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Introduction

The investigation of the symmetries of physical systems has been a guiding principle in modern physics. Symmetries help us to find conserved quantities in classical systems or to determine properties of the spectra of quantum mechanical systems. In quantum field theory (QFT) Poincaré invariance plays a fundamental role. Additional symmetries of particular importance are gauge symmetries or supersymmetry. These symmetries restrict the possible particle content of a QFT and can be used to determine some properties of the particles.

Many quantum field theories make their predictions by using perturbation theory. However, studying systems with large symmetry groups sometimes leads to the possibility of obtaining nonperturbative results. Nonperturbative aspects of QFT could mainly be studied in low-dimensional space-time. But see [25] for a recent nonperturbative result in four dimensions, to which physicists paid a lot of attention. A prominent example of theories with an infinite dimensional symmetry algebra are conformal field theories (CFT) in two dimensions. These models have fruitful applications in both mathematics and physics. They have been studied extensively.

In this thesis we will study theories with a different very high symmetry, which are in some sense orthogonal to conformal field theory as we will see. We will look at theories which are invariant under volume-preserving diffeomorphisms of space-time. Our discussion will mainly concern the two-dimensional case, i.e. two-dimensional theories which are invariant under area-preserving diffeomorphisms. We call these theories volume-preserving/area-preserving field theories.

As an example, we will study two-dimensional Yang-Mills theories. This model has been discussed in recent years for several reasons. Some mathematical papers study the cohomology of the moduli space of flat connections on a Riemann surface [28]. Other papers discuss for gauge group $U(N)$ an expansion of the partition function as a function of $\frac{1}{N}$ [18],[19]. This expansion has an interpretation in terms of strings. There are a lot of other papers on different aspects of two-dimensional Yang-Mills, e.g. [14], [15], [12], [5]. In this thesis we will concentrate on aspects which are directly connected with the area-preserving symmetry.

We will see that area-preserving QFT's are closely connected to topological field theories in two dimensions. Topological field theories are theories which are invariant under all diffeomorphisms of space-time. The main applications of topological field theories have been in mathematics (e.g. knot theory) [31]. We can get area-preserving theories by perturbing a topological theory in a suitable way.

This thesis is organized as follows: In Chapter 1 we give definitions for topological, area-preserving and conformal field theory. In Chapter 2 we work out the two-dimensional Yang-Mills example. Chapter 3 gives some more examples. Finally, Chapter 4 discusses diffeomorphisms, especially area-preserving diffeomorphisms in two-dimensional Yang-Mills theory.

Chapter 1

A Functorial Approach to Euclidean QFT

Following the functorial approach of Segal [23], [24] and Atiyah [1] [2] we want to give an axiomatic definition of a QFT. To motivate this definition we first look at some features of QFT in a less abstract and more intuitive language.

In QFT we are interested in calculating the path integral $\int D\phi e^{-S(\phi)}$, where S is the action and ϕ stand for the fields in the theory, which live on a certain space-time M. We want to assume that M is a $d+1$-dimensional compact manifold.

For a closed manifold the path-integral gives a (complex) number, independent of fields. For manifolds with boundary we have to fix the fields at the boundary (impose boundary conditions). The resulting path-integral can be seen as a functional of boundary conditions/fields at the boundary. This might lead us to see it as an element of the Hilbert space $\mathcal{H}_B$ obtained by canonical quantization near the boundary B, i.e. on a collar $B \times I$. We can also say that we associate to the boundary a Hilbert space (obtained by canonical quantization on $B \times I$) and to the manifold a state in the Hilbert space associated to its boundary.

If M has two disconnected boundary components B_1 and B_2 of codimension one we furthermore want to have the possibility to label them as "incoming" and "outgoing".

To each of the two boundary components we associate a vector space of quantum states. The space-time manifold M is interpolating between the two boundary components B_1 and B_2. We want to associate to M a linear operator

$$\Phi_M : \quad \mathcal{H}_{B_1} \longrightarrow \mathcal{H}_{B_2} \tag{1.1}$$

Let us describe the evolution operators Φ_M in terms of path-integrals: The classical fields of the theory are functions ϕ on the boundary B. A quantum state

$\psi(\phi)$ is a wave function defined on the classical fields. We can now write the evolution operator Φ_M schematically as:

$$(\Phi_M\psi)(\phi_2) = \int K_M(\phi_1,\phi_2)\psi(\phi_1)D\phi_1 \tag{1.2}$$

The kernel of the evolution operator equals the path-integral on M:

$$K_M(\phi_1,\phi_2) = \int e^{-S(\phi)}D\phi, \tag{1.3}$$

where the path-integral is over fixed boundary values: $\phi\mid_{B_1}=\phi_1, \phi\mid_{B_2}=\phi_2$. The operator Φ_M can be seen as a field-theoretic analogue of the time-evolution operator $U(t)$ in quantum mechanics. Quantum mechanics can be seen as a $0+1$-dimensional Minkowskian QFT. The time evolution operators $U(t)$ form a one-parameter group of unitary operators:

$$U(t_1+t_2) = U(t_1)\circ U(t_2) \tag{1.4}$$

We can conclude from this that there exists a Hermitian operator H such that:

$$U(t) = e^{-itH} \tag{1.5}$$

We want to demand a composition rule for the Φ_M in euclidean QFT similar to the quantum mechanical case. Therefore, we imagine that we cut the manifold M along a codimension one submanifold B into two halves M_1 and M_2.

On each of the two halves we can do the path-integral:

$$K_{M_1}(\phi_1,\phi') = \int e^{-S(\phi)}D\phi \tag{1.6}$$

with integration over fixed boundary values $\phi\mid_{B_1}=\phi_1, \phi\mid_B=\phi'$ and

$$K_{M_2}(\phi',\phi_2) = \int e^{-S(\phi)}D\phi \tag{1.7}$$

with integration over fixed boundary values $\phi\mid_B=\phi', \phi\mid_{B_2}=\phi_2$.
To calculate the path-integral over M, we have to integrate over all intermediate states ϕ':

$$K_M(\phi_1,\phi_2) = \int K_{M_1}(\phi_1,\phi')K_{M_2}(\phi',\phi_2)D\phi' \tag{1.8}$$

So we can motivate the existence of a composition law analogous to quantum mechanics:

$$\Phi_M = \Phi_{M_2}\circ\Phi_{M_1} \tag{1.9}$$

By now we have seen that a QFT can be seen as a map which associates Hilbert spaces to d-dimensional manifolds B and transition amplitudes or states to d+1-dimensional manifolds M. Furthermore, the transition amplitudes satisfy a composition law.

We will turn the observations made in this section using intuitive path-integral language into axioms. For this we want to use the language of categories and functors. These concepts will be introduced in the next section. After that we will give the promised axioms for some examples of QFT's.

1.1 Categories and Functors

We start with the definition of a category, which can be found in textbooks on topology, e.g. [20]:

Definition: A category $\mathcal{C}$ consists of the following data:

1. a class $Ob(\mathcal{C})$ of mathematical objects which are called the objects of the theory.

2. a set $Mor(X,Y)$ for any ordered pair (X,Y) of objects. Its elements are called the morphisms of X to Y. Furthermore, the morphisms have to fulfill:

$$X \neq Y \quad \text{or} \quad X' \neq Y' \Rightarrow Mor(X,Y) \cap Mor(X',Y') = \emptyset$$

3. a composition

$$\begin{aligned} \circ : Mor(X,Y) \times Mor(Y,Z) &\rightarrow Mor(X,Z) \\ (f,g) &\mapsto f \circ g \end{aligned}$$

   for any ordered triple (X,Y,Z) of objects of $\mathcal{C}$.

Together these data form a category if the following axioms are fulfilled:

(a) associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for any triplet (h,g,f) in $Mor(X,Y) \times Mor(Y,Z) \times Mor(Z,W)$

(b) identities: For any object X there is an identity morphism $\mathbf{1}_X \in Mor(X,X)$ such that

$$\mathbf{1}_X \circ f = f \quad \text{and} \quad g \circ \mathbf{1}_X = g \quad \forall f \in Mor(Y,X)\, \forall g \in Mor(X,Z)$$

We will give some examples for categories:

- example 1: The category of finite dimensional complex vector spaces Vec
  1. finite dimensional complex vector spaces $\mathcal{H}$
  2. linear maps Φ
  3. the usual composition law of linear maps

- example 2
  1. differentiable manifolds
  2. differentiable maps
  3. usual composition law

It is clear that the axioms for a category are fulfilled in these examples. The set of differentiable manifolds can be made into a category in another way: We can use cobordisms as morphisms. The next two examples will be of this type. These examples will be of particular importance for us in the following chapters.

- example 3: The category of topological field theory $\mathcal{T}$ in dimension $d+1$

  - d-dimensional, differentiable, compact, oriented, parametrized closed manifolds B and disjoint unions
  - The elements of $Mor(B_1, B_2)$ are equivalence classes of cobordant d+1-dimensional manifolds M with identifications i

$$i : B_2 \cup B_1^* \to \partial M \tag{1.10}$$

where B_1^* denotes B_1 with reversed orientation. (The orientation of M agrees with the orientation of B_2 and is reversed to that of B_1^*. We will also refer to these boundary orientations as "outgoing" and "incoming", respectively.) Two cobordisms are considered equivalent, if they differ by a diffeomorphism which respects the boundary parametrization.

$$(M, i) \sim (M', i') \Leftrightarrow \exists f : M \to M' \quad f \circ i = i' \tag{1.11}$$

  - The composition is given by sewing together cobordisms along a common boundary B.

$$\begin{aligned} M_1 \in Mor(B_1, B) &\quad\Leftrightarrow\quad \partial M_1 = B_1^* \cup B \\ M_2 \in Mor(B, B_2) &\quad\Leftrightarrow\quad \partial M_2 = B^* \cup B_2 \end{aligned}$$

sewing along B gives a new manifold M with boundary

$$\partial M = B_1^* \cup B_2 \Leftrightarrow M \in Mor(B_1, B_2)$$

We have to check the axioms:

(a) okay

(b) $\mathbf{1}_B = B \times I$ (the cylinders)

example 4: The category of volume preserving QFT $\mathcal{V}$

1. as in the previous example
2. as in the previous example, but with the following modification: On M a volume element ω is defined and the diffeomorphism f in the equivalence relation has to respect this:
$$(M,i) \sim (M',i') \Leftrightarrow \exists f: M \to M' \quad f \circ i = i' \quad f^*\omega' = \omega \tag{1.12}$$
3. as in the previous example
   (a) okay
   (b) There is no unit element. So $\mathcal{V}$ is not a category in a mathematically strict sense. But we can "come close" to $\mathbf{1}$ by letting the volumes of the cylinders shrink. $\mathbf{1}$ is the zero-volume limit of the cylinders.

The canonical thing to be defined next are the maps between categories preserving the structure of the category. These are called functors.

Definition (covariant functor): Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor
$$\mathcal{F}: \mathcal{C} \to \mathcal{D} \tag{1.13}$$
associates to an object X in $\mathcal{C}$ an object $\mathcal{F}(X)$ of $\mathcal{D}$ and to a morphism
$$f: X \to Y \quad \text{of}\, \mathcal{C} \tag{1.14}$$
a morphism $\mathcal{F}(f): \mathcal{F}(X) \to \mathcal{F}(Y)$ of $\mathcal{D}$ such that

1. $\mathcal{F}(\mathbf{1}_X) = \mathbf{1}_{\mathcal{F}(X)}$
2. $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

Definition (contravariant functor) : In this case $\mathcal{F}$ reverses the direction of the morphisms. So for
$$f: X \to Y \tag{1.15}$$
we have $\mathcal{F}(f): \mathcal{F}(Y) \to \mathcal{F}(X)$ with

1. $\mathcal{F}(\mathbf{1}_X) = \mathbf{1}_{\mathcal{F}(X)}$
2. $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$

An example for functors are "forgetful" functors. These go from a category with a particular structure to a category with less structure by simply forgetting the extra structure. For example we can map the category $\mathcal{V}$ to the category $\mathcal{T}$ by "forgetting" that the cobordisms in $\mathcal{V}$ have a volume. We will see more examples for functors in the following sections, because our definition of QFT's will make use of this concept.

1.2 Topological QFT

We are now ready to give a definition for a topological field theory. [1], [2]
Definition : A topological field theory is a contravariant functor from the category of topological field theory $\mathcal{T}$ into the category of finite dimensional vector spaces Vec (where the arrows are linear maps) subject to some further axioms:

- T1: if a d-dimensional manifold B consists of several disconnected boundary components the corresponding vector space is a tensor product:

$$B = B_1 \cup B_2 \Rightarrow \mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \tag{1.16}$$

- T2: If B^* denotes B with the reversed orientation we associate

$$\mathcal{H}_{B^*} = \mathcal{H}_B^* \tag{1.17}$$

  where $\mathcal{H}_B^*$ denotes the dual space to $\mathcal{H}_B$.

- T3: The space corresponding to the empty set are the complex numbers:

$$\mathcal{H}_\emptyset = \mathbf{C} \tag{1.18}$$

- T4: The collapsing axiom: If we glue two boundary components of a connected manifold M this corresponds to taking the trace in the category of vector spaces.

$$\Phi_M \to \mathrm{tr}\, \Phi_M \tag{1.19}$$

Let us make a few remarks on these axioms. The first additional axiom shows that the vector space associated to the empty set $B = \emptyset$ must be idempotent. So it must be $\mathbf{C}$ or zero. Associating $\mathbf{C}$ makes the theory nontrivial. This means that we associate a complex number to any closed d+1-dimensional manifold. This number is called the partition function Z and is an invariant of the manifold. It can be calculated by cutting a manifold into pieces and using functoriality to glue it together again. Functoriality means here that gluing together two manifolds along a common boundary corresponds to a composition of the two maps associated by the functor.

$$M = M_1 \cup M_2 \Rightarrow \Phi_M = \Phi_{M_2} \circ \Phi_{M_1} \tag{1.20}$$

A manifold with only one boundary component corresponds to a state in the vector space associated to its own boundary.

$$\partial M = B \Rightarrow \Phi_M : \mathbf{C} \to \mathcal{H}_B \tag{1.21}$$

If the orientation of B is reversed, we obtain an element of the dual vector space.

$$\partial M = B^* \Rightarrow \Phi_M : \mathcal{H}_B \to \mathbf{C} \tag{1.22}$$

So if we cut a closed manifold into two pieces we obtain a vector and an element of the dual space. In this case we can consider the composition rule of the functor ("gluing") as a duality pairing.

If we look at a manifold M with one boundary component B we can also see an action of the mapping class group of B: Suppose that we apply a diffeomorphism to B. How does the state Φ_M transform? We considered manifolds M as equivalent if they are related by a diffeomorphism which preserves the boundary structure. This means that we get the same state back if the diffeomorphism extends to a map $M \to M$. This will be the case, if the diffeomorphism applied to B can be continuously deformed to the identity, because then we can deform it over a collar $B \times I$. Otherwise, the state might transform. Thus, we obtain a representation of the diffeomorphism group of B modulo diffeomorphisms which can be continuously deformed to the identity – in other words, a representation of the mapping class group Γ_B.

The axioms for a functor force that the cylinder $B \times I$ is mapped to the identity map in the vector space $\mathcal{H}_B$. We shall regard operators corresponding to cylinders as imaginary time evolution operators e^{-tH}, H the Hamiltonian. This means that $H = 0$ for a topological theory. There is no dynamics, and interesting effects arise only if the topology of the cobordism M is nontrivial. If we apply the collapsing axiom to the cylinder, we calculate the dimension of the vector space $\mathcal{H}_B$.

$$\mathrm{tr}\,\Phi_{B\times I} = \mathrm{tr}\,\mathbf{1} = \dim \mathcal{H}_B \tag{1.23}$$

This formula also indicates that difficulties arise if we want to admit infinite dimensional vector spaces.

So far we have only given a meaning to a reversion of the orientation of the boundary B. Of course, the orientation of the d+1-dimensional manifolds can also be reversed. We can see a relation between Φ_M and Φ_{M^*} if the vector spaces $\mathcal{H}_B$ possess a Hermitian form and a corresponding complex conjugation. Then we can add the following (optional) axiom

- T5: $\Phi_{M^*} = \Phi_M^\dagger$ (the adjoint of Φ_M)

We will discuss this structure in detail for the two dimensional case later on.

