define $\bar{\psi} := \psi^\dagger\gamma_0$ and show that the bilinear covariants

$$(i)\ \bar{\psi}\psi \quad (ii)\ \bar{\psi}\, i\gamma_5\psi \quad (iii)\ \bar{\psi}\gamma^\mu\psi \quad (iv)\ \bar{\psi}\gamma^\mu\gamma_5\psi \quad (v)\ \bar{\psi}\, i[\gamma^\mu, \gamma^\nu]\psi$$

transform under a Lorentz transformation as a scalar, pseudoscalar, vector, pseudovector, (2,0)-tensor, respectively.

(f) Define the projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5), \quad P_R = \frac{1}{2}(1 + \gamma_5).$$

Show that $P_{L/R}$ project onto the left/right-chiral Weyl spinor.

(g) Show that the Dirac equation is equivalent to the pair of equations

$$i\gamma^\mu\partial_\mu\psi_R - m\psi_L = 0, \quad i\gamma^\mu\partial_\mu\psi_L - m\psi_R = 0\ .$$

Note that for $m = 0$, the 2 equations decouple.