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Exercise 4
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Exercises on Elementary Particle Physics

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1. *Weyl spinors - part II*

   (a) We want to rewrite the transformation laws for Weyl spinors under Lorentz transformations in the standard notation:

   $$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right).$$

   Therefore, we define the generalized Pauli matrices

   $$\sigma^\mu := (\mathbb{1}, \sigma^i), \quad \bar{\sigma}^\mu := (\mathbb{1}, -\sigma^i).$$

   Then, we can define the following quantities:

   $$\sigma^{\mu\nu} := \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu), \qquad \bar{\sigma}^{\mu\nu} := \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu).$$

   We denote the left-chiral Weyl spinor $(1/2, 0)$ by Ψ_L and the right-chiral Weyl spinor $(0, 1/2)$ by Ψ_R. Show that the Weyl spinors transform as

   $$\psi_L \mapsto D_L\psi_L = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi_L,$$

   $$\psi_R \mapsto D_R\psi_R = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\psi_R.$$

   Hint: You know T_L and T_R explicitly. Using the definitions, express first the K's and J's in terms of T_L and T_R. Second, express the $M^{\mu\nu}$'s in terms of K's and J's. Now identify the components of $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ with the components of $M^{\mu\nu}$. You will see that they are equal.

   (b) Prove the following equations:

   $$\begin{aligned} D_L^{-1} &= D_R^\dagger \\ \sigma_2 D_L \sigma_2 &= D_R^* \\ \sigma_2 &= (D_L)^T\sigma_2 D_L \end{aligned}$$

   Comparing the last equation to $\eta = \Lambda^T\eta\Lambda$, we find that σ_2 acts as a metric on the space of the spinor components. We will learn more about it later...

(c) Show that $\sigma_2 \Psi_L^*$ transforms in the $(0, 1/2)$ representation and $\sigma_2 \Psi_R^*$ transforms in the $(1/2, 0)$ representation.

(d) Let Φ_L, Φ_R, Ψ_L and Ψ_R be Weyl spinors. Show that the following expressions are invariant under Lorentz transformations:

$$\begin{array}{ll} i(\Phi_L)^T \sigma_2 \Psi_L & \qquad \Phi_R^\dagger \Psi_L \\ i(\Phi_R)^T \sigma_2 \Psi_R & \qquad \Phi_L^\dagger \Psi_R \end{array}$$

(e) Choose $\Phi_L = \Psi_L$ and compute $i(\Psi_L)^T \sigma_2 \Psi_L$. What can you say about spinor components?

(f) Show that the parity operator acts as follows on the generators of the Lorentz algebra:

$$J^i \mapsto J^i, \qquad K^i \mapsto -K^i$$

Hint: Use $M^{\mu\nu} \mapsto \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}$, where $\Lambda^\mu{}_\nu$ is now the parity operator.

(g) Show that under parity transformation a representation (m, n) of the Lorentz algebra goes to (n, m), e.g. parity maps $(1/2, 0)$ to $(0, 1/2)$.

Therefore, if $m \neq n$, the parity transformation maps an element of the vector space of the representation to an element that is not part of the vector space.

(Since parity maps left to right in the usual sense and the parity operator maps left-chiral Weyl spinors to right-chiral and vice versa, these names for Weyl spinors make sense.)

(h) Show that the dimension of the representation (m, n) is $(2m + 1)(2n + 1)$.

(i) Show that the (4 dim.) Minkowski space is the vector space of the $(1/2, 1/2)$ representation.

Hint: Use the fact that parity maps a four-vector to a four-vector, i.e. you do not leave the Minkowski space if you act with parity on a four-vector.

(j) Use your knowledge about representations of $su(2)$ to show that:

$$\begin{aligned} (1/2, 0) \otimes (0, 1/2) &= (1/2, 1/2) \\ (1/2, 0) \otimes (1/2, 0) &= (1, 0) \oplus (0, 0) \end{aligned}$$

Interpret the result.

Hint: In the language of Exercise 3.1(i) check that: $\mathbf{2} \otimes \mathbf{1} = \mathbf{2}$ and $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$.

