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Exercise 3
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Exercises on Elementary Particle Physics

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1. *Non-Abelian Gauge Symmetry and the Adjoint Representation*

   (a) A Lie algebra is defined via the commutation relations of the algebra elements:

   $$[T^i, T^j] = i f^{ijk} T^k$$

   The f^{ijk} are called *structure constants*. Show that the structure constants, viewed as matrices $(T^i)^{kj} := i f^{ijk}$, furnish a representation of the algebra. This representation is called the *adjoint representation*. Hint: Use the Jacobi identity.

   (b) Let us take a free Dirac field Lagrangian

   $$\mathcal{L}_0 = \bar{\Psi}(x) \left(i\gamma^\mu \partial_\mu \right) \Psi(x)$$

   where the Dirac field transforms under (global) $SU(N)$ transformations as

   $$\Psi \mapsto \Psi' = U\Psi, \qquad U = \exp(-i\alpha^a T^a), \qquad U^\dagger U = \mathbb{1}.$$

   Show that $\mathcal{L}_0$ is invariant under the transformation.

   (c) Next, we have a look at *local* $SU(N)$ transformations

   $$\Psi \mapsto \Psi' = U(x)\Psi, \qquad U(x) = \exp(-i\alpha^a(x) T^a), \qquad U^\dagger(x) U(x) = \mathbb{1}.$$

   Show that the transformation of $\mathcal{L}_0$ now leads to an extra term

   $$\bar{\Psi}(x) U^\dagger i\gamma^\mu \left(\partial_\mu U(x) \right) \Psi(x).$$

   Thus, $\mathcal{L}_0$ is not invariant under local $SU(N)$ transformations.

   (d) Therefore, we want to *gauge the symmetry*: We introduce a (gauge) covariant derivative by minimal coupling to a gauge field and identify the the gauge field's transformation properties. The covariant derivative is defined via the requirement that $D_\mu \Psi$ transforms in the same way as Ψ itself:

   $$D_\mu \Psi := \left(\partial_\mu - i g A_\mu^a T^a \right) \Psi$$

and demand

$$(D_\mu \Psi) \mapsto (D_\mu \Psi)' = U(x)(D_\mu \Psi).$$

Show that this is equivalent to demanding that the gauge boson transforms as

$$A^a_\mu \mapsto A^{a\prime}_\mu = A^a_\mu + f^{abc}\alpha^b A^c_\mu - \frac{1}{g}\partial_\mu \alpha^a.$$

Hint: Expand the exponential at the appropriate place in the calculation.

(e) Show that

$$\mathcal{L} = \bar{\Psi}(x)\,(i\gamma^\mu D_\mu)\,\Psi(x)$$

is now gauge invariant.

(f) Define the field strength tensor by

$$(D_\mu D_\nu - D_\nu D_\mu)\,\Psi =: -ig\left(F^a_{\mu\nu}T^a\right)\Psi$$

and find for the components of F:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

(g) Note that the covariant derivative was constructed such that $D_\mu(U(x)\Psi) = U(x)(D_\mu \Psi)$ holds. Therefore

$$\left[(D_\mu D_\nu - D_\nu D_\mu)\,\Psi\right]' = U(x)(D_\mu D_\nu - D_\nu D_\mu)\Psi$$

is valid. Derive the transformation property of the field strength tensor:

$$F^a_{\mu\nu} \mapsto F^a_{\mu\nu}{}' = F^a_{\mu\nu} + f^{abc}\alpha^b F^c_{\mu\nu}$$

(h) Because of the last equation, the field strentgh tensor itself is not gauge invariant. Verify that the product

$$\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) := \mathrm{tr}\left(\left(F^a_{\mu\nu}T^a\right)\left(F^{\mu\nu a}T^a\right)\right)$$

is gauge invariant. The trace is taken over the matrix entries of the generators.

As this term is gauge invariant, we have to add it to the Lagrangian. It gives rise to self couplings of the gauge bosons. The final result for the gauge invariant Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(x)\,(i\gamma^\mu D_\mu)\,\Psi(x) - \frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

2. *Representations of SU(n) and Young tableaux*

For representations of SU(n), there is a convenient bookkeeping technique. In this exercise, we do not present derivations, but give the rules for the practical-minded reader. Details can be found in *T. P. Cheng and L. F. Li, "Gauge Theory Of Elementary Particle Physics", p. 104,* or *J. J. Sakurai, "Modern Quantum Mechanics", p. 370.*

(i) Each Young tableau corresponds to an irreducible representation (irrep) of SU(n) and vice versa.

(ii) The boxes $\boxed{i}$ are the basic primitive objects of SU(n). Each box has an index number i in it which runs from $1, \ldots, n$.

(iii) A general Young tableau is an arrangement of boxes in rows and columns such that the length of the rows does not increase from top to bottom.

(iv) The index numbers do not decrease when going from left to right and increase when going from top to bottom.

(v) Any column with n boxes can be crossed out.

