Cosmological Perturbations
in
Multifield Inflationary Models

Marc H. Thormeier

Diplomarbeit in Physik

auswärtig angefertigt an der
Ludwig-Maximilians-Universität München,
Institut für Theoretische Physik

vorgelegt der

Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

Mai 1999

Ich versichere, daß ich diese Arbeit selbständig verfaßt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie die Zitate kenntlich gemacht habe.

Referent:	Prof.Dr. Viacheslav Mukhanov
Korreferent:	Prof.Dr. Hans-Peter Nilles

Abstract

Inflation, a period of accelerated expansion of the Universe, explains its homogeneity and flatness. In the most natural inflationary scenario such an expansion is driven by the potential of a slowly decaying scalar field φ. After inflation the field is then supposed to quickly decay into radiation. Quantum fluctuations happening during the inflationary stage provide the seeds for inhomogeneities in today's cosmic microwave background radiation temperature anisotropy. The standard assumption is that the fluctuations of the φ-field are the most important ones, causing an almost scale-invariant spectrum of curvature perturbations with Gaussian statistics. In this text the perturbations of an additional scalar field, the σ-field, were calculated: the field's energy density is so low that it has no influence on the evolution of φ and the scale factor, the fluctuations are thus of isocurvature nature; furthermore the average value of this field is supposed to vanish, so that the created density perturbations obey nongaussian statistics. If the σ-field is stable after inflation, its energy density will decay less rapidly than the radiation produced by the inflaton field and will determine the temperature anisotropies. One finds promising results for the spectrum of the perturbations with the two fields coupled and uncoupled to each other, showing an alternative mechanism for the formation of the seeds of structure formation.

Notations and conventions throughout this text

We use units with $\hbar = c = 4\pi G = k_B = 1$.

The scale factor a is normalised so that it is unity at the end of inflation, i.e. $a_f = 1$.

The Latin indices i and j run over the three spatial coordinate labels $1, 2, 3$.

The Greek indices μ, ν and λ run over the four spacetime coordinate labels $0, 1, 2, 3$.

We follow Einstein's convention that repeated indices are summed.

The trace of the Minkowski metric $\eta_{\mu\nu}$ is positive.

A comma behind a quantity denotes the partial derivative, a semicolon the covariant derivative.

A hat ($\hat{}$) denotes an operator.

$|A|$ denotes the absolute value of A.

$\langle A \rangle$ denotes the spatial average/expectation value of $A/\hat{A}$.

A^* denotes the complex conjugate of a A, and likewise $\hat{A}^\dagger$ denotes the adjoint of $\hat{A}$.

"$\sim$" means "same order of magnitude".

":=" means "defined to be equal".

Contents

1 Introduction: Classical cosmology and its flaws **13**
1.1 The standard cosmological model 13
1.2 Problems of the standard cosmological model 17

2 The chaotic inflationary scenario **21**
2.1 The basic equations 21
2.2 The period with the kinetic term dominating 23
2.3 The period of inflation 24
2.4 Will inflation get started? 27
2.5 The period of oscillation and reheating 28
2.6 Other inflationary models 30

3 Cosmological perturbations **33**
3.1 Inhomogeneities of today's Universe 33
3.2 Concepts of cosmological perturbation theory 34
3.3 The Sachs-Wolfe effect 36

4 Quantum fluctuations of the inflaton **39**
4.1 The k-range of interest for inflationary perturbations 39
4.2 The initial conditions for the fluctuations 41
4.3 The power spectrum for the fluctuations of the inflaton 42
4.4 The self-reproducing Universe 47

5 Multi-field inflationary models **49**
5.1 Introducing a second scalar field 49
5.2 A useful approximation for H_k 52
5.3 The power spectrum for the fluctuations of the σ-field 52
5.4 Calculating the integral in the expression of the power spectrum 54

6 Four useful transformations of the Klein-Gordon equation for the $\delta\sigma$-field **59**
6.1 The scale factor as a variable 59

6.2 Conformal time as a variable . . . 60
6.3 $\ln \frac{aH}{k}$ as a variable . . . 61
6.4 $\frac{3H}{M}$ as a variable . . . 64

7 Some useful approximations to the Klein-Gordon equation of the $\delta\sigma$-field 67
7.1 The long and short wavelength approximations . . . 67
7.2 The slow roll approximation . . . 68

8 A toy model: the $\delta\sigma$-field in de Sitter space 71
8.1 The spectrum of the $\delta\sigma$-field . . . 71
8.2 The energy density and the power spectrum . . . 74

9 The $\delta\sigma$-field in quasi de Sitter space, not coupled to the inflaton 79
9.1 Preliminary remarks . . . 79
9.2 The k-range of LWA-validity . . . 80
9.3 The spectrum of the $\delta\sigma$-field . . . 82
9.4 The energy density and the power spectrum for $m < M$ . . . 88
9.5 The energy density and the power spectrum for $m > M$ . . . 90

10 Quintessential inflation 93
10.1 The model without radiation . . . 93
10.2 The model with radiation . . . 95

11 The $\delta\sigma$-field in quasi de Sitter space, strongly coupled to the inflaton 101
11.1 The k-range of LWA . . . 101
11.2 The spectrum of the $\delta\sigma$-field . . . 102
11.3 The energy density and the power spectrum . . . 103

12 The $\delta\sigma$-field in a radiation-dominated background 105
12.1 The evolution of the field . . . 105
12.2 The energy density and the power spectrum . . . 110

13 Applications 113
13.1 Constraints on the model . . . 113
13.2 Remarks to *Inflation and Large Internal Dimensions* . . . 114

14 Summary and conclusion 117

A Tables of symbols, abbreviations etc. 121

Bibliography, Acknowledgements 129

Chapter 1

Introduction: Classical cosmology and its flaws

1.1 The standard cosmological model

Newton's theory of gravitation may be used as long as one has for the mass $\mathcal{M}$ on a certain length scale $\mathcal{R}$

$$\frac{\mathcal{M}}{\mathcal{R}} \ll 1. \tag{1.1}$$

Otherwise it has to be replaced by its generalisation, Einstein's theory of general relativity (GR). This is the case for very compact objects like neutron stars and black holes, or in the Universe on scales $> 200Mpc$, taking the average cosmic density to be $\sim 10^{-26}\frac{kg}{m^3}$ (10^{-120} in dimensionless units). Einstein's field equations are a set of differential equations to determine the components of the metric tensor $\mathbf{g}$. They read

$$\mathbf{G} = 2\mathbf{T} + \Lambda\mathbf{g}, \tag{1.2}$$

with $\mathbf{G}$ being the Einstein tensor (constructed in a nonlinear way out of $\mathbf{g}$'s components and their derivatives up to 2^{nd} order), $\mathbf{T}$ the energy-momentum tensor of the fluid considered and Λ the cosmological constant. As the tensors in (1.2) are symmetric, one has a collection of 10 instead of 16 equations.

To describe the dynamics of the Universe in big scales with (1.2) we use the following observation: *On large scales the Universe looks the same in all directions.* It's *postulated* that this is the case from every point in space[1], which implies a homogeneous matter distribution.[2]

[1] We would not be able have a science concerning the Universe as a hole if we did not assume it to be everywhere like where we are located!

[2] Furthermore we must assume the matter distribution to be regular; otherwise one has a stochastically fractal Universe, highly inhomogeneous and hence not leading to (1.3).

This is called the *cosmological principle*,[3] a very powerful assumption, from which one can find that the Universe is described by the maximally-symmetric Robertson-Walker metric with the line element

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\mathcal{K}r^2} + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \; d\phi^2\right), \tag{1.3}$$

a being the scale factor of the Universe, t the cosmic time, r a radial coordinate, ϑ and ϕ the polar and azimuthal angle, respectively; r is chosen in a way that $\mathcal{K}$ is the sign of the curvature, ± 1 or 0, giving a closed, open or flat (Euclidean) Universe. The first has a finite size, the latter two are infinite in extent. The only dynamical variable left is $a(t)$, being a measure of the distances between objects. So (1.3) allows a Universe with time-varying distances. This was shown to be the case in 1929 by Hubble who interpreted the redshift of the absorption lines of distant galaxies due to an *expanding* Universe.

To describe the matter/energy content itself one uses the approximation that it behaves as a perfect fluid so that in the momentarily comoving reference frame of (1.3) one has

$$T_\mu{}^\nu = \begin{pmatrix} -\varrho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}, \tag{1.4}$$

ϱ and p being the energy density and pressure of the fluid, respectively.

One combines the Robertson-Walker metric and (1.4) with Einstein's field equations to find out how a evolves in time. All off-diagonal components vanish. The 00-equation gives the *Friedmann equation*[4]

$$\frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2} = \frac{2}{3}\varrho + \frac{\Lambda}{3}, \tag{1.5}$$

with the overdot denoting the derivative with respect to time. The diagonal spatial components give

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}}{a^2} = \Lambda - 2p. \tag{1.6}$$

The ratio $\frac{\dot{a}}{a}$ is called *Hubble parameter* (since it is a measure for the rate of the expansion of the Universe) and usually denoted by H. Plugging (1.5) into (1.6) one gets

$$\ddot{a} + \Big(\frac{\varrho - \Lambda}{3} + p\Big)\; a = 0. \tag{1.7}$$

[3]To be more precise, one actually speaks of the *Copernican* cosmological principle (we are nothing special), to make a distinction towards its two generalisations, contrary to each other, the *perfect* cosmological principle and the *anthropic* cosmological principle. The former postulates that the Universe looks the same throughout all time, resulting in the so-called steady-state model contrary to the big bang theory, while the latter claims that we are living at a special point in time so that the evolution of intelligent life was possible.

[4]This equation can also be derived (less rigorously) from Newtonian mechanics.

To eliminate $\ddot{a}$ and to get rid of the Λ one multiplies (1.5) by a^2, differentiates with respect to time and substitutes the resulting $\ddot{a}$ with the help of the equation above, giving[5]

$$\dot{\varrho} = -3(\varrho + p)\frac{\dot{a}}{a} \quad \text{or} \quad d(\varrho a^3) = -pd(a^3); \tag{1.8}$$

we re-derived the 1^{st} law of thermodynamics, local energy conservation. From this equation together with the equation of state $p = p(\varrho)$ we can find ϱ, and from ϱ and (1.5) one gets a complete description of the evolution of the Universe.[6,7] If the equation of states looks like

$$p = \alpha\varrho, \tag{1.9}$$

α being a constant specific to the fluid considered, we get

$$\varrho \propto a^{-3(\alpha+1)}. \tag{1.10}$$

The fluids we observe today in the Universe are nonrelativistic gases, called "dust" (galaxies)[8], and ultrarelativistic gas (radiation). For these one has[9]

$$\begin{aligned} \alpha_{dust} &= 0, \\ \alpha_{rad} &= \frac{1}{3}. \end{aligned} \tag{1.11}$$

From this and (1.10) we get

$$\begin{aligned} \varrho_{dust} &\propto a^{-3}, \\ \varrho_{rad} &\propto a^{-4}. \end{aligned} \tag{1.12}$$

[5]This result can also be reached using (1.3) and the fact that the divergence of $\mathbf{G}$ vanishes. With (1.2) $\mathbf{T}$ is hence conserved, i.e.

$$T^{\mu\nu}{}_{;\mu} = 0,$$

with the semicolon denoting the covariant derivative.

[6]Due to the Bianchi identities (1.6) is not independent from (1.8) and (1.5).

[7]In case of the inflationary Universe we will later find ϱ from the Klein-Gordon equation.

[8]Due to their low relative particular velocities galaxies have low kinetic energies.

[9]To calculate α we start from the ideal gas law. Since the absolute temperature $\mathcal{T}$ of a gas with no interior degrees of freedom is equal one third of the time-average of the kinetic energy E_{kin} of a particle, and E_{kin} is the difference between the total energy E_{tot} and the rest energy E_{rest}, we get with

$$\varrho = \frac{N}{V}\,E_{tot},$$

V being the volume and N the particle number enclosed in this volume, that

$$p = \frac{1}{3}\left(1 - \frac{E_{rest}}{E_{tot}}\right)\varrho.$$

For particles with very low velocities (dust) one has $E_{tot} \approx E_{rest}$, while for an ultra-relativistic gas one has $E_{tot} \gg E_{rest}$. So radiation is "harder" than dust.

Today one sees a Universe with $\varrho_{rad} \ll \varrho_{dust}$. But from the two equations above we follow that when we go backwards in time we eventually arrive at a radiation-dominated Universe with $\varrho_{rad} \gg \varrho_{dust}$ (if we are allowed to go back that far!).

The only other quantity to consider is Λ. It is very small, $\leq 10^{-120}$, but recent results indicate that it is bigger than zero, see [1] and [2].[10] One can identify Λ with an energy density $2\varrho_\Lambda$ (see (1.5)). So if one starts with $\varrho_{dust} \gg \varrho_\Lambda$, eventually one arrives at a Universe dominated by Λ, or equivalently *a Universe dominated by a constant energy-density* (such a spacetime is called *de Sitter space*). In fact this seems to be the case today. But since this text will deal with the very early Universe with energy densities much bigger then ϱ_Λ, we can easily neglect the cosmological constant for that period.[11] As a and $\dot{a}$ are both positive, we get from (1.7) that

$$\ddot{a} < 0 \tag{1.13}$$

all the time in the past for both radiation- and dust-dominance. So within a finite time (~ 15 billion years) there has to have been the instant with $a = 0$, the initial singularity, called the *big bang*. Actually this extrapolation to the very very beginning is not correct because classical general relativity is not valid during the Planck epoch (denoted by the subscript$_p$) due to quantum gravitational effects. To solve (1.5) for the early Universe we neglect the term $\frac{\mathcal{K}}{a^2}$, because in the beginning the expansion was sufficiently fast so that $H^2 \gg \frac{\mathcal{K}}{a^2}$. One gets

$$a \;\propto\; t^{\frac{2}{3}\left(\frac{1}{1+\alpha}\right)}. \tag{1.14}$$

So H decreases with a increasing. Hence later one has the evolution determined by $\mathcal{K}$ and Λ.[12]

By observing[13] a uniform cosmic microwave background radiation (CMBR) with a temperature of about $3K$, it was established that the big bang was a *hot* one, with a huge excess of photons over baryonic matter. This is furthermore confirmed by the agreement of the

[10] In [3] this is considered to be one of the three observational milestones of 20^{th} century cosmology, starting a golden age.

[11] The dynamics of an empty Universe (hence $\varrho = p = 0$) with only the cosmological constant can be described using (1.5) and (1.7), resulting in

$$a(t) = \frac{e^{Ht} \;+\; \mathcal{K}\, e^{-Ht}}{2H},$$

an exponentially expanding space for $t > H^{-1}$ with $H = \sqrt{\frac{\Lambda}{3}} = \textit{constant}$. For a Euclidean Universe H is the Hubble parameter. We'll deal with a similar case when we treat the inflationary Universe.

[12] Cosmological models with $\Lambda = 0$ and positive curvature predict a final recollapsing of the Universe (*big crunch*).

[13] First by Penzias and Wilson in 1964, later with very high precision by the satellite COBE, and soon by the satellites Planck-Surveyor and MAP

observed relative abundances of helium, hydrogen and deuterium with the predictions from the theory of primordial nucleosynthesis. The CMBR is the earliest signal detected by men, an "echo of a scorching past" [4], from a time about 300,000 years after the big bang, when the proton-electron plasma underwent a phase transition to become neutral hydrogen, so that matter and radiation –so far in thermal equilibrium– decoupled and the Universe thus became transparent.[14] Until shortly before, the energy content of the Universe was primarily given by ultrarelativistic material, hence giving a radiation-dominated Universe. The theory is able to make predictions on extremely firm ground beyond the moment of plasma/hydrogen phase transition up to $10^{-5}s$ after the big bang. For earlier times one reaches energy densities of the scale of chromodynamics, and even further back electroweak and grand unification scales where the exact numbers depend on the particular model.[15]

1.2 Problems of the standard cosmological model

Although we used the cosmological principle to reduce the complicated coupled Einstein field equations to a relatively easy set of two equations, the question remains *why* the cosmological principle is valid after all. Of course this cannot be answered in the context of the standard cosmological model, since the cosmological principle is incorporated there as an axiom. However, it is far more natural to assume a spatially wildly varying matter/energy distribution as the initial condition at $t = t_p \sim 10^{-43}s$. In that case, in order to get isotropy around every point in space there have to be interactions between the regions of different densities. But no action can propagate faster than the speed of light. So today's observable Universe must have emerged from a causally connected (hence homogeneous) region at $t = t_p$. In other words, only if *at the time $t = t_p$ the size $\mathcal{R}_{cc}$ of the causally connected region coincides with the size $\mathcal{R}_U$ of today's observed Universe at that time, the cosmological principle could be taken as a corollary.* To check this we write, with the subscript$_0$ denoting today,

$$\frac{\mathcal{R}_U(t_p)}{\mathcal{R}_{cc}(t_p)} = \frac{\mathcal{R}_U(t_0)\frac{a_p}{a_0}}{t_p} \sim \frac{\frac{1}{H_0}\frac{a_p}{a_0}}{t_p} = \frac{\frac{a_p}{\dot{a}_0}}{t_p} \sim \frac{\dot{a}_p}{\dot{a}_0}, \tag{1.15}$$

[14] If we had very sensitive detectors for neutrinos and gravitational waves, we could look even deeper into the past back to the moment when electroweak phenomena stopped dominating and neutrinos and electrons decoupled; we could see even further back to the Planck epoch if we were able to detect gravitons.

[15] However, the common idea is that the energy of the particles initially was higher than any rest mass and interaction scale. As the Universe expanded, the energy density decreased, giving rise to a sequence of eras with each having a certain interaction (and hence the particles exhibiting this interaction) to be dominant. An era ends when the particular interaction rate drops below the expansion rate, the interaction "freezes", the corresponding particles decouple. So one first had the era of GUT-symmetry breaking, then CP-violation, the hadronic era, the leptonic era, the era of nucleosynthesis, the photon-dominated era, and finally the dust- and Λ dominated eras.

with the last step being only a rough approximation. Using (1.14) one gets

$$\frac{\dot{a}_1}{\dot{a}_2} = \left(\frac{t_2}{t_1}\right)^{1-\frac{2}{3}\frac{1}{1+\alpha}}, \tag{1.16}$$

so that with (1.11) one gets, t_{eq} being the moment when radiation dominance ends,[16]

$$\frac{\dot{a}_p}{\dot{a}_0} \sim \left(\frac{t_{eq}}{t_p}\right)^{\frac{1}{2}} \left(\frac{t_0}{t_{eq}}\right)^{\frac{1}{3}} \sim \left(10^{56}\right)^{1/2} \left(10^{4}\right)^{1/3} \sim 10^{29} \tag{1.17}$$

which gives

$$\frac{\mathcal{R}_U^3(t_p)}{\mathcal{R}_{cc}^3(t_p)} \sim 10^{87}. \tag{1.18}$$

Today's observable Universe is made of a giant number of initially causally disconnected regions, the cosmic isotropy is not explainable by mere "diffusion" during the period of quantum gravity. This problem is called the *isotropy* or *horizon problem*.

The second problem is the one concerning the flatness and expansion rate of the Universe. Investigating (1.5) one finds that the curvature of the Universe is determined by the *density parameter* Ω

$$\Omega := \frac{2\varrho}{3H^2}, \tag{1.19}$$

so that

$$\Omega - 1 = \frac{\frac{2}{3}\varrho - H^2}{H^2} = \frac{\mathcal{K}}{a^2H^2}. \tag{1.20}$$

$\Omega < 1$ gives an open Universe, $\Omega > 1$ results in a closed Universe. We *observe* a Universe that is highly Euclidean and consequently flat to a very high precision, so the Universe must have a very low curvature or

$$\Omega_0 \approx 1, \tag{1.21}$$

with only a small range to vary. But from (1.20) one gets

$$\Omega_0 - 1 = \frac{\mathcal{K}}{a_p^2H_p^2}\left(\frac{a_pH_p}{a_0H_0}\right)^2 = \left(\Omega_p - 1\right)\left(\frac{a_pH_p}{a_0H_0}\right)^2 \tag{1.22}$$

or

$$\Omega_0 = 1 + \left(\Omega_p - 1\right)\left(\frac{\dot{a}_p}{\dot{a}_0}\right)^2. \tag{1.23}$$

The ratio in brackets is the same we encountered with the horizon problem, so one can have Ω_0 close to unity only if $\Omega_p = 1$ to a high accuracy.[17] In other words, the initial energy density

[16] Notice that we shall very often just make order-of-magnitude estimates and hence omit numerical prefactors that are $\mathcal{O}(1)$.

[17] Ω is an unstable fixed point.

had to balance extremely well the initial expansion rate, God had only a very tiny volume in phase space to start His Universe from. But on the other hand Ω is a free parameter of the standard model. So not to invoke new physics, the allowed range for this ratio to vary should be large. This is called the *flatness problem* or the *problem of initial velocities*.[18,19]

Surprisingly both problems root in the same thing, namely $\dot{a}_p/\dot{a}_0$. One could avoid its high value if there existed an interval between the Planck time and the radiation dominated period so that

$$\frac{\dot{a}_p}{\dot{a}_0} \sim \left(\frac{t_r}{t_p}\right)^{1-\frac{2}{3}\frac{1}{1+\alpha}} \left(\frac{t_{eq}}{t_r}\right)^{\frac{1}{2}} \left(\frac{t_0}{t_{eq}}\right)^{\frac{1}{3}} \sim 1. \tag{1.24}$$

But this inequality can only be fulfilled if the fluid that dominates the early Universe exhibits the strange quality

$$\alpha \leq -\frac{1}{3} \quad \Leftrightarrow \quad p \leq -\frac{1}{3}\varrho. \tag{1.25}$$

Examining (1.7) we see that this is equivalent to

$$\ddot{a} > 0. \tag{1.26}$$

But this means an *accelerated expansion, a blowing-up with an increasing rate. This is why a period during which this condition is fulfilled is called INFLATIONARY PERIOD.*[20,21] From (1.17) one finds, if one expresses $\dot{a}$ as a function of a, that one needs at least 60 to 70 e-folds of inflation.

The fluid giving us (1.26) is required to have some more properties besides (1.25): it must be stable enough to grand inflation for a sufficient period; on the other hand we can't have this material forever because we finally need our Universe to emerge naturally from the inflationary period (so-called graceful-exit problem); it should also lead to something causing the presently observed CMBR and provide seeds for structure formation (see also Ch.3), because all inhomogeneities before inflation are wiped out and cannot be the cause of it.[22]

---

[18] It is intimately connected with the *oldness problem*: Ω only slightly bigger than unity gives a closed Universe that would have collapsed a long time ago, while Ω only slightly smaller than one gives an open Universe with the present energy density negligibly small, life would not have been possible to form.

[19] Also: the natural length scale for the curvature is the Planck length, which is many orders of magnitude below the lower limit of the radius of the Universe.

[20] From (1.2), $g_\mu{}^\nu = \delta_\mu{}^\nu$ and (1.4) one finds an equation of state $\varrho_\Lambda = -p_\Lambda$ for the cosmological constant. So Λ's equation of state does fulfil condition (1.25), we are hence looking for something acting as an effective cosmological constant. Since the cosmological term creates an accelerated expansion, it is often considered to be the cause of antigravity.

[21] From (1.20) one gets with (1.14) that

$$\Omega - 1 \propto \mathcal{K}\, a^{3\alpha+1}.$$

So during radiation- or matter-dominance Ω flows away from one. However, during an inflationary epoch Ω flows towards one.

[22] On first glance this seems paradoxical, since we want inflation to get rid of the inhomogeneities. We'll later see how this problem is solved by inflation, too, although its solution initially wasn't intended.

