Universität Bonn
Physikalisches Institut
Theoretische Physik

Winter term 2006/07
Example sheet 11
2007-01-15

General Relativity

Prof. Dr. H.-P. Nilles

1. *Friedmann models*

Using Einsteins field equations we will derive models to describe the time-evolution of the universe. Due to the *cosmic principle* of an spatially *homogeneous* and *isotropic* universe we can use the Robertson-Walker metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left\{\frac{\mathrm{d}r^2}{1-\alpha r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right\} \ . \tag{1}$$

The matter distribution (galaxies) of the universe can be viewed as a perfect fluid and we can use (see exercise 7)

$$T^{\mu\nu} = (p+\rho)\,U^\mu U^\nu + p\,g^{\mu\nu} \ . \tag{2}$$

(a) Verify that geodesics of the space, defined by Eq. (1), are given by $x^i = const.$. Galaxies move according to $x^i = const. \Leftrightarrow U^i = 0$. *(Hint: Use* ${\Gamma^\mu}_{00} = 0$*.)*

(b) What can you say about the matter density $\rho(r,t)$ and the pressure $p(r,t)$, using the cosmological principle?

(c) Show that energy-momentum conservation $D_\nu T^{\mu\nu} = 0$ leads to the hydrodynamic equation

$$g^{\mu\nu}\partial_\nu p + g^{-1/2}\partial_\nu[g^{1/2}(\rho+p)U^\mu U^\nu] + {\Gamma^\mu}_{\nu\lambda}(\rho+p)U^\nu U^\lambda = 0 \ . \tag{3}$$

(d) Show that for the Robertson-Walker metric Eq. (3) is trivially fulfilled for $\mu = r, \theta, \phi$ and for $\mu = t$ leads to

$$a(t)^2\frac{\mathrm{d}p(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[a(t)^3(\rho(t)+p(t))] \ . \tag{4}$$

(e) Calculate $T^{\mu\nu}$ and ${T^\mu}_\mu$ and write down the Einstein field equations

$$3\ddot{a} = -4\pi G(\rho+3p)a \ , \tag{5}$$

$$\ddot{a}a + 2\dot{a}^2 + 2\alpha = 4\pi G(\rho-p)a^2 \ . \tag{6}$$

Hint: Use (a) and (exercise 10, problem 2)

$$\begin{array}{rclrcl} R_{00} & = & 3\frac{\ddot{a}}{a} \ , & R_{11} & = & -\frac{2\alpha+2\dot{a}^2+\ddot{a}a}{1-\alpha r^2} \\ R_{22} & = & -(2\alpha+2\dot{a}^2+\ddot{a}a)r^2 \ , & R_{33} & = & -(2\alpha+2\dot{a}^2+\ddot{a}a)r^2\sin^2\theta \ . \end{array}$$

(The above components of the Ricci tensor have the correct sign, compared with our first definition on exercise sheet 5, problem 2.)

(f) Why do we need a third equation to solve for $a(t)$?

(g) Eliminate $\ddot{a}$ from the first equation, using the second one and derive the first order differential equation for $a(t)$

$$\dot{a}^2 + \alpha = \frac{8\pi G}{3}\rho a^2 \ . \tag{7}$$

(h) Use the *equation of state* $p = p(\rho)$ for the two cases of a

$$\begin{array}{rll} (i) & \text{matter dominated universe:} & p = 0 \ , \\ (ii) & \text{radiation dominated universe:} & p = \frac{\rho}{3} \end{array}$$

and show how ρ depends on $a(t)$. *(Hint: Use Eq. (4).)*

Knowing ρ as a function of $a(t)$, we can determine $a(t)$ for all time by solving Eq. (7). The fundamental equations of dynamical cosmology are thus the *Einstein equations* Eq. (7), the *energy-conservation* equation Eq. (4) and the *equation of state*. The cosmological models, based on a Robertson-Walker metric, in which $a(t)$ is derived in this way, are known as *Friedmann models*.