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Exercise 8
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Exercises on Elementary Particle Physics

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1. *The Standard Model Higgs Effect - part II*

   After the vev of the neutral Higgs component has broken the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge symmetry to $\mathrm{U}(1)_Q$ the covariant derivative, using the mass- and charge-eigenstates A_μ, $W_\mu^\pm$ and Z_μ, has the form:

$$D_\mu = \partial_\mu + ieQA_\mu + iZ_\mu \frac{1}{\sqrt{g'^2+g^2}} \left(g^2 T_3 - g'^2 \frac{Y}{2} \right) + \frac{ig}{\sqrt{2}} \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix}$$

   Together with the definitions:

$$e := \frac{g'g}{\sqrt{g'^2+g^2}} \qquad \text{and} \qquad Q := T_3 + \frac{Y}{2}$$

   (a) Consider the following terms of the Lagrangian:

$$\mathcal{L} \supset \bar{R} i\gamma^\mu D_\mu R + \bar{L} i\gamma^\mu D_\mu L$$

   Find the interaction terms of the fermions with the gauge bosons. For the weak interaction, analyse its V-A structure $\frac{1}{2}(c_V + c_A\gamma^5)$. Draw the corresponding Feynman diagrams (Note: use $i\mathcal{L}$, drop all fields and you get the vertex factor).

   (b) Show that the mass of the electron is $m_e = \frac{G_e v}{\sqrt{2}}$.
   Hint: Using the unitary gauge, insert the shifted Higgs field

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + \eta(x)) \end{pmatrix}$$

   into the so called Yukawa couplings of the Lagrangian:

$$\mathcal{L} \supset -G_e \left(\bar{L}\phi R + \bar{R}\phi^\dagger L \right)$$

2. *Dynkin Diagram of $SU(n)$ - part I*

Consider the space of all $n \times n$ matrices and regard it as a Lie algebra (of $GL(n)$). We choose as a basis the elements e_{ab} with components

$$(e_{ab})_{ij} = \delta_{ai}\,\delta_{bj}.$$

(a) Verify the multiplication rule and thus the commutator operation on the algebra

$$e_{ab}\,e_{cd} = e_{ad}\,\delta_{bc}\,, \qquad [e_{ab}, e_{cd}] = e_{ad}\,\delta_{bc} - e_{cb}\,\delta_{ad}.$$

In order to deal with the Lie algebra of $SU(n)$, we restrict ourselves to traceless $n \times n$ matrices (e_{ab} are the step operators, see part (i) for an example).

Therefore, the algebra consists of linear combinations of e_{ab} for $a \neq b$ (in order to be traceless) and of elements $h = \sum_i \lambda_i e_{ii}$ where $\sum_i \lambda_i = 0$.

(b) Show that H forms a subalgebra, i.e. show that

$$[h, g] = 0 \qquad \text{for } h, g \in H$$

The dimension of the Cartan subalgebra is called the **rank** of the algebra. We notice that the rank of the $SU(n)$ algebra is $n-1$ and therefore also call the algebra A_{n-1}.

(c) Calculate the commutation relation of the elements of the Cartan subalgebra with the other elements

$$[h, e_{ab}] = (\lambda_a - \lambda_b)\,e_{ab}. \tag{1}$$

We can regard this equation as an eigenvalue equation where the operation $[h, .]$ acts on the eigenvector e_{ab} to reproduce e_{ab} with the eigenvalue $(\lambda_a - \lambda_b)$. This operator is called the adjoint of h:

$$\begin{aligned} \mathrm{ad}h(e_{ab}) &:= [h, e_{ab}] \\ \mathrm{ad}h(e_{ab}) &= (\lambda_a - \lambda_b)\,e_{ab} \end{aligned}$$

Or we can regard the equation 1 (for e_{ab} fixed) as a prescription for how to associate to each $h \in H$ a number $(\lambda_a - \lambda_b)$. We can write this prescription as

$$\alpha_{e_{ab}}(h) = \lambda_a - \lambda_b.$$

We call $\alpha_{e_{ab}}$ a **root**. The roots live in the dual space of the Cartan subalgebra H. This dual space is denoted by H^*.