One desires that the existence of such a substance comes up naturally without artificial modelling as we go backwards in time to extremely high densities and pressures. There one is finally forced to describe the evolution of the content of the Universe in terms of fields. Since grand unified theories (GUT's), supersymmetry (SUSY), supergravity (SUGRA) and superstring theory provide a superabundance of scalar fields[23], we shall check whether such a field can produce the required equation of state. So we have to find its pressure and energy density. From the Lagrangian for the field φ with mass M

$$\mathcal{L} = -\frac{1}{2}\varphi_{,\mu}\;\varphi^{,\mu} - \frac{M^2}{2}\varphi^2 \tag{1.27}$$

we get the equation of motion

$$\varphi^{,\mu}_{\;\;,\mu} - M^2\varphi = 0 \tag{1.28}$$

and the energy-momentum tensor

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \frac{\eta_{\mu\nu}}{2}\Big(\varphi^{,\lambda}\varphi_{,\lambda} + M^2\varphi^2\Big), \tag{1.29}$$

$\eta_{\mu\nu}$ being the Minkowski metric. Comparing the last equation with (1.4) we find from the 00-equation that for a homogeneous field one has

$$\varrho = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}M^2\varphi^2, \tag{1.30}$$

likewise the spatial components give

$$p = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}M^2\varphi^2. \tag{1.31}$$

$\frac{1}{2}\dot{\varphi}^2$ is called the kinetic energy density, and $\frac{M^2}{2}\varphi^2$ the potential energy density. Using the two equations above in (1.25) we get that a homogeneous Klein-Gordon field which fulfils the condition

$$\dot{\varphi} < \sqrt{\frac{3}{2}}M\varphi \tag{1.32}$$

will cause (1.26). *So an ordinary scalar field is a substance that could solve all our problems if (1.32) is given for a sufficient time*. To check whether this is the case we solve the homogeneous Klein-Gordon equation (which reduces to the equation of an undamped harmonic oscillator)

$$\ddot{\varphi} + M^2\varphi = 0 \quad \Rightarrow \quad \varphi(t) \propto e^{\pm iMt}; \tag{1.33}$$

(1.32) is fulfilled only for a short time of one oscillation. So it seems at first sight that the Klein-Gordon field will not help us with our quest.[24] Fortunately the equation of motion is a little more complicated than (1.28), because we're not dealing with *a Minkowski spacetime but instead we have a Robertson-Walker spacetime*.

[23]Note that the scalar field might also be a non-fundamental field, like a condensate of fermions.

[24]In the next chapter it'll be shown more rigorously that an oscillating field cannot provide inflation.

Chapter 2

The chaotic inflationary scenario

2.1 The basic equations

We saw in the last chapter that the usual Minkowski Klein-Gordon equation gives hope to find a natural way to get an accelerated expansion. A Universe dominated by a scalar field is completely described by the Friedmann equation and the field's equation of motion. However, using (1.28) is not correct as we're dealing with a GR-background. To perform the necessary generalisation we simply replace all derivatives by covariant derivatives (denoted by semicolons)[1], so that

$$\varphi^{;\mu}{}_{;\mu} - M^2\varphi = 0. \tag{2.1}$$

Now for scalar fields the covariant derivative reduces to a partial derivative. Furthermore one has

$$\varphi^{,\mu}{}_{;\mu} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\; g^{\mu\nu}\varphi_{,\nu}\right)_{,\mu} \qquad , \tag{2.2}$$

$g_{\mu\nu}$ being the components of the metric tensor and g being the absolute value of its determinant. Taking the metric to be the Robertson-Walker metric (1.3), and *considering from now the case with*[2] $\mathcal{K} = 0$ so that

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix}, \tag{2.3}$$

we get for a homogeneous field that (2.1) reads[3]

$$\frac{1}{a^3}\frac{d}{dt}\left(a^3\dot{\varphi}\right) + M^2\varphi = 0 \quad \Rightarrow \quad \ddot{\varphi} + 3H\dot{\varphi} + M^2\varphi = 0. \tag{2.4}$$

[1] We can do that easily because no factor-ordering problems arise for scalar fields, see [5]. Note that we work with a minimally coupled field, not allowing any curvature coupling.

[2] As we'll find accelerated expansion, the term containing $\mathcal{K}$ soon becomes negligible.

[3] Note that the number of $H\dot{\varphi}$ in (2.4) is the number of dimensions we consider.

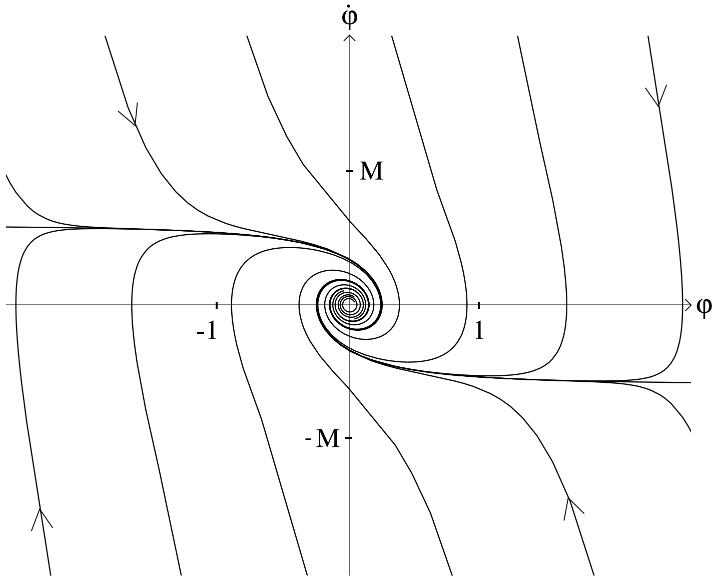

Figure 2.1: The phase diagram for (2.9).

For a nonexpanding universe ($H = 0$) this expression reduces to the usual Klein-Gordon equation for a homogeneous field. Performing the same generalisation for (1.29) one gets

$$T_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{g_{\mu\nu}}{2}\Big(\varphi^{;\lambda}\varphi_{;\lambda} + M^2\varphi^2\Big). \tag{2.5}$$

If we calculate ϱ and p from this for a homogeneous field we get the same results as in Minkowski space. So our starting equations are

$$\begin{aligned} H^2 &= \frac{1}{3}\Big(\dot\varphi^2 + M^2\varphi^2\Big),\\ \ddot\varphi &+ 3H\dot\varphi + M^2\varphi = 0. \end{aligned} \tag{2.6}$$

In solving these equations we'll follow [6]. Plugging the first equation[4] into the second one gives

$$\ddot\varphi + \sqrt{3}\Big(\dot\varphi^2 + M^2\varphi^2\Big)^{\frac{1}{2}}\dot\varphi + M^2\varphi = 0, \tag{2.7}$$

a nonlinear equation for $\varphi(t)$. The solution to this second order differential equation is determined fully by its initial conditions φ_i and $\dot\varphi_i$. To see how for a given initial conditions this system evolves, one draws a phase diagram with $\dot\varphi$ vs. φ. With this we can find out

[4]Note that while taking the square root of the first equation we explicitly assume the Hubble parameter to be positive, corresponding to an expanding Universe!

about the general behaviour of the differential equation, determine attractors and divide the phase space into regions in which one has different approximations to (2.6). Hence we need to transform (2.7) as an equation for $\varphi(t)$ into an equation for $\dot\varphi(\varphi)$. We have

$$\frac{d}{dt} = \dot\varphi \frac{d}{d\varphi} \quad \Rightarrow \quad \ddot\varphi = \frac{d\dot\varphi}{d\varphi}\dot\varphi, \tag{2.8}$$

so that (2.7) now reads

$$\frac{d\dot\varphi}{d\varphi} = -\sqrt{3}(\dot\varphi^2 + M^2\varphi^2)^{\frac{1}{2}} - M^2\frac{\varphi}{\dot\varphi} \ . \tag{2.9}$$

This is a first order differential equation for $\dot\varphi(\varphi)$. So for every $(\varphi/\dot\varphi)$-pair we can find $\frac{d\dot\varphi}{d\varphi}$ which is a function of those two values; in other words, for every point in phase space we can find the slope of the trajectory passing through[5], giving us the phase diagram shown in Fig.2.1. The symmetry under inversion through the origin is based on the invariance of (2.6) under[6]

$$\varphi \to -\varphi. \tag{2.10}$$

2.2 The period with the kinetic term dominating

Starting with an initial point in phase space where $|\varphi| \gg 1$ but

$$\dot\varphi^2 \gg M^2\varphi^2 \tag{2.11}$$

so that the kinetic energy dominates over the potential energy, we have from (2.9)

$$\frac{d\dot\varphi}{d\varphi} = -\sqrt{3}\dot\varphi, \tag{2.12}$$

so that

$$\ln\dot\varphi \propto -\sqrt{3}\varphi \quad \Rightarrow \quad \dot\varphi \propto e^{-\sqrt{3}\varphi}. \tag{2.13}$$

So $|\dot\varphi|$ changes much faster than $|\varphi|$.[7] This behaviour is easily understood consulting (1.30) and (1.31): with (2.11) we find that during this time we have[8]

$$p \approx \varrho \quad \Rightarrow \quad \alpha_{kin} \approx 1. \tag{2.14}$$

[5]Note that this technique would not be possible if we still carried the curvature term.

[6]This is equivalent to the fact that the trajectory is determined by its initial potential and kinetic energies.

[7] Equivalently we could have truncated (2.6) by skipping all potential terms to get

$$H^2 = \frac{\dot\varphi^2}{2} \quad \text{and} \quad \ddot\varphi + 3H\dot\varphi = 0.$$

[8]With (1.14) one finds that $a \propto t^{1/3}$.

This kind of matter is called *ultrahard matter*. With (1.10) we get

$$\dot{\varphi}^2 \sim \varrho_{kin} \propto a^{-6}, \tag{2.15}$$

i.e. the kinetic energy density decays very rapidly. But the potential energy and hence φ changes much slowlier: from (2.13) and (1.14) we find

$$e^{\sqrt{3}\varphi} \propto t \propto a^{1/3} \quad \Rightarrow \quad \varphi \propto \ln a. \tag{2.16}$$

Finally the kinetic energy becomes small compared to the potential energy, *a condition able to produce an inflationary period as we saw at the end of the last chapter.* This is why the φ-field is also called *the inflaton*.

2.3 The period of inflation

Suppose that after kination we reach the attractor in the "non-curly" region with

$$\dot{\varphi}^2 \approx \frac{M^2}{3} \ll \varphi^2, \tag{2.17}$$

but still with $|\varphi| \gg 1$. With (1.30) and (1.31) one finds that for some time we have

$$\alpha = \frac{\frac{1}{3} - \varphi^2}{\frac{1}{3} + \varphi^2}. \tag{2.18}$$

So φ needs to be only slightly bigger than unity to have α close to -1. Furthermore one can see that

$$\ddot{\varphi} \approx 0 \ \text{ and } \ \varrho \approx \frac{1}{2}M^2\varphi^2. \tag{2.19}$$

In Minkowski spacetime we saw that the Klein-Gordon field gives inflationary conditions only for a short time, but now we get inflation for a long time. Referring to (1.32), *the motion along the attractor is equivalent to inflation* (as long as we are not in the region where it is curly). One has

$$\frac{d\dot{\varphi}}{d\varphi} \approx 0 \quad \Leftrightarrow \quad -\sqrt{3}M\varphi \approx M^2\frac{\varphi}{\dot{\varphi}} \quad \Leftrightarrow \quad \dot{\varphi} \approx \frac{M}{\sqrt{3}}. \tag{2.20}$$

The kinetic term hardly changes and is very small compared to the potential term. So (2.6) can be approximated to

$$\begin{aligned} H^2 &= \frac{M^2}{3}\varphi^2 \\ 3H\dot{\varphi} + M^2\varphi &= 0. \end{aligned} \tag{2.21}$$

Due to (2.19) this approximation is called *slow roll approximation* (SR)[9]. It is valid up to the moment when φ is of the order of unity and inflation ceases (denoted by the subscript$_f$). SR and inflation are synonyms in the particular model described here. To solve (2.21), we first perform a variable transformation:

$$H = \frac{\dot{a}}{a} \quad \Rightarrow \quad \frac{d}{dt} = aH\frac{d}{da}. \tag{2.22}$$

The derivative with respect to a will be henceforth denoted by a prime. So we get

$$\begin{aligned} H^2 &= \frac{M^2}{3}\varphi^2 \\ 3aH^2\varphi' + M^2\varphi &= 0 \end{aligned} \tag{2.23}$$

Plugging the first equation into the second one gives

$$\begin{aligned} \varphi\varphi' &= -\frac{1}{a} \\ \Rightarrow \quad \varphi^2 - \varphi_f^2 &= 2\ln\frac{a_f}{a}, \end{aligned} \tag{2.24}$$

the subscript$_f$ denoting the end of inflation. Since $\varphi = \mathcal{O}(1)$ at the end of inflation we choose $\varphi_f = 1$. With this and remembering the convention in the very beginning of this text about normalising the scale factor ($a_f = 1$), the last equation reads

$$\varphi = (1 - 2\ln a)^{\frac{1}{2}}. \tag{2.25}$$

From here it follows that, using the subscript$_i$ to denote the beginning of inflation,

$$\varphi_i^2 \approx 2\ln\frac{a_f}{a_i} \quad \Rightarrow \quad \frac{a_f}{a_i} = \text{total expansion factor} \approx e^{\frac{\varphi_i^2}{2}}, \tag{2.26}$$

which shows a tremendous expansion during inflation for φ sufficiently large at the beginning of that period;[10] this is why one speaks of *chaotic* inflation: once φ_i is high enough to solve

[9]This simplified version is usually (but less rigorously) reached by comparing the Klein-Gordon equation to the equation of a *damped* harmonic oscillator. The $3H$ is hence viewed as a frictional term, depending on the displacement: the oscillator is inside a bowl of liquid with *viscosity increasing in height*. So starting from very high ($\varphi \gg 1$) the friction is so strong that fast objects are slowed down, one does not get any positive acceleration (to be verified later on) and only a small velocity. Hence the second derivative in time of the inflaton can be neglected and the potential energy dominates over the kinetic energy which is consequently also neglected (in other words, the logarithmic variation of $\dot{\varphi}$ is slow on the Hubble time scale). The friction prevents the oscillations which we saw in the last chapter do not lead to inflation.

[10]Note that the number by which the scale factor increases during inflation depends only on the change of the inflaton, and not on its mass determining the slope of the potential. However, the magnitude of the density perturbations produced during inflation depends on M, see Ch.4.

our problems ($\sqrt{140} \sim 12$), it can take on *any* value.[11] Note that the value of φ at the beginning of the inflationary period should not be too high, because its energy density should be below the Planck scale so that quantum gravitational effects are unimportant. So one demands $\varphi_i < \frac{1}{M} \sim 10^6$, see section 4.3; the maximal expansion factor[12] due to (2.26) is $10^{10^{12}}$, our observable Universe is hence only a tiny fraction of a giant homogeneous region.[13] (2.23) and (2.25) give $H(a)$, namely

$$H^2 = H_f^2(1 - 2\ln a) \quad \text{with } H_f = \frac{M}{\sqrt{3}}, \tag{2.27}$$

and hence

$$H' = -\frac{H_f^2}{aH}. \tag{2.28}$$

To find how φ varies with time, we again start from (2.21) and substitute the first equation into the second one. If we set the origin of time to be at the beginning of inflation $(t_i = 0)$ this gives

$$\varphi(t) = \varphi_i - \frac{M}{\sqrt{3}}t. \tag{2.29}$$

From this we see that after having started off with a very high value, φ decreases only with a minimal slope. So the energy density stays almost constant, which means that H varies only little with time, too. From (2.29) and (2.21) we get

$$H = \frac{M}{\sqrt{3}}\varphi_i - \frac{M^2}{3}t \tag{2.30}$$

and hence

$$\dot{H} \ll H^2. \tag{2.31}$$

This formula states nothing but that the change of H during the characteristic Hubble time $\frac{1}{H}$ is small compared to H, so H is nearly constant.[14] Recall that de Sitter space has $\varrho = constant$ and hence $H = constant$, so that one has an *eternally* exponentially fast expanding Universe. The spacetime considered here is called *quasi* de Sitter space, for which one has, using (2.25) and (2.29),[15]

$$a \propto a_i\, e^{\frac{2\varphi_i M}{\sqrt{3}}t - \frac{M^2}{3}t^2}. \tag{2.32}$$

[11]So the terminology has nothing to do with nonlinear dynamics and chaotic systems! That the arbitrariness of the initial value of φ is stressed so much is due to the first inflationary model, *old* inflation proposed by Guth in 1981, see the last section of this chapter.

[12]See, however, the section about the self-reproducing Universe in Ch.4.

[13]Due to quantum fluctuations this region is smaller than the calculation here shows, but nevertheless the homogeneous region by far exceeds the horizon.

[14]Other approaches to inflation start with demanding that the event horizon grows so slowly that inhomogeneities inside can leave, which is equivalent to (2.31); so this condition is typical to all inflationary models.

[15]Use this in $\Omega = 1 + (\Omega_i - 1)\left(\frac{a_i}{a}\frac{H_i}{H}\right)^2$ to see how the fixed point $\Omega_0 = 1$ is stabilized.

So initially a expands just like in de Sitter space, increasing by a factor of e after a time $\frac{1}{M\varphi_i} \approx H_i$, but later the term containing t^2 starts to dominate and ends inflation. For the duration of inflation Δt we find from (2.29)

$$1 = \varphi_i - \frac{M}{\sqrt{3}\Delta t} \quad \Rightarrow \quad \Delta t = \frac{\sqrt{3}\varphi_i}{M}, \tag{2.33}$$

which makes sense easily because φ_i is a measure of the total expansion and M is a measure of the expansion rate. Remembering that we work in Planckian units and taking $M \sim 10^{-6}$, the duration of inflation is at least $10^{-35}s$.

2.4 Will inflation get started?

We saw that even if the field is initially kinetically dominated, one sooner or later reaches the attractor where, if we are not within the curly region but follow an almost straight line instead (so that we have SR), one gets inflation. In the curly region one does not have SR and hence one will not find inflation (see next section). What is the condition on the initial conditions so that the ultrahard matter goes over to inflationary matter and does not follow a fast falling trajectory that directly hits the attractor in the curly region? In other words, when is $\dot{\varphi_\mathcal{I}}$ so large that the decay of both φ and $\dot{\varphi}$ takes so long that afterwards φ is too weak to still cause inflation but hit the curly region instead? We get from (2.13) that

$$\dot{\varphi} = \dot{\varphi}_\mathcal{I}\ e^{\sqrt{3}(|\varphi - \varphi_\mathcal{I}|)}, \tag{2.34}$$

the subscript$_\mathcal{I}$ denoting the initial conditions of kination. To start inflation we need to have more potential energy than kinetic energy. This gives

$$M\varphi_i \overset{!}{\geq} \dot{\varphi}_i = \dot{\varphi}_\mathcal{I} e^{\sqrt{3}(|\varphi_i - \varphi_\mathcal{I}|)}$$

$$\Rightarrow \quad \dot{\varphi}_\mathcal{I}^2 \leq M^2 \varphi_\mathcal{I}^2 e^{2\sqrt{3}(|\varphi_\mathcal{I} - \varphi_i|)}. \tag{2.35}$$

Initial conditions fulfilling (2.35) are (hopefully) given as the outcome of the period before, after the realm of quantum cosmology and quantum gravity has been left.

Another point to have in mind is whether the initial inhomogeneity of the Universe allows inflation to start at all! If the Universe is inhomogeneous before inflation, the Friedmann equation is not valid anymore and the inflaton field itself might easily be inhomogeneous, which so far we assumed to be homogeneous. For an inhomogeneous field one gets that ϱ and p depend on one more term $\propto (\vec{\nabla}\varphi)^2$ that could be much bigger than $M^2\varphi_i^2$. So both equations in (2.6) might not be valid. The general idea is that the term containing the gradient decays $\propto a^{-2}$, it hence decays much faster than the average φ-value. And one just needs small homogeneous islands to develop to start inflation: once a quasi de Sitter space got started, all surrounding inhomogeneities get pushed away and have no influence on the interior of the Hubble radius $\mathcal{R}_H = \frac{1}{H}$ anymore (cosmic no-hair theorem). So after a while the system has self-regulated itself to start inflation by letting a few homogeneous spots crystallise where (independent of each other) inflation starts.

2.5 The period of oscillation and reheating

At the end of inflation we have

$$\dot{\varphi} \approx \frac{M}{\sqrt{3}} \text{ and } \varphi \approx 1, \tag{2.36}$$

so that from (2.9) we get as a very rough approximation

$$\frac{d\dot{\varphi}}{d\varphi} = -M^2 \frac{\varphi}{\dot{\varphi}} \quad \Rightarrow \quad \dot{\varphi}^2 + M^2\varphi^2 = constant, \tag{2.37}$$

which is equivalent to the oscillation we found for the Minkowski Klein-Gordon equation. How does the scale factor evolve with this oscillatory field? It is natural to approach this problem in polar coordinates. We make a slightly different variable transformation because we want to take into account the decreasing radius of the trajectory in phase space (see Fig.2.1) and also have the most general expression for the argument of the trigonometric function. So

$$\begin{aligned} M\varphi &:= r(t)\ \sin(\psi(t)) \\ \text{and}\ \dot{\varphi} &:= r(t)\ \cos(\psi(t)). \end{aligned} \tag{2.38}$$

To find how $\psi(t)$ looks like, we differentiate the first equation of (2.38) with respect to time to get

$$\begin{aligned} M\dot{\varphi} &= \dot{r}\sin(\psi) + r\cos(\psi)\ \dot{\psi} \\ \Rightarrow \quad \dot{\psi} &= \frac{M\dot{\varphi} - \dot{r}\sin(\psi)}{r\cos(\psi)} \\ &= M + \sqrt{3}\ r\sin(\psi)\cos(\psi). \end{aligned} \tag{2.39}$$

We average over time and get

$$\psi(t) = \psi_f + Mt, \tag{2.40}$$

so that we find a simple oscillation in time. To find the behaviour of the amplitude r we get from (2.38) that

$$3H^2 = M^2\varphi^2 + \dot{\varphi}^2 = r^2(t). \tag{2.41}$$

Differentiating with respect to time yields

$$\begin{aligned}
\dot{r} &= \frac{\dot{\varphi}\ddot{\varphi} + M^2\varphi\dot{\varphi}}{r} \\
&= \ddot{\varphi}\cos(\psi) + Mr\sin(\psi)\cos(\psi) \quad \Big| \quad (2.7) \;\Leftrightarrow\; \ddot{\varphi} = -\sqrt{3}\; r\; r\;\; \cos(\psi) - Mr\sin(\psi) \\
&= -\sqrt{3}r^2\cos^2(\psi).
\end{aligned} \tag{2.42}$$

If we now average over time again, the second term vanishes and we find

$$\dot{r} = -\frac{\sqrt{3}}{2}r^2 \;\;\Rightarrow\;\; r \propto \frac{2}{\sqrt{3}t}, \tag{2.43}$$

so that we get from (2.41)

$$H^2 = \frac{4}{9t} \;\;\Rightarrow\;\; \frac{\dot{a}}{a} = \frac{2}{3t} \;\;\Rightarrow\;\; a \propto t^{2/3}. \tag{2.44}$$

Comparing this with (1.14) we find that the scale factor behaves as if the Universe is filled with dust:[16]

$$\alpha_{osc} = 0. \tag{2.45}$$

One arrives at a coherent wave of non-relativistic particles, the amplitude decaying with $a^{-3/2}$, the particle number density n being given by

$$n = \frac{1}{2}M\varphi^2. \tag{2.46}$$

Comparing the behaviour of the generalised Klein-Gordon equation with the usual Klein-Gordon equation, we find that for *small* H the field behaves as in Minkowski space, oscillating to and fro, not causing an inflationary period. But for higher φ the frictional term (which distinguishes the Minkowski and the generalised Klein-Gordon equation) does provide an accelerated expansion. So although it became apparent easily in the first chapter that a scalar field can exhibit the required equation of state, without the GR-modification one would not get the desired result.

The cold φ-dust is now supposed to decay into ultra-relativistic material, so that we have a transition to a radiation dominated Universe. This process is called *reheating* (though the prefix "re" is rather ill-chosen[17]). Of course we should have included a coupling term in (2.4) that allows for the decay. This would have meant including an equation of motion for the

[16] Handwavingly this can be understood by noticing that half of the time the field is kinetically dominated ($\alpha = 1$) and half of the time it is potentially dominated ($\alpha = -1$), so that the average is given by $\alpha = 0$.

