Physikalisches Institut | Exercise 5
University Bonn | Nov 18,20 2003
Theoretische Physik | WS 03/04

Exercises on Elementary Particle Physics

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1. *The Standard Model Higgs Effect*

The Glashow-Weinberg-Salam theory is the part of the Standard Model which describes the electroweak interactions by a nonabelian gauge theory with the gauge group $SU(2)_L \times U(1)_Y$. In the one family approximation, the Standard Model contains the following $SU(2)_L$ doublets and singlets:

$$\ell_L := \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \quad q_L := \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \Phi := \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad e_R, \quad u_R, \quad d_R$$

where the subscripts in the Higgs doublet denote the electromagnetic charges.

(a) By imposing that the *Gell-Mann-Nishijima relation*

$$Y = 2(Q - T_3)$$

is valid, find the correct hypercharges Y for all fields from the information about their electromagnetic charges Q and $SU(2)_L$ eigenvalues T_3.

(b) We want to discuss how the gauge bosons acquire mass via the Higgs effect in the Standard Model. The relevant part of the Lagrangian for us is then

$$\begin{aligned} \mathcal{L} &= (D_\mu \Phi)^\dagger (D^\mu \Phi) - \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 \\ &\quad - \frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \end{aligned}$$

where $F^i_{\mu\nu}$ and $G_{\mu\nu}$ are the field strenth tensors belonging to the $SU(2)_L$ and $U(1)_Y$, respectively.

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

Which parts from the Standard Model Lagrangian have been left out?

(c) The covariant derivative D_μ for an $SU(2) \times U(1)$ gauge theory is given by

$$D_\mu = \left(\partial_\mu - igA^a_\mu T^a - ig' \frac{Y}{2} B_\mu\right)$$

Write down how the covariant derivative acts on one of the left-handed lepton doublets, on one of the right-handed leptons and on the Higgs doublet.

(d) Find the minimum of the Higgs potential. Show that we can choose the physical vacuum corresponding to the vacuum expectation value in the form

$$\langle\Phi\rangle_0 := \langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Make yourself clear that by giving the electromagnetically uncharged part of the Higgs field a vev, the $U(1)_Q$ symmetry will survive.

(e) We want to reparametrize the shifted Higgs field (which contains two complex fields $\phi^+(x)$ and $\phi^0(x)$) in terms of the four real fields $\xi^a(x)$, where $a = 1, 2, 3$, and $\eta(x)$ in the following way:

$$\Phi(x) = \Phi'(x) + \langle\Phi\rangle_0 = U^{-1}(x)\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}, \quad U(x) = \exp\left(\frac{i}{v}\xi^a(x)\tau^a\right)$$

After this redefinition, we apply a gauge transformation by defining new fields. This gauge is called the unitary gauge.

$$\Phi \mapsto \Phi' = U(x)\Phi$$

Show that the Higgs potential in this gauge now reads

$$V(\Phi') = \mu^2\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4.$$

Compare the degrees of freedom in the Higgs sector to the situation before symmetry breakdown.

(f) We want to see that the gauge bosons have acquired mass. Find the mass terms for A'_μ and B'_μ in the transformed Lagrangian and show that the result is

$$\frac{g'^2v^2}{8}B'_\mu B'^\mu - \frac{g'gv^2}{4}B'_\mu A'^{\mu 3} + \frac{g^2v^2}{8}A'^a_\mu A'^{\mu a}$$

(g) Write this equation as a matrix equation with the vectors $(A'^1_\mu, A'^2_\mu, A'^3_\mu, B'_\mu)$. The (A'^1_μ, A'^2_μ) block is already diagonal. We are thus free to choose as the physical fields the $W^\pm$ bosons which are given by the the linear combinations

$$W^\pm_\mu := \frac{1}{\sqrt{2}}\left(A'^1_\mu \mp iA'^2_\mu\right)$$

where plus and minus indicate the $U(1)_Q$ charges. Identify their masses.

(h) The (A'^3_μ, B'_μ) block of the matrix still has to be diagonalized. Find the eigenvalues and eigenvectors and parametrize them in terms of the *Weinberg angle* θ_W, defined by $\tan\theta_W := \frac{g}{g'}$ The result are the rotated fields

$$Z_\mu = \cos\theta_W A'^3_\mu - \sin\theta_W B'_\mu$$
$$A_\mu = \sin\theta_W A'^3_\mu + \cos\theta_W B'_\mu$$

These fields are the Z-boson and the photon. Compare the degrees of freedom in the gauge sector to the situation before symmetry breakdown. How about the total amount of degrees of freedom?