2. *Dirac spinor - part I*

Since the vector spaces of the left- and the right-chiral Weyl spinors are not mapped to themselves under parity, we consider the following (reducible) representation of the Lorentz algebra:

$$(1/2, 0) \oplus (0, 1/2)$$

In words: you take a left-chiral Weyl spinor Ψ_L and a right-chiral Weyl spinor Φ_R and take them as the components of a new four-component spinor, called the Dirac spinor:

$$\Psi = \begin{pmatrix} \Psi_L \\ \Phi_R \end{pmatrix}$$

Remark: Only when we use the chiral representation of the Clifford algebra, we can write the Dirac spinor as two Weyl spinors in this easy way!

(a) Show that a Dirac spinor transforms under a Lorentz transformation as

$$\Psi \mapsto D\Psi = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right)\Psi$$

with $\gamma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

Hint: Use the results of 1(a) and use the chiral representation of the γ-matrices.

(b) Show that in the chiral representation the chirality operator $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ can be written as:

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

(c) Prove the following relations without using any specific representation:

$$\gamma^{5\dagger} = \gamma^5 \quad \text{and} \quad \left(\gamma^5\right)^2 = \mathbb{1}$$

(d) Show that the following operators are projection operators (i.e. $P^2 = P$)

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma_5) \quad \text{and} \quad P_R = \frac{1}{2}(\mathbb{1} + \gamma_5)$$

What is their action on a Dirac spinor (in the chiral representation)?

3. *Non-Abelian Gauge Symmetry*

(a) A Lie algebra is defined via the commutation relations of the algebra elements

$$[T^i, T^j] = i f^{ijk} T^k$$

The f^{ijk} are called *structure constants*. Show that the structure constants, viewed as matrices $(T^i)^{kj} := i f^{ijk}$, furnish a representation of the algebra. This representation is called the *adjoint representation*. Hint: Use the Jacobi identity.

(b) Let us take a free Dirac field Lagrangian

$$\mathcal{L}_0 = \bar{\Psi}(x) \left(i\gamma^\mu \partial_\mu \right) \Psi(x)$$

where the Dirac field transforms under (global) $SU(N)$ transformations as

$$\Psi \mapsto \Psi' = U\Psi, \qquad U = \exp(i\alpha^a T^a), \qquad U^\dagger U = \mathbb{1}.$$

Show that $\mathcal{L}_0$ is invariant under the transformation.

(c) Next, we have a look at *local* $SU(N)$ transformations

$$\Psi \mapsto \Psi' = U(x)\Psi, \qquad U(x) = \exp(i\alpha^a(x) T^a), \qquad U^\dagger(x)U(x) = \mathbb{1}.$$

Show that the transformation of $\mathcal{L}_0$ now leads to an extra term

$$\bar{\Psi}(x) U^\dagger i\gamma^\mu \left(\partial_\mu U(x) \right) \Psi(x)$$

Thus, $\mathcal{L}_0$ is not invariant under local $SU(N)$ transformations.

(d) Therefore, we want to *gauge the symmetry*: We introduce a (gauge) covariant derivative by minimal coupling to a gauge field and identify the gauge field's transformation properties. The covariant derivative is defined via the requirement that $D_\mu \Psi$ transforms in the same way as Ψ itself:

$$D_\mu \Psi := \left(\partial_\mu + igA_\mu^a T^a \right) \Psi$$

and demand

$$(D_\mu \Psi) \mapsto (D_\mu \Psi)' = U(x)(D_\mu \Psi)$$

Show that this is equivalent to demanding that the gauge boson transforms as

$$A_\mu^a \mapsto A_\mu^{a\prime} = A_\mu^a - f^{abc}\alpha^b A_\mu^c - \frac{1}{g}\partial_\mu \alpha^a.$$

Hint: Expand the exponential at the appropriate place in the calculation.

(e) Show that

$$\mathcal{L} = \bar{\Psi}(x) \left(i\gamma^\mu D_\mu \right) \Psi(x)$$

is now gauge invariant.