[17] It also derives from the first inflationary models where one needs a high temperature before inflation, see next section.

decay products, too. However, for simplicity we assumed that this decay is negligible before the oscillation. "The couplings [between the inflaton and the radiation] have to be extremely weak, to protect the inflaton potential which is necessary for a successful inflation picture, but at the same time strong enough to allow efficient thermalization of the inflaton energy density" [7]. "The key questions are by what mechanisms and how fast the decay of the coherent state takes place. It is important to determine the temperature of the produced particles at the end of the reheating period. The answers are relevant to many questions regarding the post-inflationary evolution. For example, it is important to know whether the temperature after reheating is high enough to allow GUT baryogenesis and the production of GUT-scale topological effects. In supersymmetric models, the answer determines the predicted abundances of gravitinos and other moduli fields." [8]. To describe the inflaton's decay, the simplest models add a phenomenological exponential decay, with the decay constant related to the coupling constant. As long as H is larger than the decay constant, the production of radiation can be neglected. With the inflaton oscillating H drops $\propto t^{-1}$ till the decay to radiation becomes relevant. But these simple models miss an important point: *the equation of motion for the decay products during the oscillations of the φ-field looks like the one for a harmonic oscillator with a mass depending on φ, so that one has a time-dependent frequency resulting in a Mathieu equation (see [9]). Hence one may find the phenomenon of parametric resonance.* In this case one gets a rapid explosive production of radiation, almost transferring the entire energy of the inflaton to radiation.[18] As this radiation is not a thermal state, one speaks of *preheating*, with thermalisation happening afterwards.

To summarise the the idea of chaotic inflation, we note that even starting from a kinetically dominated field one fairly quickly finds a potential-dominated period where the field and its derivative with respect to time only slowly changes in time, one has slow roll; this period causes an accelerated expansion. When inflation ends the field oscillates and behaves like cold matter, decaying and thus producing the radiation and entropy of the Universe.

2.6 Other inflationary models

Finally it must be mentioned that the scenario presented above is by far not the only one to get an accelerated expansion.

In Guth's *old* inflationary model[19] (see [10]) the potential was assumed to be like a *dented* Mexican hat (see Fig.2.2), similar to the one assumed for a Higgs field. This causes inflation as long as the field stays in the false vacuum at $\varphi = 0$, having been trapped there by supercooling. Finally the field tunnels to get to the absolute minimum, the true vacuum at

[18]The rest of the oscillating inflaton may still be present today in the form of cold dark matter, see next chapter.

[19]GUT-symmetry breaking creates textures, walls, strings and magnetic monopoles between domains of different vacua. One of the motivations for the old model was to show that the magnetic monopoles would be diluted so that their mass cannot force the Universe to recollapse prematurely.

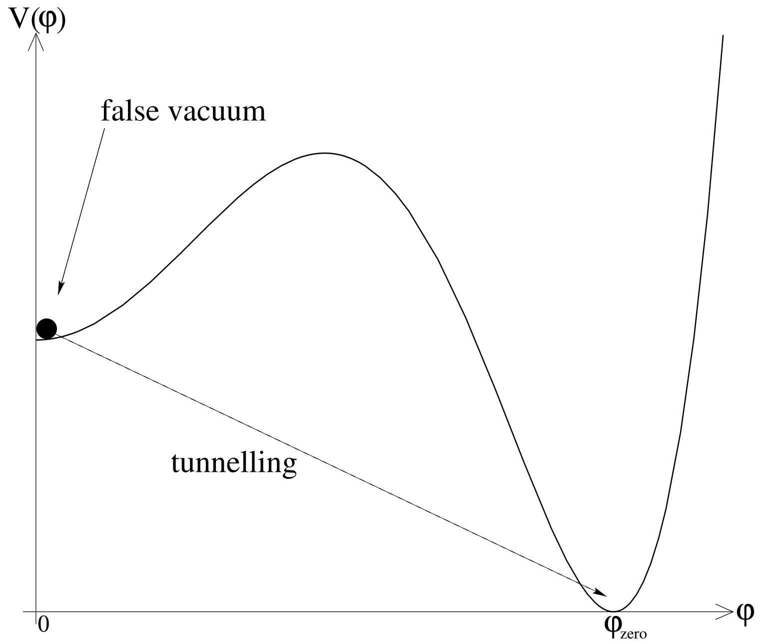

Figure 2.2: The old inflationary model

$\varphi = \varphi_{zero}$; inflation is ended by a first order phase transition. *So here the initial value of φ was confined to only a small region.* The model was soon discarded, because the tunnelling does not happen at the same moment everywhere, creating bubbles of $\varphi = \varphi_{zero}$, being much smaller than our present horizon; hence this model is observationally unacceptable.

Instead a more refined model was introduced (see [11], [12]) that used a different form of the potential (see Fig.2.3)[20], like a "flattened" Mexican Hat, very flat behind the relative maximum which presents only a small barrier to the tunnelling of the field at zero. So the field slowly rolls on the "plateau", causing inflation till it starts oscillating near the minimum. The main idea is hence that not a trapped field but a slowly rolling field causes inflation, so that the potential barrier might well not exist at all. As the nucleation of bubbles happens long before inflation ends, the problems of the old model are avoided this way. This model is called the *new* inflationary model.

There are also other chaotic scenarios: if one chooses another potential to work with, one has to make the replacements

$$\frac{M^2}{2}\varphi^2 \;\rightarrow\; V(\varphi)$$

[20]The model was not based on some arbitrary potential like the previous one but on the $SU(5)$-symmetric Coleman-Weinberg potential.

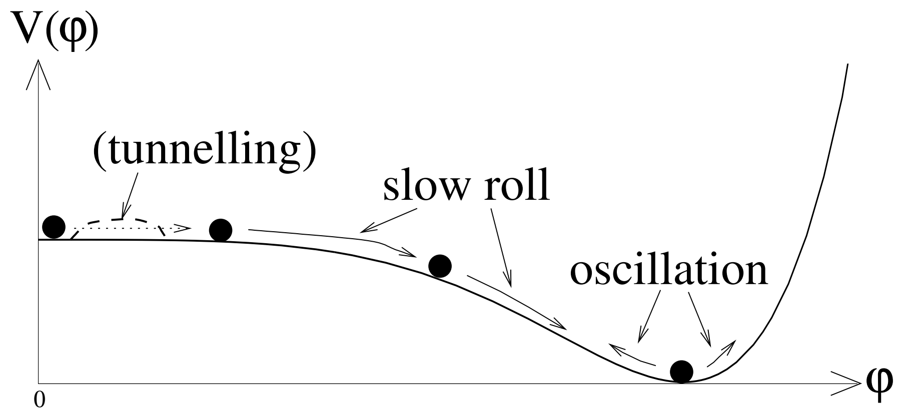

Figure 2.3: The new inflationary model

$$\text{and}\quad M^2\varphi \;\rightarrow\; \frac{\partial V}{\partial\varphi} \tag{2.47}$$

in (2.6) or (2.21). For a complete discussion with the phase diagram technique for a general potential see [13]. A very popular potential is $V = \lambda\varphi^4$, with similar features as the one discussed in the first four sections. Following the same steps as we did from (2.21), it is easy to show that for a potential $V = \lambda\varphi^n$ one gets

$$\varphi^2 = 1 - n \ln a. \tag{2.48}$$

An exponential potential, $V \propto e^{\lambda\varphi}$, gives $a \propto t^{4/\lambda^2}$, see [14]. One has *never ending power law inflation* if $4/\lambda^2 < 1$.

The idea of chaotic inflation is implemented in a modified version, *quintessential inflation* which we will get to know in Ch.10.

The list of inflationary models is by far not exceeded: "...there are roughly fifty different forms of inflation that have been named and studied. The list includes double, triple, and hybrid inflation [several scalar fields all contributing to inflation, the general idea being chaotic inflation], not to mention mutated hybrid inflation, tilted hybrid inflation, and hyperextended inflation. Cosmologists have also studied inflation that is gravity driven, spin driven, and vector field driven, as well as inflation that is warm, soft, tepid, and natural. ...I have recently been working ...on a new version of inflation ...we decided to call it *supernatural inflation*. ...[But] ...the name had in fact already been used – not once but twice!", [4]. One can also use vacuum polarisations to get the desired equation of state. There is even a model inspired by string theory where inflation is kinetically-driven, k-inflation.

But the scenario explained in the first four sections, proposed by A.D.Linde in [17], appears to be the easiest and most natural one, which is why it is the base for all further calculations in this text, except in Ch.10.

Chapter 3

Cosmological perturbations

3.1 Inhomogeneities of today's Universe

Despite the homogeneity on very large scales, on small scales the Universe is undeniably inhomogeneous: mankind exists on Earth, we can see stars, galaxies, clusters and superclusters. This visible matter is most probably not the only form of matter in the Universe: as the dynamics of clusters and galaxies seem not to make sense if one assumes the visible matter to be the only source of gravity, and one furthermore observes otherwise unexplainable effects of gravitational lensing, the existence of a lot of *cold* dark matter (CDM) is postulated. It is assumed to make the big part of the Universe's matter content[1] and to consist of dead stars, black holes and mainly clouds of nonrelativistic exotic particles, the latter having almost no interactions with photons and ordinary baryonic matter.[2] So visible matter is like "the cream on the chocolate pie" [18], indicating large scale inhomogeneities of the CDM background.

Moreover, a very precise measurement of the CMBR has revealed slight quadrupole temperature anisotropies,

$$\left.\frac{\delta\mathcal{T}}{\mathcal{T}}\right|_{today} \sim 10^{-5}. \tag{3.1}$$

Since the CMBR represents a picture of the Universe 300,000 years after the big bang, it is natural to take the matter fluctuations corresponding to the temperature fluctuations (see the last section of this chapter) as initial conditions for calculations and simulations to reproduce today's indirectly observed CDM structure[3] measured from cluster abundances.

[1]As on the one hand the expansion and flatness of space and on the other hand the observed density of luminous matter do not agree, the existence of dark matter was postulated, too.

[2]The *hot* dark matter models (HDM) propose the dark mass to be hidden in huge neutrino clouds. Meanwhile observations favour a bottom-up scenario of structure formation, which is not possible in HDM-models, so CDM appears to be more likely than HDM.

[3]The density contrast of CDM started to evolve – unlike the contrast for baryonic matter – from a contrast of 10^{-5} even *before* recombination, since the exotic particles only couple weakly to photons, whereas gravitational contraction was not possible for baryonic matter before recombination due to the

3.2 Concepts of cosmological perturbation theory

Cosmological perturbations –"linearised gravitational perturbations in a expanding Universe", [13]– are classified according to their properties before they enter our observable Universe, i.e. before horizon entry. If the metric of the Universe gets distorted due to density perturbations, one speaks of *curvature perturbations*. The other type are the so-called *isocurvature perturbations*, which have the same density everywhere, but the equation of state varies from place to place due to different compositions out of *several components*.[4] Isocurvature perturbations can induce curvature perturbations later on.

To follow the evolution of the perturbations in time, "one might try to divide it into volumes which initially evolve independently of each other. Fairly soon, however, this independence would no longer hold ... It is therefore not a good idea to think of a generic perturbation as a sum of spatial components. It is much better to think of the perturbations as a superposition of plane waves which have the advantage that *they evolve independently* while the fluctuations are still linear", [19]. So it is best to work in Fourier space where in the linear regime the modes decouple, having assumed only slight departures from homogeneity. As we're dealing with an expanding background, it is useful to introduce the concept of the comoving scale. We define

$$\lambda_{phys}(t) = \frac{a(t)\lambda_{com}}{a_f} = \frac{a}{a_f\ k}, \tag{3.2}$$

where $\lambda_{phys}(t)$ is the *physical wave length which we actually measure*. λ_{com} is called the *comoving wavelength* which stays constant in time, it's the constant of proportionality between the real distance and the expanding space, so that all time dependence is put into $a(t)$. At the end of inflation, according to our choice of a_f, the comoving wavenumber equals the physical wavenumber. k is the time-independent *comoving wave number*. In this text most quantities depending on scales will be written in terms of comoving quantities.

The most important quantity in the context of perturbation theory is the *power spectrum*, being a measure for the amplitude of the relative density fluctuations around the average density in Fourier space. With

$$\delta\varrho(\vec{x}) := \varrho(\vec{x}) - \langle\varrho\rangle \tag{3.3}$$

and, V being the volume of the region we consider, and

$$\delta\varrho_k := \frac{1}{(2\pi)^{3/2}V^{1/2}} \int \delta\varrho(\vec{x})\ e^{-i\vec{k}\vec{x}} d\vec{x}, \tag{3.4}$$

intensive radiation pressure (the Universe was opaque). So after recombination the baryonic matter just dropped into the potential walls provided by the CDM.

[4]This is why they are often called *entropy perturbations* or *isothermal perturbations*; equivalently curvature perturbations are thus also called *adiabatic perturbations.* Originally this terminology was used for a Universe filled with dust and radiation.

the power spectrum is given as

$$\mathcal{P}(k) := k^{3/2}\left|\frac{\delta\varrho_k}{\langle\varrho\rangle}\right|. \tag{3.5}$$

Now with

$$|\delta\varrho_k|^2 = \delta\varrho_k\ \delta\varrho_k^* = \frac{1}{(2\pi)^{3/2}V^{1/2}}\int \delta\varrho(\vec{y})e^{-i\vec{k}\vec{y}}d\vec{y} \quad \times \quad \frac{1}{(2\pi)^{3/2}V^{1/2}}\int \delta\varrho(\vec{x})e^{i\vec{k}\vec{x}}d\vec{x} \tag{3.6}$$

and combining the two integrals and setting $\vec{r} = \vec{y} - \vec{x}$ one gets

$$\begin{aligned}\left|\frac{\delta\varrho_k}{\langle\varrho\rangle}\right|^2 &= \frac{1}{(2\pi)^3 V}\int\int \frac{\delta\varrho(\vec{x})}{\langle\varrho\rangle}\ \frac{\delta\varrho(\vec{x}+\vec{r})}{\langle\varrho\rangle}e^{-i\vec{k}\vec{r}}d\vec{x}d\vec{r} \\ &= \frac{1}{(2\pi)^3}\int\left\langle\frac{\delta\varrho(\vec{x})}{\langle\varrho\rangle}\ \frac{\delta\varrho(\vec{x}+\vec{r})}{\langle\varrho\rangle}\right\rangle e^{-i\vec{k}\vec{r}}d\vec{r}.\end{aligned} \tag{3.7}$$

so that

$$\mathcal{P}^2(k) = \frac{k^3}{(2\pi)^3}\int \xi(\vec{r})\ e^{-i\vec{k}\vec{r}}d\vec{r} \tag{3.8}$$

with

$$\xi(\vec{r}) = \left\langle\frac{\delta\varrho(\vec{x})}{\langle\varrho\rangle}\ \frac{\delta\varrho(\vec{x}+\vec{r})}{\langle\varrho\rangle}\right\rangle, \tag{3.9}$$

called the *equal time autocorrelation function*, or, more precise, the equal time autocorrelation function of the density contrast $\delta\varrho/\langle\varrho\rangle$ (often denoted by δ), being nothing but the Fourier transform of $(2\pi)^{3/2}\frac{\mathcal{P}^2(k)}{k^3}$. [5] $\xi(\vec{r})$ is a measure of the influence of the density contrast at $\vec{x}$ on the one at $\vec{x}+\vec{r}$. Equivalently $\xi(\vec{r})$ can be seen as the joint distribution of finding a perturbation in the two volume elements dV_1 and dV_2 at a separation $\vec{r}$

$$dP = \langle\varrho\rangle\ [1+\xi(\vec{r})]\ dV_1\ dV_2. \tag{3.10}$$

In this sense $\xi(\vec{r})$ is also called *two-point correlation function*. This can be generalised to higher order correlation functions, three-, four-, etc.-correlation functions. The importance of those quantities is that they characterise the statistics of the field considered: if the randomly fluctuating field $\delta\varrho(\vec{x})/\langle\varrho\rangle$ obeys a Gaussian probability distribution, it can be shown that the phase of the complex quantity $\delta\varrho_k/\langle\varrho\rangle$ is randomly distributed. The central limit theorem then makes $\delta\varrho(\vec{x})/\langle\varrho\rangle$ (as a sum of many $\delta\varrho_k/\langle\varrho\rangle$'s) to obey Gaussian statistics. If $\delta\varrho(\vec{x})/\langle\varrho\rangle$ is a Gaussian random field, *then all correlation functions can be expressed in terms of the two-point correlation function (and hence the power spectrum), see for example* [19]. Of

[5] Sometimes this quantity is identified with the power spectrum (then denoted by $|\delta_k|^2$), although the term *power spectral density function* would be more precise. With this definition (3.8) is also known as the Wiener-Khinchin theorem.

course one can define correlation functions for other field variables, too, for instance for the fluctuations of the φ-field (see next chapter) one has

$$\langle \delta\varphi(\vec{x})\ \delta\varphi(\vec{x}+\vec{r})\rangle. \tag{3.11}$$

Note that in this text $\xi(\vec{r})$ shall always denote the correlation function of the density contrast.

Although we considered $\xi(\vec{r})$ so far as a spatial average, "we may also consider it as an ensemble average, in which we repeat the same measurement many times, and compute averages", [20]: "One imagines that the matter distribution has been determined by some physical process involving a lengthy and complex sequence of events, so that what is found at any particular place is the result of many slight variations of parameters within the sequence. Samples from well separated spots are uncorrelated, and the collection of such samples is a statistical ensemble generated by many independent applications of the process. Statistics such as the n-point correlation functions can be considered to be averages across the ensemble. The *fair sample hypothesis* states first that it makes sense to think of well separated parts of the Universe as being independent realizations of the same physical process, second that within the visible part of the Universe there are many independent samples that can be lumped to approximate a statistical ensemble, and third that averages across the ensemble are unaffected by rotation.", [21].

A measure how strong the power spectrum depends on k is given by the so-called *spectral index* n,

$$n := 1 + \frac{2k}{\mathcal{P}}\ \frac{d\mathcal{P}}{dk}. \tag{3.12}$$

For a power law, i.e. $\mathcal{P} \propto k^{\nu}$, this reads

$$n = 1 + 2\nu. \tag{3.13}$$

A scale-invariant spectrum is called Harrison-Zel'dovich spectrum and has $\nu = 0$ and hence $n = 1$.

3.3 The Sachs-Wolfe effect

To calculate today's power spectrum of cluster abundances, one starts with a Harrison-Zel'dovich spectrum for large scales ($\sim 1000 Mpc$) and a transfer function[6] providing a cut-off for small scales ($\sim 80 Mpc$, [22]). [7] The theoretical predictions head in the right directions

[6] "In the absence of physical effects, [the primordial power spectrum] would simply scale with time in accord with the linear growth law for each perturbation modes ... The spectral shape, however, does not survive unaltered to the present epoch because of various physical processes ... the transfer function ... relates the processed power spectrum to its primordial form", [19].

[7] The underlying formula is the differential equation (valid in the linear regime) for the evolution in time for the density contrast:

$$\ddot{\delta} + 2H\dot{\delta} - \langle\varrho\rangle\ \delta = 0;$$

but are by far not perfect yet. So the theory of large scale structure formation is on the right track, but how are the temperature anisotropies of the CMBR related to the perturbations of matter? In 1967, shortly after the discovery of the CMBR, Sachs and Wolfe found an easy, intuitively logical relation between the primordial perturbations and the temperature anisotropies of the CMBR (see [23])

$$\left.\frac{\delta\,\mathcal{T}}{\mathcal{T}}\right|_{today} \sim \Phi_{rec}, \tag{3.14}$$

where Φ is the gauge-invariant metric potential (see section 4.4); the index$_{rec}$ marks the moment of recombination. (3.14) is neither taking into account Doppler temperature perturbations due to peculiar velocities of either observer or the recombining atoms nor density perturbations on the path of the light ray. "The potential Φ on a scale k is proportional to the magnitude of the energy-density perturbation on that scale at the time when the corresponding wavelength crosses the Hubble radius. Thus, on an angular scale $[\vartheta]$ the temperature fluctuation $[\delta\mathcal{T}]$ is proportional to the amplitude of $[\frac{\delta\varrho}{\varrho}]$ on the scale k which dominates the contribution to $[\delta\mathcal{T}(\vartheta)]$ evaluated at the time ... when k enters the Hubble radius, and the observational limits on the temperature fluctuations translate into constraints on the amplitude of the primordial density density perturbations [and hence the power spectrum].", see [13]. *So within an order of magnitude Φ and $\mathcal{P}$ are comparable.*[8]

So how were the metric perturbations generated, and how does their power spectrum look like? It would be especially nice to obtain a spectrum *slightly* depending on scale: if $n > 1$, then the amplitude decreases for decreasing k (large scales). Such a power spectrum, called a *blue* spectrum, would allow to have small CMBR anisotropies on large scales and nevertheless high anisotropies on smaller scales to easily get the formation of galaxies.

But *what* provides the initial perturbations of the homogeneity? Before the period of radiation that precedes the moment of recombination there was inflation, so it is natural to search for the source of inhomogeneity there.

by the time one has $\delta \approx 1$ the evolution becomes nonlinear, speeding up the gravitational collapse.

[8]Of course theses two quantities are not equal, because Φ is defined in real space, whereas the power spectrum makes sense only in Fourier space.

Chapter 4

Quantum fluctuations of the inflaton

4.1 The k-range of interest for inflationary perturbations

We deliberately avoided the term quantum field theory (QFT) in the second chapter, because we treated φ like a classical field. But (1.33) actually is supposed to be an equation for an *operator* $\hat{\varphi}$ with expectation value $\langle\hat{\varphi}\rangle = \varphi$.[1] *So we are faced inevitably with the phenomenon of quantum fluctuations, occurring randomly and messing up homogeneity*; we have

$$\varphi_{hom} := \varphi \to \varphi + \delta\varphi, \tag{4.1}$$

with $\delta\varphi$ being a scalar field perturbation, with the corresponding operator $\hat{\delta\varphi}$. How do these perturbations evolve in time? In a spatially flat, isotropic and homogeneous universe the Hubble radius $1/H$ is the maximal scale for two events to be causally connected, so inside "the microphysics in operation at that epoch will therefore be relevant" [24]. The microphysics are well known and we can calculate the amplitude of the fluctuations. As an inflationary spacetime per definition has an accelerated scale factor, the *comoving* Hubble radius $(aH)^{-1} = \dot{a}^{-1}$ decreases during inflation, so that the constant comoving scales will become larger than the comoving horizon, they "cross" the horizon, i.e. physical scales will grow faster than the horizon! In other words, *inflation enables physical scales to catch up and overtake the horizon*. No other period can be the prerequisite for this effect, no other spacetime is able to stretch quantum fluctuations to a size beyond the horizon. Outside the horizon microphysical effects (and hence the amplitude of the fluctuations) are "frozen".[2] [3]

[1]This is why one actually always speaks of *classical* SR instead of SR.

[2]"One can utilise the Heisenberg picture of quantum theory to say that the operators obey the classical equation of motion, and so the evolution of the vacuum state can be followed until it crosses outside the Hubble radius.", [24].

[3]Note that compared to the initial vacuum, the state reached at the moment of crossing is not a vacuum state but one containing particles: in Minkowski space there is a privileged coordinate system

Once inflation ends, the horizon grows faster than the expanding space, and the frozen modes re-enter, so that the microphysics of earlier times appear as large scale perturbations. *By making the perturbations cross the horizon twice, the inflationary theory is able to explain the origin of curvature perturbations.* So inflation on the one hand provides homogeneity on large scales, but on the other hand induces small inhomogeneities on smaller scales. The classical standard cosmological model is able to accommodate these perturbations but nevertheless it is unable to explain their origin.[4] So when speaking about k, we will be interested in those modes that were initially $(t = t_i)$ inside the horizon but are outside the horizon later on:

$$\lambda_{phys}(t_i) < \frac{1}{H_i} \quad \rightarrow \quad \lambda_{phys}(t) > \frac{1}{H(t)}. \tag{4.2}$$

Combining these two formulas with the help of (3.2) gives

$$\frac{a_i}{a_f} H_i < k < \frac{a(t)}{a_f} H(t). \tag{4.3}$$

So at the end of inflation $(t = t_f)$ the relevant k-range is given by

$$\frac{a_i}{a_f} H_i < k < H_f. \tag{4.4}$$

As the moment of horizon crossing is of such a great importance[5], we shall define a_k to be the scale factor at the moment when a mode of wave number k crosses the horizon

$$k = a_k H_k, \tag{4.5}$$

with $H_k := H(a_k)$.

