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Exercise 10
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Exercises on Elementary Particle Physics

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1. *Representations of SU(n) - part II*

   (a) We consider a representation ρ of SU(n). The generators are denoted by $\rho(t_a)$. For elements of the Cartan subalgebra, we also may write $\rho(h)$. Follow from the commutator
   $$[\rho(t_a), \rho(t_b)] = i f_{abc} \rho(t_c)$$
   that also $-\rho(t_a)^*$ forms a representation, called the complex conjugate of ρ. We denote it by a bar, $\bar{\rho}$. ρ is said to be a real representation if it is equivalent to its complex conjugate.

   (b) Show that if M^i is a weight in ρ, $-M^i$ is a weight in $\bar{\rho}$.

   Hint: Use the fact that the Cartan generators are hermitian and the definitions of Ex.9.2.

   (c) As an example, consider SU(3). Draw the weights of the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ (see Ex.9.2(d)). Confirm that the highest weight of the representation $\bar{\mathbf{3}}$ is $(0, 1)$ in Dynkin coefficients.

2. *Symmetry Breaking and Branching Rules*

The basic idea of Grand Unification is that the Standard Model gauge group can be embedded in a simple group, e.g. SU(5). From a mathematical point of view, the problem is to determine the subalgebras of the simple group under consideration.

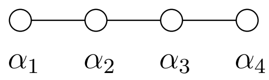

Abbildung 1: Dynkin diagram of SU(5).

Dynkin's Symmetry Breaking

To each simple root one assigns an integer number, called the Kac-label a_i. They are given as the coefficients of the decomposition of the highest root in the basis of simple roots. Deleting any node with Kac-label $a_i = 1$ from the Dynkin diagram gives a maximal regular subalgebra plus a U(1) factor.

(a) In the case of SU(5), all Kac-labels are 1. Apply Dynkin's rule to find the symmetry breaking yielding the Standard model gauge group, i.e.

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

(b) The lowest dimensional representation of SU(5) is given by the highest weight $(1, 0, 0, 0)$. Using the highest weight construction, calculate the weights in this representation.

(c) The irreducible representation corresponding to the highest weight $(1, 0, 0, 0)$ of SU(5) is a reducible representation of SU(3)×SU(2). In part (a) you have learned that α_1 and α_2 correspond to SU(3) and α_4 corresponds to SU(2). Consequently, every weight λ decomposes into

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \to (\lambda_1, \lambda_2|\lambda_4) = (\mu|\nu).$$

As a first step, write down all weights $(\mu|\nu)$. Next, find the highest weight μ. Determine the weights and the dimension of the corresponding representation. Consider now the values of ν belonging to this μ-representation. What is the dimension of the ν-representation? Repeat these steps starting with the highest weight ν. The result reads

$$\mathbf{5} \to (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}).$$

(d) Repeat the analysis for the representation corresponding to the highest weight $(0, 1, 0, 0)$. The result reads

$$\mathbf{10} \to (\mathbf{1}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}).$$

Note: All weights which appear in the calculation have multiplicity 1.

(e) (Optional!) Repeat the analysis for the representation corresponding to the highest weight $(1, 0, 0, 1)$. The result reads

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\bar{\mathbf{3}}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}).$$

Note: All weights which appear in the calculation have multiplicity 1, except for $(0, 0, 0, 0)$ in $\mathbf{24}$ of SU(5) with multiplicity 4, and $(0, 0)$ in $\mathbf{8}$ of SU(3) with multiplicity 2.

3. *Spontaneous Symmetry Breaking in SU*(5)

We want to describe the SU(5) breaking by introducing a Higgs field in the adjoint, denoted as a 5×5 hermitian traceless matrix. (Note that this is not the Higgs field of the Standard Model, which would reside in the $\bar{\mathbf{5}}$.) Then we can write the scalar potential in the form

$$V(H) = -m^2 \mathrm{Tr}\left(H^2\right) + \lambda_1 \left(\mathrm{Tr}\left(H^2\right)\right)^2 + \lambda_2 \mathrm{Tr}\left(H^4\right),$$

where we have imposed a symmetry $H \to -H$ to remove a cubic term for simplicity.

(a) Show that H can be transformed into a real diagonal traceless matrix

$$H = U H_d U^\dagger \qquad \text{with} \quad H_d := \mathrm{diag}(h_1, h_2, h_3, h_4, h_5).$$

Hint: Use the SU(5) transformation property $H \to H' = U H U^\dagger$

(b) Find that at the minimum of the scalar potential all h_i's satisfy the same cubic equation.

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2.$$

where μ is a Lagrange multiplier which accounts for the constraint $\sum_i h_i = 0$.

This means that the vacuum expectation values of the h_i's can at most take on three different values ϕ_1, ϕ_2, ϕ_3. Let n_1, n_2, n_3 be the number of times ϕ_1, ϕ_2, ϕ_3 appear in $\langle H_d \rangle$:

$$\langle H_d \rangle := \mathrm{diag}(\phi_1, .., \phi_2, .., \phi_3) \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0.$$

Next, we want to determine the symmetry breaking due to the vev of the Higgs.

(c) First, consider the case of a Higgs boson H in the fundamental representation $\mathbf{n}$ of a gauge group G (e.g. $\mathbf{2}$ of SU(2), $\mathbf{3}$ of SU(3) etc). We write the Higgs field as a vector

$$H = (h_1, \ldots, h_n)^T$$

with n entries and the generators T^a of G as $n \times n$ matrices. Assume that the Higgs H acquires a vev $\langle H \rangle$ due to some potential.

Show that from the kinetic term of the Higgs,

$$(D_\mu H)^\dagger (D^\mu H) = \left(\partial_\mu H + igT^a A^a_\mu H\right)^\dagger \left(\partial^\mu H + igT^b A^{b\mu} H\right),$$

follows that a gauge boson A^a_μ is massless, if $T^a \langle H \rangle = 0$. Therefore, T^a belongs to the resulting gauge group G'.

(d) Next, consider the case of a Higgs boson H in the adjoint representation with the kinetic term $\mathrm{Tr}(D_\mu H)^\dagger (D^\mu H)$. Follow the discussion of part (c) and deduce that

$$\begin{array}{lll} T^a \in G' & \text{if} & [T^a, \langle H \rangle] = 0 \\ T^a \notin G' & \text{if} & [T^a, \langle H \rangle] \neq 0 \end{array}$$

Discuss also the case, where some linear combination of generators commutes with the Higgs vev!

Back to the SU(5) case, the result of part (d) implies that the most general form of symmetry breaking is

$$\mathrm{SU}(5) \to \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{SU}(n_3)$$

as well as additional U(1) factors which leave $\langle H_d \rangle$ invariant. It turns out that, depending on the relative magnitude of the parameters λ_1 and λ_2, the combinations $(3, 2, 0)$ or $(4, 1, 0)$ for (n_1, n_2, n_3) minimize the potential. Thus,

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

or

$$\mathrm{SU}(5) \to \mathrm{SU}(4) \times \mathrm{U}(1),$$

which would give restrictions on phenomenologically desirable values of λ_1, λ_2.