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Exercise 8
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Exercises on Elementary Particle Physics

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1. *Dynkin Diagram of SU(n)*

Consider the space of all $n \times n$ matrices and regard it as a Lie algebra (of $GL(n)$). We choose as a basis the elements e_{ab} with components

$$(e_{ab})_{ij} = \delta_{ai}\,\delta_{bj}.$$

(a) Verify the multiplication rule and thus the commutator operation on the algebra

$$e_{ab}\,e_{cd} = e_{ad}\,\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\,\delta_{bc} - e_{cb}\,\delta_{ad}.$$

In order to deal with the Lie algebra of $SU(n)$, we restrict ourselves to traceless $n \times n$ matrices. Therefore, the algebra consists of linear combinations of e_{ab} for $a \neq b$ and of elements $h = \sum_i \lambda_i e_{ii}$ where $\sum_i \lambda_i = 0$. We say that the set of all h forms the **Cartan subalgebra** H. The dimension of the Cartan subalgebra is called the **rank** of the algebra. We notice that the rank of the $SU(n)$ algebra is $n-1$ and call it also A_{n-1}.

(b) Show that the elements of the Cartan subalgebra mutually commute. Calculate also the commutation relation with the other elements

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\, e_{ab}.$$

We can regard the above equation as an eigenvalue equation where the operation $[h, .]$ acts on the eigenvector e_{ab} to reproduce e_{ab} with the eigenvalue $(\lambda_a - \lambda_b)$. Or we can regard the above equation (for e_{ab} fixed) as a prescription how to associate to each $h \in H$ a number $(\lambda_a - \lambda_b)$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b.$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra, H^*. Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be a fixed basis of roots so every element of H^* can be written as $\rho = \sum_i c_i \alpha_i$. We call ρ **positive** ($\rho > 0$) if the first non-zero coefficient c_i is positive. If the first non-zero coefficient c_i is negative, we call ρ negative. For $\rho, \sigma \in H$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$. A **simple root** is a positive root which cannot be written as the sum of two positive roots.

(c) We choose as a basis in the root space the functionals

$$\alpha_i(h) = \lambda_i - \lambda_{i+1} \quad i = 1, 2, ..., n-1.$$

Verify the simple fact that as these roots are a basis, they are positive with $\alpha_1 > \alpha_2 \cdots > \alpha_{n-1}$. Show that these roots are simple roots. Hint: Show first that every root can only have coefficients $c_i \in \{0, 1\}$.

Next, we define a structure that resembles a scalar product on the algebra. Let t_i be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra.

$$[x, [y, t_i]] = \sum_j K_{ij} t_j.$$

The **Killing form** is then defined as $\mathcal{K}(x, y) := \mathrm{Tr}(K)$.

(d) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h, h')$, where $h = \sum_i \lambda_i \, e_{ii}$, $h' = \sum_j \lambda'_j \, e_{jj}$.

The Killing form enables us to make a connection between the Cartan subalgebra, H, and its dual H^*: One can prove that if $\alpha \in H^*$, there exists a unique element $h_\alpha \in H$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \qquad \forall \, h \in H.$$

(e) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find h_{α_i} from comparison with your result from (d).

With the help of the h_α, we are now able to define a scalar product on H^*:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \qquad \text{where} \quad \alpha_i, \alpha_j \in H^*.$$

(f) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2 \, \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root α_i, we associate a small circle and join the small circles i and j with $A_{ij} A_{ij}$ (no summation) lines.

(g) Draw the Dynkin diagram for A_n (i.e. $SU(n+1)$).

2. *The Exceptional Lie Algebra G_2*

The Dynkin diagram completely encodes the properties of a semi-simple Lie algebra. We take the exceptional Lie algebra G_2 as an illustrative example.

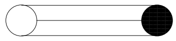

Figure 1: Dynkin diagram of G_2

(a) Using the Dynkin diagram given in fig. 1, derive the Cartan matrix A of G_2. Hint: From the definition of the Cartan matrix, it follows that the diagonal entries are equal to 2. To find the remaining 2 entries, remember that they are non-positive integers, the number of lines between the ith and the jth root in the Dynkin diagram is $A_{ij}A_{ji}$, and the second root is short.

Dynkin's Algorithm

Start with the highest weight λ. For each positive Dynkin label $\lambda_i > 0$, construct the sequence of weights $\lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - \lambda_i\alpha_i$. (Note that λ_i denotes a number, whereas α_i is the ith simple root given by the ith row of the Cartan matrix.) This process is repeated with λ replaced by each of the weights just obtained, and iterated until no more weights with positive Dynkin labels are produced. Note that this algorithm tells you nothing about the multiplicities of the weights.

(b) The special representation where the highest weight is the highest root is called the *adjoint representation.* The highest root of G_2 is given by $\theta = (1, 0)$. Using Dynkin's algorithm described above, calculate the roots of G_2. How many roots are there? What is the dimension of the Lie algebra?

(c) Determine the positive roots of G_2. Hint: Write each root as a linear combination of the 2 simple roots. By definition, a root is positive if the first non-vanishing entry is positive.

Scalar product in Dynkin labels

In Dynkin labels, the scalar product is not simply given by the "usual" scalar product of 2 vectors, but by the expression

$$(\alpha, \beta) = \sum_{i,j} \alpha_i Q_{ij} \beta_j,$$

where Q is the quadratic form matrix whose components are given by

$$Q_{ij} = (A^{-1})_{ij} \frac{\alpha_j^2}{2}.$$

(d) Determine the quadratic form matrix. Hint: By convention, the long root has length 2. Using the definition of the Cartan matrix, you can infer the length of the short root.

(e) Calculate the length of the 2 simple roots, and their products. As a crosscheck, calculate the Cartan matrix and compare with exercise (a).

(f) The lowest dimensional representation of G_2 is given by the heighest weight $\lambda = (0, 1)$. Calculate the weights of this representation. Assuming the multiplicities of all weights to be 1, determine the dimension of the irrep.

Freudenthal Recursion Formula

The multiplicity of λ' in the representation given by the highest weight λ can be calculated in terms of the multiplicities of all the weights above it:

$$\left[|\lambda + \rho|^2 - |\lambda' + \rho|^2\right] \text{mult}_\lambda(\lambda') = 2 \sum_{\alpha>0} \sum_{k=1}^{\infty} (\lambda' + k\alpha, \alpha) \text{mult}_\lambda(\lambda' + k\alpha)$$

$\rho = (1, 1, \ldots, 1)$ is the so-called Weyl vector, the round brackets denote the scalar product, and the absolute value squared is given by the scalar product.

(g) Dynkin's algorithm does not keep track of the multiplicities of the weights. Use the Freudenthal recursion formula to calculate the multiplicity of the weight $\alpha = (0, 0)$ in the irrep given by $\lambda = (0, 1)$.