(vi) The dimension of the irrep is given by the number of ways you can fill the boxes obeying the rules.

(vii) **Dimension of the irrep**

There is an easier way to compute the dimension of the associated SU(n) irrep which we may call the "hook rule". For this, we introduce 2 definitions. For any box in the tableau, draw 2 perpendicular lines in the shape of a hook, one going to the right, the other going downward. The total number of boxes that this hook passes (including the original box itself) is the hook length h_i associated with the ith box. The distance d_i to the first box is defined to be the number of steps going from the box in the upper-left corner of the tableau to the ith box with each step to the right counted as +1 and each step downward as -1. The dimension D of the irreducible representation is then given by

$$D = \prod_i \frac{n + d_i}{h_i}.$$

(viii) **Tensor products of irreps**

We want to calculate the tensor product of 2 irreps given by their Young tableaux.

I. In the tableau for the first irrep, assign the symbol 'a' to all boxes in the first row, the symbol 'b' to all boxes in the second row, etc.

II. Attach boxes labelled 'a' to the tableau of the second factor in all possible ways, subject to the rules that the resultant tableau is still a Young tableau and no 2 a's appear in the same column. Repeat this process with the b's, etc.

III. Read the symbols a, $b, \ldots$ from right to left (!) in the first row, second row, etc. Concatenate the strings you obtain from each row. To the left of any symbol there must be at least as many a's as b's, at least as many b's as c's, etc. Tableaux which do not fullfill these criteria are discarded.

(ix) **Conjugate representations**

Take the Young tableau for a representation. Complement this tableau with as many boxes as necessary to obtain a rectangular tableau with n rows. These additional boxes, rotated by 180°, give the Young tableau for the conjugate representation.

Use the foregoing rules to solve the following exercises. Take $n = 3$, i.e. we are dealing with representations of SU(3).

(a) Which of the following Young tableaux are admissible?

$$\begin{array}{|c|c|}\hline 2&2\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&1\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 1&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 4&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

(b) Using rule (2vi), determine the dimensions of the following 2 representations:

$$\begin{array}{|c|}\hline \\\hline\end{array}\qquad \begin{array}{|c|c|}\hline &\\\hline\end{array}$$

Write down the Young tableaux for the corresponding conjugate representations (rule 2ix).

(c) Calculate the dimension of the irrep corresponding to the Young tableau

$$\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$$

using rule (2vi), and then verify that the hook rule (2vii) gives the same result.

(d) We want to calculate the tensor product of 2 irreps. As a preliminary step, consider the following diagrams.

$$\begin{array}{|c|c|c|c|c|}\hline &&a&a&b\\\hline &\multicolumn{4}{c}{}\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|}\hline &&a\\\hline &a&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline &&a&b\\\hline &a&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline &&a&a\\\hline &b&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline &&a\\\hline &a&b\\\hline\end{array}$$

Using rule 2(viii)III, find out which diagrams are admissible and which are discarded.

(e) Using rule (2viii), calculate $3 \otimes 3$, $3 \otimes \bar{3}$, and $8 \otimes 8$. Write down the final answer in terms of the dimensions of the irreps (and not in terms of the tableaux).

3. *Dynkin's approach to group theory and tensor products of irreps*

This exercise is voluntary and requires knowledge of group theory at an intermediate level.

Our goal is to calculate $3 \otimes \bar{3}$ using some more advanced methods in group theory.

(a) Draw the Dynkin diagram of SU(3) and derive from it the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(b) The fundamental representation of SU(3) is given by the heighest weight vector $\lambda = (1, 0)$ and the conjugate representation is given by $\bar{\lambda} = (0, 1)$ (both in Dynkin labels). Use Dynkin's algorithm to construct all the weights of the fundamental and conjugate representation.

Dynkin's algorithm: Start with the highest weight λ. For each positive Dynkin label $\lambda_i > 0$, construct the sequence of weights $\lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - \lambda_i\alpha_i$. (Note that λ_i denotes a number, whereas α_i is the ith simple root given by the ith row of the Cartan matrix.) This process is repeated with λ replaced by each of the weights just obtained, and iterated until no more weights with positive Dynkin labels are produced.

Note of caution: This algorithm tells you nothing about the multiplicities of the weights. In the present case, the multiplicities are 1.

(c) Calculate the weights of $R := 3 \otimes \bar{3}$. Hint: Exercise sheet 1.

(d) Find the heighest weight of R. To find out to which irrep it corresponds, apply Dynkin's algorithm. Substract the set of weights of the irrep from the set of weights of R. Repeat the procedure with the remaining weights of R. You will finally obtain the decomposition of R into irreducible representations.

Note: You need to know the multiplicities of the weights. In general, the multiplicities can be calculated using a recursion relation known as the Freudenthal formula. In the present case, all multiplicities in the irrep corresponding to the highest weight you find in the first step are 1, except for the weight $(0, 0)$ whose multiplicity is 2.