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Exercise 7
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Exercises on Elementary Particle Physics

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1. *Non-Abelian Gauge Symmetry - part III*

   In Exercise 5, we discussed non-abelian gauge symmetries. We showed that the Lagrangian

   $$\mathcal{L} = \bar{\Psi}(x)\,(i\gamma^\mu D_\mu)\,\Psi(x) - \frac{1}{2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$$

   is invariant under local gauge transformations, where $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and the generators of the algebra T^a are normalized to $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

   (a) Show that, in the case of SU(2), the normalization condition is fulfilled by the Pauli-matrices $T^a = \frac{1}{2}\sigma^a$.

   (b) Prove the following equation:

   $$\frac{1}{2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$$

2. *The Standard Model Higgs Effect - part I*

In the Standard model, the electroweak interactions of leptons are described by the Lagrangian

$\mathcal{L} =$	$\bar{R} i\gamma^\mu D_\mu R + \bar{L} i\gamma^\mu D_\mu L$	kinetic energy of leptons and interactions with gauge bosons
	$-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$	kinetic energy of gauge bosons and self-interactions
	$+ (D_\mu \phi)^\dagger (D^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2$	Higgs field with potential
	$-G_e \left(\bar{L} \phi R + \bar{R} \phi^\dagger L\right)$	electron mass and coupling to Higgs

with the covariant derivative

$$D_\mu = \partial_\mu + ig' \frac{Y}{2} B_\mu + igT^a W^a_\mu$$

and the particle content

	Hypercharge Y	rep. of $\mathrm{SU}(2)_L$	rep. of Lorentz algebra
$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	-1	$\mathbf{2}$	$\left(\frac{1}{2}, 0\right)$ (*)
$R = e_R$	-2	$\mathbf{1}$	$\left(0, \frac{1}{2}\right)$ (*)
$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}$	1	$\mathbf{2}$	$(0, 0)$
$T^a W^a_\mu$	0	$\mathbf{3}$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
B_μ	0	$\mathbf{1}$	$\left(\frac{1}{2}, \frac{1}{2}\right)$

(*) for now, L and R contain Dirac spinors

(a) Write down how the covariant derivative acts on the left-handed lepton doublet, on the right-handed lepton and on the Higgs doublet.

(b) Show that the Lagrangian is Lorentz invariant.

(c) Show that the Lagrangian is gauge invariant.

(d) Find the minimum of the Higgs potential for $\mu^2 < 0$ and choose the vacuum expectation value (vev) to be

$$\langle \Phi \rangle_0 := \langle 0 | \Phi | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

(e) We want to reparametrize the shifted Higgs field (which contains two complex fields ϕ_+ and ϕ_0) in terms of the four real fields ξ^a ($a = 1, 2, 3$) and η in the following way:

$$\phi(x) = \phi'(x) + \langle\phi\rangle_0 = U^{-1}(x) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}, \quad U(x) = \exp\left(-\frac{i}{v}\xi^a(x)T^a\right)$$

After this redefinition, we apply a $\mathrm{SU}(2)_L$ gauge transformation to all $\mathrm{SU}(2)_L$ non-singlets (L, ϕ and $T^a W^a_\mu$). This gauge is called the unitary gauge. In this gauge, the Higgs field has the simple form:

$$\phi \mapsto \phi' = U(x)\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}$$

In the following, we skip the prime for all fields. Show that the Higgs potential in this gauge now reads

$$V(\phi) = \mu^2\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4.$$

What is the mass of the η field? Compare the degrees of freedom in the Higgs sector to the situation before symmetry breakdown.

(f) Next, we consider the kinetic energy of the Higgs field. Show that:

$$\begin{aligned} (D_\mu\phi)^\dagger (D^\mu\phi) &= \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{4}g^2 (v + \eta)^2 W^-_\mu W^{\mu+} \\ &\quad + \frac{1}{8}(v + \eta)^2 \left(W^3_\mu, B_\mu\right) \begin{pmatrix} g'^2 & -g'g \\ -g'g & g^2 \end{pmatrix} \begin{pmatrix} W^{\mu 3} \\ B^\mu \end{pmatrix} \end{aligned}$$

where the matrix will lead to the mass matrix in the following.

(g) The masses are given by the terms that are quadratic in the fields, e.g.

$$\frac{1}{4}g^2v^2 W^-_\mu W^{\mu+} = m_W^2 W^-_\mu W^{\mu+}$$

so $m_W = \frac{1}{2}vg$. Obviously, this is not so easy for W^3_μ and B_μ. To see the masses of these fields we have to diagonalize the mass matrix

$$\frac{1}{8}v^2 \left(W^3_\mu, B_\mu\right) O^T O \begin{pmatrix} g'^2 & -g'g \\ -g'g & g^2 \end{pmatrix} O^T O \begin{pmatrix} W^{\mu 3} \\ B^\mu \end{pmatrix} =: \frac{1}{2}\left(Z_\mu, A_\mu\right) \begin{pmatrix} m_Z^2 & 0 \\ 0 & m_A^2 \end{pmatrix} \begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix}$$

with an orthogonal matrix. Determine this orthogonal matrix by computing the eigenvalues and the eigenvectors of the mass matrix.

What are the masses of the Z_μ and A_μ fields?

Compare the degrees of freedom in the gauge sector to the situation before symmetry breakdown. How about the total amount of degrees of freedom?

(h) On the other hand, one can always write an orthogonal 2×2 matrix in the following way:

$$O = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}$$

Write $\cos\theta_W$ in terms of g' and g. Prove that the ratio of the masses of the W- and the Z-boson is:

$$\frac{m_W}{m_Z} = \cos\theta_W$$

This is a prediction of the standard model, which has been experimentally confirmed within a small error.

(i) Next, we consider the covariant derivative.

$$D_\mu = \partial_\mu + ig'\frac{Y}{2}B_\mu + igT^a W_\mu^a$$

Substitute the fields B_μ and W_μ^a by $W_\mu^\pm$, Z_μ and A_μ. Show that it follows:

$$D_\mu = \partial_\mu + iA_\mu \frac{g'g}{\sqrt{g'^2+g^2}}\left(T_3 + \frac{Y}{2}\right) + iZ_\mu \frac{1}{\sqrt{g'^2+g^2}}\left(g^2 T_3 - g'^2\frac{Y}{2}\right) + \frac{ig}{\sqrt{2}}\begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix}$$

Now, we can define the electric charge:

$$e := \frac{g'g}{\sqrt{g'^2+g^2}} \qquad \text{and} \qquad Q := T_3 + \frac{Y}{2}$$