Universität Bonn
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Theoretische Physik

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Example sheet 9
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General Relativity

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1. *Motion in Schwarzschild solutions*

(a) Write the equations of motion for the Schwarzschild solution.
(Use the Christoffels from exercise sheet 6 with general A, B.)

(b) Set $\theta = \pi/2$ (why can this be done?), and integrate suitably to get

$$\frac{dt}{dp} = \frac{1}{B(r)}\,, \qquad r^2\,\frac{d\phi}{dp} = J = const\,. \tag{1}$$

(Multiply the equations for ϕ and t in (a) with $dp/d\phi$ and dp/dt respectively.)
The integral of the equation for r should now give

$$A(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E = const\,, \tag{2}$$

where p is the parameter along the worldline.

(c) Show that $d\tau^2 = E\,dp^2$, and that therefore $E = 0$ must hold for photons, while $E > 0$ for other matter.

(d) Eliminate dp from the integrals of motion obtained in (b) to get a relation between r and ϕ. Show that

$$\phi = \pm \int \frac{\sqrt{A(r)}\,dr}{r^2\sqrt{\frac{1}{J^2\,B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}} \tag{3}$$

is a solution.

2. *Precession of perihelia*

In ***1.*** we showed that the shape of Schwarzschild trajectories is given by

$$\phi = \pm \int \frac{\sqrt{A(r)}\mathrm{d}r}{r^2\sqrt{\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}}\,, \tag{4}$$

where E and J^2 are constants of the motion. In weak fields we can expand

$$A(r) = 1 + \frac{2MG}{r}\,, \quad B(r) = 1 - \frac{2MG}{r}\,.$$

(a) Determine E and J^2 by looking at the aphelion $r = r_+$ and perihelion $r = r_-$ of a planet in bound orbit around the sun, for general $B(r)$. *(At $r_\pm$, $dr/d\phi$ vanishes.)*

(b) Show that the amount of orbital precession per revolution is

$$\Delta\phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi \ ,$$

where $\phi(r_+) - \phi(r_-)$

$$= \int_{r_-}^{r_+} \left[\frac{r_-^2 (B^{-1}(r) - B^{-1}(r_-)) - r_+^2 (B^{-1}(r) - B^{-1}(r_+))}{r_+^2 r_-^2 (B^{-1}(r_+) - B^{-1}(r_-))} - \frac{1}{r^2} \right]^{-1/2} \times \frac{\sqrt{A(r)}\mathrm{d}r}{r^2} \ . \tag{5}$$

(c) Show that for weak fields we can use

$$B^{-1}(r) = 1 + \frac{2MG}{r} + \frac{4M^2G^2}{r^2} \ ,$$

which makes the first term in Eq. (5) quadratic in $1/r$ and that we can then write

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} \left[C \left(\frac{1}{r_-} - \frac{1}{r} \right) \left(\frac{1}{r} - \frac{1}{r_+} \right) \right]^{-1/2} \times \frac{\sqrt{A(r)}\mathrm{d}r}{r^2} \ . \tag{6}$$

(d) Determine C in the limit $r \to \infty$. You should get

$$C = 1 - \frac{4MG}{L} + \ldots \ ,$$

where $\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-} \right)$. Show that Eq. (5) now reduces to

$$\phi(r_+) - \phi(r_-) = \left(1 + \frac{2MG}{L} \right) \times \int_{r_-}^{r_+} \frac{(1 + \frac{MG}{r})\mathrm{d}r}{r^2 \sqrt{\left(\frac{1}{r_-} - \frac{1}{r} \right) \left(\frac{1}{r} - \frac{1}{r_+} \right)}} \ . \tag{7}$$

(e) Calculate $\Delta\phi$. *(You can approximate the result of the integral with $(1 + \frac{MG}{L})\pi$.)*

(f) Determine the total precession $\Delta\phi$ for Mercury over the time of a century. (415 revolutions per century; $L = 55.3 \times 10^9$m; $MG = 1475$m).
The observed value is (43.11 ± 0.45) arcseconds.