Physikalisches Institut
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Theoretische Physik

Exercise 6
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WS 03/04

Exercises on Elementary Particle Physics

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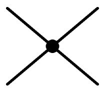
1. *Dimensional Regularization of* φ^4 *Theory*

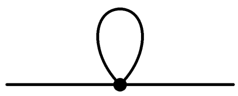
   Consider the Lagrangian of a real scalar field:

   $$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,m^2\,\varphi^2 - \frac{1}{4!}\,\lambda\,\varphi^4$$

   The Feynman rules for φ^4 theory are very simple:

    $\sim$ $\frac{i}{p^2-m^2+i\epsilon}$  $\sim$ $-i\lambda$

   (a) At 1-loop order, the propagator will receive a correction:

   To calculate this loop graph, we need one final *Feynman rule*: Integrate over the momentum of the particle running in the loop. If it is a fermion, pick up an extra factor of -1.

   Using the Feynman rules given above, calculate the amplitude for the correction to the propagator, and call the resulting integral $I(m)$.

   (b) The amplitude $I(m)$ is divergent in $d = 4$ dimensions. Generally, the integrals are "less divergent" when the dimension d decreases. Perform the integration in d dimensions:

   $$I(d, m) = \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\epsilon}$$

   We will later take $d \to 4$. Now, perform a Wick rotation:

   i. View k^0 as a complex variable. Draw the complex k^0-plane. The integration is along the real axis. Mark the positions of the poles of the propagator.

   ii. Rotate the axes by 90°. The integration which ran from $-\infty$ to $+\infty$ now runs from $-i\infty$ to $+i\infty$. Explain why this rotation does not change the value of the integral.

iii. Change variables: $k^0 = iq^0$, $k^i = q^i$. Perform the variable substitution in the integral. The q^0 integration runs from $-\infty$ to $+\infty$. Rename the variables $q \to k$.

Your final result should read

$$I(d,m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}.$$

(c) The integral only depends on the absolute value of k. Change to radial coordinates. You will need the area of the sphere in d dimensions which is given by

$$\mathrm{Vol}(\partial B^d) = \frac{d\pi^{d/2}}{\Gamma(d/2+1)}.$$

Then substitute

$$x = \frac{m^2}{|k|^2 + m^2} \quad \leftrightarrow \quad |k|^2 + m^2 = \frac{m^2}{x} \quad \leftrightarrow \quad 1 - x = \frac{|k|^2}{|k|^2 + m^2}.$$

Your final result should read

$$I(d,m) = \frac{d(4\pi)^{-d/2}(m^2)^{d/2-1}}{2\Gamma(d/2+1)} \int_0^1 dx (1-x)^{d/2-1} x^{-d/2}.$$

(d) Use the definition of Euler's beta function

$$B(\alpha, \beta) = \int_0^1 dx\ x^{\alpha-1}(1-x)^{\beta-1}$$

and the relations

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Gamma(z+1) = z\Gamma(z) \tag{1}$$

to arrive at the following expression for the 1-loop correction of the propagator:

$$I(d,m) = (4\pi)^{-d/2}(m^2)^{d/2-1}\Gamma(1-d/2). \tag{2}$$

(e) The gamma function $\Gamma(z)$ has 1st order poles at negative integers and 0. The case $d \to 4$ corresponds to $1 - d/2 \to -1$, so the result has a pole as expected. To isolate the pole, we have to expand $\Gamma(1-d/2)$ near $d = 4$. Define $\epsilon = 4 - d$. Using eq. (1), first show that

$$\Gamma(1-d/2) = \frac{2}{\epsilon} \frac{\Gamma(1+\epsilon/2)}{-1+\epsilon/2}.$$

Now we can expand around $\Gamma(1) \neq \pm\infty$. Taylor expand the gamma function in the numerator and its denominator (geometric series). Discarding all terms linear and higher order in ϵ, your final result should read

$$\Gamma(1-d/2) = -\left(\frac{2}{\epsilon} + 1 + \Gamma'(1)\right). \tag{3}$$

(f) The first 2 factors in eq. (2) also have to be expanded in powers of ϵ, because if their expansion contains terms of order ϵ, there is a non-vanishing contribution from their product with the pole of the gamma function. First, substitute $\epsilon = 4 - d$ to obtain

$$(4\pi)^{-d/2}\ (m^2)^{d/2-1} = (4\pi)^{-2}\ m^2\ (4\pi)^{\epsilon/2}\ (m^2)^{-\epsilon/2}.$$

A dimensionfull quantity cannot be expanded in a power series. We introduce the so-called *renormalization point* μ which has the dimension of mass:

$$(4\pi)^{-d/2}\ (m^2)^{d/2-1} = (4\pi)^{-2}\ m^2\ (\mu^2)^{-\epsilon/2}\ (4\pi)^{\epsilon/2}\ \left(\frac{m^2}{\mu^2}\right)^{-\epsilon/2} \tag{4}$$