1.2.1 The Two-Dimensional Theory

We shall now describe a topological theory in 1+1 dimensions [8]. A lot of the above considerations simplify:

- The only compact connected one-dimensional manifold is the circle S^1 (up to diffeomorphisms). So we only have one vector space $\mathcal{H} = \mathcal{H}_{S^1}$ and can build all other spaces by taking tensor products. The two possible orientations of the circle correspond to taking the dual.

- All two-dimensional manifolds can be obtained by gluing together spheres with one, two and three holes.

So if we give the morphisms associated with these spheres we have determined the whole theory.
We start with the three-holed sphere: We choose our orientations such that two boundary components are incoming and one is outgoing. Then the three-holed sphere gives rise to a bilinear map:

$$c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \tag{1.24}$$

This makes $\mathcal{H}$ into an algebra. This structure allows us to identify states with operators. If we choose a basis $\phi_0 \ldots \phi_{N-1}$ of $\mathcal{H}$ we can calculate the coefficients ${c_{ij}}^k$ by calculating three-point functions on the sphere:

$${c_{ij}}^k = \langle \phi_i \phi_j \phi^k \rangle \tag{1.25}$$

$$\phi_i \, \phi_j = \sum_k c_{ij} \;{}^k \phi_k \tag{1.26}$$

Here, lower indices refer to incoming states, upper indices to outgoing states. Our algebra is commutative due to the permutation symmetry of the holes. It is furthermore associative because we can decompose the four-holed sphere in two different ways. The result must not depend on the particular way of cutting due to our axioms.

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{1.27}$$

Next we consider the two-holed sphere. If we label one boundary component as incoming and one as outgoing then we associate:

$$\Phi_{S^1 \times I} = \mathbf{1} \tag{1.28}$$

as above. If we label both boundaries as incoming the two-holed sphere gives rise to a bilinear form η on $\mathcal{H}$.

$$\eta : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbf{C} \tag{1.29}$$

The matrix elements can be calculated as:

$$\eta_{ij} = \langle \phi_i \phi_j \rangle \tag{1.30}$$

We can use this bilinear form to raise and lower indices, in other words to identify incoming with outgoing states. Similarly, we can label both boundary components as outgoing:

$$\eta^* : \mathbf{C} \rightarrow \mathcal{H} \otimes \mathcal{H} \tag{1.31}$$

By gluing these two cylinders to one cylinder with one incoming and one outgoing boundary component (i.e. the identity) we see that our bilinear form is nondegenerate.

If we glue η to the three-holed sphere we see that η and c have to be compatible in the sense that

$$\eta(c(x, y), z) = \eta(x, c(y, z)) \quad \forall x, y, z \in \mathcal{H} \tag{1.32}$$

Finally, we have to describe the one-holed sphere. If we label its puncture as outgoing it gives rise to a special state in $\mathcal{H}$ the "vacuum" $\mid 0 >$. If we fill the punctures of a manifold M by gluing disks to them we have:

$$\Phi_M \mid 0, ..., 0 >= Z(\overline{M}) \tag{1.33}$$

where $\overline{M}$ is the closed manifold obtained by removing the punctures and $Z(\overline{M})$ the partition function on $\overline{M}$. This equation explains why we call the disk the vacuum.

Given the above data (where η and c have to obey the compatibility conditions (1.32) and (1.27)) we can calculate all partition functions by cutting and gluing.

1.2.2 Hermitian Structures

In this section we will give our two-dimensional theory some more structure. So far we only had a bilinear form η on our vector space. Now we want to work out the consequences of axiom T5 in the case of a two-dimensional theory, so we have to assume that we also have a nondegenerate Hermitian form $< \quad , \quad >$. In this section $< \quad , \quad >$ does not necessarily have to be positive definite. In our theory we want to have that:

$$\eta(Cx, y) =< x, y > \quad \forall x, y \in \mathcal{H} \tag{1.34}$$

where C is a complex conjugation on states (i.e. an antiunitary involution). Equation (1.34) can for example be achieved if we demand an antilinear involution C on states and then define the Hermitian form by (1.34). If we want to start with an arbitrary Hermitian form on the state space we have to look for a relation between η and the Hermitian form. We know:

$$\forall x \in \mathcal{H} \quad \exists!\, \tilde{x} \in \mathcal{H}: \quad \eta(x, y) =< \tilde{x}, y > \forall y \in \mathcal{H} \tag{1.35}$$

$\tilde{x}$ is unique if $< \quad , \quad >$ is nondegenerate. We can define an antilinear operator A by:

$$Ax = \tilde{x} \tag{1.36}$$

or – if we have an antilinear involution on $\mathcal{H}$ – a linear operator B by

$$Bx = C\tilde{x} \tag{1.37}$$

This gives

$$\eta(x, y) =< CBx, y > \tag{1.38}$$

Ker $B = 0$ because η is nondegenerate. If $CB = BC$ we can redefine:

$$<< x, y >>=< Bx, y > \tag{1.39}$$

This is antilinear in x, linear in y and nondegenerate because Ker $B = 0$. If C is antiunitary with respect to $< \quad , \quad >$ it is also antiunitary with respect to the redefined form. Using the symmetry of η we can then also proof that

$$<< x, y >>= \overline{<< y, x >>} \tag{1.40}$$

Thus, our redefinition gives a Hermitian form fulfilling (1.34). We will work with this redefined form. A complex conjugation on states induces a complex conjugation on operators:

$$\Phi : \mathcal{H} \to \mathcal{H} \tag{1.41}$$

Since we know what $C(\Phi x)$ is and want a complex conjugation to fulfill:

$$C(\Phi x) = C(\Phi) Cx \tag{1.42}$$

it is natural to conjugate operators by:

$$C(\Phi) = C\Phi C \tag{1.43}$$

Generally, we will also denote a complex conjugation by a "bar": $\bar{x}, \bar{\Phi}$.... The complex conjugation on $\mathcal{H}$ gives us the possibility to give a relation between invariants $Z(M)$ and $Z(M^*)$ for a closed manifold M. $*$ denotes again a reversion of orientation. We demand:

$$Z(M) = \overline{Z(M^*)} \tag{1.44}$$

If we cut the manifolds M, M^* into two parts we obtain the rules which relate the states associated to manifolds with boundaries and different orientations.

$$\tag{1.45}$$

By cutting we obtain an element Φ_{M_1} of the vector space and an element φ of the dual such that we can calculate the partition function $Z(M) = \varphi(\Phi_{M_1})$ by duality pairing. With the help of our Hermitian form we can write the linear functional in terms of an element v of the vector space:

$$\varphi(\quad) =< v, \quad > \tag{1.46}$$

We can draw the picture of the cut manifold using the bilinear form η:

$$\tag{1.47}$$

Here we can calculate:

$$Z(M) = \eta(\Phi_{M_1}, \Phi_{M_2}) \tag{1.48}$$

By the relation between η and $< \quad , \quad >$ we have:

$$\eta(\Phi_{M_1}, \Phi_{M_2}) =< C\Phi_{M_2}, \Phi_{M_1} > \tag{1.49}$$

So that we see that the linear map in (1.46) is

$$\varphi(\quad) =< C\Phi_{M_2}, \quad > \tag{1.50}$$

We conclude: If we reverse the boundary of a manifold M_2 corresponding to a vector Φ_{M_2} we obtain the functional

$$< \bar{\Phi}_{M_2}, \quad > \tag{1.51}$$

Now we reverse the orientation of our whole manifold and repeat the cutting procedure:

$$\tag{1.52}$$

We demanded that the partition functions of M and M^* are related by a complex conjugation. This gives us:

$$Z(M^*) = \overline{< \bar{\Phi}_{M_2}, \Phi_{M_1} >} =< \Phi_{M_2}, \bar{\Phi}_{M_1} > \tag{1.53}$$

So if we reverse both the orientation of the manifold M_1 and its boundary (and thus obtain a vector again) we assign:

$$\Phi_{M_1} \to \bar{\Phi}_{M_1} \tag{1.54}$$

If we reverse the orientation of a manifold keeping the boundary orientation fixed (and thus obtain an element of the dual space) we assign:

$$\Phi_{M_2} \to < \Phi_{M_2}, \quad > \tag{1.55}$$

We can now ask what happens if we reverse the orientation of a manifold with an incoming and an outgoing boundary:

$$\partial M_3 = B_1^* \cup B_2, \quad \Phi_{M_3} : \mathcal{H}_{B_1} \to \mathcal{H}_{B_2} \tag{1.56}$$

Again we can look at a closed manifold:

and calculate the partition function of that manifold:

$$Z(M) =< \bar{\Phi}_{M_2}, \Phi_{M_3} \Phi_{M_1} > \tag{1.57}$$

Again we reverse the orientation of the whole manifold:

The partition function is:

$$Z(M^*) =< \Phi_{M_1}, \Phi_{M_3'} \bar{\Phi}_{M_2} > \tag{1.58}$$

where M_3' denotes M_3 with the reversed orientation while the boundary orientation is kept fixed. By (1.44) this is:

$$Z(M^*) = < \Phi_{M_3}\Phi_{M_1}, \bar{\Phi}_{M_2} > \tag{1.59}$$

So a reversion of the orientation of the manifold M_3 (keeping the boundary fixed) leads to the adjoint (see T5):

$$\Phi_{M_3'} : \mathcal{H}_{B_2} \to \mathcal{H}_{B_1} \tag{1.60}$$

$$\Phi_{M_3'} = \Phi_{M_3}^{\dagger} \tag{1.61}$$

What happens if we only reverse the boundary orientation of M_3(and obtain a manifold M_3'')? We can see this from the following picture:

$$Z(M) = < \bar{\Phi}_{M_1}, \Phi_{M_3''}\Phi_{M_2} > = < \bar{\Phi}_{M_3''}\bar{\Phi}_{M_2}, \Phi_{M_1} > \tag{1.62}$$

By comparing this to (1.57) we see:

$$\bar{\Phi}_{M_3''} = \Phi_{M_3}^{\dagger} \Rightarrow \Phi_{M_3''} = \Phi_{M_3}^{t} \tag{1.63}$$

So $\Phi_{M_3''}$ should correspond to the transpose with respect to η. This makes sense in pictures:

We have one more possibility to cut M^*:

From

$$Z(M^*) = < \Phi_{M_2}, \Phi_{M_3'''}\bar{\Phi}_{M_1} > \tag{1.64}$$

we can conclude that reversing both the orientation of the manifold and its boundary corresponds to a complex conjugation on operators.

Of course we can also give a Hermitian structure to higher dimensional topological theories. In this case we demand a Hermitian form and a corresponding complex conjugation on $\mathcal{H}$. As in two dimensions, we connect a complex conjugation with the reversion of an orientation.

1.3 Volume-Preserving Theories

We want to define a theory which is invariant under volume-preserving diffeomorphisms of space time. In analogy to the definition of topological field theory we make the following definition:
(Pre-) Definition : A volume preserving theory is a functor $\mathcal{F}^{vol}$ from the category of volume preserving field theory to the category of finite dimensional vector spaces subject to the axioms T1-T4. We will impose some further axioms.

How should the functor $\mathcal{F}^{vol}$ depend on the volume? By definition, we consider two manifolds as equivalent if they are related by a diffeomorphism f that relates the volume elements:

$$f^*\omega_1 = \omega_2 \tag{1.65}$$

We can conclude that the total volumes are equal:

$$\int \omega_1 = \int \omega_2 \tag{1.66}$$

It is a theorem of Moser that conversely, if there are two volume elements defined on a closed manifold, the existence of a diffeomorphism relating the volume elements in the above way follows from the equality of the total volumes. This motivates us to assign different maps to manifolds of different total volume v. To stress the volume dependence of our maps we shall write:

$$M \to \Phi_M^{v(M)} \tag{1.67}$$

Now we look at the composition law. If we cut M into two parts we get:

$$\Phi_M^{v(M)} = \Phi_{M_2}^{v(M_2)} \circ \Phi_{M_1}^{v(M_1)} \tag{1.68}$$

In order to make the theory volume invariant we have to add the following to this composition law: Suppose M_1' (resp.M_2') is topologically equivalent to M_1 (resp. M_2') but has a different volume $v(M_1')$ (resp. $v(M_2')$). If

$$v(M_1') + v(M_2') = v(M) \tag{1.69}$$

then the composition of the maps corresponding to M_1' and M_2' should again give Φ_M:

- V1: $\Phi_M^{v(M)} = \Phi_{M_2}^{v_2} \circ \Phi_{M_1}^{v_1} \quad \text{whenever} \quad v_1 + v_2 = v(M), \quad v_1,\ v_2 > 0$

We will also make some regularity assumptions on the volume dependence of Φ_M^v. It seems natural to impose that this dependence is continuous or better:

- V2: Φ_M^v is continuous in v.

In our new theory manifolds of the form $B \times I$ again correspond to propagators P_v, but this time the propagators have a volume v. They satisfy the composition law

$$P_{v_1+v_2} = P_{v_1} \circ P_{v_2} \tag{1.70}$$

This composition law looks "almost" like the composition law of a one-parameter-semigroup. The only thing that is missing is the identity element. By the semigroup law this would have to correspond to $v = 0$. This would correspond to a cylinder of volume zero in our geometrical picture. The zero-volume cylinder is not an element of our the category, but it makes sense in pictures that gluing a cylinder of volume zero to a manifold means performing the identity operation. So we impose the following axiom:

- V3: $\lim_{v\to 0} P_v = \mathbf{1}$

This gives the structure of a semigroup to the operators P_v, $v \in \mathbf{R}_0^+$. It enables us to see P_v again as a time evolution operator e^{-tH}. This is a motivation to define the Hamiltonian of the theory to be the generator of this semigroup:

- V4: $\frac{d}{dv} P_v \mid_{v=0} = H$

We see that the volume v has an interpretation as time t. Therefore, we will also use the notation t for volume in the following.

We can give our theory a Hermitian structure as in T5, see also section 1.2.2. Then we want to impose the reality condition:

- V5: $\overline{P_t} = P_t$

as will be motivated for the two-dimensional case later on. Altogether, we can give the following definition:
Definition: A volume preserving field theory is a functor from $\mathcal{V}$ to Vec subject to axioms T1-T4, V1-V4 (optional: T5, V5).

We will now look at a special class of theories. So far we have only made assumptions about the zero area limit of the cylinders. For the other operators, we have only assumed that they are continuous for $v > 0$, but we did not say anything about the zero area limit. An interesting class of theories are those fulfilling the following extra axiom

- V6: $\lim_{v\to 0} \Phi_M^v$ exists for all M

These theories are closely related to topological field theories. Using the "forgetful" functor between $\mathcal{V}$ and $\mathcal{T}$ and the limit demanded in V6, we can define a topological theory such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V} & \stackrel{\mathcal{F}^{vol}}{\longrightarrow} & Vec,\ \Phi^t_M \\ {\scriptstyle forget}\big\downarrow & & \big\downarrow{\scriptstyle lim_{t\to 0}} \\ \mathcal{T} & \stackrel{\mathcal{F}^{top}}{\longrightarrow} & Vec,\ \Phi^0_M \end{array}$$

We have to check that $\mathcal{F}^{top}$ defined in this way fulfills the axioms of topological field theory. We check the axioms for a functor:

1.

$$\mathcal{F}^{top}(cyl) = \lim_{t\to 0} \mathcal{F}^{vol}(cyl_t) = \lim_{t\to 0} P_t = \mathbf{1},$$

as demanded above.

2.

$$\begin{aligned} \mathcal{F}^{top}(M_1 \cup M_2) &= \lim_{t\to 0} \mathcal{F}^{vol}((M_1 \cup M_2)_t) \\ &= \lim_{t\to 0} \Phi^t_{M_1 \cup M_2} \\ &= \lim_{t_1,t_2\to 0} \Phi^{t_2}_{M_2} \circ \Phi^{t_1}_{M_1} \\ &= \Phi^0_{M_2} \circ \Phi^0_{M_1} \\ &= \mathcal{F}^{top}(M_2) \circ \mathcal{F}^{top}(M_1) \end{aligned}$$

The other axioms for a topological field theory follow directly from the fact that they are fulfilled in the volume-preserving theory. In the following, we will mainly discuss theories with a topological limit. In this special class of theories the partition function can be calculated by assigning the total volume to a certain part of the manifold. This means that in the composition law V1 one of the volumes v_1, v_2 is allowed to be zero. We check:

$$\begin{aligned} \Phi^{v(M)}_{M_2} \circ \Phi^0_{M_1} &= \Phi^v_{M_2} \circ \lim_{v_1\to 0} \Phi^{v_1}_{M_1} \\ &= \lim_{v_1\to 0} \Phi^{v+v_1}_{M} \\ &= \Phi^{v(M)}_{M} \end{aligned}$$

Here, we used the continuity of the Φ_M.

