Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 12
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General Relativity

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1. *Age of the universe*

Consider the Einstein equations for the Robertson-Walker metric:

$$\frac{\dot{a}^2}{a^2} + \frac{\alpha}{a^2} = \frac{8\pi G}{3}\rho \ , \tag{1}$$

$$-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\alpha}{a^2} = 8\pi G p \ . \tag{2}$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + 3pa^2\dot{a} = 0 \ , \tag{3}$$

$$-\frac{4\pi}{3}(\rho + 3p)G = \frac{\ddot{a}}{a} \ , \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^{3(1+w)}) = 0 \ , \tag{5}$$

where w in the last relation is defined by the equation of state, $p = w\rho$.

(b) Discuss the evolution of the scale factor a for $p = \rho/3$ and $\alpha = 0, \pm 1$ with initial conditions $\lim_{t\to 0} a(t) = 0\,,\ \lim_{t\to 0} \dot{a}(t) > 0$.

(c) Discuss the evolution of the scale factor a for $p = 0$ and $\alpha = 0, \pm 1$ with initial conditions $\lim_{t\to 0} a(t) = 0\,,\ \lim_{t\to 0} \dot{a}(t) > 0$.

(d) Calculate the age of the universe for $\alpha = 0$ and (i) $p = \rho/3$ and (ii) $p = 0$.
Hint: The following integrals may be useful:

$$\begin{aligned} \int dx \tfrac{\sqrt{x}}{\sqrt{b+x}} &= \sqrt{x(b+x)} - b \ln\left(\sqrt{x} + \sqrt{b+x}\right) \ , \\ \int dx \tfrac{\sqrt{x}}{\sqrt{b-x}} &= -\sqrt{x(b-x)} + b \arctan\left(\tfrac{\sqrt{x}}{\sqrt{b-x}}\right) \ , \qquad 0 < x < b \ . \end{aligned}$$

The present day Hubble expansion rate is

$$H_0 = \left.\frac{\dot{a}}{a}\right|_{\text{today}} \simeq 0.073\,\mathrm{Gyr}^{-1} \ . \tag{6}$$