Let $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ be a fixed basis of roots so every element of H^* can be written as $\rho = \sum_i c_i \alpha_i$.

We call ρ **positive** ($\rho > 0$) if the first non-zero coefficient c_i is positive. Note, that the basis roots α_i are positive by definition. If the first non-zero coefficient c_i is negative, we call ρ negative. For $\rho, \sigma \in H^*$, we shall write $\rho > \sigma$ if $\rho - \sigma > 0$.

A **simple root** is a positive root which cannot be written as the sum of two positive roots.

(d) We choose a basis α_i for the root space:

$$\alpha_i(h) = \lambda_i - \lambda_{i+1} \quad i = 1, 2, ..., n-1.$$

Verify that these roots are a basis and that they are positive with $\alpha_1 > \alpha_2 \cdots > \alpha_{n-1}$. Show that these roots are simple roots.

Next, we define a structure that resembles a scalar product on the algebra. Let t_i be a basis of the algebra, then the double commutator with any two algebra elements will be a linear combination in the algebra.

$$[x, [y, t_i]] = \sum_j K_{ij} t_j.$$

The **Killing form** is then defined as $\mathcal{K}(x, y) := \mathrm{Tr}(K)$.

(e) Prove that the Killing form on the Cartan subalgebra is bilinear and symmetric. (It is, however, in general not positive definite and thus not a scalar product.) Determine $\mathcal{K}(h, h')$, where $h = \sum_i \lambda_i \, e_{ii}$, $h' = \sum_j \lambda'_j \, e_{jj}$.

The Killing form enables us to make a connection between the Cartan subalgebra, H, and its dual H^*: One can prove that if $\alpha \in H^*$, there exists a unique element $h_\alpha \in H$ such that

$$\alpha(h) = \mathcal{K}(h_\alpha, h) \qquad \forall\, h \in H.$$

(f) Calculate $\mathcal{K}(h_{\alpha_i}, h)$ with the help of the above theorem and find h_{α_i} from comparison with your result from (e).

With the help of the h_α, we are now able to define a scalar product on H^*:

$$\langle \alpha_i, \alpha_j \rangle := \mathcal{K}(h_{\alpha_i}, h_{\alpha_j}), \qquad \text{where} \quad \alpha_i, \alpha_j \in H^*.$$

(g) Calculate the **Cartan matrix**, defined by

$$A_{ij} := \frac{2\, \langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

The information about the algebra that is encoded in the Cartan matrix is complete in the sense that it is equivalent to knowing all structure constants. There is one more equivalent way of depicting the algebra information in drawing a **Dynkin diagram**: To every simple root α_i, we associate a small circle and join the small circles i and j with $A_{ij} A_{ij}$ (no summation, $i \neq j$) lines.

(h) Draw the Dynkin diagram for A_n (i.e. $SU(n+1)$).

(i) As an example, consider the Lie algebra of SU(2)
(cf. exercise sheet 3.1).
The step operators are given by

$$J_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \qquad J_- = \frac{1}{2}(\sigma_1 - i\sigma_2)$$

and the Cartan subalgebra consists of the single element

$$h = J_3 = \frac{1}{2}\sigma_3 .$$

- Confirm that

$$e_{12} = J_+, \qquad e_{21} = J_- \quad \text{and} \quad h = \frac{1}{2}e_{11} - \frac{1}{2}e_{22} .$$

- Calculate $\alpha_{J_\pm}(J_3)$.
- Choose $\alpha_1 = \alpha_{J_+}$ as the basis root, which is positive and simple. For $\alpha_1 \in H^*$, find the unique element $h_{\alpha_1} \in H$ such that

$$\alpha_1(h) = \mathcal{K}(h_{\alpha_1}, h) \qquad \forall\, h \in H.$$

  Solution: $h_{\alpha_1} = \frac{1}{2}J_3$
- Calculate the Killing form $\mathcal{K}(h_{\alpha_1}, h_{\alpha_1})$ and draw the Dykin diagram.