We start with a discussion of the two-dimensional case in the next section.

1.3.1 The Two-Dimensional Theory

What happens to the spheres with one, two and three holes if we give them a volume and assume that the theory has a topological limit? The cylinder $S^1 \times I$

corresponds to the propagator e^{-tH} as in our $d+1$-dimensional theory. All other morphisms can be obtained from the topological theory consisting of zero-area objects by gluing a cylindrical piece containing the total area to the zero-area-object. This is possible in two dimensions due to our modified decomposition law, as we will explain.

The bilinear forms η_t of area t can be decomposed in the following way:

in equations:

$$\eta_t(x,y) = \eta(P_t x, y) = \eta(x, P_t y) = \eta(P_{\frac{t}{2}} x, P_{\frac{t}{2}} y) = ... \tag{1.71}$$

The three-holed sphere can be similarly decomposed:

in equations

$$x \cdot_t y = (P_t x) \cdot y = x \cdot (P_t y) = P_t(x \cdot y) \tag{1.72}$$

Differentiation of this equation gives the result that the Hamiltonian commutes with the operator product in the above sense. A decomposition of the one-holed sphere is:

Altogether we can state that a two-dimensional area-preserving theory is a topological theory built up from manifolds of area 0 plus a Hamiltonian that commutes with the operator product. As our vector space has a Hermitian form we would certainly like the Hamiltonian to be symmetric with respect to this form (to be self-adjoint). From the decomposition of the two holed sphere we have already seen that the propagator is symmetric with respect to η, see (1.71). In other words:

$$P_t^t = P_t \tag{1.73}$$

This implies

$$P_t^\dagger = \overline{P_t} \tag{1.74}$$

If we postulate

$$\overline{P_t} = P_t \tag{1.75}$$

we can conclude

$$P_t^\dagger = P_t \tag{1.76}$$

and consequently

$$H^{\dagger} = H \tag{1.77}$$

So we can derive the Hermiticity of the Hamiltonian by imposing a reality condition on the propagator.

1.3.2 The Operator Algebra $\mathcal{H}_{S^1}$

In this subsection we want our Hermitian form to be a (positive definite) inner product. Furthermore, we want the conjugation C to respect the algebra structure. This means:

$$\overline{x \cdot y} = \overline{x} \cdot \overline{y} \quad \forall x, y \in \mathcal{H} \tag{1.78}$$

We want to investigate the properties of the operator algebra (=space of states with the product coming from the topological three-holed sphere) under these assumptions. A discussion of the operator algebra in topological field theory can be found in [8].

In the following we will repeat some well-known arguments from linear algebra to derive that our algebra (a C^*-algebra with positive definite Hermitian form) has a basis of idempotents.

One can easily see that real states, i.e. states which fulfill $Cx = x$, cannot be nilpotent, i.e. $x^n \neq 0$ for all positive integers n . States interpreted as operators are always symmetric with respect to η , see eq.(1.32). Real states are thus self-adjoint (with respect to $< \; , \; >$) . Note that this is only true, if (1.78) holds. Take now a real state $x \neq 0$ and assume $x^n = 0$. Let n_0 be the smallest n for which $x^n = 0$ and ϕ_0 be the vacuum. For n_0 even we have:

$$x^{n_0} = 0 \Rightarrow < x^{n_0}, \phi_0 >= 0 \Rightarrow < x^{\frac{n_0}{2}}, x^{\frac{n_0}{2}} >= 0 \Rightarrow x^{\frac{n_0}{2}} = 0 \tag{1.79}$$

in contradiction to n_0 minimal. In the last step we used the positive definiteness of $< \; , \; >$. A nondegenerate form allows zero-norm states. For n_0 odd we have:

$$x^{n_0} = 0 \Rightarrow x^{n_0+1} = 0 \Rightarrow < x^{n_0+1}, \phi_0 >= 0 \Rightarrow x^{\frac{n_0+1}{2}} = 0 \tag{1.80}$$

in contradiction to n_0 minimal for $n_0 > 1$ and to $x \neq 0$ for $n_0 = 1$. Of course we could have argued that self-adjoint operators are diagonalizable. If they are nilpotent the only possible eigenvalue is 0 . But I wanted to show the importance of positive definiteness on a more fundamental level.

We will proof now that our algebra has a basis of idempotents, i.e. states fulfilling:

$$x_i \cdot x_j = \delta_{ij} \lambda_i x_i \tag{1.81}$$

To proof this, we first assume that the Hamiltonian (which can be diagonalized because it is self-adjoint) has only one-dimensional eigenspaces (no degenerate eigenvalues). We choose real eigenvectors (which is always possible for Hermitian operators). We have:

$$Hx_i = E_i x_i, \quad x_i = Cx_i, \quad i \neq j \Rightarrow E_i \neq E_j \tag{1.82}$$

Using that H commutes with the operator product we get:

$$H(x_i \cdot x_j) = E_i x_i \cdot x_j = E_j x_i \cdot x_j \Rightarrow (E_i - E_j) x_i \cdot x_j = 0 \tag{1.83}$$

Because $E_i \neq E_j$ for $i \neq j$ we conclude

$$x_i \cdot x_j = 0 \tag{1.84}$$

For $i = j$ we see that $x_i \cdot x_i$ is an eigenvector with eigenvalue E_i and therefore proportional to x_i. It cannot be zero because of the above argument. So we have:

$$x_i \cdot x_j = \lambda_i x_i \quad \lambda_i \neq 0 \tag{1.85}$$

We now turn to the case of higher dimensional eigenspaces. All real states commute with the Hamiltonian and with each other. So they can be simultaneously diagonalized on the eigenspaces of H. There exists a basis of orthonormal eigenstates x_i of H and all real states. Let us determine the operator products between these states:

$$x_i \cdot x_j = \lambda_{ij} x_j \tag{1.86}$$

because x_j is an eigenstate of x_i . But we also have:

$$x_j \cdot x_i = \lambda_{ji} x_i \tag{1.87}$$

because x_i is also an eigenstate of x_j. If $i \neq j$ then x_i and x_j are linearly independent and $x_i \cdot x_j = x_j \cdot x_i$ gives $\lambda_{ij} = \lambda_{ji} = 0$. So the x_i have the following products:

$$x_i \cdot x_j = \delta_{ij} \lambda_j x_j, \quad \lambda_j \neq 0 \tag{1.88}$$

The states form a basis of idempotents.

Our proof also holds for infinite dimensional Hilbert spaces if the eigenspaces of H are finite dimensional.(See section 1.3.4 for a discussion of infinite dimensional vector spaces.)

1.3.3 The Partition Function

If we have a basis of idempotents (1.88) it is easy to calculate the partition function of a surface of genus g, $g \geq 2$. We first look at the topological case. We can cut a genus g surface into $2g - 2$ three-holed spheres:

If we glue $2g-2$ three-holed spheres together (using the inner product/ duality pairing) we get:

$$Z_g = \sum_i \lambda_i^{2g-2} \tag{1.89}$$

For the area-preserving theory we can include a cylindrical piece containing the whole area. We get

$$Z_g = \sum_i \lambda_i^{2g-2} e^{-tE_i} \tag{1.90}$$

where t is the area of the surface.

1.3.4 The Case of Infinite Dimensional Vector Spaces

To include theories with an infinite dimensional "vector space" we will have to make further modifications. It is for example a priori not clear what we mean by a dual space. A possibility is to require $\mathcal{H}$ to be a Hilbert space in the mathematical sense. Then, the Hermitian form has to be positive definite. By the dual space we mean then the topological dual space $\mathcal{H}^*$ of $\mathcal{H}$. The Riesz lemma gives us the possibility to identify $\mathcal{H}^*$ and $\overline{\mathcal{H}}$ by an isomorphism.(There is an antiisomorphism between $\mathcal{H}$ and $\mathcal{H}^*$.) We are then in a situation comparable to the finite dimensional case. But this might be a little bit too restrictive for the following reason: Often the space of physical states $\mathcal{H}_{phys}$ is only a subspace of a Hilbert space $\mathcal{H}$. It consists for example only of states invariant under a certain symmetry. Then we have the inclusion:

$$\mathcal{H}_{phys} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{H}^*_{phys} \tag{1.91}$$

We should also remember that in principle we are interested in calculating the partition function on closed manifolds. We want to do this by "cutting and gluing". For this procedure it is most important that the "states" corresponding to manifolds with boundary make sense inside "inner products", i.e. for a space consisting of wave functions under the integral sign. So we should accept to associate distribution-like objects to manifolds and not be afraid of delta functions etc. In concrete examples we have to see if we can find a "Hilbert space" such that calculations by "cutting and gluing" make sense.

A first regularity assumption for the morphisms of these Hilbert spaces is that they are bounded linear operators.

The next point to think about is then the convergence of partition functions. By the axioms given above we cannot exclude that the partition functions are divergent. Formula (1.90) holds in the infinite dimensional case if the eigenspaces of H are finite dimensional. We want to assume this in the following. Convergence depends on the coefficients λ_i, E_i in (1.90). What can we assume about these coefficients? The E_i are energy-eigenvalues. So they should be bounded from below. If the spectrum is not bounded from above (which makes sense physically) we have a good chance to get convergent partition functions. I will give two examples for how to get convergent partition functions.

- If we start with a convergent topological partition function and add an area-preserving perturbation, then the perturbed partition function is also convergent (if E is bounded from below).
- The partition function is convergent for $t > 0$ if the sequence λ_n is bounded and E_n grows faster than n.

$$Z_g \leq (max\lambda_n) \sum_n (e^{-t})^n \tag{1.92}$$

which is convergent because $e^{-t} < 1$ for $t > 0$. In this example the topological limit does not have to be convergent!

We see that the torus partition function

$$\mathrm{tr}\, \Phi_{S^1 \times I} = \mathrm{tr}\, e^{-tH} \tag{1.93}$$

always diverges in the limit $t \to 0$. This is a divergence which can not be avoided. To get convergent partition functions guaranteed by axioms we could impose the condition that the Φ_M are trace-class. We can cut open any genus g surface and associate the corresponding operator. Gluing the surface together means to take the trace of this operator. If the trace is finite, the partition function is finite. "How many" of the Φ_M should be trace-class? We can postulate that all operators $\Phi_M^{v>0}$ are trace-class or even $\Phi_M^{v\geq 0}$. (If M is not the torus.) We get theories with or without a topological limes. Our examples for convergent partition functions given above show that there are non-trivial examples for both kinds of theories. Taking a zero-area limit might be subtle. We have to add, in which sense these limits have to be taken. This is also important for the definition of the Hamiltonian, since its definition involves the zero-area limit of the cylinder. For semigroups of bounded linear operators in Banach spaces we know: Let P_t be uniformly continuous:

$$\lim_{t \to 0} \|P_t - \mathbf{1}\| = \lim_{t \to 0} \|P_t - P_0\| = 0 \tag{1.94}$$

Then

$$t \mapsto P_t \tag{1.95}$$

is continuous in $\mathbf{R}_0^+$. One can show that these operators have a continuous generator H such that

$$P_t = e^{-tH} \tag{1.96}$$

As we want to have a continuous Hamiltonian H in our theory we have to demand that P_t is uniformly continuous. For theories with a topological limit we demand that the zero-area limit for other operators exists in the same sense.

1.4 Two-Dimensional Conformal Field Theory

The axiomatic treatment of conformal field theory (CFT) in two dimensions is due to Segal (see also the review in [16]). In this section I will review some aspects of his work. In two-dimensional CFT the manifolds M are equipped with a complex structure. It can be shown that there is a one-to-one correspondence between conformal classes of metrics and complex structures. We want to consider two manifolds as inequivalent if they differ in their complex structures. Therefore we wish to describe the space of all inequivalent complex structures. We start with the space of conformally equivalent metrics. Two metrics are called conformally equivalent if they are related by a rescaling:

$$g_1 = fg_2 \tag{1.97}$$

for a smooth function f on M. So in principle we get all complex structures by dividing the space of metrics by the space of functions and we get all inequivalent complex structures by dividing this by the space of diffeomorphisms of the manifold. The diffeomorphisms do not act freely on the space of metrics (due to the existence of Killing vector fields) so that our quotient is highly singular. An alternative approach is to use the fact that one can choose in each conformal class of metrics a representative of constant gaussian curvature R_g. The Gauss-Bonnet theorem

$$\int_M R_g dA = 2\pi\chi(M) = 2\pi(2-2g) \tag{1.98}$$

shows that the curvature of this representative can be normalized to be $1, 0, -1$ for genus $g = 0, 1, \geq 2$. For genus 1 it imposes no normalization restriction on the area so that we normalize the area to one. In the next step we divide the space of constant curvature metrics by the group of diffeomorphisms continuously connected to the identity. This is a normal subgroup in the group of all diffeomorphisms which acts freely on the space of constant curvature metrics. The space we have obtained by dividing is a smooth manifold called the Teichmüller space. In the last step we obtain singularities due to fixed points. We now have to divide the Teichmüller space by the quotient of $Diff(M)$ by our normal subgroup. We obtain the moduli space $\mathcal{M}$. It is the space of inequivalent complex structures on M.

In CFT one further assumes that all surfaces with boundaries come with a parametrization of their boundary circles. Surfaces are only equivalent if they are isomorphic by a map which preserves the boundary parametrization. Again we consider a circle as outgoing if its parametrization agrees with the orientation induced by the complex structure on M, otherwise as incoming. Conjugation of the complex structure of a surface is related to taking the adjoint similar to section 1.2.2 .

One important class of surfaces consists of manifolds which are topologically annuli. The space of annuli $\mathcal{A}$ forms a semigroup where the composition of two annuli is given by sewing the outgoing boundary of one annulus to the incoming of the next. Every annulus (without boundary parametrization) can be mapped uniquely (up to rigid rotations T) to a standard annulus

$$A_r = \{z \in \mathbf{C}, r \leq \mid z \mid \leq 1\} \tag{1.99}$$

So we have: $\mathcal{A}$ is diffeomorphic to

$$(0,1) \times (Diff^+S^1 \times Diff^+S^1)/T \tag{1.100}$$

where $Diff^+S^1$ are the orientation preserving diffeomorphisms of S^1. $\mathcal{A}$ is furthermore a complex manifold as shown by Segal. The tangent space T_A to $\mathcal{A}$ at a particular element A consists of complex vector field along ∂A modulo those vector fields which extend holomorphically over A, in formulas:

$$T_A = Vect\partial A/VectA = VectS^1 \oplus VectS^1/VectA \tag{1.101}$$

The axioms give us a representation of $\mathcal{A}$ on the Hilbert space. Segal shows that we can conclude from this that we also have a representation of $Diff^+S^1$ in the following way: If A is an annulus write φA for the annulus obtained by changing the outgoing parametrization by φ. Φ_A is the operator associated to A. Then we get a representation of $Diff^+S^1$ by

$$\varphi \to \Phi_{\varphi A}\Phi_A^{-1} \tag{1.102}$$

If we look at a standard annulus A_r with induced parametrizations and write $r = e^{-t}$ for $t \in \mathbf{R}$ we can relate the amplitude Φ_{A_r} to the Hamiltonian analogous to the area-preserving theory:

$$\Phi_A = e^{-tH} \tag{1.103}$$

In conformal field theory we have however a richer algebra which corresponds to the generators of infinitesimal changes of the complex structure of a surface. These changes correspond to elements of $Vect\partial M/VectM$ as in (1.101).

To finish this section we mention that one usually looks at theories which depend slightly on the metric. (Theories with a "conformal anomaly".) The transition amplitudes depend on the metric in the following way: If the metric is scaled by e^{σ}, σ a real smooth function on M vanishing near the boundary, the transition amplitude is changed by:

$$\Phi_M \to \exp(\frac{ci}{24\pi}\int[\frac{1}{2}\partial\sigma \wedge \partial\overline{\sigma} + R_g\sigma])\,\Phi_M \tag{1.104}$$

c is the central charge.

