Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 5
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General Relativity

Prof. Dr. H.-P. Nilles

1. *Geodesics of S^2*

On exercise sheet 3 (problem 2) we showed that the trajectories of a freely moving particle in a gravitational field are the geodesics of the curved spacetime. Therefore let's compute the geodesics of S^2!

(a) Write the equations for geodesics of S^2 (equations of motion for a free particle on a sphere of fixed radius $R = 1$):

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{1a}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0\,. \tag{1b}$$

(Hint: On exercise sheet 3 (problem 3) we computed the non-vanishing Christoffel symbols for spherical coordinates:

$$\begin{array}{lllllllll} \Gamma^1{}_{22} & = & -r\,, & \Gamma^2{}_{12} & = & \frac{1}{r}\,, & \Gamma^3{}_{13} & = & \frac{1}{r}\,, \\ \Gamma^1{}_{33} & = & -r\sin^2\theta\,, & \Gamma^2{}_{33} & = & -\sin\theta\cos\theta\,, & \Gamma^3{}_{23} & = & \cot\theta\,.\,) \end{array}$$

(b) Let $\theta = \theta(\phi)$ be the equation of the geodesic. Show that the two equations of (1) lead to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2} - 2\cot\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 - \sin\theta\cos\theta = 0\,. \tag{2}$$

(c) Substitute $f(\theta) = \cot\theta$ and write (2) as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\phi^2} + f = 0\,. \tag{3}$$

(d) What is the general solution of (3) in spherical coordinates? Show that the solution can be rewritten in cartesian coordinates as

$$z = \alpha\, x + \beta\, y\,, \qquad x^2 + y^2 + z^2 = 1\,,$$

where α and β are suitably chosen constants.
What form do the trajectories of a free particle on a sphere take?

2. *Riemann Tensor*

The Christoffel symbols are *not* tensors, and thus are not suitable to describe a curved geometry in a coordinate-invariant way. The only tensor that can be constructed from the metric and its first and second derivatives is the *Riemann tensor*

$$R^{\sigma}{}_{\mu\nu\lambda} = \frac{\partial \Gamma^{\sigma}{}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\sigma}{}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\kappa}{}_{\mu\nu}\Gamma^{\sigma}{}_{\kappa\lambda} - \Gamma^{\kappa}{}_{\mu\lambda}\Gamma^{\sigma}{}_{\kappa\nu} \,. \tag{4}$$

Through self-contractions we get the *Ricci tensor* $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$ and the *curvature scalar* $R \equiv g^{\mu\kappa} R_{\mu\kappa}$.

(a) Using the metric, the Riemann tensor can be made fully covariant (for details see [Weinberg], p.141):

$$R_{\sigma\mu\nu\lambda} = g_{\sigma\rho} R^{\rho}{}_{\mu\nu\lambda} = \frac{1}{2}\left(\frac{\partial^2 g_{\sigma\nu}}{\partial x^{\mu}\partial x^{\lambda}} + \frac{\partial^2 g_{\mu\lambda}}{\partial x^{\sigma}\partial x^{\nu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma}\partial x^{\lambda}} - \frac{\partial^2 g_{\sigma\lambda}}{\partial x^{\mu}\partial x^{\nu}}\right) \tag{5}$$
$$+ g_{\alpha\beta}\left(\Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\beta}{}_{\lambda\mu} - \Gamma^{\alpha}{}_{\sigma\lambda}\Gamma^{\beta}{}_{\mu\nu}\right) \,.$$

Check the symmetry properties $R_{\sigma\mu\,\nu\lambda} = -R_{\sigma\mu\,\lambda\nu}$, $R_{\sigma\mu\,\nu\lambda} = -R_{\mu\sigma\,\lambda\nu}$, $R_{\sigma\mu\,\nu\lambda} = +R_{\nu\lambda\,\sigma\mu}$, and $R_{\sigma\mu\nu\lambda} + R_{\sigma\nu\lambda\mu} + R_{\sigma\lambda\mu\nu} = 0$.

(b) Show that the number of independent components of the Riemann tensor $R_{\sigma\mu\nu\lambda}$ is $\frac{1}{12}N^2(N^2-1)$ for $N \geq 4$ and $\frac{1}{8}N(N-1)(N^2-N+2)$ otherwise. How many independent curvature tensor components are there for $N \leq 4$?

(c) Calculate the components of $R^{\ell}{}_{mnk}$, R_{mk} and the curvature scalar R for a space with coordinates (θ, ϕ) and metric $g_{mn} = diag(a^2, a^2 \sin^2\theta)$.
(Use again the Christoffel symbols from 1.(a)!)

3. *Bianchi Identities*

(a) In the previous problem we already proved the first *Bianchi identities* $R_{\sigma\mu\nu\lambda} + R_{\sigma\nu\lambda\mu} + R_{\sigma\lambda\mu\nu} = 0$. Now verify the second *Bianchi identities*

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \,, \tag{6}$$

where $X_{;\nu}$ denotes the covariant derivative

$$X^{\mu\cdots}{}_{\nu\cdots;\rho} = \frac{\partial}{\partial x^{\rho}} X^{\mu\cdots}{}_{\nu\cdots} + \Gamma^{\mu}{}_{\rho\sigma} X^{\sigma\cdots}{}_{\nu\cdots} + \ldots - \Gamma^{\sigma}{}_{\nu\rho} X^{\mu\cdots}{}_{\sigma\cdots} - \ldots \,. \tag{7}$$

Use the fact that (6) is explicitly covariant and work in a locally inertial system where the Γ*s (but not their derivatives) vanish.*

(b) Use (b) to contract the indices in (6) multiple times to arrive at

$$(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{;\mu} = 0 \,. \tag{8}$$

What does this imply for energy-momentum conservation in General Relativity?
(Hint: Since exercise sheet 4 (problem 1.(i)) we always demand $g_{\mu\nu;\eta}$*.)*