---

due to Poincaré-invariance and hence we have definite modes (and hence a definite understanding of particles and anti-particles) which expand throughout the entire space and hence have a global nature. But if geometry forbids the infinite extension of a mode, it is not possible anymore to find a unique vacuum state [the Casimir effect is interpreted in the same manner: two perfectly conducting plates (which create boundary conditions) interrupt spatial translational invariance and are attracted to each other]. GR is not allowed to have a privileged coordinate system, hence it cannot have a privileged vacuum state but many complete orthonormal sets instead. To go from one coordinate system to another one has to perform a Bogolubov transformation combining particles and anti-particles to quasi-particles, the particles of the other vacuum. So which particles are the *real* particles? If one takes the viewpoint of the instrumentalism of the Copenhagen interpretation, the answer is simple: this question does not make any sense. We have a state and a detector, and a particle is what a detector *measures*. One cannot speak of a particle without specifying how it was measured. "A particle is merely an abstract heuristic model that provides an easy mental image of how one type of detector measurement is related to another. ...particles are what detectors are designed to measure" [25]. This interpretation also helps to understand the Unruh-effect where one finds a heat bath of particles if one places a particle detector into an accelerated system.

[4]However isocurvature perturbations may form on scale bigger than the horizon.

[5]In Ch.6 we shall even get to know a variable designed to measure how close one is to horizon crossing.

4.2 The initial conditions for the fluctuations

Deeply inside the horizon one hardly notices the expansion of the Universe, usual Minkowski physics determines the laws.[6] It thus supplies the initial conditions for the modes that were originally inside the horizon and then got stretched to exit. So way inside the horizon, we get the Klein-Gordon equation for the real $\delta\varphi$-field

$$\delta\ddot{\varphi} - \nabla^2\ \delta\varphi + M^2\delta\varphi = 0, \tag{4.6}$$

solved by

$$\delta\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2}}\Big[\delta\varphi_{\vec{\kappa}}\ e^{-i(\vec{\kappa}\vec{x}-\omega_\kappa t)} + \delta\varphi^*_{\vec{\kappa}}\ e^{i(\vec{\kappa}\vec{x}-\omega_\kappa t)}\Big]d\vec{\kappa} \tag{4.7}$$

and hence

$$\delta\dot{\varphi}(\vec{x},t) = \frac{i\omega_\kappa}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2}}\Big[\delta\varphi_{\vec{\kappa}}\ e^{-i(\vec{\kappa}\vec{x}-\omega_\kappa t)} - \delta\varphi^*_{\vec{\kappa}}\ e^{i(\vec{\kappa}\vec{x}-\omega_\kappa t)}\Big]d\vec{\kappa} \tag{4.8}$$

with

$$\omega_\kappa = \sqrt{\kappa^2 + M^2}, \tag{4.9}$$

$\vec{\kappa}$ being the wavenumber in Minkowski space. The quantisation of this field takes place the same way as one gets from classical mechanics to quantum mechanics: for a discrete system of one or more particles one has

$$\Big[\hat{q}_a, \hat{p}_b\Big] = \left[\hat{q}_a, \widehat{\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_b}\right)}\right] = i\delta_{ab}, \tag{4.10}$$

δ_{ij} being the Kronecker symbol. So with (1.27) one is lead to postulate

$$\Big[\delta\hat{\varphi}(\vec{x},t), \delta\dot{\hat{\varphi}}(\vec{y},t)\Big] = i\delta(\vec{x}-\vec{y}), \tag{4.11}$$

$\delta(\vec{x}-\vec{y})$ being Dirac's Delta function. By making the transition

$$\delta\varphi_{\vec{\kappa}}\ \ \rightarrow\ \ \delta\hat{\varphi}_{\vec{\kappa}} := \delta\varphi_{\vec{\kappa}}\ \hat{a}^\dagger_{\vec{\kappa}} \tag{4.12}$$

in (4.7) and (4.8) and plugging them into (4.11), setting $t = t_i = 0$, one sees that $\hat{a}_{\vec{\kappa}}$ and $\hat{a}^\dagger_{\vec{\kappa}'}$ are annihilation and creation operators with

$$\Big[\hat{a}_{\vec{\kappa}}, \hat{a}^\dagger_{\vec{\kappa}'}\Big] = (2\pi)^{3/2}\delta(\vec{\kappa}-\vec{\kappa}') \tag{4.13}$$

so that

$$\delta\varphi_{\vec{\kappa}}(t_i) \sim \frac{1}{\sqrt{\omega_\kappa}}\ \ \text{and}\ \ \delta\dot{\varphi}_{\vec{\kappa}}(t_i) \sim i\sqrt{\omega_\kappa}. \tag{4.14}$$

[6] This is equivalent to say that GR should not be important for $k \to \infty$.

As we assume the perturbations to be isotropic, we have $\delta\varphi_{\vec{k}} = \delta\varphi_k$. Since $\vec{\kappa} = \vec{k}_{phys}$ in Minkowski space, the left equation of the two above is equivalent to

$$|\delta\varphi_{\vec{k}_{phys}}(t_i)|^2 k_{phys}^3 \sim \frac{k_{phys}^3}{\sqrt{k_{phys}^2 + M^2}}. \tag{4.15}$$

(Quasi) de Sitter space can be approximated by Minkowski spacetime inside the horizon, so that

$$k_{phys} \approx k = constant. \tag{4.16}$$

Consequently at the moment of horizon crossing one has approximately with (4.5)[7]

$$|\delta\varphi_{\vec{k}}|^2 k^3\Big|_{crossing} \sim \frac{H_k^3}{\sqrt{H_k^2 + M^2}}. \tag{4.17}$$

To a good approximation we may neglect the mass in the denominator, because $M \sim H_f \leq H_k$. (4.17)[8] will serve as our initial condition for the evolution of the modes outside the horizon in a quasi de Sitter space where the expansion of space cannot be neglected. $|\delta\varphi_k|^2 k^3$ is called the spectrum of the field. To use (4.17) is called the *horizon-crossing approximation*. A more accurate method which we'll not use but just briefly mention is the so-called *Bessel-approximation* where one finds the behaviour of the modes around horizon crossing and extends the solution deeply inside and far outside the horizon, so that the Minkowski solution and the quasi de Sitter solution are not matched directly but have an intermediate function in between, serving as an "adapter". (6.37) is used for this procedure.

4.3 The power spectrum for the fluctuations of the inflaton

To calculate $\xi_\varphi(\vec{r})$ (no matter whether we are in an expanding or nonexpanding spacetime), with the energy density ϱ_φ (approximated by the potential term) for the field $\varphi + \delta\varphi$, we start from

$$\varrho_\varphi \sim \frac{M^2}{2}(\varphi + \delta\varphi)^2 \tag{4.18}$$

[7] For light scalar fields this reduces to $|\delta\varphi_k|^2 k^3 \sim H^2$. "One can think of this effect as similar to the Hawking radiation from the event horizon of a black hole: there is an event horizon in de Sitter space and one therefore sees a thermal background at temperature $\mathcal{T}_H = H/2\pi$ which corresponds to fluctuations in the $[\delta\varphi_k]$-field", [19].

[8] We'll give a more rigorous derivation of this formula in footnote 1 in section 8.1.

and hence, because the average of $\delta\varphi$ vanishes,

$$\langle \varrho_\varphi \rangle \sim \frac{M^2}{2}\Big(\varphi^2 + \langle \delta\varphi^2 \rangle\Big). \tag{4.19}$$

So with (3.3) one gets

$$\frac{\delta\varrho_\varphi}{\langle \varrho_\varphi \rangle} = \frac{2\varphi\ \delta\varphi + \delta\varphi^2 - \langle \delta\varphi^2 \rangle}{\varphi^2 + \langle \delta\varphi^2 \rangle}. \tag{4.20}$$

Now if

$$\varphi \gg \delta\varphi, \tag{4.21}$$

the last two terms in the numerator are negligible as well as the second term in the denominator, so this reads

$$\frac{\delta\varrho_\varphi}{\langle \varrho_\varphi \rangle} = \frac{2\delta\varphi}{\varphi}. \tag{4.22}$$

The density fluctuations are proportional to the fluctuations of the field, so *their corresponding correlation functions are also proportional*. So if we find the quantum fluctuations to be Gaussian, this quality will also hold for the density perturbations.[9]

As we're dealing with an operator, *the brackets of the correlation function are to be viewed as evaluating the expectation value of field operators.* So to calculate $\xi_\varphi(\vec{r})$, we expand $\delta\varphi$ in terms of (4.7) with (4.12)

$$\begin{aligned}\langle \delta\varphi(\vec{x})\delta\varphi(\vec{x}+\vec{r})\rangle &= \langle 0|\hat{\delta\varphi}(\vec{x})\hat{\delta\varphi}(\vec{x}+\vec{r})|0\rangle \qquad (4.23)\\ &= \frac{1}{2}\,\frac{1}{(2\pi)^3}\int\!\!\int \delta\varphi_{k'}\delta\varphi^*_{k''}\langle 0|\hat{a}_{\vec{k}'}\hat{a}^\dagger_{\vec{k}''}|0\rangle e^{i(\vec{k}'-\vec{k}'')\vec{x}}e^{-i\vec{k}''\vec{r}}d\vec{k}'d\vec{k}''.\end{aligned}$$

The other three products of the creation and annihilation operators vanish: there has to be an $\hat{a}^\dagger$ at the very left and an $\hat{a}$ at the very right of each product for it not to be zero; for the same reason we need the same numbers of $\hat{a}$'s and $\hat{a}^\dagger$'s in each product (this will be important when we treat isocurvature perturbations). Using the commutation relation (4.13) gives

$$\begin{aligned}\langle \delta\varphi(\vec{x})\delta\varphi(\vec{x}+\vec{r})\rangle &= \frac{1}{2}\,\frac{1}{(2\pi)^{3/2}}\int\!\!\int \delta\varphi_{k'}\delta\varphi^*_{k''}\ \delta(\vec{k}'-\vec{k}'')e^{i(\vec{k}'-\vec{k}'')\vec{x}}e^{-i\vec{k}''\vec{r}}d\vec{k}'d\vec{k}''\\ &= \frac{1}{2}\,\frac{1}{(2\pi)^{3/2}}\int |\delta\varphi_{k''}|^2 e^{-i\vec{k}''\vec{r}}d\vec{k}''. \qquad (4.24)\end{aligned}$$

"The statistical properties of the fluctuations of the field $[\delta\varphi]$ are completely determined by its set of correlation functions. The three-point correlation function,

$$[\langle \delta\varphi(\vec{x})\delta\varphi(\vec{x}+\vec{r}_1)\delta\varphi(\vec{x}+\vec{r}_2)\rangle], \tag{4.25}$$

[9] One gets this for all polynomial potentials.

vanishes because it contains an odd number of creation and annihilation operators; the four-point correlation function reduces to products of the two-point function in equation [(4.24)] [due to Wick's Theorem, see e.g. [26]], and so on. So *the result is that the field fluctuations are those of a Gaussian random process.*", [27]. Due to the proportionality between $\delta\varrho$ and $\delta\varphi$ the Gaussianity of $\delta\varphi$ translates to the Gaussianity of $\delta\varrho$. As we'll see in the next chapter, this proportionality is not given if $\delta\varphi \gg \varphi$ (fluctuations around zero), so that in that case one does not have Gaussian density perturbations.

With (4.22), (4.24) and (3.8) we get

$$\mathcal{P}_\varphi(k) = \frac{\sqrt{|\delta\varphi_k|^2 k^3}}{(2\pi)^{3/2}\varphi}, \tag{4.26}$$

which at the end of inflation gives

$$\mathcal{P}_\varphi(k)\Big|_f \sim \sqrt{|\delta\varphi_k|^2 k^3\Big|_f}. \tag{4.27}$$

To find $|\delta\varphi_k|$ one might want to start with a generalised Klein-Gordon equation that contains spatial derivatives. But the fluctuations of the inflaton will create perturbations in the energy-momentum tensor and hence induce metric perturbations. So we can't just deal with the inflaton's inhomogeneities by simply using the inhomogeneous Klein-Gordon equation. To take the backreaction of the perturbed metric onto the inflaton field into account (and vice versa), it is best to work with gauge-invariant quantities (see [13] for an in-depth discussion): The most general form of a metric with linear perturbations is (see [13])

$$ds^2 = a^2(\eta)\Big[-(1+2\phi)d\eta^2 + 2B_{|i}dx^i d\eta + \Big((1-2\psi)\delta_{ij} + 2E_{|i|j}\Big)dx^i dx^j\Big], \tag{4.28}$$

with $|$ denoting the spatial covariant derivative, ϕ, ψ, B and E being arbitrary functions of space and time and η being the so-called conformal time, introduced to factor a^2 out of the Robertson-Walker metric.[10] One has

$$d\eta = \frac{dt}{a} \quad \Big[\Rightarrow \quad d\eta = \frac{da}{a^2 H}\Big] \text{ and hence } \frac{d}{dt} = \frac{1}{a}\frac{d}{d\eta}. \tag{4.29}$$

The potential we encountered with the Sachs-Wolfe effect is one of the two combinations of these quantities deliberately chosen to achieve gauge invariance:

$$\Phi := \phi + \frac{1}{a}\Big[(B - E^\star)a\Big]^\star \text{ and } \Psi := \psi - \frac{a^\star}{a}(B - E^\star), \tag{4.30}$$

with the star denoting the derivative with respect to conformal time. One chooses the longitudinal gauge $B = E = 0$ (which gives $\phi = \Phi$ and $\psi = \Psi$) and plugs the two potentials into

[10] η is negative, with $\eta = 0$ corresponding to the infinite future.

the metric. The result is used in (2.1) to find the linearised equation of for the inhomogeneous field. One gets

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{1}{a^2}\nabla^2\delta\varphi + M^2\delta\varphi = 4\dot{\varphi}\dot{\Phi} - 2M^2\varphi\Phi. \tag{4.31}$$

To get the linearised perturbed Einstein equations one uses the metric in (1.2) and finds from the $i \neq j$-components that $\Phi = \Psi$. The 00-equation gives[11]

$$\frac{\nabla^2}{a^2}\Phi - 3H\dot{\Phi} - 3H^2\Phi = \dot{\varphi}\dot{\delta\varphi} + M\varphi\ \delta\varphi - \dot{\varphi}^2\Phi. \tag{4.33}$$

The last two equations can be approximated to a simpler set,

$$\begin{aligned} \Phi &= -\frac{\delta\varphi}{\varphi} \\ \text{and} \quad 3H\dot{\delta\varphi} + M^2\delta\varphi &= -2M^2\varphi\Phi. \end{aligned} \tag{4.34}$$

Plugging the first equation into the second one and using (2.8) gives

$$\delta\varphi \propto \frac{1}{\varphi}. \tag{4.35}$$

With (4.17) one finally gets

$$\mathcal{P}_\varphi(k)\Big|_f \sim M \ln\frac{H_k}{k}. \tag{4.36}$$

We thus arrive at a nearly scale invariant power spectrum (with slight logarithmic deviation) of Gaussian curvature perturbations, with the density contrast slightly increasing for larger scales.[12] [13] That these perturbations should be observable today with *classical* properties and no specific quantum characteristics is far from obvious. However, it was shown that the transition from quantum physics to classical physics conserves all predictions, see [29]. Using the fact that (4.36) stays almost constant in time on scales beyond the horizon (see [13]) one gets with (3.1)

$$10^{-5} \sim M \ln\frac{H_k}{k}. \tag{4.37}$$

On large scales the logarithm is $\mathcal{O}(100)$.[14] So one finds

$$M \sim 10^{-6}, \tag{4.38}$$

[11]Alternatively one can also work (see [28]) with the $0i$-equation,

$$\dot{\Phi} + H\Phi = \dot{\varphi}\ \delta\varphi. \tag{4.32}$$

[12]So far (4.36) is in agreement with the measurements of COBE.

[13]This result is also valid for the quintessential potential (10.1).

[14]This is easily understood due to the following consideration: One needs 60 e-folds to get the observable Universe to be homogeneous, see the first chapter. Hence only the fluctuations during the last 60 e-folds play a role in today's observable very large scale structure (galactic scales are smaller by a factor of 10^5, so that the value of the logarithm has decreases by only a factor of 10).

a reasonable mass for a scalar field in SUSY-GUT's.[15] *One sees how microscopic physics leaves its fingerprint on the very big scales: from viewing large scale CMBR anisotropies one gets information about particle physics.*

But up to which value of φ can we thus work with the approximation of a homogeneous background and perturbations not disturbing the homogeneity significantly? One has to have the mean energy density of the fluctuations smaller than the energy density of the background,

$$\langle \varrho_{\delta\varphi} \rangle < \varrho_{\varphi}. \tag{4.39}$$

For the left hand side one has

$$\begin{aligned} \langle \varrho_{\delta\varphi} \rangle_f &\sim M^2 \langle \delta\varphi^2 \rangle \ \Big| \ (4.24) \text{ with } \vec{r} = 0 \\ &\sim M^2 \int |\delta\varphi_k|^2 k^2 dk \sim M^2 \int \frac{\mathcal{P}_\varphi^2(k)}{k} dk \sim M^4 \int\limits_{a_i H_i}^{H_f} \frac{\left(\ln \frac{k}{H_k}\right)^2}{k} dk. \end{aligned} \tag{4.40}$$

As H_k is a quantity that changes rather little with k, we take it to be constant. So

$$\langle \varrho_{\delta\varphi} \rangle_f \sim M^4 \int\limits_{a_i \frac{H_i}{H_k}}^{\frac{H_f}{H_k}} \left(\ln \frac{k}{H_k}\right)^2 d\left(\ln \frac{k}{H_k}\right) \sim M^4 \left(\ln \frac{k}{H_k}\right)^3 \Bigg|_{a_i \frac{H_i}{H_k}}^{\frac{H_f}{H_k}}. \tag{4.41}$$

Since the modes with k near the lower cut-off mainly determine the value of the integral, we shall use $H_k = H_i$ so that we finally get that at the end of inflation

$$\langle \varrho_{\delta\varphi} \rangle_f \sim M^4 \left(\ln \frac{a_f}{a_i}\right)^3 \sim M^4 \varphi_i^6. \tag{4.42}$$

Plugging this into condition (4.39) one finds

$$\varphi \leq M^{-\frac{1}{3}}. \tag{4.43}$$

If we want to treat a period with φ not satisfying the inequality above one has to apply different methods.

[15]For a quartic potential (see last section in Ch.2) one finds

$$\mathcal{P}_\varphi(k)\Big|_f \sim \sqrt{\lambda}\left(\ln \frac{H_k}{k}\right)^{3/2} \quad \Rightarrow \quad \lambda \sim 10^{-14}.$$

4.4 The self-reproducing Universe

As a perturbation crosses the horizon, it does not contribute any inhomogeneities on scales smaller than the horizon, it merely raises or lowers the homogeneous value of the field by the root-mean-square deviation $(\delta\varphi)_{rms} = \sqrt{\langle\delta\varphi^2\rangle}$ (the rms-deviation gives the average deviation of φ from its expectation value). The description of the classical motion of φ, resulting in inflation as found in Ch.2, is valid if the change of φ due to the fluctuations created during the characteristic time scale $1/H$ (a increases by a factor of e) is not larger than the amount $\Delta_{sr}\varphi$ by which φ decreases during that same period due to slow roll. The amount the field changes classically during the characteristic time scale is given as

$$\Delta_{sr}\varphi(t) = \varphi(t) - \varphi(t - 1/H), \tag{4.44}$$

so that we get the the condition for classical evolution dominating as

$$(\delta\varphi)_{rms} \overset{!}{<} \Delta_{sr}\varphi. \tag{4.45}$$

Plugging (2.29) into (4.44) gives

$$\Delta_{sr}\varphi \sim \frac{M}{H}. \tag{4.46}$$

We get the root-mean-square deviation with (4.24) with $\vec{r} = 0$. One is interested in a fluctuation that crosses the horizon after a characteristic time has passed so that

$$\langle\delta\varphi^2\rangle \sim \int\limits_{aH}^{eaH} \frac{H^2}{k^3} k^2 \; dk. \tag{4.47}$$

We have taken the Hubble parameter to be constant in this time interval. Integration gives

$$\langle\delta\varphi^2\rangle \sim \; H^2 \;\text{ so that }\; (\delta\varphi)_{rms} \sim \pm H. \tag{4.48}$$

To emphasise it again, $(\delta\varphi)_{rms}$ is a random variable, typically in some regions the value of the scalar field increases by H, and in other regions it decreases by H. Using the formulas for $(\delta\varphi)_{rms}$ and $\Delta_{sr}\varphi$ in (4.45) gives

$$M^2\varphi^2 \sim H^2 < M \quad \Rightarrow \quad \varphi < M^{-\frac{1}{2}}. \tag{4.49}$$

This is the condition for the evolution of φ not to be dominated by quantum fluctuations; φ decays faster by SR than it might typically grow by quantum fluctuations. So if we have an originally homogeneous field (due to the considerations in the last section) of size $1/H$, but $\varphi > M^{-\frac{1}{2}}$, we get that after a characteristic period has passed, the size of the Universe has increased about a factor of e, so that the volume has increased by a factor of 20, it consists of 20 regions that have the size of the original one. Now all of these regions are homogeneous, but half of them have a field with $\varphi(t+1/H) \approx \varphi(t)+(\delta\varphi)_{rms}$ and others with $\varphi(t+1/H) \approx \varphi(t) - (\delta\varphi)_{rms}$. The regions with the higher φ-value expand even more rapidly than their "predecessor", producing again a volume that can be divided into regions of the original size. Again for half of the regions the φ-field is higher than it was before. This process can be repeated up to the Planck scale for all regions with φ high enough that quantum fluctuations are important. Hence if we had started off with a Universe dominated by a scalar field being above the critical value it would decrease in some regions by negative fluctuations until the SR motion dominates and determines the motion, giving the inflationary scenario described in the last chapter. But a large fraction would give rise to new regions with inflationary conditions. So our original Universe would constantly produce new "baby-Universes", ad infinum. Those baby-Universes then undergo an analogous evolution, producing another generation of baby-Universes, so that one can conclude to have an "*eternally existing, [fractal] self-reproducing inflationary Universe*", [30].[16] Once a region has left the critical regime where quantum fluctuations take over, it evolves according to chaotic inflation (Ch.2) and then the way described in the first chapter. The regions produced by SR are nicely homogeneous, but the very large scale structure is fractal. So although inflation provides an explanation for the observed homogeneity and hence the cosmological principle, it can be viewed only as a local fact in the context of the self-reproducing Universe (although the homogeneous region is many orders of magnitude larger than the observed Universe); if one speaks about "the end of inflation" one merely means the end of classical SR in that part of the Universe we inhabit. After the era of self-reproduction stopped in our region there are still quantum fluctuations too large to neglect till φ drops below $M^{-\frac{1}{3}}$ (see (4.43)). Summarised we find the following ranges:

$$\begin{aligned} M^{-\frac{1}{2}} \leq \varphi \leq M^{-1}, &\qquad \text{self-reproduction} \\ M^{-\frac{1}{3}} \leq \varphi \leq M^{-\frac{1}{2}}, &\qquad \text{large fluctuations} \\ 1 \leq \varphi \leq M^{-\frac{1}{3}}, &\qquad \text{small fluctuations.} \end{aligned} \tag{4.50}$$

So one says that $M^{-\frac{1}{2}}$ is the *self-reproduction scale* and $M^{-\frac{1}{3}}$ the *backreaction scale*. The fluctuations in the second region are pushed out of our observable Universe. (4.50) is the reason why we won't take $\varphi_i = \frac{1}{M}$ as our initial condition in the next chapters: With $\mathcal{N}$ being the number of e-folds of inflation one has $\mathcal{N} \sim \varphi_i^2$; as one needs to have $\varphi_i < \frac{1}{\sqrt{M}}$ to stay below the self-reproduction scale, and on the other hand inflation has to solve the horizon problem, one gets

$$60 < \mathcal{N} < \frac{1}{M} \sim 10^6. \tag{4.51}$$

[16]So in a way globally the Universe is a steady state Universe (see footnote 3 in Ch.1). There's hope to avoid the necessity of the big bang via this way.

