Universität Bonn
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Theoretische Physik

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Example sheet 8
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General Relativity

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1. *Schwarzschild solution (Episode II)*

The metric outside a spherical symmetric mass distribution can be written in the so-called *standard form*

$$ds^2 = B(r)\, dt^2 - A(r)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{1}$$

On exercise sheet 6 (problem 2) we already computed the components of the Ricci tensor for this metric. We will now try to determine $A(r)$ and $B(r)$ from the Einstein equations.

Reminder: (non-vanishing $R_{\mu\nu}$)

$$\begin{aligned} R_{tt} &= -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} & \qquad R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \\ R_{rr} &= \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} & \qquad R_{\theta\theta} &= -1 - \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) + \frac{1}{A} \end{aligned}$$

(a) What are the Einstein field equations for empty space?
Use them to solve for $A(r)$ and $B(r)$.
*(Hints: Calculate $R_{rr}/A + R_{tt}/B$ to get a relation between A and B.
What form does $g_{\mu\nu}$ take for $r \to \infty$?)*

(b) Fix the remaining integration constant by using the Newtonian limit for the metric $g_{tt} = 1 - 2GM/r$. Why is only g_{tt} affected by Newtonian considerations?

(c) Write the Schwarzschild line element ds^2 in the standard form.
Use the substitution $r = \rho\left(1 + \frac{GM}{2\rho}\right)^2$ to write ds^2 in its *isotropic* form.

(d) Discuss the standard form. What can you say about singularities?
The *Schwarzschild radius* r_S is defined to be $r_S \equiv 2GM$. Why are stars with $R \leq r_S$ called *black holes*?

(e) Is the deviation from the Minkowski metric large for the gravitational field of the sun?
(Hint: $R_\odot \approx 7 \cdot 10^5\, km\,,\ \ r_{S,\odot} \approx 3\, km\,.$)

(f)* To further discuss the geometric properties of the Schwarzschild-metric it is convenient to consider an appropriately chosen surface, embedded in $\mathbb{R}^3$.
Write down the Schwarzschild metric on the plane $\theta = \pi/2$, $t =$ const. and determine a two dimensional surface in $\mathbb{R}^3$ which has the same metric.
(Hint: Parametrise the surface in cylindrical coordinates!)
Discuss your result.

First in 1960 M.D. Kruskal showed that it is possible to define a coordinate system (Kruskal coordinates), which covers the whole manifold.