Chapter 2

Two-Dimensional Yang-Mills Theory

In this chapter we want to treat two-dimensional Yang-Mills theory (YM_2) in some detail.

The quantum field theory is exactly solvable as has first been pointed out by Migdal [21]. Here we will treat YM_2 in the context of area-preserving QFT and follow the lines given in the previous chapter to calculate the partition function. We first have to check that YM_2 is indeed an area-preserving QFT. The action of the theory is:

$$S = \frac{1}{2e^2} \int_\Sigma \operatorname{tr} F \wedge *F \tag{2.1}$$

where we take Σ to be a compact, orientable, two-dimensional manifold. F is the field strength, the curvature of a connection A. (F can locally be interpreted as a Lie-algebra valued two-form.) The gauge group is taken to be a compact Lie-group. tr denotes an invariant inner product on the Lie-algebra. $*$ is the Hodge-$*$-operator. Altogether the action can be read as the square of a norm of F, where the norm is a combination of the norm on the Lie-algebra given by trace and the norm on the space of two-forms coming from the scalar product $(\alpha, \beta) = \int \alpha \wedge *\beta$ (for closed manifolds). The definition of the Hodge-star between zero- and two-forms in two dimensions depends only on a measure, not on the full metric. This shows that the action is invariant under area-preserving diffeomorphisms.

To discuss the theory along the lines of area-preserving QFT we have to determine the topological limit of the theory, the Hilbert space corresponding to the circle and the morphisms associated to the spheres with one, two and three holes. Then we will be able to calculate the partition function and the expectation values of observables. A discussion of YM_2 in this sense can for example be found in [7].

2.1 A Related Topological Theory

Let us look at the integrand of the path-integral $\int e^{-S[A]} DA$:

$$e^{-S} = \exp -\frac{1}{2e^2}\int \mathrm{tr}\, F \wedge *F \tag{2.2}$$

We can perform a "functional Gaussian Fourier-transform" and obtain:

$$\int D\phi \exp(-i\int \mathrm{tr}\, \phi F - \frac{e^2}{2}\int \mathrm{tr}\, \phi^2 \omega) \tag{2.3}$$

where ϕ is a Lie-algebra valued function and ω the volume form of the manifold. So the action

$$S = i\int \mathrm{tr}\, \phi F + \frac{e^2}{2}\int \mathrm{tr}\, \phi^2 \omega \tag{2.4}$$

leads to the same partition function. The first part of this action is the action of a known topological field theory. This "ϕF-model" has been considered for example in [27], where also the topological nature of the quantum theory is shown. The second part of the action gives a perturbation of the topological theory that depends on the measure ω . In fact we see that the coupling constant e^2 and the measure ω always enter together so that the theory is invariant under the scaling $e^2 \to \lambda e^2, \quad \omega \to \frac{\omega}{\lambda}$. The partition function Z will be a function of the invariant combination $e^2 a$, a the total area. We will also write t as a shorthand for $\frac{e^2}{2}a$. We see that the weak coupling limit $e^2 \to 0$ is the same as letting the area of the surface shrink to zero. In this limit $t \to 0$ we obtain the topological theory. The partition function of the topological theory can be seen to calculate the volume of the moduli space of flat connections on Σ:

$$Z_{top} = \int DAD\phi e^{i\int \mathrm{tr}\, \phi F} = \int DA\delta(F) \tag{2.5}$$

This localizes the path-integral to connections which fulfill $\delta(F) = 0$. (Actually a one parameter ambiguity arises because of different regularization methods, see [27]). In [27] the volume of the moduli space is studied from different points of views. It is found that the results coming from field theory agree with the results coming from mathematically more precise calculations. Other aspects of the geometry of the moduli space and its relation to gauge theories are studied in [28].

2.2 The Hilbert Space $\mathcal{H}_{S^1}$

According to the philosophy of the axioms given in the last chapter the Hilbert space associated to the circle S^1 can be obtained by canonical quantization on the cylinder. All other Hilbert spaces appearing in the theory are tensor products of this space. Yang-Mills on a cylinder has been studied in [22] . We refer the reader to this for a detailed treatment in the Hamiltonian formalism without using gauge conditions. We will give an argument to motivate the results obtained there.

The Hilbert space of our theory has to reflect the physical degrees of freedom in some way. So apart from an action of the Hamiltonian on the Hilbert space we have to consider the Wilson loop on the circle. The conjugacy class of the holonomy on the circle is the only gauge invariant quantity of a two-dimensional gauge-theory. So it seems natural to take as a Hilbert space the space of class functions on the gauge group. This space contains only gauge invariant states.

Let us also look at the action in temporal gauge $A_0 = 0$. This gauge can be imposed if space-time is a cylinder.

$$S = i \int \operatorname{tr} \phi \partial_0 A_1 + \frac{e^2}{2} \int \operatorname{tr} \phi^2 \omega \tag{2.6}$$

This is of the form

$$S = p\dot{q} - H \tag{2.7}$$

We can read off that ϕ can be seen as the conjugate momentum to A_1. A_1 and ϕ therefore have canonical commutation relations. The Hamiltonian is proportional to the Laplacian on the group. We could now proceed canonically and look at wave functionals $\Psi[A]$ and implement the action of the canonical momentum by $\frac{1}{i}\frac{\delta}{\delta A_1}$. This space contains states that are not gauge invariant. We would have to implement the Gauß-law as an operator equation on the space. The physical states would have to solve the constraint equation. This approach can be seen as a "quantize first" procedure. The other possibility is to impose the constraints before quantization. This corresponds to working with class functions in our case. In general the two approaches can be nonequivalent. (The gauge transformations do not act freely on the space of connections. "Quantize first" and "reduce first" are equivalent if the action of the symmetry group is free.) In the following we work with the space of class functions. This space has a basis consisting of the characters of the irreducible unitary representations. We have an inner product on that space given by

$$< f_1, f_2 >= \int f_1^*(g) f_2(g) dg \tag{2.8}$$

The measure appearing in this equation is the Haar-measure on the group. The characters are orthonormal with respect to this product. (The volume of the group is normalized to one.) The Laplacian acts on a character as the quadratic casimir, so the characters are eigenfunctions of the Hamiltonian.

Our states associated to surfaces with one boundary component can be seen as path-integrals as functionals of the boundary conditions. If we want to fix the boundary conditions in a gauge invariant way we only have to give the conjugacy class of the boundary holonomy. This kind of fixing boundary conditions fits with the reduce-first quantization procedure.

2.2.1 Operators, Bases

Let us discuss the operators of the theory (and the corresponding bases of eigenstates) more detailed on a slightly more abstract level. This discussion is copied from quantum mechanics, see e.g. [6] . We have two different kinds of operators:

- The Hamiltonian H with a discrete spectrum
- The conjugacy classes of the holonomies. The space of conjugacy classes can be parametrized by the maximal torus of the Lie-group modulo the Weyl group. This is a continuous set (with an orbifold singularity). In this sense the conjugacy classes can be compared with the position variable in ordinary quantum mechanics.

These operators form an algebra acting on a state space (which does not have to be further specified at the moment). If the operators correspond to physical observables we can apply the spectral theorem for self-adjoint operators. This states that it should be possible to give an eigenvector expansion of the states ϕ in the space of states.

$$\phi = \sum |E_n >< E_n|\phi > \tag{2.9}$$

$$\phi = \int d\Theta|\Theta >< \Theta|\phi > \tag{2.10}$$

(The measure will be the Haar-measure on the Lie group.) The systems $|E_n >$, $|\Theta >$ (eigenvectors of the Hamiltonian/the loop operator) are (different types of) "basis systems".$< E_n|\phi >$ and $< \Theta|\phi >$ are components of ϕ "in direction" of the basis vectors. The $< E_n|\phi >$ are discrete components while $< \Theta|\phi >$ depends continuously on Θ and is, thus, a function of the conjugacy class Θ : $< \Theta|\phi >= \phi(\Theta)$. There is a difference between the systems $|\Theta >, |E_n >$. The $|E_n >$ are states in our space while the $|\Theta >$ correspond to linear functionals. (The word "basis" makes sense if we think of a basis expansion as an integration (2.10).) The "pairing" $< \Theta_1|\Theta_2 >$ has an interpretation as a delta-function:

$$< \Theta_1|\phi >= \int d\Theta_2 < \Theta_1|\Theta_2 >< \Theta_2|\phi > \tag{2.11}$$

So $< \Theta_1|\Theta_2 >$ maps the value of the function $< \Theta|\phi >$ at the position Θ_2 to its value at another position Θ_1. I assume that the $|E_n >$ (eigenvectors of a Hermitian operator) have been chosen such that:

$$< E_n|E_m >= \delta_{n.m} \tag{2.12}$$

So we have the choice between two different kinds of basis systems. We choose now the continuous basis system which means that we describe our states in terms of wave functions:

$$\phi \leftrightarrow < \Theta|\phi >= \phi(\Theta) \tag{2.13}$$

Our space of states is now mapped to a space of wave functions. The eigenvectors $|E_n>$ are particular elements of the space of states and thus must have an expansion

$$|E_n>=\int d\Theta|\Theta><\Theta|E_n> \tag{2.14}$$

$<\Theta|E_n>=E_n(\Theta)$ are "eigenfunctions of H" since $<\Theta|H|E_n>=E_n<\Theta|E_n>$. The pairing between states is in terms of wave functions

$$<\phi|\psi>=\int d\Theta<\phi|\Theta><\Theta|\psi>=\int d\Theta\phi^*(\Theta)\psi(\Theta) \tag{2.15}$$

which is of course the usual inner product on the space of class functions. This can also be seen by using the expansion $|f>=\int f(\Theta)|\Theta>d\Theta$ and the delta-function pairing between the Θ's. We know already that the eigenfunctions on the space of class functions of H (which is represented by Δ on the space of class functions) are the characters of the irreducible unitary representations. The eigenvalues are the values of the quadratic casimir in the particular representation,

$$<\Theta|E_n>=\chi_R(\Theta), \quad E_n=c_2(R) \tag{2.16}$$

For any wave function we have an expansion in energy eigenfunctions:

$$\phi(\Theta)=<\Theta|\phi>=\sum_R<\Theta|R><R|\phi>=\sum_R\chi_R(\Theta)<R|\phi> \tag{2.17}$$

$<R|\phi>$ is a representation representation of states. $<R|\phi>$ is the projection of the state ϕ along $|R>$. The formula relates the representation of the state in terms of the two basis systems. It is valid for states in the cylinder-Hilbert space and thus – if we return to our interpretation of surfaces as states associated to its own boundary – for surfaces with boundary S^1. We can also consider surfaces with more than one boundary circle. The corresponding states are functions of n conjugacy classes (for n boundary circles) $\phi(\Theta_1,\ldots,\Theta_n)$. We can give a generalization of equation (2.17):

$$\phi(\Theta_1,\ldots,\Theta_n)=\sum_{R_1,\ldots,R_n}\phi(R_1,\ldots,R_n)\chi_{R_1}(\Theta_1),\ldots,\chi_{R_n}(\Theta_n) \tag{2.18}$$

Here, $\phi(R_1,\ldots,R_n)$ corresponds to the surface where the boundary circles are labeled by representations $R_1,\ldots,R_n$. If one wants to treat certain boundary components as outgoing one has to label them by dual representations. In the following we will sometimes think in terms of conjugacy classes and sometimes in terms of representations. Thinking in terms of conjugacy classes is closer to the geometrical pictures. We can translate both formulations into each other using the above formulas.

2.3 The Basic Amplitudes

In this section we will study the morphisms associated to the spheres with one, two and three holes [7],[9],[28]. According to the observations in the previous chapter it is sufficient to determine the morphisms of the one and three-holed sphere in the ϕF -model. We can glue a cylindrical piece to them afterwards to include the area-dependence and find the morphisms of the Yang-Mills theory.

Let us start with the disk in the topological theory. The wave function should only be nonvanishing for flat connections on the disk. This means that the holonomy at the boundary must be conjugate to one, i.e. we can describe the state assigned to the disk by:

$$\mathrm{disk}(\Theta) = \delta(\Theta|\mathbf{1}) = <\Theta|\mathbf{1}> \tag{2.19}$$

An expansion in the character basis gives:

$$\mathrm{disk}(\Theta) = \sum_R <\Theta|R><R|\mathbf{1}> = \sum_R \chi_R(\Theta)\dim R \tag{2.20}$$

For the two-holed sphere we have two possibilities to orient the boundary circles. This will give the bilinear form η or the propagator. We will first look at the propagator:

For a flat connection the holonomies should be conjugate to each other and the wave function becomes a delta-function on conjugacy classes.

$$\mathrm{cyl}(\Theta_1,\Theta_2) = \delta(\Theta_1,\Theta_2) = <\Theta_1|\Theta_2> \tag{2.21}$$

Again we expand:

$$<\Theta_1|\Theta_2> = \sum_R <\Theta_1|R><R|\Theta_2> = \sum_R \chi_R(\Theta_1)\,\chi_R^*(\Theta_2) \tag{2.22}$$

For a propagator e^{-tH} in the area-preserving theory with H the Laplacian we get:

$$\mathrm{cyl}(\Theta_1,\Theta_2) = \sum_R \chi_R(\Theta_1)\,\chi_R^*(\Theta_2)\,e^{-tc_2(R)} \tag{2.23}$$

(Remember that we have absorbed the coupling $\frac{e^2}{2}$ into the factor t.) The other possibility to orient the boundary circles is:

We will look again at the topological case: This time we have:

$$\eta(\Theta_1,\Theta_2) = \delta(\Theta_1,\Theta_1^{-1}) \tag{2.24}$$

We also want to know which bilinear form is induced by this on class functions. A class function f can be expanded as a state:

$$f = \int d\Theta f(\Theta)|\Theta > \tag{2.25}$$

($| >$ indicates that Θ is to be read as a state.) We get :

$$\eta(f_1, f_2) = \int f_1(\Theta_1) f_2(\Theta_2)\, \delta(\Theta_1, \Theta_2^{-1})\, d\Theta_1 d\Theta_2 = \int f_1(\Theta_1) f_2(\Theta_2^{-1})\, d\Theta_1 \tag{2.26}$$

This form is bilinear in f_1, f_2 and symmetric by the properties of the Haar measure. In the previous chapter we have introduced a complex conjugation C relating the bilinear form η and the Hermitian form $< \quad , \quad >$ as in equation (1.34). If we compare the Hermitian and the bilinear form in the Yang-Mills case we see that

$$(Cf)(g) = f^*(g^{-1}) \tag{2.27}$$

So the characters fulfill

$$C\chi = \chi \tag{2.28}$$

We come to the three-holed sphere and the product on the space of states.

We will see that this product corresponds on the space of class functions to the convolution product

$$(f_1 * f_2)(g) = \int f_1(gh^{-1}) f_2(h)\, dh \tag{2.29}$$

If we want to use the same argument as in the other two cases we have to say: The wave function should be nonvanishing whenever the product of an element in the conjugacy class of Θ_1 with an element of the conjugacy class of Θ_2 is conjugate to an element in the conjugacy class of Θ_3. Unfortunately, I don't know a definition of a product for conjugacy classes for general compact Lie groups. I will write symbolically:

$$\mathrm{pants}(\Theta_1, \Theta_2, \Theta_3) = \delta(\Theta_3, \Theta_1 \cdot \Theta_2) \tag{2.30}$$

This should be read as explained above. All calculations with this expression should be read with care ... To get another expression for this I transform formally to the representation basis. Using the expansion

$$|R_i >= \int d\Theta |\Theta >< \Theta | R_i > \tag{2.31}$$

I get:

$$\begin{aligned} \mathrm{pants}(R_1, R_2, R_3) &= \int d\Theta_1 d\Theta_2 d\Theta_3\, \chi_{R_1}(\Theta_1)\chi_{R_2}(\Theta_2)\chi^*_{R_3}(\Theta_3)\, \delta(\Theta_3, \Theta_1 \cdot \Theta_2) \\ &= \int d\Theta_1 d\Theta_2\, \chi_{R_1}(\Theta_1)\chi_{R_2}(\Theta_2)\chi_{R_3}\left((\Theta_1\Theta_2)^{-1}\right) \end{aligned} \tag{2.32}$$

The characters diagonalize the convolution product:

$$\chi_{R_1} * \chi_{R_2} = \frac{1}{\dim R_1} \chi_{R_1} \delta_{R_1,R_2} \tag{2.33}$$

If we use this and the orthonormality property we get

$$\text{pants}(R_1, R_2, R_3) = \frac{1}{\dim R_1} \delta_{R_1,R_2,R_3} \tag{2.34}$$

This means that the pants represent the convolution product. If we label one of the states as outgoing we can write:

$$|R_1 > *|R_2 >= \frac{1}{\dim R_1} \delta_{R_1,R_2} |R_1 > \tag{2.35}$$

We can retransform this using formula (2.18). A short calculation gives:

$$\text{pants}(\Theta_1, \Theta_2, \Theta_3) = \sum_R \frac{1}{\dim R} \chi_R(\Theta_1) \chi_R(\Theta_2) \chi_R^*(\Theta_3) \tag{2.36}$$

The arguments used in this description of the three-holed sphere were mathematically not too nice. Therefore, I will rederive (2.36) using a lattice calculation. We will also see that (2.36) has a precise meaning in terms of products between conjugacy classes for a theory with finite groups.