(i) A few more remarks to clarify some things: Go to a part of the fermionic kinetic Lagrangian of the Standard Model

$$\bar{\ell}'_L \left(i\gamma^\mu D_\mu\right) \ell'_L$$

From here you can make yourself clear that the $U(1)_Q$ charge assignment to the $W^\pm$ is consistent. Hint: Expand the $SU(2)_L$ structure of this term and read off the fermion-gauge field interactions. Verify the charges of $W^\pm$ from charge conservation at each vertex.

(j) Now let us have a look at the part of the Lagrangian which has the *Yukawa* couplings for the leptons:

$$G_1 \bar{\ell}'_L \Phi' e'_R + \text{h.c.}$$

Observe that the Higgs effect has also led to fermion mass terms.
The $SU(2)_L \times U(1)_Y$ symmetry has indeed been broken. Show this by identifying terms in this part of the Lagrangian that explicitly violate this symmetry.

2. *Crossing symmetry*

Suppose that the reaction

$$A + B \to C + D.$$

is known to occur. Crossing symmetry tells you that any of these particles can be brought to the other side of the equation, turning it into its antiparticle. The resulting interaction will be allowed *provided that it is kinematically permissible.* For example,

$$A + \bar{C} \to \bar{B} + D.$$

The power of crossing symmetry lies in the fact that the calculations involved in these various interactions are practically identical.

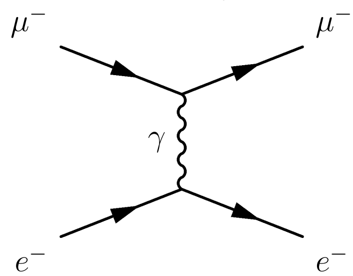

Figure 1: Electron-muon scattering

Consider now the electron-muon scattering amplitude which has been calculated on the last exercise sheet:

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2} \left[\bar{u}(3)\gamma^\mu u(1)\right] \left[\bar{u}(4)\gamma_\mu u(2)\right] \tag{1}$$

After averaging over the initial spins and summing over the final spins, the result reads:

$$\langle|\mathcal{M}|^2\rangle = \frac{8g_e^4}{(p_1-p_3)^4}\big((p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)-m_\mu^2(p_1\cdot p_3)-m_e^2(p_2\cdot p_4)+2m_e^2m_\mu^2\big)$$

(a) Write down all possible reactions which are related to electron-muon scattering

$$e^- + \mu^- \to e^- + \mu^-$$

by crossing symmetry and draw their Feynman diagrams. Which reactions are kinematically allowed?

(b) One of the reactions listed in (2a) is *electron-positron annihilation.*

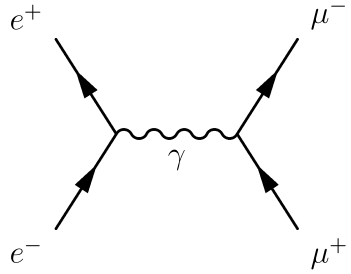

Figure 2: Elektron-positron annihilation

Using the Feynman rules of QED, find the absolute value squared, initial spin averaged and final spin summed amplitude for this reaction. Label the external particles and the momenta as in the case of electron-muon scattering.

(c) Compare your result obtained in (2b) with eq. (1). Show that the amplitudes for electron-positron annihilation and electron-muon scattering are related by

$$p_2 \leftrightarrow -p_3.$$

Where does the minus sign come from? How are the Feynman graphs related?

3. *One more Feynman rule: Antisymmetrization*

Include a minus sign between diagrams that differ only in the interchange (i) 2 incoming electrons, (ii) 2 outgoing electrons, (iii) 2 incoming positrons, (iv) 2 outgoing positrons, (v) an incoming electron and an outgoing positron, (vi) an incoming positron and an outgoing electron.

(a) Consider electron-electron scattering. There are 2 diagrams that contribute. One is given by fig. 1, the muon replaced by an electron. Write down the result for $\langle\mathcal{M}^2\rangle$.

(b) Find the second diagram. Show that the result for $\langle\mathcal{M}^2\rangle$ can be obtained from the first graph by substituting $p_3 \leftrightarrow p_4$.

(c) According to the antisymmetrization rule, are these diagrams to be substracted or added?