Chapter 5

Multi-field inflationary models

5.1 Introducing a second scalar field

The result at the end of section 4.3 (a power spectrum of curvature perturbations fitting the observed CMBR anisotropies, the inflaton mass being reasonable) is giving hope that the origin of the seeds of today's large scale structure can be traced back to QFT-fluctuations during a quasi de Sitter period in combination with a realistic particle model. However, it is naïve to consider only *one* scalar field, "as in unified theories of elementary particles there exist many scalar fields, some of which (let us denote them by [σ]) interact extremely weakly with other particles (the so-called "hidden sector")", [31]. Examples are axion[1] and Polonyi[2] fields.

The simplest generalisation of the scenario just explained (with φ and a influencing each other) is performed by adding a second scalar, the σ-field with mass m, being driven by the former two but on the other hand being too weak to have vice versa an impact on their evolution. In other words, the energy-momentum tensor of the σ-field is very small from the beginning of the period of inflation until the end of it, so that the total energy-momentum tensor for a long time is dominated by φ (which hence is responsible for inflation), we have to have

$$\varrho_\varphi \gg \varrho_\sigma. \tag{5.1}$$

We shall work with the approximation that the energy densities are given by their potential terms. The Lagrangian for the two fields in Minkowski space reads

$$\mathcal{L} = \dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 + \dot{\sigma}^2 - (\vec{\nabla}\sigma)^2 - \Big(M^2\varphi^2 + m^2\sigma^2\Big), \tag{5.2}$$

[1] One proposal to circumvent the strong CP-problem is made by introducing a global symmetry (the Peccei-Quinn symmetry). It is supposed to be spontaneously broken; for that one needs a further bosonic field, the axion.

[2] "A necessary ingredient of all models of elementary particles based on $N = 1$ supergravity", [32].

having assumed the simplest nontrivial coupling between the two fields, with g being the coupling constant. The σ-field can be decomposed as

$$\sigma(\vec{x},t) = \langle\sigma(t)\rangle + \delta\sigma(\vec{x},t) := \langle\sigma\rangle + \delta\sigma. \tag{5.3}$$

where $\langle\sigma\rangle$ satisfies the homogeneous Klein-Gordon equation. We will especially investigate the special case where initially the average σ-field vanishes compared to the fluctuations of the σ-field,[3]

$$\langle\sigma\rangle \ll \delta\sigma \quad \Rightarrow \quad \sigma \approx \delta\sigma, \tag{5.4}$$

so that we'll look for $\mathcal{P}_{\delta\sigma}(k)$ instead of $\mathcal{P}_{\langle\sigma\rangle}(k)$ (*the subscript always denotes the component that is larger than the other one*). Condition (5.1) now reads

$$\varrho_\varphi \gg \langle\varrho_{\delta\sigma}\rangle. \tag{5.5}$$

This is somewhat analogous to (4.39). We'll see that (5.5) will put restrictions on some of the parameters. *Due to the low energy density of the σ-field we are going to deal with isocurvature perturbations, unlike the perturbations in the last chapter.*[4] *Furthermore, since the average σ-field is supposed to vanish, the density perturbations will be nongaussian.*

With (5.5) the σ-field's fluctuations' mean energy density is far smaller than the mean energy density of the fluctuations of the φ-field. It does hence not look like that the power spectrum of the σ-field is going to play an important role, compared to $\mathcal{P}_\varphi(k)$. But (I) if after inflation the φ-field strongly interacts with other fields to decay into ultra-relativistic particles and its energy density hence quickly decays $\sim a^{-4}$ (see (1.12)), and (II) if the σ-field does not decay but instead oscillates and behaves like cold dust $\sim a^{-3}$ (as we found in Ch.2 for the inflaton field for the period of oscillation), then later on the σ-field will become dominating, and hence induce metric perturbations, which is why *the σ-field's power spectrum could be of prime importance for the large scale structure of the Universe.* The σ-dust can have survived till today, contributing substantially to the CDM ("What could be more natural to expect that the hidden mass of the Universe is connected with hidden fields!", [32])[5].

The equation of motion for the σ-field in a time-dependent background is easily found. First we apply (5.3). For the homogeneous field we can use (2.4) with the replacements

$$\begin{aligned} \varphi &\rightarrow \langle\sigma\rangle, \\ M &\rightarrow m, \end{aligned} \tag{5.6}$$

[3]This could be due to high-temperature effects, see [31] Moreover, if we are dealing with a very massive or a strongly coupled field, it automatically strongly decreases.

[4]As these perturbations do not depend on the fluctuations of the inflaton, one also finds the terminology *isoinflaton* perturbations.

[5]Once we introduced *one* additional field, we are free to take over its results to other scalar fields (not interacting with each other) obeying the same conditions as the σ-field. The sum of all the energy densities of course has to stay way below of the energy density of the inflaton. Note that all these fields might decay at another point in time.

and an added coupling term so that

$$\langle\ddot{\sigma}\rangle + 3H\langle\dot{\sigma}\rangle + \left(m^2 + \alpha H^2\right)\langle\sigma\rangle = 0, \tag{5.7}$$

having take into account that $\varphi^2 = \frac{H^2}{H_f^2}$ (see (2.21)) and using the abbreviation

$$\alpha = \frac{3g^2}{M^2}. \tag{5.8}$$

For the inhomogeneous field we note again that the σ-field does not influence the energy density during inflation, *so we can neglect its gravitational influence on spacetime and hence don't have to take metric perturbations into account*[6], unlike the case for the fluctuations of the inflaton. So we just take (4.31), use

$$\delta\varphi \;\rightarrow\; \delta\sigma, \tag{5.9}$$

add a coupling term and chop off all terms on the right hand side. One gets[7]

$$\ddot{\delta\sigma} + 3H\dot{\delta\sigma} - \frac{1}{a^2}\nabla^2\delta\sigma + \left(m^2 + \alpha H^2\right)\delta\sigma = 0 \tag{5.10}$$

as the equation of motion. *Quantisation and initial conditions are analogous to the results we obtained for $\delta\varphi$.* We again expand the operator related to $\delta\sigma(\vec{x}, t)$ in terms of plane waves using creation and annihilation operators (which fulfil the same commutation relation as in section 4.2) to get

$$\hat{\delta\sigma}(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2}}\left[\delta\sigma_{\vec{k}}^*(t)e^{i\vec{k}\vec{x}}\hat{a}_{\vec{k}} + \delta\sigma_{\vec{k}}(t)e^{-i\vec{k}\vec{x}}\hat{a}_{\vec{k}}^\dagger\right]\, d\vec{k}. \tag{5.11}$$

One gets that the $\delta\sigma_k$-modes evolve according to

$$\ddot{\delta\sigma}_k + 3H\dot{\delta\sigma}_k + \left(\frac{k^2}{a^2} + m^2 + \alpha H^2\right)\delta\sigma_k = 0. \tag{5.12}$$

with (4.17) as the initial condition with the replacements (5.6) and (5.9). Note that in general we may no longer neglect the mass term in the denominator, because one might have $m > H_k$. Solving (5.12) (mostly approximately) will be an important topic in the following chapters.

[6] The density perturbations with two fields inducing metric perturbations is treated in [28].

[7] Equally valid we could have performed the same generalisation for the $\delta\sigma$-field (coupled to the inflaton) as we did for the φ-field (see (2.1) up to (2.4)), but without further complications taking along the spatial gradients, arriving at

$$\frac{1}{a^3}\left[\frac{d}{dt}\left(a^3\dot{\delta\sigma}\right) - a\nabla^2\delta\sigma\right] + \left(m^2 + g^2\varphi^2\right)\delta\sigma = 0.$$

5.2 A useful approximation for H_k

By fixing the constants of integration with (4.17) we get $|\delta\sigma_k|^2 k^3$ depending on H_k, which dependence on k can be found with (4.5). Since $|\delta\sigma_k|^2 k^3$ will appear in an integration over k, it is necessary to know the k-dependence of H_k. But with (2.27) the equation (4.5) is a transcendental equation. Fortunately one can find an approximation. For de Sitter space H is the same for all k. For quasi de Sitter space H_k is a slowly varying function of k, so that to 0^{th} order we could take H_k to be constant. But realizing that (see (2.27))

$$H_k^2 = H_f^2\Big(1 - 2\ln a_k\Big) = 2H_f^2 \ln \frac{\sqrt{e}}{a_k} \tag{5.13}$$

and using (4.5) we get

$$H_k^2 \sim H_f^2 \ln \frac{\sqrt{e}H_k}{k}. \tag{5.14}$$

So we don't take the "bare" H_k but the H_k in the argument of the logarithm to be constant. Thus, by using (5.14), we extracted a weak k-dependence out of H_k before setting it constant.[8] For big scales (where $a_k \ll 1$) one has $-\ln a_k \gg 1$ so that

$$H_k^2 \approx H_f^2 \ln \frac{H_k}{k} \tag{5.15}$$

for $k \ll H_f$. We shall mostly use this formula. Of course this procedure could be repeated to get an approximation to higher orders, but integrals over logarithms of logarithms are quite hard to solve so that we shall stop here. One is fairly flexible which value to choose for the constant H_k in the argument of the logarithm: if the modes with k near the lower bound of (4.4) determine the value of the integral, we take $H_k \sim H_i$ (as already done in section 4.3); if on the other hand the modes with k near the upper cut-off are more important, we choose $H_k \sim M$.

5.3 The power spectrum for the fluctuations of the σ-field

We can take over all results from (4.18) to (4.26) with the corresponding replacements. So if we consider the average field to be much larger than the fluctuations, we first solve the equations of motion, (5.12) and (5.7). The result is then plugged into (4.26). If on the other hand the fluctuations dominate, that is if we have (5.4), we get from (4.20) that

$$\frac{\delta\varrho_{\delta\sigma}}{\langle\varrho_{\delta\sigma}\rangle} = \frac{\delta\sigma^2 - \langle\delta\sigma^2\rangle}{\langle\delta\sigma^2\rangle}. \tag{5.16}$$

[8] We actually tacitly used this step already in section 4.3.

Note that $\delta\sigma$ *and* $\delta\varrho$ *are no longer proportional, so that Gaussianity of* $\delta\sigma$ *will not translate into Gaussianity of* $\delta\varrho$, we have *nongaussian isocurvature perturbations*. From (5.16) we find

$$\xi_{\delta\sigma}(\vec{r}) = \frac{\langle \delta\sigma^2(\vec{x}+\vec{r})\ \delta\sigma^2(\vec{x})\rangle}{\langle \delta\sigma^2\rangle^2} - 1. \tag{5.17}$$

Using (4.24) with $r = 0$ we get that

$$\langle \delta\sigma^2\rangle = \frac{1}{2}\ \frac{1}{(2\pi)^{3/2}}\ \int |\delta\sigma_{k''}|^2 d\vec{k}''. \tag{5.18}$$

To calculate the term in the numerator of $\xi_{\delta\sigma}(\vec{r})$ we use (5.11) to get

$$\begin{aligned}\langle \delta\sigma^2(\vec{x}+\vec{r})\delta\sigma^2(\vec{x})\rangle &= \langle 0|\hat{\delta\sigma}^2(\vec{x}+\vec{r})\ \hat{\delta\sigma}^2(\vec{x})|0\rangle \\ &= \frac{1}{4(2\pi)^6}\int\int\int\int \delta\sigma^*_{k'}\ \delta\sigma_{k''}\ \delta\sigma^*_{k'''}\ \delta\sigma_{k''''} e^{i(\vec{k}'-\vec{k}''+\vec{k}'''-\vec{k}'''')\vec{x}} \\ &\times\ e^{i(\vec{k}'''-\vec{k}'''')\vec{r}}\langle 0|\hat{a}_{\vec{k}'}\hat{a}^\dagger_{\vec{k}''}\hat{a}_{\vec{k}'''}\hat{a}^\dagger_{\vec{k}''''}|0\rangle d\vec{k}'d\vec{k}''d\vec{k}'''d\vec{k}'''' \\ &+\ \frac{1}{4(2\pi)^6}\int\int\int\int \delta\sigma^*_{k'}\ \delta\sigma^*_{k''}\ \delta\sigma_{k'''}\ \delta\sigma_{k''''} e^{i(\vec{k}'+\vec{k}''-\vec{k}'''-\vec{k}'''')\vec{x}} \\ &\times\ e^{-i(\vec{k}'''+\vec{k}'''')\vec{r}}\langle 0|\hat{a}_{\vec{k}'}\hat{a}_{\vec{k}''}\hat{a}^\dagger_{\vec{k}'''}\hat{a}^\dagger_{\vec{k}''''}|0\rangle d\vec{k}'d\vec{k}''d\vec{k}'''d\vec{k}''''.\end{aligned} \tag{5.19}$$

All other 14 products of the creation and annihilation operators vanish. The commutation relation gives

$$\langle 0|\hat{a}_{\vec{k}'}\hat{a}^\dagger_{\vec{k}''}\hat{a}_{\vec{k}'''}\hat{a}^\dagger_{\vec{k}''''}|0\rangle = (2\pi)^3\ \delta(\vec{k}'-\vec{k}'')\ \delta(\vec{k}'''-\vec{k}'''') \tag{5.20}$$

and

$$\langle 0|\hat{a}_{\vec{k}'}\hat{a}_{\vec{k}''}\hat{a}^\dagger_{\vec{k}'''}\hat{a}^\dagger_{\vec{k}''''}|0\rangle = (2\pi)^3\ \delta(\vec{k}'-\vec{k}''')\ \delta(\vec{k}''-\vec{k}'''') + (2\pi)^3\ \delta(\vec{k}'-\vec{k}'''')\ \delta(\vec{k}''-\vec{k}'''). \tag{5.21}$$

Using this in (5.19) we find

$$\begin{aligned}\langle \delta\sigma^2(\vec{x}+\vec{r})\delta\sigma^2(\vec{x})\rangle &= \frac{1}{4(2\pi)^3}\left[\int |\delta\sigma_k|^2 d\vec{k}\right]^2 \\ &+\ 2\ \times\ \frac{1}{4(2\pi)^3}\int\int |\delta\sigma_{\vec{k}'}|^2|\delta\sigma_{k''}|^2 e^{i(\vec{k'}+\vec{k''})\vec{r}}d\vec{k}'d\vec{k}''.\end{aligned} \tag{5.22}$$

The first term cancels the "-1" in (5.17). So we get[9]

$$\xi_{\delta\sigma}(\vec{r}) \sim \frac{1}{\langle\delta\sigma^2\rangle^2}\int\int |\delta\sigma_{\vec{k}'}|^2|\delta\sigma_{k''}|^2 e^{i(\vec{k'}+\vec{k''})\vec{r}}d\vec{k}'d\vec{k}''. \tag{5.23}$$

[9]Notice that

$$\frac{1}{2(2\pi)^3}\int\int |\delta\sigma_{\vec{k}'}|^2|\delta\sigma_{k''}|^2 e^{i(\vec{k'}+\vec{k''})\vec{r}}d\vec{k}'d\vec{k} = 2\left[\frac{1}{2(2\pi)^{\frac{3}{2}}}\int |\delta\sigma_{k'}|^2 e^{-i\vec{k}'\vec{r}}d\vec{k}'\right]^2 = 2\langle\ \delta\sigma(\vec{x})\ \delta\sigma(\vec{x}+\vec{r})\rangle^2.$$

In order to get a term $e^{i\vec{k}\vec{r}}$ we set $\vec{k} := \vec{k}' + \vec{k}''$ so that

$$\xi_{\delta\sigma}(\vec{r}) \sim \frac{1}{\langle\delta\sigma^2\rangle^2} \int\int |\delta\sigma_{|\vec{k}-\vec{k}'|}|^2 |\delta\sigma_{k'}|^2 e^{i\vec{k}\vec{r}} d\vec{k} d\vec{k}'. \tag{5.24}$$

Now we can plug (5.24) into (3.8), finally giving

$$\mathcal{P}^2_{\delta\sigma}(k) \sim \frac{m^4\ k^3}{\langle\varrho_{\delta\sigma}\rangle^2} \int |\delta\sigma_{|\vec{k}-\vec{k}'|}|^2 |\delta\sigma_{k'}|^2 d\vec{k}'\ , \tag{5.25}$$

with

$$\langle\varrho_{\delta\sigma}\rangle \sim m^2 \int |\delta\sigma_{k'}|^2 k'^2 dk'. \tag{5.26}$$

5.4 Calculating the integral in the expression of the power spectrum

In order to perform the integration in (5.25), we will introduce several new symbols in this section so that the equations become more transparent. We'll keep the treatment fairly general so that we later may easily apply the final result at the end of this section to specific cases. We first set, after having incorporated (5.15) in (5.27),

$$|\delta\sigma_k|^2 k^3 := S\ \cdot\ \varsigma\left(\frac{k}{H_k}\right), \tag{5.27}$$

S containing all prefactors and ς the functional dependence on k/H_k. We furthermore introduce

$$\vec{l} := \frac{\vec{k}}{H_k}. \tag{5.28}$$

As an example, from $|\delta\sigma_k|^2 k^3 \sim C\ k^\nu$ one gets $\varsigma(l) \sim l^\nu$ and $S \sim CH_k^\nu$. Furthermore one gets from (4.4)

$$\varepsilon := a_i \frac{H_i}{H_k} \le l \le \frac{H_f}{H_k} =: L. \tag{5.29}$$

This result was to be expected according to [34]: "Since $[\delta\sigma]$ is a Gaussian random process with zero mean value, we have

$$[\langle\delta\sigma^2(\vec{x})\delta\sigma^2(\vec{x}+\vec{r})\rangle = \langle\delta\sigma^2\rangle^2 + 2\langle\delta\sigma(\vec{x})\delta\sigma(\vec{x}+\vec{r})\rangle^2]$$

so the dimensionless mass autocorrelation function is

$$\left[\frac{\langle\delta\varrho(\vec{x})\ \delta\varrho(\vec{x}+\vec{r})\rangle}{\langle\varrho\rangle^2} = 2\frac{\langle\delta\sigma(\vec{x})\ \delta\sigma(\vec{x}+\vec{r})\rangle^2}{\langle\delta\sigma^2\rangle}\right].$$

That is, the mass autocorrelation function is proportional to the square of the field autocorrelation function."

We make the approximation that

$$H_k \approx H_{k'} \approx H_{|\vec{k}-\vec{k}'|}, \tag{5.30}$$

so one has

$$\langle \varrho_{\delta\sigma} \rangle^2 \ = S^2 \ m^4 \Bigg[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \Bigg]^2 \tag{5.31}$$

and

$$\begin{aligned} \frac{\mathcal{P}^2_{\delta\sigma}(l)}{l^3} \sim \Bigg[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \Bigg]^{-2} \int \frac{\varsigma(l')}{l'^3} \ \frac{\varsigma(|\vec{l}-\vec{l}'|)}{|\vec{l}-\vec{l}'|^3} \ d\vec{l}', \\ \text{with} \ \ d\vec{l}' = l'^2 \ \sin\vartheta \ d\phi \ d\vartheta \ dl'. \end{aligned} \tag{5.32}$$

We choose the angle between $\vec{l}$ and $\vec{l}'$ to be identical with ϑ, so the integration over $d\phi$ just gives a prefactor of 2π. In the (l/l')-plane (see Fig.5.1) the calculation of $\mathcal{P}^2_{\delta\sigma}(k)$ takes place in form of an integration along some line $l = constant$. (5.29) is valid for l' and $|\vec{l}-\vec{l}'|$, too. The restrictions on l and l' cause the region of interest to shrink to the square (in Fig.5.1) bordered by $l=\varepsilon, \ l=L, \ l'=\varepsilon$ and $l'=L$. Furthermore

$$|\vec{l}-\vec{l}'| = \sqrt{l^2 + l'^2 - 2ll'\cos\vartheta} \quad \Rightarrow \quad |l-l'| \leq |\vec{l}-\vec{l}'| \leq l + l'. \tag{5.33}$$

With (5.29) this gives a "tube" of width 2ε around the line $l = l'$, and one cuts off everything above the line $l' = L - l$, leaving behind the grey area in Fig.5.1. Making the substitution

$$\Delta := |\vec{l}-\vec{l}'| \quad \Rightarrow \quad \frac{\Delta \ d\Delta}{2ll'} = \sin\vartheta \ d\vartheta, \tag{5.34}$$

one finds (5.32) to be[10]

$$\frac{\mathcal{P}^2_{\delta\sigma}(l)}{l^2} \sim \Bigg[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \Bigg]^{-2} \int \frac{\varsigma(l')}{l'^2} \Bigg[\int\limits_{|l-l'|}^{l+l'} \frac{\varsigma(\Delta)}{\Delta^2} d\Delta \Bigg] \ dl' \qquad \text{or} \tag{5.35}$$

$$\Bigg[\int\limits_{\varepsilon}^{L} \ \frac{\varsigma(l')}{l'} dl' \Bigg]^2 \ \frac{\mathcal{P}^2_{\delta\sigma}(l)}{l^2} \sim \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l')}{l'^2} \Bigg[\int\limits_{l-l'}^{l+l'} \frac{\varsigma(\Delta)}{\Delta^2} d\Delta \Bigg] \ dl' + \int\limits_{l+\varepsilon}^{L-l} \frac{\varsigma(l')}{l'^2} \Bigg[\int\limits_{l'-l}^{l+l'} \frac{\varsigma(\Delta)}{\Delta^2} d\Delta \Bigg] \ dl'$$

for the range

$$2\varepsilon \leq l \leq \frac{L-\varepsilon}{2}. \tag{5.36}$$

[10]Note the symmetry of the integrals over l' and Δ.

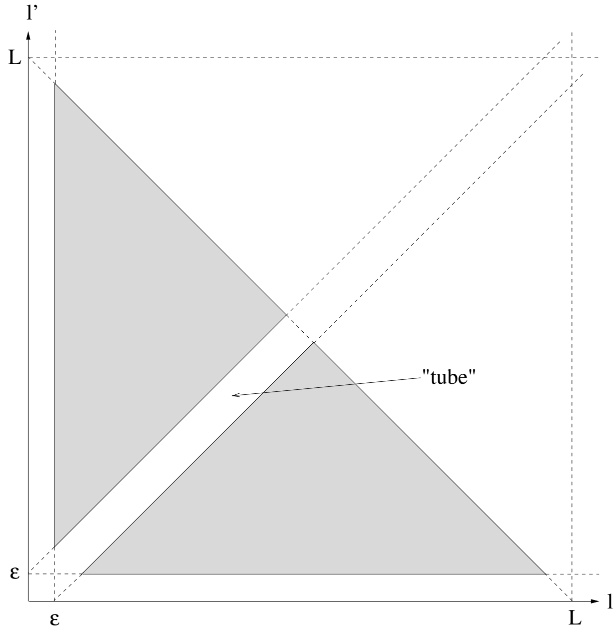

Figure 5.1: The region of interest in the (l/l')-plane.