2.4 Lattice Calculation

We will look at YM_2 from a lattice point of view in this section [27], [21]. In lattice gauge theory one triangulates the space time Σ. A connection on a lattice is given by an assignment of a group element g_{ij} for every link L_{ij} connecting two vertices x_i and x_j of the lattice. This group element should describe parallel transport along a link. If we compare a lattice theory with the corresponding continuum theory we should regard the group element g_{ij} as the path-ordered exponential of the continuum connection along the link L_{ij}. A gauge transformation associates to every vertex x_i a group element g_i. The link variables g_{ij} transform under a gauge transformation as:

$$g_{ij} \to g_i g_{ij} g_j^{-1} \tag{2.37}$$

For any plaquette (=interior of a closed polygon of the lattice) one can calculate a group element g_p by multiplying the link variables. Suppose the vertices of the polygon are $x_{i_1}, \ldots x_{i_n}$, then

$$g_p = g_{i_1 i_2} \cdot \ldots \cdot g_{i_{n-1} i_n} \tag{2.38}$$

The conjugacy class of this group element is gauge invariant. We want to define a partition function for the lattice theory. The integrand $exp(-\int_\Sigma L)$ of the path-integral factorizes into factors corresponding to each plaquette:

$$e^{-\int_\Sigma L} = \prod_p e^{-\int_p L} = \prod_p e^{-S_p} = \prod_p Z_p \tag{2.39}$$

So we have to associate a local factor Z_p to every plaquette. This local factor should depend on the conjugacy class of the group element g_p and the area of the plaquette. After we have chosen the local factors we can calculate the partition function by:

$$Z = \int \prod_{L_{ij}} dg_{ij} \prod_p Z_p(g_p) \tag{2.40}$$

Our considerations in the previous sections suggest the wave function associated to the disk as local factors. That is

$$Z_p(g_p) = \sum_R \dim R\, \chi_R(g_p) \tag{2.41}$$

for a topological theory. For the nontopological case we glue a cylindrical piece containing the area to our disk.

$$\begin{aligned} Z_p\left(g_p, t\right) &= \int dg \sum_R \dim R\, \chi_R\left(g\right) \sum_{R'} \chi^*_{R'}\left(g\right) \chi_{R'}\left(g_p\right) e^{-tc_2(R')} \\ &= \sum_R \dim R\, e^{-tc_2(R)} \chi_R\left(g_p\right) \end{aligned} \tag{2.42}$$

This is the local factor originally used by Migdal [21]. To get a well-defined partition function it is necessary to show that it is independent of the triangulation of the surface. For two triangulations it is always possible to find a finer triangulation that contains both triangulations. So it has to be shown that the local factors are invariant under subdivisions of the plaquettes. This can be shown by the following argument:

We divide the plaquette p into two parts p_1 and p_2. The group element associated to the dividing link is V. The group elements U, W fulfill $UW = g_p$. To establish invariance under subdivisions we have to show:

$$Z_p(g_p) = \int dV\, Z_{p_1}(UV) Z_{p_2}(V^{-1}W) \tag{2.43}$$

This follows immediately from

$$\int dV\, \chi_R(UV) \chi_{R'}(V^{-1}W) = \delta_{R,R'} \frac{1}{\dim R}\, \chi_R(UW) \tag{2.44}$$

As a first application we will once again calculate the partition function of the three-holed sphere. We cut the pair of pants the following way:

Gluing this disk together to a sphere again corresponds to the following calculation:

$$Z_3(U_1, U_2, U_3) = \int dV dW dX \sum_R \dim R \, \chi_R \left(U_1 V W^{-1} U_2 W X^{-1} U_3 X V^{-1} \right) \quad (2.45)$$

applying

$$\int dV \, \chi_R(UVBV^{-1}) = \frac{1}{\dim R} \chi_R(U) \chi_R(B) \quad (2.46)$$

for $B = W^{-1} U_2 W X^{-1} U_3 X$ and $U = U_1$ gives

$$Z_3 = \int dW dX \sum_R \chi_R(U_1) \, \chi_R(W^{-1} U_2 W X^{-1} U_3 X) \quad (2.47)$$

Remembering that χ_R is a class function and applying the same identity again gives for the W-integration

$$\begin{aligned} Z_3 &= \int dX \sum_R \frac{1}{\dim R} \chi_R(U_1) \chi_R(U_2) \chi_R(X^{-1} U_3 X) \\ &= \sum_R \frac{1}{\dim R} \chi_R(U_1) \chi_R(U_2) \chi_R(U_3) \end{aligned} \quad (2.48)$$

This verifies our result of the previous section.

2.5 Partition Functions

We are now ready to calculate the YM_2 partition function for any surface using gluing techniques.

1. The sphere.
   We can for example cut the sphere into two one-holed spheres and give the whole area to one of the halves. Using the formula (2.42) for the disk with

area we get:

$$Z = \int dg \sum_R \dim R \, e^{-tc_2(R)} \chi_R(g) \sum_{R'} \dim R' \, \chi^*_{R'}(g)$$
$$= \sum_R (\dim R)^2 \, e^{-tc_2(R)} \tag{2.49}$$

This is divergent for $t \to 0$.

2. The torus.
To calculate the torus partition function we can for example use

$$Z = \operatorname{tr} e^{-tH} \tag{2.50}$$

and obtain

$$Z = \sum_R e^{-tc_2(R)} \tag{2.51}$$

which is also divergent for $t \to 0$. One can easily check that the same result is obtained if we cut the torus in the following way:

or if we glue the edges of a square:

To give an example, we calculate the torus partition function for $SU(2)$. For a spin-j-representation the quadratic casimir is

$$c_2 = j(j+1) = \frac{1}{4}[(2j+1)^2 - 1] = \frac{1}{4}(n^2 - 1) \tag{2.52}$$

where n is the dimension of the representation. Inserting this we get

$$Z = e^{\frac{t}{4}} \sum_n e^{-\frac{t}{4}n^2} \tag{2.53}$$

3. Genus $g \geq 2$.
We have seen that the three-holed sphere corresponds to the convolution product on the space of class functions which is diagonalized by the characters. So we can simply use the result obtained in section 1.3.3. We have

to plug in $\frac{1}{\dim R}$ for the λ's and the quadratic casimir values for the E_i and get:

$$Z_g = \sum_R (\dim R)^{2-2g}\, e^{-tc_2(R)} \tag{2.54}$$

Other possibilities to calculate this result by gluing techniques (different ways to cut a genus g surface) are discussed in [27].

The YM_2 partition function has also been computed using other methods. For example Fine [14], [15] discusses the continuum path-integral.

2.6 Expectation Values of Wilson Loops

In this section we will calculate the expectation values of the observables in our theory, the Wilson loops. See for example the review [7] for general formulas. There can be different kinds of non-self-intersecting loops on a manifold. We have contractible loops and noncontractible loops where the noncontractible loops can be divided into those which cut a surface into two disconnected pieces and those which leave the surface connected. We will explain how to calculate the three different types of loops by giving an easy example for each one.

We start with the Wilson loop on the sphere:

Take a loop γ on the sphere. This will always be contractible. We calculate the expectation value by cutting the sphere along the loop γ into two disks and then integrating the Wilson loop operator $W_\gamma^R = \mathrm{tr}_R \exp \oint_\gamma A = \chi_R(U)$ against the disk amplitudes

$$\begin{aligned} \langle W_\gamma^R \rangle &= \int dU \sum_{R'} e^{-t_1 c_2(R')} \dim R'\, \chi_{R'}(U) \chi_R(U) \sum_{R''} e^{-t_2 c_2(R'')} \dim R''\, \chi_{R''}(U^{-1}) \\ &= \sum_{R'R''} \dim R'\, \dim R''\, e^{-t_1 c_2(R') - t_2 c_2(R'')}\, N_{R'R}^{R''} \end{aligned} \tag{2.55}$$

where t_1 and t_2 are the areas of the two different disks. $N_{R'R}^{R''}$ are the fusion numbers appearing in the reduction of the tensor product

$$R_1 \otimes R_2 = \bigoplus N_{R_1 R_2}^{R_3}\, R_3 \tag{2.56}$$

In terms of characters this equation can be written as

$$\chi_{R_1} \chi_{R_2}(U) = \sum_{R_3} N_{R_1 R_2}^{R_3}\, \chi_{R_3}(U) \tag{2.57}$$

This means that the fusion numbers fulfill:

$$N_{R_1 R_2}^{R_3} = \int dU \chi_{R_1} \chi_{R_2} \chi_{R_3}^*(U) \tag{2.58}$$

We have used this relation when integrating over the group.

We can easily generalize this calculation to a contractible loop on a more complicated manifold. If we cut along a contractible loop we get a disk and a manifold with one extra boundary circle. We do the same calculation as before replacing one of the disks by the more complicated manifold. For surfaces with boundaries the expectation value of our Wilson loop becomes a function of the boundary holonomies.

We will now look at Wilson loops on the cylinder. Here we have one nontrivial loop:

To calculate its expectation value we cut the cylinder into two pieces:

and calculate:

$$\begin{aligned} & \langle W_\gamma^R \rangle (V, W) \\ = & \int dU \sum_{R'} \chi_{R'}(V^{-1}) \chi_{R'}(U)\, e^{-t_1 c_2(R')} \chi_R(U) \sum_{R''} \chi_{R''}(U^{-1}) \chi_{R''}(W)\, e^{-t_2 c_2(R'')} \\ = & \sum_{R'R''} N_{R'R}^{R''} e^{-t_1 c_2(R') - t_2 c_2(R'')} \chi_{R'}(V^{-1}) \chi_{R''}(W) \end{aligned} \tag{2.59}$$

Our next example is the torus. Here we have two homologically nontrivial loops. To calculate the expectation value of the Wilson loop along one of them, we simply identify the two ends of the cylinder.

$$\int dW \sum_{R'R''} N_{R'R}^{R''}\, e^{-t_1 c_2(R') - t_2 c_2(R'')} \chi_{R'}(W^{-1})\, \chi_{R''}(W) = \sum_{R'} e^{-t c_2(R')} N_{R'R}^{R'} \tag{2.60}$$

The calculations for each of the two different nontrivial loops on the torus are the same because cutting along each of them leads to an object which is topologically a cylinder. We see that the expectation value is only nonvanishing if there are representations R' such that the tensor product of R' and R contains R' again. In other words, R' has to appear on both sides of the equation:

$$R' \otimes R = \bigoplus N_{R'R}^{R''} R'' \tag{2.61}$$

Or:

$$\exists R' : \quad N_{R'R}^{R'} \geq 1 \tag{2.62}$$

2.6.1 Wilson Loops on the Torus

In this section we want to answer the question for which representations the expectation value of the Wilson loop does not vanish.

$$W_\gamma^R \neq 0 \tag{2.63}$$

γ one of the two noncontractible loops on the torus. This reduces to the question for which R condition (2.62) can be fulfilled. We will start with the $SU(2)$-case and then go to the $SU(n)$-case.

SU(2)

In the $SU(2)$-case we are able to give concrete formulas for the expectation values in all representations.

All representations can be labeled by their spin j. A tensor product of two representations can be decomposed as

$$\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} = \mathcal{H}^{j_1+j_2} \oplus \mathcal{H}^{j_1+j_2-1} \oplus \ldots \oplus \mathcal{H}^{|j_1-j_2|} \tag{2.64}$$

We want to calculate

$$\langle W^{j_1} \rangle = \sum_{j_2} N_{j_1 j_2}^{j_2} \quad e^{-t j_2(j_2+1)} \tag{2.65}$$

If j_1 is half-integer, $j_1 = \frac{n}{2}$, $n \in 2\mathbf{N}+1$ we have

$$\mathcal{H}^{\frac{n}{2}} \otimes \mathcal{H}^{j_2} = \mathcal{H}^{\frac{n}{2}+j_2} \oplus \ldots \oplus \mathcal{H}^{|\frac{n}{2}-j_2|} \tag{2.66}$$

If j_2 is integer all Hilbert spaces on the rhs carry half-integer spin and if j_2 is half-integer they carry integer spin. Thus, $\mathcal{H}^{j_2}$ can never appear on the right handside. We conclude that the expectation value of a Wilson loop in a half-integer spin representation j_1 always vanishes. Now we take j_1 to be integer.

$$\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} = \mathcal{H}^{j_1+j_2} \oplus \ldots \oplus \mathcal{H}^{|j_1-j_2|} \tag{2.67}$$

The representations on the rhs are labeled by spins

$$j_1 + j_2 - m, \quad m \in \{0, \ldots, m_{max}\} \tag{2.68}$$

where

$$m_{max} = j_1 + j_2 - |j_1 - j_2| = 2\, min\,(j_1, j_2) \tag{2.69}$$

This gives

$$m_{max} = 2j_2 \quad \text{for} \quad j_1 \geq j_2, \quad m_{max} = 2j_1 \quad \text{for} \quad j_1 \leq j_2 \tag{2.70}$$

For $\mathcal{H}^{j_2}$ to appear on the right hand side we need $m = j_1$ for $m \leq m_{max}$.

$$\begin{aligned} j_2 \geq j_1 \quad &\Rightarrow \quad m = j_1 < 2j_1 = m_{max} \\ j_1 \geq j_2 \quad &\Rightarrow \quad m = j_1 \leq m_{max} = 2j_2 \quad \text{for} \quad j_2 \geq \frac{1}{2} j_1 \end{aligned}$$

We conclude that $\mathcal{H}^{j_1} \oplus \mathcal{H}^{j_2}$ contains $\mathcal{H}^{j_2}$ once for each $j_2 \geq \frac{1}{2}j_1$. Therefore, the expectation value is

$$\langle W^{j_1} \rangle = \sum_{j_2=\frac{1}{2}j_1}^{\infty} 1 \cdot e^{-tj_2(j_2+1)} \tag{2.71}$$

where the summation is in half-integer steps. We can rewrite this as a sum over dimensions $n = 2j_2 + 1$:

$$\langle W^{j_1} \rangle = \sum_{n=j_1+1}^{\infty} e^{-\frac{t}{4}[n^2-1]} \tag{2.72}$$

We see that we have obtained "a piece" of the torus partition function Z:

$$\langle W^{j_1} \rangle = Z - \sum_{n=1}^{j_1} e^{-\frac{t}{4}[n^2-1]} \tag{2.73}$$

SU(n)

The calculation of Wilson loop expectation values on the torus becomes more difficult for other groups. But for the groups $SU(n)$ we can say which expectation values vanish and which do not. We proceed in two steps:

1. We will name irreducible representations (irreps) R for which the expectation value vanishes. In group theory language: We will find representations R for which $N_{RR'}^{R'} = 0 \quad \forall R'$.

2. For all other representations we will see that the expectation value does not vanish. In group theory language: We will see that for these representations we can always find representations R' for which $N_{RR'}^{R'} \neq 0$.