Next, use the relation

$$x = \exp\log x = 1 + \log x + \frac{1}{2!}(\log x)^2 + \frac{1}{3!}(\log x)^3 + \ldots$$

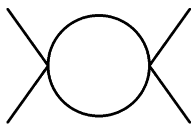
to expand the last 2 factors in eq. (4). Your final result should read

$$(4\pi)^{-2}\ m^2\ (\mu^2)^{-\epsilon/2}\left(1 + \frac{\epsilon}{2}\log 4\pi - \frac{\epsilon}{2}\log\frac{m^2}{\mu^2} + \ldots\right). \tag{5}$$

(g) Substituting your results obtained in eqs. (3), (5) into the expression for the 1-loop correction of the propagator given in eq. (2), show the following result:

$$\boxed{I(d,m) = -(4\pi)^{-2}\ m^2\ (\mu)^{-\epsilon/2}\ \left(\frac{2}{\epsilon} + 1 + \Gamma'(1) - \log\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\epsilon)\right)}$$

(h) At 1-loop, the 4-point-function will receive a correction which is of 2nd order in λ:



Calculate the amplitude for the correction to the 4-point-function, and call the resulting integral $J(d,m,q)$. Note that the resulting amplitude depends on the sum of incoming momenta $q = p_1 + p_2$. Do not forget to impose 4-momentum conservation at the vertices.

(i) Prove the so-called *Feynman trick:*

$$\frac{1}{ab} = \int_0^1 dx\ \frac{1}{[x\,a + (1-x)\,b]^2}$$

Hint: Solve the integral by variable substitution $y = (a-b)x + b$.

(j) Use the Feynman trick to rewrite the correction to the amplitude:

$$J(d,m,q) = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k+xq)^2 - \Delta\right]^2}, \qquad \Delta = -x(1-x)q^2 + m^2 \qquad (6)$$

(k) It is clear that we may shift the integration variable k in eq. (6) to get

$$J(d,m,q) = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2}.$$

Differentiating eq. (2) with respect to m^2, derive

$$\frac{d}{dm^2} I(d,m) = \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k^2 - m^2 + i\epsilon)^2} = (4\pi)^{-d/2}(m^2)^{d/2-2}\Gamma(2-d/2).$$

Use this result to obtain

$$J(d,m,q) = \int_0^1 dx\ i(4\pi)^{-d/2}\Delta^{d/2-2}\,\Gamma(2-d/2).$$

(l) Proceeding analogously to (1e)-(1g), show the following result:

$$\boxed{J(d,m,q) = i(4\pi)^{-2}(\mu)^{-\epsilon/2}\left(\frac{2}{\epsilon} + \Gamma'(1) - \log\frac{m^2}{4\pi\mu^2} - \int_0^1 dx\,\log\frac{\Delta}{m^2}\right)}$$

Note: The integral can be solved explicitly, see e.g. *Gradshteyn, Ryzhik, "Table of Integrals, Series, and Products", p. 250, eq. 2.733.*

So far, we have **regularized** the divergent integrals, i.e. we have separated the infinite and finite parts. The procedure to get rid of the infinities using physical input data is called **renormalizing** the theory. We will deal with renormalization in one of the next exercise sheets.

2. *Triplet Higgs and the ρ parameter*

On the last exercise sheet, we learned that the GWS theory makes use of a Higgs *doublet* to break the $SU(2)_L \times U(1)_Y$ gauge symmetry. After diagonalization of the mass matrices with the help of the Weinberg angle $\tan\theta_W = \frac{g'}{g}$, the vector boson masses were found to be

$$M_{W^\pm}^2 = \frac{1}{4}v^2 g^2, \qquad M_Z^2 = \frac{1}{4}v^2(g^2 + g'^2), \qquad M_A^2 = 0.$$

(a) Calculate the ρ parameter for the GWS model, where ρ is defined by

$$\rho := \frac{M_{W^\pm}^2}{M_Z^2 \cos^2\theta_W}$$

Now consider again the Lagrangian from the last exercise sheet

$$\mathcal{L} \;=\; (D_\mu\Phi)^\dagger (D^\mu\Phi) + \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

but Φ is now a triplet $(\phi^{++}, \phi^+, \phi^0)^T$ under $SU(2)_L$ and the 3 dimensional generators of $SU(2)_L$ are given by

$$T^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) Calculate the hypercharge of Φ and write down the covariant derivative.

(c) Calculate the vacuum expectation value $\left\langle \Phi^\dagger\Phi \right\rangle =: \frac{v^2}{2}$ of the Higgs field and define the shifted field.

(d) Find the vector boson masses from the shifted Lagrangian. Calculate the ρ parameter for this model.

Experiment says $\rho = 1$. Which of the two models can you rule out?