For the rest of this text all calculations of power spectra shall be performed in this range! Introducing the following abbreviation for the indefinite integral

$$\Sigma(\Delta) := \int \frac{\varsigma(\Delta)}{\Delta^2} d\Delta, \tag{5.37}$$

one gets for the right hand side of (5.35)

$$\int_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l')}{l'^2} \left[\Sigma(l+l') - \Sigma(l-l')\right] \, dl' + \int_{l+\varepsilon}^{L-l} \frac{\varsigma(l')}{l'^2} \left[\Sigma(l+l') - \Sigma(l'-l)\right] \, dl', \tag{5.38}$$

again abbreviated by

$$\int_{\varepsilon}^{l-\varepsilon} I_1(l') \, dl' \; + \; \int_{l+\varepsilon}^{L-\varepsilon} I_2(l') \, dl'. \tag{5.39}$$

Unfortunately there is only a limited class of $\varsigma(l)$ where one can solve the integral above exactly, so one has to find some approximation scheme. A case we shall encounter several times is with Σ having a singularity (and consequently also $\frac{\varsigma(l)}{l^2}$ having a singularity) if its argument is zero (a case we shall encounter several times). Then the integrands above have singularities at $l' = 0$ and $l' = l$. Both are not within the integration range due to the cut-offs, nevertheless the modes with l near the cut-off determine the values of the integrals. The first integral depends on the regions near both singularities, while the second one is determined only by the region near the singularity at $l = l'$. Hence, if the two singularities are not too close, one may approximate the integrands by

$$I_1 \approx \left. I_1 \right|_{l' \to 0} + \left. I_1 \right|_{l' \to l} \text{ and } I_2 \approx \left. I_2 \right|_{l' \to l}. \tag{5.40}$$

Due to the symmetry of $\Sigma(|l - l'|)$ around $l = l'$ one finds

$$\int\limits_{l+\varepsilon}^{L-\varepsilon} \left. I_2 \right|_{l' \to l} dl' \approx \int\limits_{\varepsilon}^{l-\varepsilon} \left. I_1 \right|_{l' \to l} dl', \tag{5.41}$$

so that one gets

$$\left[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \right]^2 \frac{\mathcal{P}^2_{\delta\sigma}(l)}{l^2} \sim \int\limits_{\varepsilon}^{l-\varepsilon} \left. I_1 \right|_{l' \to 0} dl' + 2 \int\limits_{\varepsilon}^{l-\varepsilon} \left. I_1 \right|_{l' \to l} dl'. \tag{5.42}$$

Making a Taylor expansion up to first order of $\left[\Sigma(l + l') - \Sigma(l - l') \right]$ with respect to l' around zero gives

$$\int_{\varepsilon}^{l-\varepsilon} \left. I_1 \right|_{l' \to 0} dl' \approx 2 \frac{\varsigma(l)}{l^2} \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l')}{l'} dl'. \tag{5.43}$$

For $l' \approx l$ we make the approximation

$$\begin{aligned}
\int_{\varepsilon}^{l-\varepsilon} \left. I_1 \right|_{l' \to l} dl' &\approx \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l)}{l^2} \left[\Sigma(2l) - \Sigma(l - l') \right] dl', \\
&\approx \frac{\varsigma(l)}{l^2} \left[\Sigma(2l) \int\limits_{\varepsilon}^{l-\varepsilon} dl' - \int\limits_{\varepsilon}^{l-\varepsilon} \Sigma(l - l')\, dl' \right] \\
&\approx \frac{\varsigma(l)}{l^2} \left[l\, \Sigma(2l) - \int\limits_{\varepsilon}^{l-\varepsilon} \Sigma(l')\, dl' \right].
\end{aligned} \tag{5.44}$$

so that

$$\mathcal{P}^2_{\delta\sigma}(l) \sim \frac{2\varsigma(l)}{\left[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \right]^2} \left[\int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l')}{l'} dl' + l\, \Sigma(2l) - \int\limits_{\varepsilon}^{l-\varepsilon} \Sigma(l') dl' \right]. \tag{5.45}$$

If one deals with ς being a power law of l (with exponent only slightly bigger than zero), or being logarithmically dependent on l (or both of them together), the second term in brackets can be neglected (this will become obvious when we treat the de Sitter case), and the third and the first one are approximately equal up to the sign so that one may even write for $\varepsilon \ll l \ll L$

$$\mathcal{P}^2_{\delta\sigma}(l) \sim \frac{4\varsigma(l)}{\left[\int\limits_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl'\right]^2} \int\limits_{\varepsilon}^{l} \frac{\varsigma(l')}{l'} dl'. \tag{5.46}$$

So the problem of finding $\mathcal{P}^2_{\delta\sigma}(k)$ is reduced to just one fairly simple integration.[11]

[11] Note that the condition (5.29) on $|\vec{l} - \vec{l'}|$ does not have a strong influence on the final result.

Chapter 6

Four useful transformations of the Klein-Gordon equation for the $\delta\sigma$-field

To solve (5.12)[1], it is very convenient to use other variables instead of t. Depending on the functional form of H and a, each variable has its advantages for a different background.

6.1 The scale factor as a variable

First we treat $a(t)$, which appears explicitly in the term with the gradient. Using (2.22) one gets

$$Ha\frac{d}{da}\Big(Ha\frac{d}{da}\delta\sigma_k\Big) + 3H^2 a\frac{d}{da}\delta\sigma_k + \left(\frac{k^2}{a^2} + m^2 + \alpha H^2\right)\delta\sigma_k = 0$$

$$\Rightarrow \quad \delta\sigma_k'' + \Big(\frac{4}{a} + \frac{H'}{H}\Big)\delta\sigma_k' + \left(\frac{k^2}{H^2a^4} + \frac{m^2}{H^2a^2} + \frac{\alpha}{a^2}\right)\delta\sigma_k = 0. \tag{6.1}$$

To get a simpler differential equation where we only have the function and its second derivative we set $\delta\sigma_k(a) = \frac{u_k(a)}{A(a)}$ and find

$$u_k'' + \left(2\frac{A'}{A} - \frac{4}{a} - \frac{H'}{H}\right) u_k' + \left[\frac{m^2}{H^2a^2} + \frac{k^2}{H^2a^4} + \frac{\alpha}{a^2} - \frac{A''}{A} - \frac{A'}{A}\left(2\frac{A'}{A} - \frac{4}{a} - \frac{H'}{H}\right)\right] u_k = 0. \tag{6.2}$$

Demanding that $A(a)$ is constructed such that the coefficient for $u'(a)$ vanishes gives

$$2\frac{A'}{A} = \frac{4}{a} + \frac{H'}{H} \tag{6.3}$$

[1] As in the integrals for $\mathcal{P}_{\delta\sigma}(k)$ and $\langle\varrho_{\delta\sigma}\rangle$ one always encounters $|\delta\sigma_k|^2$ instead of $\delta\sigma_k$, we had a look at the differential equation directly for $|\delta\sigma_k|^2$. But this equation is nonlinear and hence much more difficult to solve than the original one.

$$\text{and} \qquad u_k'' + \left(\frac{m^2}{H^2 a^2} + \frac{\alpha}{a^2} + \frac{k^2}{H^2 a^4} - \frac{A''}{A} \right) u_k = 0. \tag{6.4}$$

If H is not constant, (6.3) is solved by

$$A(a) \propto H^{1/2} a^2, \tag{6.5}$$

so that one gets

$$\delta\sigma_k = \frac{u_k}{H^{1/2} a^2} \tag{6.6}$$

with u_k satisfying

$$u_k'' + \left(\frac{m^2}{H^2 a^2} + \frac{\alpha}{a^2} + \frac{k^2}{H^2 a^4} - \frac{(H^{1/2} a^2)''}{H^{1/2} a^2} \right) u_k = 0. \tag{6.7}$$

Now if we have $H = constant$, (6.3) looks like

$$2\frac{A'}{A} = \frac{4}{a}, \quad \text{so one gets} \quad A(a) \propto a^2 \quad \text{and} \quad \delta\sigma_k = \frac{u_k}{a^2} \tag{6.8}$$

with

$$a^2 u_k'' + \left(\frac{k^2}{a^2 H^2} + \frac{m_{eff}^2}{H^2} - 2 \right) u_k = 0. \tag{6.9}$$

The coupling term to the inflaton is in this case equivalent to curvature coupling, due to the constant Hubble parameter. It results in a constant, increased effective mass $m_{eff}^2 = m^2 + g^2$. We have a whole chapter devoted on the solution of (6.9).

6.2 Conformal time as a variable

Another useful transformation is the one to conformal time for which one has from (4.29) that

$$\frac{d^2}{dt^2} = \frac{1}{a^2} \frac{d^2}{d\eta^2} - \frac{1}{a^3} \frac{da}{d\eta} \frac{d}{d\eta}. \tag{6.10}$$

Consequently (5.12) reads

$$\delta\sigma_k^{\star\star} + 2\frac{a^\star}{a} \delta\sigma_k^\star + \left(k^2 + a^2 m^2 + \alpha a^2 H^2 \right) \delta\sigma_k = 0, \tag{6.11}$$

the star again denoting the derivative with respect to conformal time. Again we want a differential equation where the coefficient of the first derivative disappears, so we set $\delta\sigma_k(\eta) = \frac{v_k(\eta)}{E(\eta)}$, which gives us

$$v_k^{\star\star} + 2\left(\frac{a^\star}{a} - \frac{E^\star}{E} \right) v_k^\star + \left[2\frac{E^\star}{E} \left(\frac{E^\star}{E} - \frac{a^\star}{a} \right) - \frac{E^{\star\star}}{E} + k^2 + a^2 m^2 + \alpha a^2 H^2 \right] v_k = 0, \tag{6.12}$$

so that we demand

$$a = E. \tag{6.13}$$

Hence

$$\delta\sigma_k = \frac{v_k}{a} \tag{6.14}$$

with

$$v_k^{\star\star} + \Big(k^2 - \frac{a^{\star\star}}{a} + a^2 m^2 + a^2 \alpha H^2\Big) v_k = 0. \tag{6.15}$$

From the definition of conformal time on gets

$$a^{\star\star} = a(\dot{a}^2 + \ddot{a}a) \quad \Rightarrow \quad \frac{a^{\star\star}}{a} = a^2 H^2 + a\ddot{a}, \tag{6.16}$$

and from the definition of H we have

$$\dot{H} = \frac{a\ddot{a} - \dot{a}^2}{a^2} \quad \Leftrightarrow \quad a\ddot{a} = a^2 H^2 + a^2 \dot{H}^2 \quad \Rightarrow \quad \frac{a^{\star\star}}{a} = 2a^2 H^2 + a^2 \dot{H}. \tag{6.17}$$

So for de Sitter space (6.15) reads

$$v_k^{\star\star} + \Big(k^2 + a^2\big(m_{eff}^2 - 2H^2\big)\Big) v_k = 0, \tag{6.18}$$

while for a a chaotic inflationary period with quadratic potential one has $\dot{H} = H_f$, so that one finds

$$v_k^{\star\star} + \Big(k^2 + a^2\big(m^2 + \alpha H^2 + H_f^2 - 2H^2\big)\Big) v_k = 0. \tag{6.19}$$

6.3 $\ln \frac{aH}{k}$ as a variable

To improve the horizon approximation (see section 4.2) one would like to use something in between the Minkowski limit deeply inside the horizon and the limiting solution way outside the horizon, so one needs an approximate solution around horizon crossing. The easiest way to do so is by approximating quasi de Sitter space with de Sitter space *inside the horizon* up to horizon crossing (see footnote 1 in Ch.8). But one can get an even more refined approximation. One first introduces a measure of the deviation of the quasi de Sitter equation-of-state from the de Sitter equation-of-state:

$$\epsilon := \frac{3}{2}\Big(\frac{\varrho + p}{\varrho}\Big). \tag{6.20}$$

One can see that $\epsilon_{de\ Sitter} = 0$; for quasi de Sitter space inflation ends at $p = -\frac{1}{3}\varrho$ where one has $\epsilon_f = 1$. The smaller ϵ, the slowlier it changes. From (1.30) and (1.31) one gets that during SR (this is why ϵ *is also called the SR-parameter*) one has

$$\epsilon \approx \frac{3\dot{\varphi}^2}{M^2 \varphi^2} = \frac{H_f^2}{H^2}. \tag{6.21}$$

So (6.19) reads

$$\frac{v_k^{\star\star}}{a^2H^2}+\left(\frac{k^2}{a^2H^2}+\Big(\frac{m^2}{H_f^2}+1\Big)\epsilon+\alpha-2\right)v_k=0. \tag{6.22}$$

We are interested in the behaviour of this equation near horizon crossing ($\frac{a_kH_k}{k}=1$). The solution can then be extended deeply inside and way outside the horizon. We introduce a second variable, measuring how close one is to the moment of horizon crossing,

$$z:=\ln\frac{aH}{k}. \tag{6.23}$$

So $z|_{crossing}=0$ and

$$dz=\Big(\frac{a}{H}\frac{\partial H}{\partial a}+1\Big)\frac{da}{a}=(1-\epsilon)\frac{da}{a}\quad\Rightarrow\quad\frac{da}{dz}=\frac{a}{1-\epsilon}. \tag{6.24}$$

Together with (4.29) one finds

$$dz=(1-\epsilon)aH\ d\eta\quad\Rightarrow\quad\frac{d^2}{d\eta^2}=a^2(1-\epsilon)\frac{d}{dz}\Big(aH(1-\epsilon)\frac{d}{dz}\Big). \tag{6.25}$$

Using (2.28) and (6.24) this reads

$$\frac{d^2}{d\eta^2}=a^2H^2(1-\epsilon)^2\left[\left(1+\frac{d}{dz}\Big(\ln(1-\epsilon)\Big)\right)\frac{d}{dz}+\frac{d^2}{dz^2}\right]. \tag{6.26}$$

So (6.22) looks like

$$\tilde{\tilde{v}}_k+\left(1+\frac{d}{dz}\Big(\ln(1-\epsilon)\Big)\right)\tilde{v}_k+(1-\epsilon)^{-2}\left[e^{-2z}+\left(\frac{m^2}{H_f^2}+1\right)\epsilon+\alpha-2\right]v_k=0, \tag{6.27}$$

where the tilde denotes the derivative with respect to z. Note that this equation is still exact, except for the little approximation we introduced with (6.21). For $z\ll 1$ (near horizon crossing) one can expand $\epsilon(z)$ around $z=0$. We use the abbreviation $\epsilon(0)=\epsilon_{crossing}=H_f^2/H_k^2=:\epsilon_k$. We get

$$\begin{aligned}\frac{d}{dz}\Big(\ln(1-\epsilon)\Big)&\approx\frac{d}{dz}\Big(\ln(1-\epsilon)-\frac{\tilde{\epsilon}_k}{1-\epsilon_k}z\Big)=-\frac{\tilde{\epsilon}_k}{1-\epsilon_k}\\ \text{and}\quad\frac{1}{(1-\epsilon)^2}&\approx\frac{1}{(1-\epsilon_k)^2}.\end{aligned} \tag{6.28}$$

Since $\epsilon_k\ll 1$, we can make a second expansion, this time in ϵ_k:

$$\frac{d}{dz}\Big(\ln(1-\epsilon)\Big)\approx-\tilde{\epsilon}_k\quad\text{and}\quad\frac{1}{(1-\epsilon)^2}\approx 1. \tag{6.29}$$

But

$$\tilde{\epsilon} = H_f \frac{a}{1-\epsilon}\left(\frac{1}{H'}\right)' = \frac{2}{1-\epsilon}\epsilon^2, \tag{6.30}$$

so that $\tilde{\epsilon}_k$ is a second order quantity. Hence $\frac{d}{dz}\Big(\ln(1-\epsilon)\Big)$ can be neglected entirely. We get

$$\tilde{\tilde{v}}_k + \tilde{v}_k + \left[e^{-2z} + \left(\frac{m^2}{H_f^2}+1\right)\epsilon_k + \alpha - 2\right] v_k = 0. \tag{6.31}$$

To get rid of the term with the first derivative, we set

$$v_k(z) = Z(z)\cdot \mathcal{V}_k(z), \tag{6.32}$$

so that

$$\tilde{\tilde{\mathcal{V}}}_k + \left(2\frac{\tilde{Z}}{Z}+1\right)\tilde{\mathcal{V}}_k + \left[\frac{\tilde{\tilde{Z}}}{Z} + \frac{\tilde{Z}}{Z} + e^{-2z} + \left(\frac{m^2}{H_f^2}+1\right)\epsilon_k + \alpha - 2\right]\mathcal{V}_k = 0. \tag{6.33}$$

Demanding that

$$2\frac{\tilde{Z}}{Z} = -1, \tag{6.34}$$

one finds that

$$Z = e^{-\frac{z}{2}} = \sqrt{\frac{k}{aH}}. \tag{6.35}$$

So one gets

$$\delta\sigma_k(a) = \sqrt{\frac{k}{a^3 H}}\ \ \mathcal{V}_k \tag{6.36}$$

with $\mathcal{V}_k$ obeying

$$\tilde{\tilde{\mathcal{V}}}_k + \left[e^{-2z} + \left(\frac{m^2}{H_f^2}+1\right)\epsilon_k + \alpha - \frac{9}{4}\right]\mathcal{V}_k = 0 \tag{6.37}$$

in quasi de Sitter space near horizon crossing. (6.37) is solved in terms of Bessel functions:

$$\begin{aligned} \mathcal{V}_k \propto J_{\pm\chi}\Big(\sqrt{e^{-2z}}\Big) \quad &= \quad J_{\pm\chi}\left(\frac{k}{aH}\right), \\ \text{with}\ \ \chi = \sqrt{\frac{9}{4} - \Big(\frac{m^2}{H_f^2}+1\Big)\epsilon_k - \alpha} \quad &= \quad \sqrt{\frac{9}{4} - \frac{m^2+H_f^2}{H_k^2} - \frac{g^2}{H_f^2}}. \end{aligned} \tag{6.38}$$

Compared with (8.1) and (8.2), H is not constant and one has simply to replace $\frac{m^2}{H^2}$ by $\frac{m^2+H_f^2}{H_k^2} + \frac{g^2}{H_f^2}$. For applications of the Bessel approximation and further discussions see [33].

6.4 $\frac{3H}{M}$ as a variable

In case we are not dealing with de Sitter space, i.e. we are having a time-varying Hubble parameter, one can work with $\delta\sigma_k$ as a function of H. This is especially useful if under certain circumstances one may neglect the k^2-dependence of (5.12) (see next chapter). From (2.30) one gets

$$\frac{d}{dt} = -\frac{M^2}{3}\frac{d}{dH}. \tag{6.39}$$

So (5.12) reads

$$\frac{d^2\delta\sigma_k}{dH^2} - \frac{9}{M^2}H\frac{d\delta\sigma_k}{dH} + \left(9\frac{m^2}{M^4} + \frac{9}{M^4}\alpha H^2\right)\delta\sigma_k = 0. \tag{6.40}$$

We introduce

$$h = \frac{3H}{M} = \frac{\sqrt{3}H}{H_f} = \sqrt{3}\varphi \quad \text{and hence get} \quad \frac{d}{dt} = -M\frac{d}{dh}. \tag{6.41}$$

So h is a positive number, initially quite high and $\mathcal{O}(1)$ at the end of inflation. We get

$$\delta\sigma_k^{\diamond\diamond} - h\ \delta\sigma_k^{\diamond} + \left(\frac{m^2}{M^2} + \frac{\alpha}{9}h^2\right)\delta\sigma_k = 0, \tag{6.42}$$

the diamond denoting the derivative with respect to h. Setting $\delta\sigma_k(h) = \mathcal{H}(h)\ w_k(h)$ one gets

$$w_k^{\diamond\diamond} + \left(2\frac{\mathcal{H}^{\diamond}}{\mathcal{H}} - h\right)w_k^{\diamond} + \left(\frac{m^2}{M^2} + \frac{\alpha}{9}h^2 + \frac{\mathcal{H}^{\diamond\diamond}}{\mathcal{H}} - h\frac{\mathcal{H}^{\diamond}}{\mathcal{H}}\right)w_k = 0. \tag{6.43}$$

By demanding that

$$2\frac{\mathcal{H}^{\diamond}}{\mathcal{H}} = h \tag{6.44}$$

we get that

$$w_k^{\diamond\diamond} + \left(\frac{m^2}{M^2} + \frac{\alpha}{9}h^2 + \frac{\mathcal{H}^{\diamond\diamond}}{\mathcal{H}} - \frac{h^2}{2}\right)w_k = 0. \tag{6.45}$$

From (6.44) we find that

$$\mathcal{H} \propto e^{\frac{h^2}{4}} \quad \Rightarrow \quad \frac{\mathcal{H}^{\diamond\diamond}}{\mathcal{H}} = \frac{1}{2} + \frac{h^2}{4}. \tag{6.46}$$

So

$$\delta\sigma_k \propto e^{\frac{h^2}{4}} w_k \tag{6.47}$$

with

$$w_k^{\diamond\diamond} + \left(\frac{m^2}{M^2} + \frac{\alpha}{9}h^2 + \frac{1}{2} - \frac{h^2}{4}\right)w_k = 0. \tag{6.48}$$

If the coupling to the inflaton can be neglected or is nonexistent, one has

$$w_k^{\diamond\diamond} + \left(\frac{m^2}{M^2} + \frac{1}{2} - \frac{h^2}{4}\right) w_k = 0. \tag{6.49}$$

This equation is known as *Weber's equation*. We shall deal with its solutions in Ch.9.

Chapter 7

Some useful approximations to the Klein-Gordon equation of the $\delta\sigma$-field

Even with the transformations we got to know in the last chapter, in most cases one still is not able to solve (5.12). So we shall look for approximations by dropping some terms.

7.1 The long and short wavelength approximations

The first approximation is the limiting case of the so-called *long wavelength approximation* (LWA): if k^2 is small enough (or λ_{com} is large enough) so that we are dealing with modes with long wavelengths, we can drop the k-dependence, thereby treating the $\delta\sigma$-field as homogeneous (as we already did when deriving (6.49)). Starting from (6.19) one gets

$$v_k^{\star\star} + a^2\Big(m^2 + \alpha H^2 + H_f^2 - 2H^2\Big)\ v_k = 0. \tag{7.1}$$

For de Sitter space one equivalently skips the term k^2 in (6.18). So (5.12) looks like[1]

$$\ddot{\delta\sigma}_k + 3H\dot{\delta\sigma}_k + \Big(m^2 + \alpha H^2\Big)\delta\sigma_k = 0. \tag{7.2}$$

On the contrary, if k^2 is large compared to the rest, i.e. λ_{com} is rather small, one speaks of the *short wavelength approximation* (SWA). One gets

$$v_k^{\star\star} + k^2 v_k = 0, \tag{7.3}$$

so that

$$\delta\sigma_k \propto \frac{e^{\pm ik\eta}}{a(\eta)}. \tag{7.4}$$

[1]Note that this is the same equation as for $\langle\sigma\rangle$.

Unless the terms don't cancel each other, one may apply the LWA in de Sitter space if

$$\frac{k^2}{a^2} < \min\Big\{m_{eff}^2, 2H^2\Big\}, \tag{7.5}$$

and likewise one has LWA in quasi de Sitter space if

$$\frac{k^2}{a^2} < \min\Big\{m^2, \alpha H^2, H_f^2, 2H^2\Big\}. \tag{7.6}$$

But this is always given for the modes outside the horizon (where one has $k < aH$), except for a few modes where the terms cancel each other. We'll deal with these modes when treating a concrete case, and we will find out that those SWA-at-crossing modes hardly matter.