First we have to say some things about the reduction of the tensor product of two irreducible representations R, R' of $SU(n)$. An irrep $R = D(k_1, \ldots, k_{n-1})$ of $SU(n)$ is characterized by a highest weight

$$\lambda = \sum_{i=1}^{n-1} k_i \lambda^i \quad k_i \in \mathbf{N} \tag{2.74}$$

$n-1$ is the dimension of the Cartan subalgebra. λ_i are the fundamental weights. These are defined to be the unique weights fulfilling

$$\lambda^i(h_j) = \delta^i_j \tag{2.75}$$

The h_j are the usual basis of the Cartan subalgebra of $SU(n)$, in matrix form

$$h_j = diag(0, \ldots, 0, 1, -1, 0, \ldots, 0) \tag{2.76}$$

where the 1 is at position j. To each irrep $R = D(k_1, \ldots, k_{n-1})$ we can associate a Young tableau T_R.

It has $r_i = k_i + \ldots + k_{n-1}$ boxes in row i. Or: k_i gives the difference of the number of boxes in row i and row $i+1$, $k_i = r_i - r_{i+1}$. Vice versa, we can associate an irrep of $SU(n)$ to a Young tableau with $\leq n$ rows by reading off the appropriate highest weight from the tableau. It is important for us to note that Young tableaux with n rows also represent irreps of $SU(n)$. They are equivalent to tableaux where the columns with n boxes are "crossed out". Young tableaux with more than n rows do not represent $SU(n)$ representations. By now I have also established some notation and I should add that for an irrep R' I call the corresponding quantities $k'_i, r'_i, T_{R'}$ etc.

There is a simple algorithm for the reduction of $R \otimes R'$ with the help of Young tableaux. I will give the algorithm without any proof.

- Find the Young tableaux $T_R, T_{R'}$ for R and R' as described above.
- Fill the Young tableau T_R with numbers in the following way: fill in a one into each box of the first row, a two in each box of the second row, a k in each box of the k'th row,... till n-1.
- Attach the boxes labeled one to the second Young tableau $T_{R'}$ in all possible ways but according to the following rules
  - The resultant diagram is a Young tableau
  - No two one's appear in the same column

  If you get the same tableau in several ways cross out all except one.
- Repeat the process with the $2's, 3's, \ldots, n-1's$.
- For each tableau obtained in this way read the sequence of numbers in the tableau from right to left starting in the first row. Kill the tableau if the sequence is not a "lattice permutation". "Lattice permutation" means: Left of any number in the sequence there are no fewer 1's than 2's, no fewer 2's than 3's,..., i's than i+1's,... In other words: If m is a certain point of the sequence and $h(i, m)$ is the number of i's that have appeared in the sequence so far, then we have

$$h(i, m) \geq h(i+1, m) \tag{2.77}$$

- For Young tableaux with n rows "cross out" the columns with n boxes.
- If there are diagrams that are exactly identical (including the numbers) cross out all except one.
- associate the corresponding $SU(n)$ representations to the remaining diagrams.

The Young tableaus obtained by this procedure are characterized by the following properties:

1. No number appears twice in the same column
2. The sequence of numbers one obtains reading the rows from right to left starting with the first row is a lattice permutation.
3. The sequence in each row is increasing from left to right.

We have to decide now under which circumstances we can obtain a Young tableau which is identical to the tableau $T_{R'}$. By attaching boxes from T_R to $T_{R'}$ we "increase the number of boxes". To get the tableau $T_{R'}$ back we necessarily have to make use of the rule that columns with n boxes can be crossed out. This means that the net effect of attaching the boxes T_R to $T_{R'}$ must be that we get z columns with n boxes. So the number of boxes Z of T_R must be a multiple of n, $Z = nz$. We can conclude:

A necessary condition for $N_{RR'}^{R'} \geq 1$ is that the number of boxes Z of the Young tableau T_R is a multiple of n, i.e.

$$Z = \sum_{i=1}^{n-1} r_i = \sum_{i=1}^{n-1} i k_i = nz \quad z \in \mathbf{N} \tag{2.78}$$

In the following we will look at irreps R which fulfill this basic condition. Next we show that for these representations we can always find an irrep R' such that $N_{RR'}^{R'} \geq 1$. This means that we have to find a tableau $T_{R'}$ which allows that we attach z boxes of T_R to each row such that the resulting diagram is a Young diagram with the characterizing properties 1,2,3 mentioned above. If we take $T_{R'}$ with $k'_i \geq z$ for $i = 1, \ldots, n-1$ we can be sure that the diagram we obtain from gluing z boxes to each row is again a Young tableau. There is only one box with a number in each column, so it is also clear that no number appears twice in a column. Condition 1 is fulfilled. We have to check now that we can attach the boxes in a way such that also conditions 2 and 3 are fulfilled. This is equivalent to filling a rectangular diagram with n rows and z columns with the numbers in T_R such that 2 and 3 are fulfilled. We give an algorithm for this:

- Take the biggest numbers in T_R (= the numbers j appearing in row j if T_R has j rows) and fill them in from right to left starting in the last row of the rectangle.

- Proceed with the numbers $j-1$ of row $j-1$ in the following way: Fill in a $j-1$ on top of each j. (We have enough $j-1$'s to do this because T_R is a Young tableau, $r_{j-1} \geq r_j$.) If there are $j-1$'s left go to the row nearest to the bottom of the rectangle and fill them in from right to left.

- Proceed like this with the other rows: Recursively: Suppose the numbers i are already filled in. Fill now in the $i-1$'s: Put an $i-1$ on top of each i and fill in the remaining $i-1$'s from right to left starting with the row closest to the bottom.

This process will always give the desired result. So the Young-tableaux $T_{R'}$ with $k_i' \geq z$ give $N_{RR'}^{R'} \geq 1$ (if R fulfills the basic condition $Z = nz$).
Altogether we have obtained the following result:

Let $R = D(k_1, \ldots, k_n)$ be a representation of $SU(n)$ with highest weight $\lambda = \sum_{i=1}^{n-1} k_i \lambda^i$. The expectation value of the Wilson loop W_γ^R does not vanish if and only if $\sum_{i=1}^{n-1} i k_i = nz$ with $z \in \mathbf{N}$.

2.6.2 Geometrical Pictures of the Products on the Algebra of Class Functions

On the algebra of class functions we have two different products: The convolution product

$$(f * g)(U) = \int dV f(UV^{-1}) g(V) \tag{2.79}$$

and the pointwise product between functions

$$(fg)(U) = f(U)g(U) \tag{2.80}$$

In the character basis we have

$$\begin{aligned} \chi_{R_1} * \chi_{R_2} &= \delta_{R_1 R_2} \frac{1}{\dim R_1} \chi_{R_1} && (2.81) \\ \chi_{R_1} \chi_{R_2} &= \sum_{R_3} N_{R_1 R_2}^{R_3} \chi_{R_3} && (2.82) \end{aligned}$$

In the following we want to discuss the geometrical meaning of these algebraic structures in the context of YM_2 in the zero-area limit. We have already seen that the convolution product is connected to the zero-area three-holed sphere. We stress this once more by calculating the expectation values of three Wilson loops inserted at the three boundaries (all labeled incoming):

$$
\begin{aligned}
&\langle W_{R_1} W_{R_2} W_{R_3} \rangle = \dots \\
= \ & \int dW dV dU \sum_R \frac{1}{\dim R} \chi_R(U^{-1}) \chi_R(V^{-1}) \chi_R(W^{-1}) \chi_{R_1}(U) \chi_{R_2}(V) \chi_{R_3}(W) \\
= \ & \frac{1}{\dim R_1} \delta_{R_1 R_2 R_3} \qquad (2.83)
\end{aligned}
$$

Using our result for the expectation value of a Wilson loop on the cylinder we can now also calculate the fusion coefficients $N_{R_1 R_2}^{R_3}$ as an expectation value of Wilson loops:

We have to integrate the amplitude of the Wilson loop on the cylinder with open ends against the boundary loops. If we label the R_3-boundary as "outgoing".

$$\langle W_{R_1} W_{R_2} W_{R_3} \rangle = N_{R_1 R_2}^{R_3} \qquad (2.84)$$

We can give a one-to-one translation of the convolution product and the point-product in the character basis in the following way: The geometrical picture for a character is the expectation value of a Wilson loop at one boundary as a function of the holonomy of the other boundary:

$$\langle W_R \rangle (V) = \chi_R(V) \qquad (2.85)$$

So (2.81) is in pictures

and (2.82) is in pictures

Associativity of the convolution product is geometrically the possibility to decompose the four-holed sphere into two three-holed spheres in two different ways as discussed before. For the other product it reflects the fact that if we glue three cylinder pieces it does not matter if we first glue the first two pieces or the last two pieces:

The two different products give us the possibility to interpret states as operators in two different ways:

$$\chi_{R_1} * |R_2> \;\equiv\; \chi_{R_1} * \chi_{R_2} \tag{2.86}$$

$$\chi_{R_1}|R_2> \;\equiv\; \chi_{R_1}\,\chi_{R_2} \tag{2.87}$$

where $|R_2>$ is to be interpreted as a state and χ_{R_1} as an operator. If we let the characters act on the topological disk "in the *-way" we can "create" all basis states:

$$\chi_R * |disk>=|R> \tag{2.88}$$

Another possibility to create the basis states is to let the characters act on the trivial representation character "in the $\cdot$ - way"

$$\chi_R|R_0>=|R> \tag{2.89}$$

Note that in the area-preserving theory with Hamiltonian not equal to zero $|R_0>$ is an eigenstate of H of zero energy.

Chapter 3

Further Examples

In this chapter we will see some other examples of topological field theories and their area-preserving perturbations.

3.1 A Theory for Finite Groups

3.1.1 The Topological Theory

In this section I want to discuss a toy theory which is a two-dimensional gauge theory with finite gauge group. We will first look at the topological case. To define the theory one can simply use the lattice construction given in section 2.4. Integrations over the group have to be replaced by summations. To calculate the partition function we have to perform a summation over link variables. The inner product on the space of functions on a finite group is

$$< f_1, f_2 >= \frac{1}{|G|} \sum_{g \in G} f_1^*(g) f_2(g) \tag{3.1}$$

Using lattice arguments we can calculate all partition functions in essentially the same way as before. Note that our theory is invariant under subdivisions because of the same argument used before and therefore well-defined.

The Operator Algebra

We want to discuss the three-holed sphere once more in the finite group case. This time we have a well-defined product between conjugacy classes C:

$$C_i C_j = \sum_k \mathcal{N}_{ij}{}^k C_k \tag{3.2}$$

For an element $g \in C_k$ the factor $\mathcal{N}_{ij}{}^k$ counts for how many combinations $x, y, x \in C_i$, $y \in C_j$ the product xy gives g. $\mathcal{N}_{ij}^k$ is independent of the choice of g which makes the product well-defined. So $\mathcal{N}_{ij}^k$ does exactly what our partition function of the three-holed sphere is supposed to do. We can give formulas for $\mathcal{N}_{ij}^k$. To do this we define an operator T_R for a representation R and a conjugacy class C_i by

$$T_R = \sum_{g \in C_i} R(g) \tag{3.3}$$

where $R(g)$ is g in the representation R. This operator commutes with every operator $R(h)$. By Schur's lemma it is proportional to the identity:

$$T_R = \lambda_i^{(R)} \cdot \mathbf{1} \tag{3.4}$$

Taking the trace gives

$$n_i \, \chi_R(g) = \lambda_i^{(R)} \dim R \tag{3.5}$$

where n_i is the number of elements in C_i and g is an (arbitrary) element in C_i. We also write $\chi_R(g) = \chi_i^R$. We conclude

$$\lambda_i^{(R)} = \frac{n_i}{\dim R} \chi_i^R \tag{3.6}$$

The conjugacy classes C_i can be mapped into the center of the group algebra (= the algebra of formal linear combinations of group elements) by

$$C \to \sum_{g \in C} g \tag{3.7}$$

These elements form a basis of the center of the group algebra. The multiplication of conjugacy classes reads in the group algebra

$$\sum_{g_i \in C_i} g_i \sum_{g_j \in C_j} g_j = \sum_k \mathcal{N}_{ij}{}^k \sum_{g_k \in C_k} g_k \tag{3.8}$$

Any representation R extends canonically to a representation of the group algebra by

$$R\left(\sum_g \lambda_g g\right) = \sum_g \lambda_g \, R(g) \tag{3.9}$$

If we apply R to the multiplication law and use $T_R = \lambda_i^{(R)}\mathbf{1}$ we obtain:

$$\lambda_i^{(R)} \lambda_j^{(R)} = \sum_k \mathcal{N}_{ij}{}^k \lambda_k^{(R)} \tag{3.10}$$

The $\lambda_i^{(R)}$ form themselves a representation of the group algebra. We can now calculate the $\mathcal{N}_{ij}^k$ using the explicit formulas for the $\lambda_i^{(R)}$ and the character orthogonality relations. We get:

$$\mathcal{N}_{ij}{}^k = \sum_R \frac{n_i n_j}{|G|} \frac{\chi_i^R \, \chi_j^R \, \chi_k^{R*}}{\dim R} \tag{3.11}$$

So we obtained the partition function of the three-holed sphere as it was given before for a finite group. We have seen what it means mathematically.

Let us make some remarks about the numbers χ_i^R. For finite groups the number of conjugacy classes equals the number of irreducible representations. So the χ_i^R form a quadratic matrix $\left(\chi_i^R\right)_{Ri}$. We let this matrix act on the center of the group algebra. An application to one of the basis vectors $\sum_{g\in C_i} g$ gives

$$\phi_j = \sum_i \chi_{R_j}(C_i) \sum_{g\in C_i} g = \sum_{g\in G} \chi_{R_j}(g)\, g \tag{3.12}$$

The product of two of these elements gives

$$\phi_j\, \phi_k = \delta_{jk} \frac{1}{\dim R_j}\, \phi_j \tag{3.13}$$

So the elements ϕ_j of the center of the group algebra diagonalize the product on the group algebra. The matrix $(\chi_i^R)_{Ri}$ transforms between these elements and the conjugacy-class-basis. The factor $\frac{1}{\dim R_i}$ appearing in the operator product of the ϕ's is the inverse of the matrix element which evaluates the character on the unit element of the group. The diagonalization of the operator product has been described for more general topological field theory in [8]. Of course we want to see how the ϕ_i are connected with the characters as functions on the group. There is an isomorphism between the space of functions $\mathcal{R}$ on the group and the group algebra given by:

$$f \to \sum_{g\in G} f(g) g \tag{3.14}$$

The corresponding product on the space of functions is

$$f_1 * f_2(g) = \sum_{h\in G} f_1(gh^{-1}) f_2(h) \tag{3.15}$$

So we have a map

$$(functions, *) \to (group\, algebra, \cdot) \tag{3.16}$$

The center of the group algebra is the image of the space of class functions:

$$(class\, functions, *) \to (center\, of\, group\, algebra, \cdot) \tag{3.17}$$

The characters and their $*$-products are mapped to the ϕ's and their $\cdot$-product. The concept of functions on the group can be generalized to infinite groups (and is isomorphic to the group algebra in terms of formal linear combinations in the case of finite groups). For finite groups we have the nice possibility to see the conjugacy classes and the characters as basis elements in the same algebra: the center of the group algebra. This is the algebra describing our topological field theory with finite groups.

3.1.2 Area-Preserving Perturbation

We can try to give an area-preserving perturbation of this theory. In the previous chapter the area dependence of the theory was encoded in the factor $e^{-tc_2(R)}$. To preserve the invariance of the theory under subdivisions of the lattice it is important that the area dependence of the local factor is exponential. The quadratic casimir was chosen for YM_2 theory because it gives the right continuum limit. In principle it is possible to work with any function that depends only on the representation R. So for the theory with compact Lie groups other casimirs would be possible. Generally, we can look at plaquette actions of the form

$$Z_p(g_p) = \sum_R \dim R \, e^{-tf(R)} \chi_R(g_p) \tag{3.18}$$

f a function of the representation only. The $f(R)$ have an interpretation as energy eigenvalues in our theory so they should be real numbers and bounded from below. The latter condition is automatically satisfied for finite groups. Suitable functions f are for example real valued functions of the dimension of the representation.

$$f(R) = f(\dim R) \tag{3.19}$$

(This idea is copied from the quadratic casimir for SU(2).) A different type of function is

$$f(R) = \sum_{C_i} \mathrm{Re}\, \chi_R(C_i) \tag{3.20}$$

(Note that this is a finite sum.)

