Physikalisches Institut | Exercise 11
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Exercises on Elementary Particle Physics

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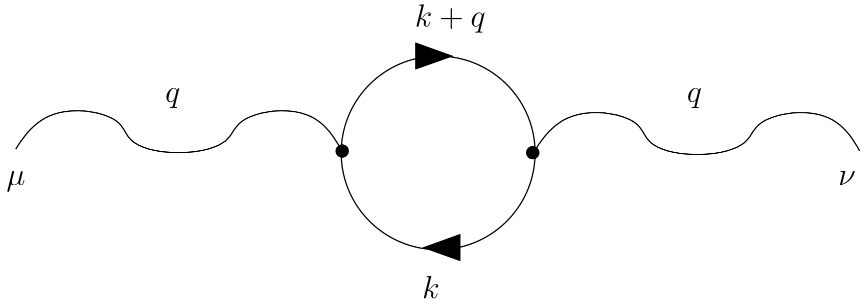
1. *Renormalization of the Electric Charge in QED - part I*

   We calculate loop corrections to the photon propagator in QED due to the vacuum polarization diagram. We will see that the correction can be interpreted as a renormalization effect on the electric charge, the QED coupling constant. The vacuum polarization diagram is given by the (amputated) Feynman diagram



   (a) Write down the matrix element $i\Pi^{\mu\nu}(q)$ for this process. Use the QED Feynman rules from Ex.6.2 plus the additional Feynman rules:

Propagator of fermions with momentum q	$i\frac{\not{q}+m}{q^2-m^2}$
Loop momentum k	$\int\frac{d^4k}{(2\pi)^4}$
Fermion loop	$\cdot(-1)$

   You will find

$$i\Pi^{\mu\nu}(p) = -e^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left(\gamma^\mu\frac{\not{k}+m}{k^2-m^2+i\epsilon}\gamma^\nu\frac{\not{k}+\not{q}+m}{(k+q)^2-m^2+i\epsilon}\right) \qquad (1)$$

   (Hint: The trace comes from the contraction of the spinor indices of the γ-matrices. $i\epsilon$ is added by hand to avoid an infinity when $k^2=m^2$ or $(k+q)^2=m^2$.)

   (b) Use the trace theorems for γ matrices from Ex.2.3 to simplify the numerator of eqn. (1). (Note that the denominator is a scalar with respect to the trace.) The result is

$$4\left\{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k_\rho(k+q)^\rho - m^2)\right\}.$$

(c) Prove the so-called Feynman trick:

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[xa + (1-x)b]^2}$$

Hint: Solve the integral by variable substitution $y = (a-b)x + b$.

(d) Use the Feynman trick to combine the two denominators of eqn. (1). The result reads

$$\int_0^1 dx\ \frac{1}{\left[\ell^2 + x(1-x)\,q^2 - m^2 + i\epsilon\right]^2}$$

where $\ell = k + xq$.

(e) Shift the integration variable from an integration over k to an integration over ℓ and argue that you can drop all terms linear in ℓ. The result is:

$$i\Pi^{\mu\nu}(q) = -4e^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{2\ell^\mu\ell^\nu + 2x^2 q^\mu q^\nu - 2xq^\mu q^\nu - g^{\mu\nu}\ell^2 - g^{\mu\nu}(x^2q^2 - xq^2 - m^2)}{\left(\ell^2 - \Delta + i\epsilon\right)^2} \tag{2}$$

where $\Delta = m^2 - x(1-x)q^2$.

(f) In QED one can prove that, due to the gauge symmetry, all terms proportional to q^μ or q^ν vanish in every S-matrix calculation. Drop the corresponding term from your result. (The proof makes use of the so-called *Ward identity* of QED.)

(g) Show that

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu\ell^\nu}{f(\ell^2)} = \frac{1}{4}\int \frac{d^4\ell}{(2\pi)^4} g^{\mu\nu} \frac{\ell^2}{f(\ell^2)}$$

(h) Recall that $\ell^2 = (\ell^0)^2 - (\ell^i)^2$. Therefore, the integral of eqn. (2) has to be performed in Minkowski Space. It is much more convenient to perform such integrals in 4-dim Euclidean space. To do so, one has to perform a Wick rotation:

i. View ℓ^0 as a complex variable. Draw the complex ℓ^0-plane. The integration is along the real axis. Mark the positions of the poles of eqn. (2)

ii. Use Cauchy's integral theorem to argue that the integral from $-i\infty$ to $+i\infty$ is equal to the integral from $-\infty$ to $+\infty$.

iii. So we define new (euclidean) coordinates: $l^0 = in^0$ and $l^i = n^i$ and rewrite the integral in terms of n^μ. At the end, rename n^μ to ℓ^μ.

iv. Now we can set $\epsilon \to 0$, because there is no divergence on the path of integration.

The result should read:

$$i\Pi^{\mu\nu}(q) = -4ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{\frac{1}{2}g^{\mu\nu}\ell^2 + g^{\mu\nu}x(1-x)q^2 + g^{\mu\nu}m^2}{\left(\ell^2 + \Delta\right)^2} \tag{3}$$