7.2 The slow roll approximation

Even if we apply the LWA, in some cases the Klein-Gordon equation still remains hard to solve. Therefore a *further* approximation has to be made, namely *slow roll (SR): the gain of "speed" during a characteristic time is small*, or

$$\left|\frac{\ddot{\delta\sigma}_k}{H\dot{\delta\sigma}_k}\right| \ll 1. \tag{7.7}$$

So we assume that the second derivative in (7.2) is negligible compared to the rest and hence get

$$3H\dot{\delta\sigma}_k + \left(m^2 + \alpha H^2\right)\,\delta\sigma_k = 0, \tag{7.8}$$

solved by $\delta\sigma_k = \delta\sigma_k^{sr}$. It has to fulfil (7.7) which gives constraints on m and g, verifying our initial assumption a posteriori. But by assuming SR we cut down a 2^{nd} order differential equation to a 1^{st} order differential equation, *so that we have lost one particular solution*, $\delta\sigma_k^d$ (we shall make use of the superscripts only if we need to distinguish between the two solutions; usually the SR-solution is meant). How to find the $\delta\sigma_k^d$? First of all we notice that to a good degree $\delta\sigma_k^{sr}$ satisfies (7.2) (if certain conditions are fulfilled). So we shall from now on take $\delta\sigma_k^{sr}$ to be one of the two *exact* particular solutions of (7.2). Now, quite generally, suppose that one has a linear 2^{nd} order differential equation of $f(x)$

$$\alpha(x)f''(x) + \beta(x)f'(x) + \gamma(x)f(x) = 0, \tag{7.9}$$

the prime denoting the derivative with respect to x *in this paragraph*. Suppose furthermore that we know one particular solution, $s_1(x)$. Hence also $c \cdot s_1(x)$, c being an arbitrary constant, solves this equation. We now make the transition[2]

$$c \to c(x) \tag{7.10}$$

[2]This is why this procedure is called *method of variation of constants.*

and demand that

$$s_2(x) := c(x)\ s_1(x) \tag{7.11}$$

solves (7.9) and hence plug it in to determine $c(x)$. One gets

$$\begin{aligned} &\alpha c'' s_1 + 2\alpha c' s_1' + \alpha c s_1'' + \beta c s_1' + \beta c' s_1 + \gamma c s_1 = 0 \\ \Rightarrow\quad &\alpha s_1 c'' + (2\alpha s_1' + \beta s_1)c' + (\alpha s_1'' + \beta s_1' + \gamma s_1)c = 0 \end{aligned} \tag{7.12}$$

Now the coefficient of c vanishes because s_1 fulfils (7.9). So one has

$$c'' + \left(2\frac{s_1'}{s_1} + \frac{\beta}{\alpha}\right)\ c' = 0 \tag{7.13}$$

which gives

$$\frac{dc'}{c'} = -2\frac{ds_1}{s_1} - \frac{\beta}{\alpha}dx \tag{7.14}$$

and hence (absorbing the lower integrational bound in the exponent into the constant of proportionality so that we may use an indefinite integral)

$$c'(x) \propto \frac{e^{-\int \frac{\beta(x)}{\alpha(x)}dx}}{s_1^2(x)}. \tag{7.15}$$

Integrating a second time (using that $constant\ \times\ s_1$ solves (7.9) so that we again may use an indefinite integral), and plugging the result into (7.11) gives,

$$s_2(x) \propto \int \frac{e^{-\int \frac{\beta(x)}{\alpha(x)}dx}}{s_1^2(x)}dx \quad \times \quad s_1(x). \tag{7.16}$$

This is known as *Liouville's formula*. With (7.15) one sees that the Wronskian of (7.9) is given by

$$W(s_1, s_2) \propto e^{-\int \frac{\beta(x)}{\alpha(x)}dx}, \tag{7.17}$$

which hence is always nonzero, so that s_1 and s_2 are a complete set of solutions of (7.9). As an example consider $f'' + f = 0$ with $s_1 = \sin(x)$. We get

$$s_2 \propto \sin(x) \cdot \int \frac{1}{\sin^2(x)}dx \propto \ \sin(x)\ \csc(x)\ =\ \cos(x). \tag{7.18}$$

In case of the $\delta\sigma$-field we have $\alpha(x) = 1$ and $s_1 \approx \delta\sigma_k^{sr}$ so that with (2.22), (7.2) and (7.16) we get

$$\delta\sigma_k^d(a) \propto \int a^{-4}\ H^{-1}(a)\ \left(\delta\sigma_k^{sr}(a)\right)^{-2}\ da \quad \times \quad \delta\sigma_k^{sr}(a) \tag{7.19}$$

as the second (approximate) particular solution to (7.8). Once we've solved the integral above we know whether $\delta\sigma_k^d$ grows or decays faster than the slow roll solution. It will turm out that $\delta\sigma_k^d$ marks the solution decaying much faster than $\delta\sigma_k^{sr}$ so that we can neglect it entirely.

Chapter 8

A toy model: the $\delta\sigma$-field in de Sitter space

To get an idea of what to expect in quasi de Sitter space we shall first consider as a toy model how the $\delta\sigma$-field evolves with H being constant, so that one has de Sitter space. Although the functional dependence of the spectrum on k will be different for the quasi de Sitter space, the underlying physics is similar.

8.1 The spectrum of the $\delta\sigma$-field

The differential equation (6.9) can be solved in an exact manner, unlike the ones we're going to encounter in the next chapter. Nevertheless we shall also apply the methods of approximation we got to know in the last chapter, to make the physics more transparent. According to [35], (6.9) can be solved in terms of Bessel functions,

$$\delta\sigma_k(t) = \delta\sigma_k(a) = a^{-\frac{3}{2}}\Big[C_1(k)J_{-\zeta}\Big(\frac{k}{Ha}\Big) + C_2(k)J_{\zeta}\Big(\frac{k}{Ha}\Big)\Big], \tag{8.1}$$

where

$$\zeta = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \quad \Rightarrow \quad 0 \leq \Re(\zeta) \leq \frac{3}{2} \tag{8.2}$$

which gives for $m \ll H$

$$\zeta \approx \frac{3}{2} - \frac{m^2}{3H^2} =: \frac{3}{2} - \frac{\tau}{2}. \tag{8.3}$$

Note that the constants of integration are functions of k. For *large arguments*, that is $k \gg aH$, one has (see [36])

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos\Big(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big), \tag{8.4}$$

so that

$$\delta\sigma_k = \frac{1}{a}\Big[\mathcal{C}_1 e^{i\frac{k}{aH}} + \mathcal{C}_2 e^{-i\frac{k}{aH}}\Big]. \tag{8.5}$$

So the field oscillates deeply inside the horizon. We thus arrive at the same result one gets using the SWA, see (7.3) and (7.4).[1] The other asymptotic limit for *small arguments* of the Bessel functions, i.e. for $k \ll aH$, is given by (see [36])

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1+\nu)}. \tag{8.6}$$

So (8.1) reads

$$\delta\sigma_k \approx C_1\ a^{-\zeta-3/2} + C_2\ a^{\zeta-3/2}. \tag{8.7}$$

The first term clearly decays much faster than the second one, so that we can neglect it. (8.7) is easily understandable if one uses (7.2): this equation is nothing but a damped harmonic oscillator, solved by

$$\delta\sigma_{k\ 1,2} \propto e^{(-3/2\pm\zeta)Ht} = a^{-3/2\pm\zeta}. \tag{8.8}$$

Choosing the solution that decays slowlier gives a "+" in the exponent. For real ζ one has the damping term dominating the evolution (H causes the amplitude to freeze, see section 4.1), the case with $\zeta = 0$ corresponds to the asymptotic limit, while for imaginary ζ ($m > \frac{3}{2}H$) one has a damped oscillation, in fact $\delta\sigma_k$ just behaves as cold dust. This oscillation is analogous to the one in Ch.s 9&12 when H drops below m. If one assumes furthermore SR one has from (7.2) that

$$3H\dot{\delta\sigma}_k + m^2\delta\sigma_k = 0, \tag{8.9}$$

and hence

$$\delta\sigma_k^{sr} \propto e^{\frac{-m^2}{3H}t}. \tag{8.10}$$

One may use SR if (7.7) is fulfilled, which gives $m^2 \ll H^2$. So the SR solution is equal to the second solution in (8.7) with (8.3). It decays only very slowly, it is frozen outside the

[1] Of course with the way we calculated (4.17), this inside-horizon solution is not valid as we assumed a Minkowski spacetime inside. To derive (4.17) more rigorously we shall match (8.5) to the initial conditions of de Sitter space *deeply inside the horizon.* Choosing the initial conditions means choosing the initial vacuum state (see also [13]): for short wavelengths, i.e. $k \to \infty$, GR should not be important so that

$$\delta\sigma_k(t_i) \sim \frac{1}{a_i\sqrt{k}} \quad \text{and} \quad \dot{\delta\sigma}_k(t_i) \sim \frac{i\sqrt{k}}{a_i}.$$

We furthermore neglected the mass term in (4.9). Setting $a = a_i$, we can match (8.5) and its derivative with respect to time to the initial conditions. One gets

$$\delta\sigma_k = \frac{iHa_i}{2a\ k^{3/2}}\left[e^{-i\frac{k}{H}(\frac{1}{a}-\frac{1}{a_i})} + \left(\frac{2k}{iHa_i} - 1\right)e^{i\frac{k}{H}(\frac{1}{a}-\frac{1}{a_i})}\right] \approx \frac{e^{i\frac{k}{H}\left(\frac{1}{a}-\frac{1}{a_i}\right)}}{a\sqrt{k}}.$$

If we extend this solution to the moment where $k = a_k H_k$ we get (4.17). With the initial conditions given above and the same method one also works with the Bessel approximation.

horizon: to a good approximation one has $\delta\sigma_k = constant$. The 2^{nd} particular solution is found using Liouville's formula. One gets

$$\begin{aligned}\delta\sigma_k^d &\propto \int \frac{e^{-3Ht}}{\delta\sigma_k^{sr\,2}} dt \;\times\; \delta\sigma_k^{sr} \\ &\propto e^{(\frac{m^2}{3H}-3H)t} = a^{\frac{m^2}{3H^2}-3} = a^{-\zeta-3/2}.\end{aligned} \tag{8.11}$$

This is precisely the the approximation to the second Bessel equation with positive coefficient which we discarded in (8.7). So $\delta\sigma_k^d$ decays much faster than (8.10), a feature we're going to find in other cases, too. This is why the second particular solution is called the *decaying mode* (hence the superscript^d).

We fix the constants of integration for (8.7) with (4.17).[2] For imaginary ζ one gets

$$\delta\sigma_k \propto \frac{1}{a^{3/2}} \;\times\; \text{oscillating function}(a) \quad\Rightarrow\quad |\delta\sigma_k|^2 k^3 \sim \frac{|C|^2}{a^3}. \tag{8.12}$$

Using (4.17) we find

$$|\delta\sigma_k|^2 k^3\Big|_{crossing} \sim \frac{|C|^2}{a_k^3} \;\sim\; \frac{H^3}{\sqrt{H^2+m^2}} \sim \frac{H^3}{m} \tag{8.13}$$

so that

$$|\delta\sigma_k|^2 k^3 \sim \frac{k^3}{m} a^{-3} \tag{8.14}$$

for all $m > \frac{3}{2}H$. If ζ is real, one has

$$\begin{aligned} & & |\delta\sigma_k^2| k^3 &= |C|^2\, a^{2\zeta-3} \\ \Rightarrow\quad \frac{H^3}{\sqrt{H^2+m^2}} &\sim |C|^2 \left(\frac{k}{H}\right)^{2\zeta-3} &\Rightarrow\quad |C|^2 &= \frac{H^3}{\sqrt{H^2+m^2}} \left(\frac{k}{H}\right)^{3-2\zeta}\end{aligned} \tag{8.15}$$

From (8.2) one gets

$$\sqrt{H^2+m^2} = H\sqrt{\frac{13}{4} - \zeta^2}, \tag{8.16}$$

so that

$$|\delta\sigma_k|^2 k^3 \sim \frac{H^2}{\sqrt{\frac{13}{4}-\zeta^2}} \left(\frac{k}{H}\right)^{3-2\zeta} a^{2\zeta-3}. \tag{8.17}$$

This result is true for $0 \leq m \leq \frac{3}{2}H$, and for $m \ll H$ it reduces to

$$|\delta\sigma_k|^2 k^3 \sim H^2 \left(\frac{k}{H}\right)^{\tau} a^{-\tau}. \tag{8.18}$$

[2] As $H_k = H = constant$ all modes have the same amplitude at the moment of horizon crossing. That one does not get a Harrison-Zel'dovich spectrum is due to the fact that we take de Sitter space to last for only a finite time.

Summarising we can say that regardless of m the field oscillates inside the horizon while decaying. After horizon crossing the field either slowly decays with no oscillations, it is "frozen" ($m \ll H$), see section 4.1, or it continues oscillating, decaying (like dust) even more rapidly than before ($m > \frac{3}{2}H$).

8.2 The energy density and the power spectrum

We first determine $\langle \varrho_{\delta\sigma} \rangle$ and see which constraints arise from (5.5). First we shall consider the case with $m > \frac{3}{2}H$. Using (8.12) in (5.26) with (4.3) we get for the energy density of the $\delta\sigma$-field during inflation

$$\langle \varrho_{\delta\sigma} \rangle \sim m\ a^{-3} \int_{a_i H}^{aH} k^2 dk \ \sim\ mH^3 \left[1 - \left(\frac{a_i}{a} \right)^3 \right]. \tag{8.19}$$

The energy density, starting from zero, grows very rapidly to reach a "plateau" so that one has a constant energy density. If one wants to interpret the constant energy density in terms of particle production (see Ch.4, footnote 3), one has to have the amount of particles exponentially growing in order to catch up with the increasing volume. *Hence at a given instant the main contribution to the particle number comes from a short time before, all earlier generations are diluted. This is why one speaks of "gravitational particle production at the end of inflation"*. Again we take de Sitter space to be "switched off" after some time, inflation ends (with $a_f = 1$) and one has

$$\langle \varrho_{\delta\sigma} \rangle_f \sim mH^3 \tag{8.20}$$

Applying (5.5) one gets

$$m\ H^3 \ll H^2, \tag{8.21}$$

so that $m \ll \frac{1}{H}$. Hence

$$\frac{3}{2}H < m \ll \frac{1}{H}, \tag{8.22}$$

which can only be fulfilled for $H < 1$. m is allowed to be very heavy, up to the Planck mass. To calculate the power spectrum[3], we note that

$$\begin{aligned} L &= 1, \\ \varepsilon &= a_i, \\ \varsigma(l') &= l'^3, \\ \Rightarrow\quad \Sigma(l') = l'^2 \quad &\text{and} \quad \int_{\varepsilon}^{L} \frac{\varsigma(l')}{l'} dl' \approx \text{unity}. \end{aligned} \tag{8.23}$$

[3] The final result of course can easily calculated from (5.25) directly, but we want to demonstrate the formulas derived in section 5.4).

Hence

$$\mathcal{P}^2_{\delta\sigma}(k) \sim l^2\ l\ \left[l'^3 \Big|_{a_i}^{l-a_i} + l'^3 \Big|_{l+a_i}^{1-l} \right] = l^3 \Big[(1-l)^3 - 3a_i^3 - 6a_i l^2 \Big]. \tag{8.24}$$

Neglecting the last two terms one has for $\varepsilon \ll l \ll 1$

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f \sim \left(\frac{k}{H}\right)^{\frac{3}{2}}, \tag{8.25}$$

similar to (9.48). This result contradicts observation (see section 9.4)!

Now we are going to do the same calculation for the case $0 < m < \frac{3}{2}H$. With (8.17) we get

$$\langle \varrho_{\delta\sigma} \rangle \sim \frac{m^2 H^2}{\sqrt{\frac{13}{4} - \zeta^2}} a^{2\zeta - 3} \int\limits_{a_i H}^{aH} \left(\frac{k}{H}\right)^{3-2\zeta} \frac{dk}{k}. \tag{8.26}$$

For $m = 0$ the energy density vanishes. Using (5.28) and (8.16) one gets[4] for $m > 0$

$$\langle \varrho_{\delta\sigma} \rangle \sim \frac{\frac{9}{4} - \zeta^2}{\sqrt{\frac{13}{4} - \zeta^2}}\ H^4\ a^{2\zeta-3} \int\limits_{a_i}^{a} l^{2-2\zeta} dl \sim \frac{3 + 2\zeta}{\sqrt{\frac{13}{4} - \zeta^2}}\ H^4 \left[1 - \left(\frac{a_i}{a}\right)^{3-2\zeta} \right], \tag{8.27}$$

the time development similar to the case before, just that the "plateau" is reached less quickly.

[4]One can see that modes with high k (i.e. $l \sim L = 1$) contribute the most to the energy density. How can we determine the highest mode that does *not* substantially contribute to the energy density. With l_L denoting this mode one has

$$\langle \varrho_{\delta\sigma} \rangle \propto l'^{3-2\zeta} \Big|_{l_L}^{1},$$

so we have to have $l_L^{3-2\zeta}$ to be one order of magnitude smaller than unity so that

$$l_L^{3-2\zeta} = e^{-1} \quad \Rightarrow \quad l_L = e^{\frac{1}{2\zeta-3}}.$$

So very large scales do not contribute to the energy density, the main contribution comes from the modes with

$$He^{\frac{1}{2\zeta-3}} < k < H \quad \Rightarrow \quad e^{-\frac{1}{\tau}} H \leq k \leq H.$$

This result is not true for the case with the σ-field and the φ-field uncoupled (next chapter); there the modes with the lowest k are most important.

At the end of inflation we have[5] [6] (with m large enough so that $a_i \ll e^{-\frac{3H^2}{2m^2}}$, see footnote 4, so that the term $a_i^{3-2\zeta}$ can be neglected)

$$\langle \varrho_{\delta\sigma} \rangle_f \sim \frac{3+2\zeta}{\sqrt{\frac{13}{4}-\zeta^2}}\, H^4. \tag{8.28}$$

Notice that (8.20) and the equation above have the same limit for $m \to \frac{3}{2}H$, as it should be. Note also that, as long as $a_i \ll e^{-\frac{3H^2}{2m^2}}$, the lower cut-off for the integration of (5.26) is not needed for the integral to be finite. We also skipped the lower integration bounds in (8.24). This does not have to be the case in general! In the next chapter we'll find integrals which value is given primarily by the lower cut-off. From (5.5) we get that

$$\frac{3+2\zeta}{\sqrt{\frac{13}{4}-\zeta^2}} H^4 \ll H^2. \tag{8.29}$$

[5]From (8.28) one also easily gets the variance and hence the root-mean-square deviation for the $\delta\sigma$-field:

$$(\delta\sigma)_{rms} = \sqrt{\langle \delta\sigma^2 \rangle} = \sqrt{\frac{\langle \varrho_{\delta\sigma} \rangle_f}{m^2}} = \frac{H^2}{m}.$$

As we found in the previous footnote the energy density and hence $(\delta\sigma)_{rms}$ (for $m \ll H$) are mainly determined on scales $\ll H^{-1} e^{\frac{3H^2}{2m^2}}$. So if one considers scales $\gg H^{-1} e^{\frac{3H^2}{2m^2}}$, [if $\frac{m}{H}$ is not too small (i.e. $e^{\frac{3H^2}{2m^2}}$ too large), we want the scales to be within our observable Universe!] the σ-fields fluctuates many times within the limits

$$-\frac{H^2}{m} \le \sigma \le \frac{H^2}{m}.$$

On the other hand, on small scales ($< H^{-1}$) the field is quasi-homogeneous. An analogous calculation also holds for quasi de Sitter space.

[6]From (4.6), (4.11), (4.12), (4.13) and (4.14) one gets in Minkowski space

$$\langle \hat{\delta\sigma}^2 \rangle = \frac{1}{(2\pi)^3} \int \frac{\frac{1}{2} + \langle \hat{a}_\kappa^\dagger \hat{a}_\kappa \rangle}{\sqrt{\kappa^2 + m^2}} d\vec{\kappa}.$$

Comparing this with the variance one gets for the number density with particles for a certain k

$$n(k) \sim \frac{H^2\sqrt{k^2+m^2}}{2k^3} \left(\frac{k}{H} \right)^{\frac{2m^2}{3H^2}},$$

which is $\gg 1$ for $k \ll m,\ H$. Hence the nonvanishing commutator of $\hat{a}_k^\dagger$ and $\hat{a}_k$ is not important. "The long wave quantum fluctuations of the [σ]-field can be considered as being a classical fluctuation [$\delta\sigma_k$] on the field [σ]. This effect is similar to Bose condensation where the number density $n(k)$ becomes infinite for $k = 0$. ...The amplitude of the [σ]-field at some point [$\vec{x}$] is given by a sum of the classical field [$\sigma(\vec{x})$], of the "frozen" long-wave semi-classical perturbations with $k < H$ and of oscillating quantum fluctuations with $k > H$. The last contribution (almost) completely disappears after the renormalization of [$\langle \delta\sigma^2 \rangle$]", [32]. The same physics also holds for quasi de Sitter space.

As the prefactor in front of H^4 is of the order of unity for $0 \leq \zeta < \frac{3}{2}$, the equation above states nothing else but that the energy density of de Sitter space, H^2, has to be way below the Planck scale. We thus arrive at the same constraint on H as for the case $m > \frac{3}{2}H$. For $m \ll H$ one gets

$$\langle \varrho_{\delta\sigma} \rangle_f \sim H^4. \tag{8.30}$$

This result was first obtained in [37] and is used commonly as the formula giving the energy density of gravitationally produced particles during inflation. However, for a quadratic-chaotic potential we are going to find a more refined expression in the next chapter. To calculate the power spectrum, we use (5.35) with

$$\varsigma(l) = l^{3-2\zeta}. \tag{8.31}$$

The integral one gets can be solved exactly in terms of Gauß' hypergeometric functions ${}_2F_1$. But for quasi de Sitter space there will not be an exact solution at hand, so that we shall apply the approximation procedure explained in Ch.5 for the case $m \ll H$ (and hence we can work with $\tau \ll 1$ instead of ζ) to understand how it works before treating more complicated cases. We use

$$\begin{aligned} \varsigma(l) &\sim l^\tau \\ \text{so that} \int \frac{\varsigma(l)}{l} dl &\sim \frac{l^\tau}{\tau}, \\ \Sigma(l) &\sim \frac{l^{\tau-1}}{\tau - 1} \\ \text{and} \quad \int \Sigma(l) dl &\sim \frac{l^\tau}{\tau(\tau-1)}. \end{aligned} \tag{8.32}$$

So (5.45) gives

$$\mathcal{P}^2_{\delta\sigma}(k) \sim \tau^2 \; 2l^\tau \left[\frac{l^\tau - 2{a_i}^\tau}{\tau} - \frac{1}{2} l^\tau + \frac{l^\tau}{\tau} \right] \sim \tau \;\; l^{2\tau}. \tag{8.33}$$

Consequently

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f^{m \ll H} \sim \frac{m}{H} \left(\frac{k}{H} \right)^{\frac{2m^2}{3H^2}}, \tag{8.34}$$

which we could have gotten straight from (5.46), thus demonstrating the validity of the step from (5.45) to (5.46). This result and the *exact* integral were compared numerically with $Mathematica^{©}$ for several values of $l,\ \tau$ and a_i, to give an agreement within an inaccuracy of 10%. Unlike (8.25), (8.34) is a very promising result, giving a spectral index slightly bigger than unity,

$$n = 1 + \frac{4m^2}{3H^2}. \tag{8.35}$$

So we arrive at a power spectrum of nongaussian isocurvature perturbation that is blue! In the hope that this nice feature survives the transition to a more realistic time-dependent Hubble parameter we shall now work on the $\delta\sigma$-field in quasi de Sitter space.

Chapter 9

The $\delta\sigma$-field in quasi de Sitter space, not coupled to the inflaton

9.1 Preliminary remarks

After having gotten a feeling of what to expect, we shall now consider the case with the $\delta\sigma$-field evolving in quasi de Sitter space. First we shall consider the case with no or very small coupling,

$$g^2\varphi^2 \ll \mathsf{min}\Big\{m^2, M^2\Big\}, \tag{9.1}$$

so that the equation of motion looks like

$$\delta\ddot{\sigma}_k + 3H\delta\dot{\sigma}_k + \left(\frac{k^2}{a^2} + m^2\right)\delta\sigma_k = 0. \tag{9.2}$$

Note that if $m \gg M$, one finds a nice effect[1] lowering the classical value of σ, *so one can initially easily have V_σ to be of the same order of magnitude as V_φ* (so that $\langle\sigma\rangle \gg \delta\sigma$ and $\sigma_i \sim \frac{M}{m}\varphi_i$): due to its larger slope the σ-field quickly rolls into the valley so that its average becomes very small, hence high temperature effects are not needed – or only to a small extend: "Even though the two fields do not interact with each other directly, they move towards the state [φ]= 0 and [σ]=0 in a coherent way. The reason is that the motion of these fields is determined by the same value of the Hubble constant", [38]. This gives

$$\begin{aligned} &3H\dot{\varphi} + M^2\varphi = 0 \qquad \text{and} \qquad 3H\dot{\sigma} + m^2\sigma = 0 \\ &\Rightarrow \quad \frac{m^2}{M^2}\frac{\dot{\varphi}}{\varphi} = \frac{\dot{\sigma}}{\sigma} \qquad \Rightarrow \qquad \sigma \sim \sigma_i\Big(\frac{\varphi}{\varphi_i}\Big)^{\frac{m^2}{M^2}} = \frac{M}{m}\varphi_i\Big(\frac{\varphi}{\varphi_i}\Big)^{\frac{m^2}{M^2}}. \end{aligned} \tag{9.3}$$

Now for $m \gg M$ "the 'heavy' field [σ] rapidly rolls down, and then from the last equation it follows, rather paradoxically, that the Hubble constant at the end of inflation is dominated

[1] We'll find the same for the case with σ and φ strongly coupled, see next chapter

by the 'light' field $[\varphi]$. Thus we can consistently consider the creation of fluctuations of the field $[\sigma]$ ...at the end of inflation.", [38].