3.2 Cohomological Field Theory

In the previous chapter we have seen an example for a topological theory: the ϕF-model. This model had an action which is invariant under all diffeomorphisms. From this action we got a topological quantum field theory. We will now look at a different kind of topological field theory, which is called cohomological field theory [32]. These theories are subsectors of a more general quantum field theory. The gauge theory discussed in [29] and topological sigma-models [30] are important examples of cohomological field theories. The two-dimensional theory has been discussed in [10],[11]. We will look at these two-dimensional models and then try to give area-preserving perturbations of the topological theory.

The crucial ingredient in a cohomological field theory is a nilpotent operator Q:

$$Q^2 = 0 \tag{3.21}$$

Q is anticommuting. The physical local observables $\phi_i(x)$ of the topological sector of the theory commute with Q:

$$\{Q, \phi_i\} = 0 \tag{3.22}$$

They are Q-closed. Quite generally we have the property that the expectation value of a Q-commutator vanishes:

$$\langle\{Q,\mathcal{O}\}\rangle = 0 \tag{3.23}$$

it follows that if $\{Q,A\}=0$ then

$$\langle A\{Q,B\}\rangle = \langle\{Q,AB\}\rangle = 0 \tag{3.24}$$

Q-commutators do not contribute inside correlation functions with Q-closed states. So our physical observables should be defined only up to Q-commutators. It makes sense to define the physical Hilbert-space as the space of Q-cohomology classes.

$$\mathcal{H}_{phys} = \frac{kerQ}{imQ} \tag{3.25}$$

A topological field theory should have the property that the correlators of a set of local observables

$$\langle\phi_{i_1}(x)\ldots\phi_{i_n}(x)\rangle_\Sigma \tag{3.26}$$

only depend on the topology of the surface Σ on which the fields are defined. They should be independent of the metric and the positions of the operators. To achieve this general covariance we claim that the stress energy tensor $T_{\mu\nu}$ (which generates the diffeomorphisms, see section 4.1 for further explanations) is a Q-commutator

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} \tag{3.27}$$

It vanishes inside correlation functions of physical observables. We can associate conserved charges P_μ, G_μ to the operators $T_{\mu\nu}$ and $G_{\mu\nu}$. This gives a fermionic extension of the ordinary translation group generated by the P_μ generated by the G_μ.

$$P_\mu = \{Q, G_\mu\}, \quad [P_\mu, P_\nu] = \{G_\mu, G_\nu\} = 0 \tag{3.28}$$

Recall the definition of the stress tensor in a theory given by an action S:

$$T_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{3.29}$$

We see that we automatically get a cohomologically trivial stress energy tensor if the Lagrangian is a Q-commutator:

$$S = \int d^2z\{Q,V\} \quad \Rightarrow \quad T_{\mu\nu} = \{Q, \frac{1}{\sqrt{g}}\frac{\delta V}{\delta g^{\mu\nu}}\} \tag{3.30}$$

We should now be able to recover some of the features of the topological field theories discussed in Chapter 1 from an axiomatic point of view. So we have to look at the operator algebra. We can define a bilinear form η by the two-point functions on the sphere:

$$\eta(\phi,\psi) = \langle\phi\psi\rangle_0 \tag{3.31}$$

or for a basis $\phi_1, \ldots, \phi_n$ of the physical subspace:

$$\eta_{ij} = \langle \phi_i \phi_j \rangle_0 \tag{3.32}$$

η is nondegenerate on the cohomology classes. We will use this form to raise and lower indices (to identify "incoming" with "outgoing" states). Next we have to discuss the possibility to calculate correlation functions by cutting and gluing. To a set of observables $\psi_1, \ldots, \psi_n$ and a surface Σ we associate a state in the physical Hilbert space by

$$\eta(\psi, |\psi_1, \ldots \psi_n >_\Sigma) = \langle \psi \, \psi_1 \ldots \psi_n \rangle_\Sigma \quad \forall \psi \in \mathcal{H}_{phys} \tag{3.33}$$

($| >$ indicates that the object is a state.)
If we look at operators $\psi_1 \ldots \psi_n$ on Σ_1 and $\psi_{n+1} \ldots \psi_m$ on Σ_2, and Σ_1 and Σ_2 such that $\Sigma_1 \cup \Sigma_2 = \Sigma$ we get

$$\eta(|\psi_1, \ldots, \psi_n >_{\Sigma_1}, |\psi_{n+1}, \ldots, \psi_m >_{\Sigma_2}) = \langle \psi_1 \ldots \psi_n \psi_{n+1} \ldots \psi_m \rangle_\Sigma \tag{3.34}$$

This is a gluing property. We can also write

$$\langle \psi_1 \ldots \psi_n \psi_{n+1} \ldots \psi_m \rangle = \sum_{jk} \langle \psi_1 \ldots \psi_n \phi_j \rangle \, \eta^{jk} \, \langle \phi_k \psi_{n+1} \ldots \psi_m \rangle \tag{3.35}$$

The three-point functions can be used to define an operator algebra

$$c_{ijk} = \langle \phi_i \phi_j \phi_k \rangle \tag{3.36}$$

By raising one of the indices with η we can write an operator product

$$\phi_i \phi_j = \sum_k c_{ij}{}^k \phi_k \tag{3.37}$$

We can apply our factorization procedure to the four-point function and get

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \sum_{mn} \langle \phi_i \phi_j \phi_m \rangle \, \eta^{mn} \, \langle \phi_n \phi_k \phi_l \rangle = \sum c_{ij}{}^m c_{mkl} \tag{3.38}$$

There are two ways of factorizing, so we have the following property of the coefficients c_{ijk}

$$\sum c_{ij}{}^m c_{mkl} = \sum c_{ik}{}^m c_{mjl} \tag{3.39}$$

So we get an associative ring which is of course also commutative. This gives a connection with the axiomatic discussion. Note however that this ring might be nilpotent. This is the case in the chiral ring in superconformal field theory. For the diagonalization of the ring as discussed in Chapter 1 we made the assumptions that we have a C^*-algebra with a positive definite Hermitian form and that the conjugation of the algebra relates η and the Hermitian form as in (1.34). In cohomological field theories one can have a positive definite Hermitian form, but then the complex conjugation relating η and the Hermitian form does not necessarily respect the algebra structure as in (1.78).

3.2.1 The Descent Equations and Nonlocal Observables

The independence of the correlation functions of the positions of the operators suggests a new class of nonlocal physical observables. If the differential of the correlation function vanishes the differential of a physical operator must be zero up to a Q-commutator.

$$d\phi^{(0)} = \{Q, \phi^{(1)}\} \tag{3.40}$$

(We will also write $\phi^{(0)}$ for our local observables ϕ from now on.) The one form $\phi^{(1)}$ can be used to define a new observable

$$\Phi(C) = \int_C \phi^{(1)} \tag{3.41}$$

where C is a closed curve on Σ. Q-invariance is easily checked

$$\{Q, \Phi(C)\} = \int_C \{Q, \phi^{(1)}\} = \int_C d\phi^{(0)} = 0 \tag{3.42}$$

We can find another expression for $\phi^{(1)}$ if we look at the algebra spanned by the conserved charges again. If we identify $P = d = \{Q, G\}$ and use $\{Q, \phi^{(0)}\} = 0$ we find

$$d\phi^{(0)} = \{Q, \{G, \phi^{(0)}\}\} \tag{3.43}$$

thus

$$\phi^{(1)} = \{G, \phi^{(0)}\} \tag{3.44}$$

(up to Q- commutators)
This suggests to define also an operator

$$\phi^{(2)} = \{G, \phi^{(1)}\} \tag{3.45}$$

for which we can also derive a descent equation

$$d\phi^{(1)} = \{Q, \phi^{(2)}\} \tag{3.46}$$

The operator $\int_\Sigma \phi^{(2)}$ again gives a physical operator. The operators $\phi^{(0)}, \phi^{(1)}, \phi^{(2)}$ can be interpreted as components of one superfield.

3.2.2 Perturbations of the Theory

Topological perturbations

In [10] the descendant fields $\phi_i^{(2)}$ are used to define a family of topological field theories by adding $\int \phi_i^{(2)}$ with coupling constants t_i to the action

$$S(t) = S(0) - \sum_n t_n \int_\Sigma \phi_n^{(2)} \tag{3.47}$$

These perturbations preserve the topological properties of the theory. Thus all quantities can be calculated by factorization. The necessary information is encoded in the perturbed three-point functions

$$c_{ijk}(t) = \langle \phi_i \phi_j \phi_k \exp\left(\sum t_n \int \phi_n^{(2)}\right)\rangle_{t=0} = \langle \phi_i \phi_j \phi_k \rangle_t \tag{3.48}$$

area-preserving perturbations

We can introduce an area-dependence into these theories by adding $\int_\Sigma \phi^{(0)}\omega$ (ω volume element) to the action.

$$S(e) = S(0) + e \int_\Sigma \phi^{(0)}\omega \tag{3.49}$$

e is a coupling constant which can be absorbed in the total area as discussed for the Yang-Mills case. If $\phi^{(0)}$ carries some indices of e.g. a target space these should be contracted. More generally, we can add any function of $\phi^{(0)}$ integrated over Σ. The perturbed theory depends only on the volume – not on the full metric – because $\phi^{(0)}$ is a physical operator of the topological theory. The stress-energy tensor gets a contribution from the volume dependent part of the action. If we use (see Chapter 4)

$$\omega = \sqrt{g}d^2x \quad \text{and} \quad \delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu} \tag{3.50}$$

we get

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} + \frac{1}{2}\phi^{(0)}g_{\mu\nu} \tag{3.51}$$

The last term is not Q-exact. So the theory gets some dynamics. The operator $\phi^{(0)}$ obtains the role of the Hamiltonian discussed in previous chapters. It is clear that it commutes with the operator product because the algebra of the topological theory is commutative.

3.2.3 YM_2 as a Perturbed Cohomological Field Theory

Two-dimensional topological Yang-Mills theory can also be described as a cohomological field theory [28]. The following explanations are taken from [9].

We introduce an extra field ψ in the Yang-Mills theory which is an anti-commuting one form which takes values in the Lie algebra. Then we define the following transformation of fields

$$\delta A = \psi \quad \delta\psi = D\phi \quad \delta\phi = 0 \tag{3.52}$$

This is nilpotent up to a gauge transformation

$$\delta^2 A = D\phi \tag{3.53}$$

δ corresponds to the charge Q of the previous sections. Q should be thought of as the conserved charge associated to δ. We have to restrict to gauge invariant quantities because δ squares to zero only for these. A δ-invariant action containing the above multiplet of fields is

$$S = i \int_{\Sigma} \mathrm{tr}\,(\phi F + \frac{1}{2}\psi \wedge \psi) \tag{3.54}$$

which is an extension of the topological Yang-Mills action. We check the invariance:

$$\delta S = \int_{\Sigma} \mathrm{tr}\, D(\phi\psi) = \int_{\partial\Sigma} \mathrm{tr}\,\phi\psi = 0 \tag{3.55}$$

if Σ is a closed surface. The extra term $\int \mathrm{tr}\,\psi \wedge \psi$ has an interpretation as a symplectic form on the space of connections $\mathcal{A}$. Observables of the theory have to be δ- closed and gauge invariant. The field ϕ is δ-closed and we can form gauge invariant polynomials:

$$\mathrm{tr}\,\phi^2, \ldots\ldots, \mathrm{tr}\,\phi^n \tag{3.56}$$

The span of these invariant polynomials corresponds to the polynomial algebra spanned by the casimir operators. We can derive the nonlocal observables by differentiating the $\mathcal{O}$'s. We will do so for the quadratic casimir element $\mathcal{O}^{(0)} = \frac{1}{2}\mathrm{tr}\,\phi^2$:

$$\frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\mathrm{tr}\,\phi^2 = \mathrm{tr}\,\phi D_{\alpha}\phi = \delta \mathrm{tr}\,\phi\psi \tag{3.57}$$

We have found

$$\mathcal{O}^{(1)} = \mathrm{tr}\,\phi\psi \tag{3.58}$$

Differentiating $\mathcal{O}^{(1)}$ gives

$$\mathcal{O}^{(2)} = \mathrm{tr}\,\phi F + \frac{1}{2}\psi \wedge \psi \tag{3.59}$$

as we have seen before. $\int \mathcal{O}^{(2)}$ was our δ-invariant action. The action of physical Yang-Mills theory is of the form

$$\int \mathcal{O}^{(2)} + e^2 \int \mathcal{O}^{(0)}\omega \tag{3.60}$$

and thus a perturbed topological cohomological theory as described in the last section. Of course we could also use other casimirs (instead of the quadratic casimir) to add an area-preserving perturbation to the topological theory.

Chapter 4

Action of Diffeomorphisms

In the first chapter we have studied three different QFT's. The difference between these theories is that they have different symmetries. In the first chapter we only considered space-time symmetries, i.e. invariance under a certain diffeomorphism group. In our YM_2-example we had an invariance under area-preserving diffeomorphisms and a gauge symmetry. These symmetries might be not unrelated, since our gauge invariant physical states were also invariant under area-preserving diffeomorphisms. In this chapter we will study some aspects of space-time symmetry and the action of diffeomorphisms groups and investigate possible connections with gauge invariance in the YM_2-case.

4.1 The Stress-Energy Tensor

If a Lagrangian is invariant under a group of diffeomorphisms, we can apply Noether's theorem. This states that we can associate a locally conserved current j to any one-parameter group of invariants. Integrating the time component of the current over a fixed time slice gives a conserved charge Q. This can be seen to generate infinitesimal transformations by $\delta_\epsilon A = \epsilon[Q, A]$, where A is a field. Charges, that generate space-time transformations, are constructed from the stress-energy-tensor $T_{\mu\nu}$. Thus, we will study which properties the stress-tensor has if the theory is invariant under a certain diffeomorphism group. The construction of the stress-tensor by the Noether procedure has the disadvantage that it gives as a first step only a stress-energy tensor, which is not symmetric and in the case of gauge theories not even gauge invariant. There is a recipe how one can get a symmetric, gauge invariant tensor by adding a total divergence as a second step. However, there is an alternative definition of the stress-tensor which might be more elegant in this respect. We will use it in the following.

The stress-tensor can also be defined as an object that describes the dependence of the action S on the metric g. In this context the action has to be interpreted as a function on the space of metrics $\mathcal{M}_M$ of a given manifold M.

