Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 5
21. Nov. 2005
WS 05/06

Exercises on Elementary Particle Physics

Prof. Dr. H.-P. Nilles

1. *Dirac spinor - part II*

Last time, we discussed Lorentz transformations $D(\Lambda)$ acting on a Dirac spinor $\Psi(x)$. We repeat the result (a little bit more detailed):

$$\Psi(x) \mapsto \Psi'(x') = D(\Lambda)\Psi(x) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}\right)\Psi(x) \tag{1}$$

D denotes here a representation of the proper orthochronous Lorentz group, i.e. $det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$. This part of the full Lorentz group contains the identity and can therefore be expressed by the exponential function.

(a) Prove the following equation:

$$[\gamma^\mu, \gamma^{\nu\sigma}] = (M^{\nu\sigma})^\mu{}_\rho \gamma^\rho \tag{2}$$

(b) Derive

$$D^{-1}\gamma^\mu D = \Lambda^\mu{}_\nu \gamma^\nu \tag{3}$$

Hint: Use infinitesimal transformations $D \approx \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\gamma^{\mu\nu}$ and $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$ and use eqn. (2).
Note: This equation holds for the full Lorentz group, i.e. also for the parity transformation.

(c) Prove the following equations:

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{and} \quad [\gamma^5, D] = 0$$

(Note: D is here a proper orthochronous Lorentz transformation and can therefore be expressed by eqn. (1). So, D cannot be the parity operation.)

(d) Show that

$$D^\dagger = \gamma^0 D^{-1} \gamma^0$$

and from this that follows

$$\bar{\Psi}(x) \mapsto \bar{\Psi}(x) D^{-1} \ .$$

(e) Next, we consider the parity operator, i.e. $(\Lambda_P)^0{}_0 = 1$ and $(\Lambda_P)^i{}_i = -1$. Show that one representation of the parity operator is:

$$D_{\mathrm{P}} = \gamma^0$$

Hint: Use eqn. (3).

(f) Examine the action of the parity operator $D_{\mathrm{P}} = \gamma^0$ on a Dirac spinor in the chiral representation.

(g) Now, we can analyse the list of five bilinear covariants (i.e. they are bilinear in the field and covariant under proper orthochronous Lorentz transformation, e.g. $\bar{\Psi}\gamma^\mu\Psi$ transforms like a four-vector). Check the covariance and the behaviour under parity:

$$\begin{array}{rl} scalar & \bar{\Psi}\Psi \\ vector & \bar{\Psi}\gamma^\mu\Psi \\ tensor & \bar{\Psi}\gamma^{\mu\nu}\Psi \\ pseudo-scalar & \bar{\Psi}\gamma^5\Psi \\ pseudo-vector & \bar{\Psi}\gamma^5\gamma^\mu\Psi \end{array}$$

2. *Non-Abelian Gauge Symmetry - part II*

In part I, we defined the covariant derivative:

$$D_\mu \Psi = (\partial_\mu + igA_\mu^a T^a)\Psi \quad \text{with} \quad (D_\mu \Psi) \mapsto U(x)(D_\mu \Psi)$$

(a) Define the field strength tensor by

$$(D_\mu D_\nu - D_\nu D_\mu)\,\Psi =: ig\left(F_{\mu\nu}^a T^a\right)\Psi.$$

and find for the components of F

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc}A_\mu^b A_\nu^c.$$

(b) Note that the covariant derivative was constructed such that $D'_\mu U(x) = U(x)D_\mu$ holds (cf. Exercise 4, 3(d)). Therefore

$$[(D_\mu D_\nu - D_\nu D_\mu)\,\Psi]' = U(x)(D_\mu D_\nu - D_\nu D_\mu)\Psi$$

is valid. Derive the transformation property of the field strength tensor

$$\begin{aligned} F_{\mu\nu} &\quad\mapsto\quad F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \\ F_{\mu\nu}^a &\quad\mapsto\quad F_{\mu\nu}^{a\;\prime} = F_{\mu\nu}^a - f^{abc}\alpha^b F_{\mu\nu}^c \end{aligned}$$

where $F_{\mu\nu} = F_{\mu\nu}^a T^a$.

(c) Because of the last equation, the field strentgh tensor itself is not gauge invariant. Verify that the product

$$\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$$

is gauge invariant. The trace is taken over the matrix entries of the generators.

As this term is gauge invariant, we have to add it to the Lagrangian. It gives rise to self couplings of the gauge bosons. The final result for the gauge invariant Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}(x)\,(i\gamma^\mu D_\mu)\,\Psi(x) - \frac{1}{2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$$

3. *Spontaneous Symmmetry Breaking in the Linear Sigma Model*

As an application of spontaneous symmetry breaking, we want to have a look at the linear sigma model which consists of N real scalar fields with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{1}{2}\mu^2\phi^i\phi^i - \frac{\lambda}{4}\left(\phi^i\phi^i\right)^2, \quad \text{sum over } i = 1,\ldots,N,$$

with $\lambda > 0$ and $\mu^2 < 0$.

(a) Let us find the symmetry group of the Lagrangian: We transform the fields $\phi^i \mapsto R^{ij}\phi^j$. What kind of matrices R are allowed such that $\mathcal{L}$ remains invariant?

(b) Find the minimum ϕ_0^i of the potential. You will find that the minimum is any ϕ_0^i that fulfills

$$\sum_i \phi_0^i\phi_0^i = -\frac{\mu^2}{\lambda}.$$

This condition determines only the length of the "vector" ϕ_0^i. We choose coordinates such that ϕ_0^i points into the N-th direction:

$$\phi_0^i(x) = (0,0,\ldots,0,v)\,, \quad v = \sqrt{-\frac{\mu^2}{\lambda}}.$$

(c) Now we break the symmetry by defining a set of shifted fields

$$\phi^i(x) := \left(\pi^k(x), v+\eta(x)\right), \quad k = 1,..,N-1.$$

Rewrite the Lagrangian in terms of the π and η fields. The result is

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \mu^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 \\ &\quad + \frac{1}{2}\partial_\mu\pi^k\partial^\mu\pi^k - \lambda v\eta\left(\pi^k\pi^k\right) - \frac{\lambda}{2}\eta^2\left(\pi^k\pi^k\right) - \frac{\lambda}{4}\left(\pi^k\pi^k\right)^2. \end{aligned}$$

(d) Have a look at the system after spontaneous symmetry breaking. How many massive and massless fields are there now? What is the symmetry of the new Lagrangian? Compare your result to *Goldstone's Theorem* which says that for every spontaneously broken continuous symmetry, the theory must contain a massless particle.