This is the space of differentiable maps from the manifold into the space of positive definite symmetric bilinear forms $PSBF$.

$$g : M \to PSBF \tag{4.1}$$

$PSBF$ is an open subset in the vector space of symmetric bilinear forms. The dependence of S on the metric is controlled by its derivative on a curve in the space of metrics:

$$\frac{d}{dt} S(g(t)) \mid_{t=0} = \partial S \mid_{g_0} v, \tag{4.2}$$

where $g(.)$ is a curve in $\mathcal{M}_M$ through $g_0 = g(0)$.

$$\begin{array}{llll} g(.): & M \times I \to & PSBF, & I \text{ an intervall} \subset \mathbf{R} \\ g(t): & M \to & PSBF & \end{array}$$

$v = \frac{d}{dt} g(t) \mid_{t=0}$ is a tangential vector to the space of metrics at the particular point g_0. v is a symmetric tensor field on the manifold M_{g_0} (where M_{g_0} denotes the manifold M equipped with a *fixed* metric g_0).
We want to define the stress-tensor T as a gradient of the action S seen as a function on the space of metrics. For this purpose we need an inner product G_g between tangential vectors of $\mathcal{M}_M$ for any point $g \in \mathcal{M}_M$. Then we can use the usual definition of a gradient and write:

$$\partial S \mid_{g_0} v =: G_{g_0}(T, v) \quad \forall v \in T_{g_0}\mathcal{M}_M \tag{4.3}$$

Thus, we have to look for a suitable G_g. We have seen that the tangential vectors $v \in T_{g_0}\mathcal{M}_M$ are symmetric tensor fields on M_{g_0}. A standard product between symmetric tensor fields on M_{g_0} is obtained as follows: If we have two vector spaces equipped with inner products then an inner product on the tensor product is given canonically. For any point $p \in M_{g_0}$ we have:

$$< v, w >_p = v_{\mu\nu} w^{\mu\nu} \tag{4.4}$$

which is a function on M_{g_0}. We can connect this product with an L^2-product between functions on M and obtain a product between tensor fields:

$$< v, w > = \int_M \sqrt{g_0}\, d^d x < v, w >_p \tag{4.5}$$

This is independent of the points p of M (of course), but it depends on the metric g_0 – that is, on the point $g_0 \in \mathcal{M}_M$. We use this product as a metric on $\mathcal{M}_M$.

$$G_{g_0}(v, w) = \int_M d^d x \sqrt{g_0} < v, w >_p \tag{4.6}$$

T is defined by:

$$\partial S \mid_{g_0} v = G_{g_0}(T, v) = \int_M d^d x \sqrt{g_0}\, T_{\mu\nu} v^{\mu\nu} \tag{4.7}$$

It is an element of $T_{g_0}\mathcal{M}_M$ or a symmetric tensor field on M_{g_0}. The definition (4.7) can equivalently be written as:

$$T_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{4.8}$$

We want to investigate its properties if the action is invariant under certain changes of the metric:

$$S(g(t)) = const \Rightarrow \partial S\mid_{g_0} v = 0 \Rightarrow \int_M d^d x\sqrt{g_0}\,T_{\mu\nu}v^{\mu\nu} = 0 \tag{4.9}$$

1. **topological field theory**
   In this case the action is invariant under arbitrary deformations of the metric. It follows that:
   $$T = 0 \tag{4.10}$$

2. **volume-preserving theory**
   Here, we have an invariance under changes of the metric which leave the volume element ω of the manifold invariant. For a point $p \in M$ we have
   $$\omega_p = \sqrt{\det g_p}\,d^d x \tag{4.11}$$
   Volume-invariance means:
   $$\begin{aligned} 0 &= \frac{d}{dt}\sqrt{\det g(t)}\mid_{t=0} \\ &= \frac{1}{2}\frac{1}{\sqrt{\det g_0}}\frac{d}{dt}\det(g_0 + tv)\mid_{t=0} \\ &= \frac{1}{2}\frac{1}{\sqrt{\det g_0}}\det g_0\,\frac{d}{dt}\det(\mathbf{1} + tg_0^{-1}v)\mid_{t=0} \\ &= \frac{1}{2}\sqrt{\det g_0}\ \mathrm{tr}\ g_0^{-1}v \\ &= \frac{1}{2}\sqrt{\det g_0}\ g^{\mu\nu}v_{\mu\nu} \end{aligned} \tag{4.12}$$
   We conclude:
   $$T \sim g_0 \tag{4.13}$$
   T is proportional to the metric. It can be fully expressed in terms of its trace.

3. **conformal field theory**
   Arguments similar to the following can be found in any introductory text on CFT, e.g. [17]. CFT's are invariant under rescalings of the metric:
   $$g_p(t) = \lambda_p(t)g_p(0) \quad \text{for arbitrary } \lambda \tag{4.14}$$

The corresponding tangential vectors are:

$$v = \frac{d}{dt} g_p(t) \mid_{t=0} = g_p(0) \frac{d}{dt} \lambda_p(t) \mid_{t=0} \tag{4.15}$$

$\frac{d}{dt}\lambda_p(t) \mid_{t=0} = \dot{\lambda}_p$ is an arbitrary function on M. So we have:

$$\begin{aligned} 0 &= G_{g_0}(T, \dot{\lambda}_p g_0) \\ &= \int d^d x \sqrt{g_0} < T, g_0 > \dot{\lambda}_p \end{aligned}$$

Because $\dot{\lambda}_p$ is arbitrary, we can conclude that:

$$< T, g_0 > = \mathrm{tr}\, T = 0 \tag{4.16}$$

This time the trace vanishes –in contrast to the volume-preserving theory. Let's look at the stress-energy tensor in two-dimensional CFT for a moment: If we are on a flat space-time,

$$g_{\mu\nu} = \delta_{\mu\nu} \tag{4.17}$$

we can transform to coordinates

$$z = x + iy \quad \bar{z} = x - iy \tag{4.18}$$

and calculate the stress-tensor in these coordinates. We get:

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4} T^{\mu}_{\mu} = 0 \tag{4.19}$$

and two non vanishing components:

$$T_{zz} = T, \quad T_{\bar{z}\bar{z}} = \overline{T} \tag{4.20}$$

The conservation law gives:

$$\partial_{\bar{z}} T = 0 \qquad \partial_z \overline{T} = 0 \tag{4.21}$$

T is holomorphic, $\overline{T}$ is antiholomorphic. The conserved charges that generate local conformal transformations are given by:

$$Q = \frac{1}{2\pi i} \oint (dz T(z) \epsilon(z) + d\bar{z} \overline{T}(\bar{z}) \bar{\epsilon}(\bar{z})) \tag{4.22}$$

If we look at the transformations $\epsilon(z) = z^{n+1}$ we get the modes of the stress-energy tensor:

$$L_n = \frac{1}{2\pi i} \oint dz\, T(z) z^{n+1} \quad T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2} \tag{4.23}$$

In a quantum theory, the L_n have the following commutation relations:

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \tag{4.24}$$

This algebra is called the Virasoro algebra and plays a fundamental role in CFT.

4.2 Two-Dimensional Yang-Mills Theory

In the previous chapters we have associated states to surfaces in YM_2. These states can be seen as a path-integral as a functional of boundary conditions. We picked gauge invariant boundary conditions in Chapter 2. Our Hilbert space consists of functions of the conjugacy classes of the holonomies. These conjugacy classes are also invariant under diffeomorphisms of the circle. The diffeomorphisms of S^1 arise as a restriction of area-preserving diffeomorphisms of a collar $S^1 \times I$. If we want to see an action of the diffeomorphisms on states, we can try to pick non gauge invariant boundary conditions. This corresponds to extending the Hilbert space with non physical states. In the following we try to find an action of the diffeomorphisms on spaces which contain states that are not gauge invariant.

4.2.1 BRST-Quantization

Faddeev-Popov-BRST-quantization [13], [4], [26] is a quantization method frequently used in gauge theories. Let me briefly describe it.

The path-integral $\int DAe^{-S[A]}$ over all connections is divergent because we integrate over gauge equivalent connections. One would like to extract the volume of the gauge group

$$\mathcal{G} = maps(\Sigma, G) \tag{4.25}$$

This means that we would like to integrate over only one representative of each class of gauge equivalent connections. Instead of our formal path-integral measure DA we would like to have a measure on the quotient space of all connections modulo gauge transformations. This aim can be achieved by the Faddeev-Popov procedure. One introduces a ghost field c, an antighost field $\overline{c}$ –these are anticommuting zero-forms– and a field w which is a commuting zero form. We define the following nilpotent BRST-transformation law:

$$\begin{aligned} \delta A_\mu &= -D_\mu c = -(\partial_\mu c + [A_\mu, c]) && (4.26)\\ \delta c &= \frac{1}{2}[c, c] && (4.27)\\ \delta \overline{c} &= iw && (4.28)\\ \delta w &= 0 && (4.29) \end{aligned}$$

The BRST-transformation acts on A as a "fermionic gauge transformation". The gauge parameter is simply replaced by the anticommuting ghost field. Let us now pick a gauge fixing condition

$$G(A) = 0 \tag{4.30}$$

and add a term δV to the action, where

$$V = \int d\mu \, \mathrm{tr} \, \overline{c} G(A) \tag{4.31}$$

For the variation δV we get:

$$\delta V = i \int d\mu \mathrm{tr}\, w G(A) + \int d\mu \mathrm{tr}\, \overline{c}\delta[G(A)] \tag{4.32}$$

The new action is

$$S = S_{YM} + \delta V \tag{4.33}$$

In the path-integral $\int DADcD\overline{c}Dw\, e^{-S}$ the integration over w gives a delta function and enforces $G(A) = 0$. So we managed to fix the gauge. Because we added a BRST-variation to the action, the whole action is now BRST-invariant instead of gauge invariant. Gauge invariance is replaced by BRST-invariance [4], [26]. We can associate a charge Q to the transformation δ. The symmetry generator acts on the Hilbert space. The Hilbert space of a theory with the new action contains of course also ghost-states. Because the BRST-transformation encodes the gauge transformation the physical subspace of the ghost-extended space consists of states fulfilling

$$Q|\psi>= 0 \tag{4.34}$$

Similar to the cohomological theories described in Chapter 3 we want to identify states that only differ by a Q-commutator. The physical Hilbert space consists of Q-cohomology classes.

If we want to say something about the action of the diffeomorphisms in this theory we can for example calculate the stress-energy tensor using

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \tag{4.35}$$

The stress tensor consists of two parts:

$$T_{\mu\nu} = \frac{1}{2} T g_{\mu\nu} + T^{ghost}_{\mu\nu} \tag{4.36}$$

The first part, which does not contain ghosts, is proportional to the metric as shown in the last section. We calculate the ghost contribution:

$$\begin{aligned} T^{ghost}_{\mu\nu} &= \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \{Q, V\} \\ &= \{Q, \frac{1}{\sqrt{g}} \frac{\delta V}{\delta g^{\mu\nu}}\} \\ &= \{Q, \Lambda_{\mu\nu}\} \end{aligned} \tag{4.37}$$

with

$$\Lambda_{\mu\nu} = \frac{1}{2} \overline{c} G(A) g_{\mu\nu} + \overline{c} \frac{\delta G}{\delta g^{\mu\nu}} \tag{4.38}$$

Thus

$$T^{ghost}_{\mu\nu} = \frac{1}{2} i w G(A) g_{\mu\nu} + \frac{1}{2} \overline{c} \left(\delta G(A)\right) g_{\mu\nu} + \overline{c} \delta \left(\frac{\delta G}{\delta g_{\mu\nu}} \right) + i w \frac{\delta G}{\delta g_{\mu\nu}} \tag{4.39}$$

The ghost contribution to the stress-energy tensor contains also nondiagonal parts. If it acts on a representative of a Q-cohomology class we get another element of the same cohomology class, because $T^{ghost}_{\mu\nu}$ is a Q-commutator. So it acts as a very special kind of gauge transformation. Note that it depends on A.

4.2.2 Diffeomorphisms in the Classical Theory

Classical Yang-Mills theory has been treated in the beautiful paper [3]. From the action

$$S = \int_\Sigma \operatorname{tr} F \wedge *F \tag{4.40}$$

one can derive the classical equations of motion

$$D * F = 0 \tag{4.41}$$

where D is the covariant derivative. This is on Lie algebra valued functions ξ given by

$$D\xi = d\xi + [A, \xi] \tag{4.42}$$

It extends to n-forms compatible with the wedge-product. In addition to the equations of motion we have the Bianchi-identity

$$DF = 0 \tag{4.43}$$

Thus, the curvature of an extremal connection is a harmonic two-form. There are two different kinds of solution to the equations of motion:

1. $f := *F = 0 \Rightarrow F = 0$
2. $f \neq 0$

Let us look at both types of solution:

1. If $F = 0$ the space of solutions is the space of flat connections. We want to see a relation between diffeomorphisms and gauge transformations acting on the flat connections. We might guess that any diffeomorphism can be made undone by a special gauge transformation because we could not see an action of diffeomorphisms on gauge invariant states. Let φ be a diffeomorphism. Our question is: Can we find a gauge transformation such that:

   $$\varphi^* A = A^g \tag{4.44}$$

   Here $\varphi^* A$ is the pullback of A by φ and A^g is the transformation of A by a gauge transformation g:

   $$A^g = gAg^{-1} + g^{-1}dg \tag{4.45}$$

An infinitesimal version of equation (4.44) is

$$L_X A = D\lambda \tag{4.46}$$

where L is the Lie derivative, X the generator of a diffeomorphism and λ the generator of a gauge transformation. Inserting Cartan's identity

$$L_X = di_X + i_X d \tag{4.47}$$

we get

$$di_X A + i_X dA = D\lambda \tag{4.48}$$

Because $F = 0$ we have

$$0 = i_X F = i_X dA + [i_X A, A] \Rightarrow i_X dA = [A, i_X A] \tag{4.49}$$

and thus

$$D(i_X A) = D\lambda \tag{4.50}$$

So $\lambda = i_X A$ does the job. The action of a diffeomorphism generated by X on a particular connection can be made undone by a gauge transformation. Note that λ depends explicitly on A.

2. Let us study the case $f \neq 0$. We will first look at the Abelian case, i.e. Maxwell theory. Here we have as an equation of motion

$$df = 0 \tag{4.51}$$

This means that f is a constant, $f = c$, c a real number. The field strength is accordingly

$$F = *f = c\omega \tag{4.52}$$

This is manifestly invariant under area-preserving diffeomorphisms, but not under the full diffeomorphism group. In this case we can only expect a relation between area-preserving diffeomorphisms and gauge transformations. Area-preserving (or symplectic) vector fields have the property that

$$L_X\omega = 0 = i_X d\omega + di_X\omega = di_X\omega \tag{4.53}$$

(ω is a two-form.) Since ω is nondegenerate, we have a one-to-one correspondence between closed one-forms α with $d\alpha = 0$ and symplectic vector fields. We now assume that X is a hamiltonian vector field, i.e. there exists a function h such that

$$i_X\omega = dh \tag{4.54}$$

We want to find a gauge parameter λ for which

$$L_X A = d\lambda \tag{4.55}$$

We calculate:

$$\begin{aligned} L_X A &= d i_X A + i_X dA \\ &= d i_X A + i_X c\omega \\ &= d(i_X A + ch) \end{aligned}$$

Thus,

$$\lambda = i_X A + ch \tag{4.56}$$

is a possible choice for a generator of a gauge transformation. To include the non-abelian case we rewrite this formula as

$$\lambda = *F\, h + i_X A \tag{4.57}$$

and verify that this formula also holds for nonabelian groups. We have to compute the covariant derivative:

$$\begin{aligned} D\lambda &= D(*F\, h) \;+\; D i_X A \\ &= (D * F)\, h \;+\; *F\, dh + d i_X A + [A, i_X A] \\ &= *F\, i_X \omega + L_X A - i_X dA + [A, i_X A] \\ &= i_X f\omega + L_X A - i_X F \\ &= L_X A \end{aligned} \tag{4.58}$$

Here we have used the equations of motion $D * F = 0$. By this we have also solved the problem for the nonabelian case.

Summary

In this thesis we work out some aspects of area-preserving QFT.
In Chapter 1 we give an axiomatic formulation of volume-preserving theories. We use the functorial approach suggested by Segal for two-dimensional conformal field theory and by Atiyah for topological field theory. We review some aspects of their work to enable a comparison in two dimensions between area-preserving, conformal and topological field theory. The two-dimensional case of area-preserving QFT is worked out. Explicit formulas for partition functions and the operator algebra are given. The relation with topological field theory is explained.
Chapter 2 contains a discussion of two-dimensional Yang-Mills gauge theory, a model, which has been discussed by many authors. Here, we stress its properties as an area-preserving QFT. We also review a lattice approach and general formulas for the calculation of expectation values of the physical observables of the theory, the Wilson loops. We apply these formulas to obtain some explicit results for nontrivial Wilson loops on the torus.
In Chapter 3 we study some further examples for area-preserving QFT. The first one starts with a formulation of topological YM_2 with finite groups. We investigate the mathematical structure and give area-preserving perturbations. We then review cohomological topological field theories and perturb them to obtain area-preserving theories.
In Chapter 4 we discuss the action of diffeomorphisms. We describe the stress-energy tensor and its properties in theories with certain invariances. In area-preserving QFT it can be fully expressed in terms of its trace whereas the trace vanishes in CFT. In this sense area-preserving and conformal field theory are "orthogonal" to each other. We had a closer look at area-preserving diffeomorphisms in YM_2. The diffeomorphisms act trivially on physical (gauge invariant) states. We looked at the action of the diffeomorphisms on non-physical states and on connections in the classical theory. We found that the action of area-preserving diffeomorphisms can be compared to certain gauge transformations.
The results of this thesis might lead one to be interested in area-preserving QFT in more generality. What happens if we look at systems without gauge symmetry? Can we formulate geometrically motivated axioms such that an action of the symmetry algebra W_∞ can be derived? What happens in higher dimensions?

Volume-preserving symmetry plays an important role in many physical systems, e.g. the quantum Hall Effect. Further study of an axiomatic formulation and applications remain interesting projects.

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