Another thing to have in mind is the following: As (4.48) is also true for $\delta\sigma$, one finds

$$\frac{d}{dt}\langle\delta\sigma^2\rangle \sim H^3.$$

With (6.39) this reads

$$\frac{d}{dH}\langle\delta\sigma^2\rangle \sim -\frac{3}{M^2}H^3 \quad\Rightarrow\quad \langle\delta\sigma^2\rangle \sim \frac{M^2}{12}\varphi_i^2 - \frac{3}{M^2}\frac{H^4}{4}.$$

After a while when $H \ll H_i$ one hence gets $\sqrt{\langle\delta\sigma^2\rangle} \sim M\varphi_i^2$ as the typical amplitude for a quasi-homogeneous field (at scale H^{-1}, see footnote 5 in the previous chapter). Demanding that this quasi-homogeneous σ-field does not grow larger than the inflaton field (otherwise with $m \ll M$ one can have the last stages of inflation being driven by the σ-field(!), see also [38]) one gets $M^2\varphi_i < 1 \Leftrightarrow \varphi_i < \frac{1}{\sqrt{M}}$, which is the same condition on the inflaton as the one to avoid self-reproduction, see section 4.4.

9.2 The k-range of LWA-validity

At the end of this section we'll have found that we can use the LWA for all $\frac{m}{M}$ for all calculations and considerations about the evolution of the modes outside the horizon during inflation. However this is not obvious on first sight.

To simplify (9.2) one would like to make use of the LWA. This is *not* possible if (see(7.1))

$$k^2 > a^2|m^2 + H_f^2 - 2H^2|. \tag{9.4}$$

For modes fulfilling the inequality above one has to apply the SWA instead; later, with H having decreased enough so that this inequality does not hold, the evolution can be described by using the LWA. So one has SWA *at the moment of horizon crossing*[2] if

$$a_k^2H_k^2 > a_k^2|m^2 + H_f^2 - 2H_k^2| \quad\Rightarrow\quad \frac{m^2+H_f^2}{3} < H_k^2 < m^2 + H_f^2. \tag{9.5}$$

If $\frac{m}{M} = 0$, one has no SWA at crossing at all for the k-range of interest. If $\frac{m}{M} \ll 1$, one has only a few modes with SWA at crossing near H_f. If $\frac{m}{M} = \frac{1}{2}$, one has SWA for modes with high k, LWA at crossing and later SWA for modes in the middle of the range, and modes that have LWA from the moment of crossing on for low k. If $\frac{m}{M} \approx \frac{3}{5}$, one has SWA for modes with high k, LWA at crossing and later SWA for modes in the middle of the range, and modes that are LWA at crossing, later SWA and then LWA again for low k (the "dip" in Fig.9.1).

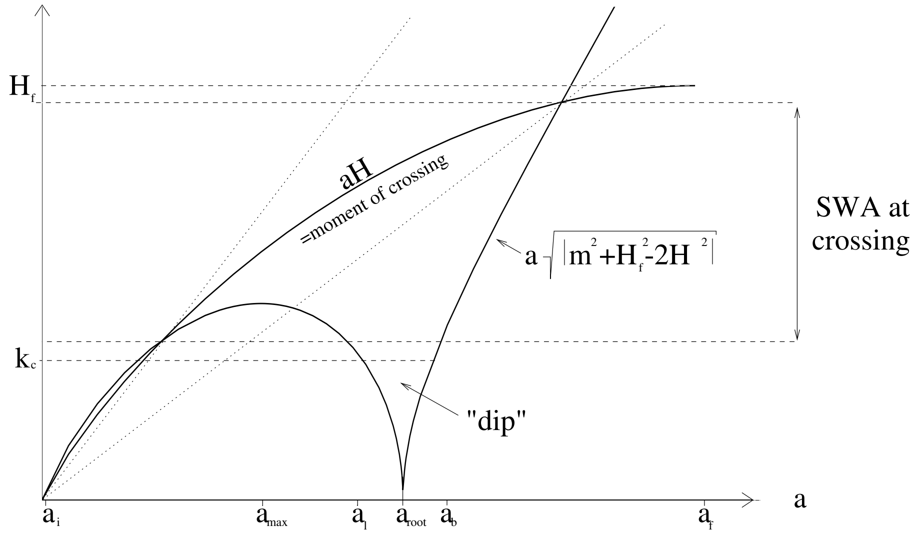

Figure 9.1: SWA at crossing for modes between the two intersections.

For $\frac{m}{M} > 1$ the situation looks like illustrated in Fig.9.1: The transition from SWA to LWA happens *before* horizon crossing if the curve $a\sqrt{|m^2 + H_f^2 - 2H^2|}$ is above the curve aH. So the SWA-at-crossing k-range is between the two intersections of the two curves, (with the transition to the LWA happening later, outside the horizon), or equivalently it is between the intersections of aH with the two dashed lines, $a\sqrt{m^2 + 3H_f^2}$ and $a\sqrt{\frac{m^2}{3} + H_f^2}$. With $\frac{m}{M}$ increasing, the dashed lines' slopes increase, *quickly making the range very narrow*. To get a quantitative understanding, we use (5.15) to get

$$\begin{aligned} \frac{m^2}{M^2} + \frac{1}{3} &< \ln \frac{H_k}{k} < 3\frac{m^2}{M^2} + 1 \\ \Rightarrow \quad H_k\, e^{-3\frac{m^2}{M^2}-1} &< k < H_k\, e^{-\frac{m^2}{M^2}-\frac{1}{3}}. \end{aligned} \tag{9.6}$$

So the k-range with modes that obey (7.3) at the moment of horizon crossing gets extremely narrow for already moderate values of $\frac{m}{M}$, so that *we shall neglect it entirely*.[3] Of course, for e.g. $\frac{m}{M} = \sqrt{2}$ the SWA-at-crossing range is still quite large. *But the crucial thing we'll notice in the following sections is that for $\frac{m}{M} < \sqrt{2}$ the integrals are mainly determined by the modes with k around a_iH_i, and for $\frac{m}{M} > \sqrt{2}$ the integrals are mainly determined by the modes with k around H_f,*[4] so the SWA-modes do not play an important role, being somewhere in the

[2]We don't care for modes inside the horizon as we have our initial conditions given at the moment of horizon crossing.

[3]Remembering that we make only order-of-magnitude calculations.

[4]Note that for $m > M\varphi_i$ the upper bound of (9.6) is below a_iH_i, so for a σ-field with such high

middle between the two extremes; also, as one can see from the figure, $\frac{a_k}{a_b}$ (see below) is not much bigger than unity, so that the SWA-oscillations are not acting for a long time.

Before we calculate $|\delta\sigma_k|^2 k^3$ we 'll first deal with the "dip" (see Figure 9.1) that appears for $\frac{m}{M} > 1$ when an LWA-at-crossing mode evolves according to SWA for some time to re-enter the LWA-regime again later on, *to show that it does not have to be taken into account, either.* For a certain k_c (see Figure 9.1) the description of the evolution of v switches back from LWA to SWA (we are "inside the dip") at $a = a_l$, and from that moment on $\delta\sigma_{k_c}$ decays $\propto \frac{1}{a}$ till the LWA-regime is entered again at $a = a_b$. *But if* $\frac{a_l}{a_b} = \mathcal{O}(1)$, *the decay will have affected* $\delta\sigma_{k_c}$ *only slightly when the LWA-regime is re-entered.* As after the root at $a = a_{root}$ the curve rises not slowlier than it descended after a_l and furthermore since $a_l \gg a_{max}$, we may just calculate $\frac{a_{max}}{a_{root}}$. If this ratio is of $\mathcal{O}(1)$, then $\frac{a_l}{a_b}$ is even closer to unity. From $m^2 + H_f^2 - 2H^2 = 0$ we get that

$$a_{root} = e^{\frac{1}{4} - \frac{3m^2}{4M^2}}. \tag{9.7}$$

From $\frac{\partial}{\partial a}\left(a\sqrt{\left|m^2 + H_f^2 - 2H^2\right|}\right)\bigg|_{a_{max}} = 0$ we get that

$$a_{max} = e^{-\frac{1}{4} - \frac{3m^2}{4M^2}} \tag{9.8}$$

Hence

$$\frac{a_{max}}{a_{root}} = \frac{1}{\sqrt{e}} \sim 1, \tag{9.9}$$

so the dip does not have to be taken into account, either.

9.3 The spectrum of the $\delta\sigma$-field

So with the "dip" being unimportant and by neglecting the narrow range of SWA, *we can work entirely in the LWA.* Hence

$$\ddot{\delta\sigma_k} + 3H\dot{\delta\sigma_k} + m^2\delta\sigma_k = 0 \tag{9.10}$$

But we know that this equation is equivalent to Weber's equation (6.49) with (6.47), see Ch.6, which is solved in terms of parabolic cylindrical functions. There are two representations, the standard one (see [39]) and Whittaker's representation (see [36]), respectively:

$$\delta\sigma_k(h) = C_1\ e^{\frac{h^2}{4}}\ U_{-\frac{m^2}{M^2}-\frac{1}{2}}(h)\ +\ C_2\ e^{\frac{h^2}{4}}\ V_{-\frac{m^2}{M^2}-\frac{1}{2}}(h)$$

$$\delta\sigma_k(h) = \mathcal{C}_1\ e^{\frac{h^2}{4}}\ D_{\frac{m^2}{M^2}}(h) + \mathcal{C}_2\ e^{\frac{h^2}{4}}\ D_{\frac{m^2}{M^2}}(-h). \tag{9.11}$$

masses one has no SWA-at-crossing k-range at all

One has[5]

$$U_{-\nu-\frac{1}{2}}(x) = D_\nu(x). \tag{9.12}$$

For small h we work with the second representation and for large h we use the first one. To get an idea how the $D_\nu(x)$ look like, we notice that (6.49) looks similar to the Schrödinger equation for a harmonic oscillator. Indeed, for positive, integer ν the parabolic cylindrical functions are equal to the Hermite functions:

$$D_\nu(x) = \frac{1}{2^\nu} e^{-\frac{x^2}{4}} \mathbf{H}_\nu\Big(\frac{x}{\sqrt{2}}\Big), \tag{9.13}$$

with $\mathbf{H}_\nu(x)$ being a Hermite polynomial. Now if $|x| < \sqrt{2\nu}$, the Hermite polynomials oscillate and can be approximated like

$$\mathbf{H}_\nu\Big(\frac{x}{\sqrt{2}}\Big) \propto e^{\frac{x^2}{4}} \quad \times \quad \text{oscillating function}(x). \tag{9.14}$$

For $|x| > \sqrt{2\nu}$ one gets

$$D_\nu(x) \propto e^{-\frac{x^2}{4}} \; x^\nu \tag{9.15}$$

and

$$V_\nu(x) \propto e^{\frac{x^2}{4}} \; x^{\nu-\frac{1}{2}}. \tag{9.16}$$

So we get decaying oscillations from (9.14) for both solutions $D_{\frac{m^2}{M^2}}(h)$ and $D_{\frac{m^2}{M^2}}(-h)$ if [6]

$$h < \sqrt{2}\frac{m}{M} \quad \text{or} \quad H < \frac{\sqrt{2}}{3}m, \tag{9.17}$$

namely

$$\delta\sigma(h)_k \propto \quad e^{\frac{h^2}{4}} \; \times \; \text{oscillating function}(h) = a^{-3/2} \times \text{oscillating function}. \tag{9.18}$$

For

$$h > \sqrt{2}\frac{m}{M} \tag{9.19}$$

one has from (9.15) and (9.16) that

$$|\delta\sigma_k(h)|^2 \propto h^{2\frac{m^2}{M^2}} \quad \text{and} \quad |\delta\sigma_k(h)|^2 \propto e^{h^2} \; h^{\frac{2m^2}{M^2}-2} \propto a^{-6} \; H^{-\frac{2m^2}{M^2}-2}, \tag{9.20}$$

[5]For completeness we also give the relation between $D_\nu(-x)$ and $V_{-\frac{1}{2}-\nu}(x)$:

$$V_{-\frac{1}{2}-\nu}(x) = \frac{\Gamma(-\nu)}{\pi}\Big(\sin[\pi(-\nu-1/2)] \; D_\nu(x) + D_\nu(-x)\Big).$$

[6]We would have found this result also by dropping the potential term in (9.10) to get $\delta\ddot{\sigma}_k + 3H\delta\dot{\sigma}_k = 0$.

$V_{-\frac{m^2}{M^2}-\frac{1}{2}}(h)$ (second equation) clearly decaying much faster than $D_{\frac{m^2}{M^2}}(h)$. The moment of transition (denoted by the subscript$_t$) from power law behaviour to oscillations with exponential decay happens at[7]

$$h_t = \sqrt{2}\frac{m}{M} \quad \text{or} \quad H_t = \frac{\sqrt{2}}{3}m. \tag{9.21}$$

This is analogous to the oscillations we found in de Sitter space for $m > \frac{3}{2}H$ (when ζ becomes imaginary), see (8.14), and in the radiation-dominated period, see (12.6). So oscillations seem to be a general feature for modes obeying the LWA; one has oscillations whenever, roughly speaking, $H < m$ (see (9.21)), no matter whether we're dealing with de Sitter inflation, quasi de Sitter inflation or a radiation-dominated universe.

So one has with (9.17), (9.18), (9.19) and (9.20) that

$$\begin{aligned} |\delta\sigma_k|^2 k^3 &\propto H^{2\frac{m^2}{M^2}} \quad \text{if} \quad H > \frac{\sqrt{2}}{3}m \\ |\delta\sigma_k|^2 k^3 &\propto e^{\frac{3H^2}{2H_f^2}} \propto a^{-3} \quad \text{if} \quad H < \frac{\sqrt{2}}{3}m, \end{aligned} \tag{9.22}$$

having neglected the decaying solution for high H in the upper equation, and the last step in the lower equation being possible due to (2.27). Clearly the second spectrum with H below m decays more rapidly with a increasing than the first one. To get a better feeling for the result above we shall apply SR to (9.10) so that

$$3H\delta\dot{\sigma}_k + m^2\delta\sigma_k = 0. \tag{9.23}$$

With (6.39) this gives

$$\frac{d\,(\delta\sigma_k)}{\delta\sigma_k} = \frac{m^2}{M^2}\frac{dH}{H} \tag{9.24}$$

so that

$$\delta\sigma_k^{sr} \propto H^{\frac{m^2}{M^2}}. \tag{9.25}$$

[7]It's nice to notice that the derivatives match at this point, too. We have for $H > \frac{\sqrt{2}}{3}m$

$$\delta\sigma_k = C_1 H^{\frac{m^2}{M^2}} \;\Rightarrow\; \delta\sigma_k' = \frac{m^2}{M^2}C_1 H^{\frac{m^2}{M^2}-1}H' = -\frac{m^2}{M^2}\frac{H_f^2}{aH^2}C_1 H^{\frac{m^2}{M^2}} = -\frac{m^2}{M^2}\frac{H_f^2}{aH^2}$$

and, using (2.27), for $H < \frac{\sqrt{2}}{3}m$

$$\delta\sigma_k = \frac{C_2}{a^{3/2}} \;\Rightarrow\; \delta\sigma_k' = -\frac{3C_2}{2a^{5/2}} = -\frac{3\delta\sigma_k}{2a}.$$

So the two derivatives are equal if

$$H^2 = \frac{2}{3}H_f^2\frac{m^2}{M^2} \;\Rightarrow\; H = H_t = \frac{\sqrt{2}}{3}m.$$

To find the decaying mode, we use (7.19). For the integral we get

$$\int a^{-4}\Big(1-2\ln a\Big)^{-\frac{1}{2}-\frac{m^2}{M^2}}da. \tag{9.26}$$

There's no known solution to this integral. However, for $a \to 0$ the a^{-4} is more divergent than the $\Big(1-2\ln a\Big)^{-\frac{1}{2}-\frac{m^2}{M^2}}$ converges, so that for small a one may work with

$$\Big(1-2\ln a\Big)^{-\frac{1}{2}-\frac{m^2}{M^2}}\int a^{-4}da \tag{9.27}$$

instead, so that

$$\delta\sigma_k^d \propto \frac{\delta\sigma_k^{sr}}{a^3\ H^{1+\frac{2m^2}{M^2}}}. \tag{9.28}$$

So for $a \ll 1$ the second particular solution clearly decays much faster than the slow roll solution (agreeing perfectly with the second solution obtained in (9.20)!). From (7.7) we get

$$\begin{aligned}\left|\frac{\ddot{\delta\sigma}_k^{sr}}{H\dot{\delta\sigma}_k^{sr}}\right| &= \frac{M^2}{3}\left|\frac{\Big(\frac{m^2}{M^2}-1\Big)\frac{m^2}{M^2}H^{\frac{m^2}{M^2}-2}}{\frac{m^2}{M^2}H^{\frac{m^2}{M^2}-1H}}\right| \\ &= \frac{H_f^2}{H^2}\Big|\frac{m^2}{M^2}-1\Big|<1 \quad\Rightarrow\quad H_f^2\Big|\frac{m^2}{M^2}-1\Big|<H^2. \end{aligned}\tag{9.29}$$

This is always fulfilled for $m \le M$. For $m \gg M$ it reads

$$\frac{m}{\sqrt{3}} < H, \tag{9.30}$$

thereby approximately re-deriving (9.19). *So one has SR as long as H is above m*, to be compared with (9.22). On has *SR all the time* if (see also (9.21))

$$m < H_f \le H \tag{9.31}$$

(strictly speaking for $m < \sqrt{\frac{3}{2}}M$). We'll call scalar fields with such a mass from now on *light scalars*. Likewise, one has *no SR at all* if

$$H \le H_i < m. \tag{9.32}$$

Scalar fields with such a mass will be called *heavy scalars*. For

$$M \sim H_f \le m \le H_i \tag{9.33}$$

the range is split (see Fig.9.2) into (I) the modes that are oscillating right from the moment

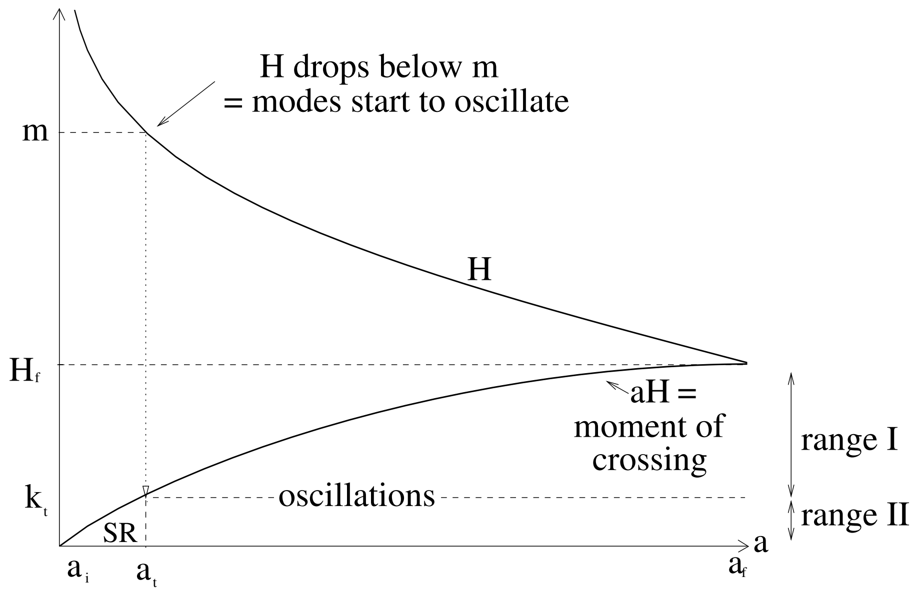

Figure 9.2: The behaviour within the LWA for modes outside the horizon.

of horizon crossing till the end of inflation; and (II) the modes that first encounter SR, and then start to oscillate when $H = H_t \sim m$. At this moment one has (see (9.21))

$$a = a_t := e^{\frac{1}{2} - \frac{m^2}{3M^2}} \tag{9.34}$$

(the $\frac{1}{2}$ in the exponent can be neglected already for moderate $\frac{m}{M}$). So for range I one has $a_k > a_t$ and vice versa, for range II we have $a_k > a_t$. The mode where the two regions border and hence the behaviour shifts has the wavenumber[8]

$$k_t = a_k H_k = a_t H_t \sim m\ e^{-\frac{m^2}{3M^2}}. \tag{9.35}$$

The higher one takes m to be, the smaller is k_t and hence the smaller range, (II), with furthermore the period during which the modes oscillate till the end of inflation becoming larger. For $m \sim H_i$ one has no SR-regime at all. The lower one takes m to be, the bigger is range II, with the period where one has oscillations getting smaller till one reaches $m \sim H_f$ from where on one has no oscillations at all (see Fig.9.2).

We fix the constants of integration by matching to (4.17). For $k > k_t$ one fits the oscillating solution (9.18), and for $k < k_t$ one fits the slow roll solution (9.25) or (9.20), to

[8]Notice that k_t decreases slowlier for $\frac{m}{M}$ increasing than the range (9.6), so the (neglected) SWA-at-crossing range always stays in range II.

which at $a = a_t$ then one glues the oscillating solutions, see also Fig.9.2. For modes that don't obey SR at the moment of crossing (range I) one has

$$\frac{|C_{osc}|^2}{a_k^3} = \frac{H_k^3}{\sqrt{H_k^2+m^2}} \quad \Rightarrow \quad |C_{osc}|^2 \sim \frac{k^3}{\sqrt{H_k^2+m^2}} \tag{9.36}$$

so that

$$|\delta\sigma_k|^2 k^3\Big|_{osc} \sim \frac{k^3}{\sqrt{H_k^2+m^2}} a^{-3}. \tag{9.37}$$

For the modes that obey the SR-condition at the moment of crossing (range (I)) we get

$$|C_{SR}|^2 H_k^{2\frac{m^2}{M^2}} \sim \frac{H_k^3}{\sqrt{H_k^2+m^2}}. \tag{9.38}$$

so that

$$|\delta\sigma_k|^2 k^3\Big|_{SR} \sim \frac{H_k^{3-2\frac{m^2}{M^2}}}{\sqrt{H_k^2+m^2}} H^{2\frac{m^2}{M^2}}. \tag{9.39}$$

For $m > M$ one has a transition at $a = a_t$ from modes with SR to oscillating modes so that

$$\frac{|C_{SR\to osc}|^2}{a_t^3} \sim \frac{H_k^{3-2\frac{m^2}{M^2}}}{\sqrt{H_k^2+m^2}} H_t^{2\frac{m^2}{M^2}} \Rightarrow \quad |C_{SR\to osc}|^2 \sim \frac{H_k^{3-2\frac{m^2}{M^2}}}{\sqrt{H_k^2+m^2}} \left(\frac{2}{9e}\right)^{\frac{m^2}{M^2}} m^{\frac{2m^2}{M^2}} \tag{9.40}$$

so that

$$|\delta\sigma_k|^2 k^3\Big|_{SR\to osc} \sim \frac{H_k^{3-2\frac{m^2}{M^2}}}{\sqrt{H_k^2+m^2}} \left(\frac{2}{9e}\right)^{\frac{m^2}{M^2}} \frac{m^{\frac{2m^2}{M^2}}}{a^3}. \tag{9.41}$$

So at the end of inflation one has, the two ranges put together,[9]

$$|\delta\sigma_k|^2 k^3\Big|_f \sim \begin{cases} \frac{k^3}{\sqrt{H_k^2+m^2}} & \text{for } \; me^{-\frac{m^2}{3M^2}} \leq k \leq H_f \\ \frac{H_k^{3-2\frac{m^2}{M^2}}}{\sqrt{H_k^2+m^2}} \left(\frac{2}{9e}\right)^{\frac{m^2}{M^2}} m^{2\frac{m^2}{M^2}} & \text{for } \; a_i H_i \leq k < me^{-\frac{m^2}{3M^2}}. \end{cases} \tag{9.42}$$

Applying (5.15) and approximating $\sqrt{m^2+H_k^2}$ one finds[10]

$$|\delta\sigma_k|^2 k^3\Big|_f \sim \begin{cases} \frac{k^3}{m} & \text{for } \; me^{-\frac{m^2}{3M^2}} \leq k \leq H_f \\ M^2\left(\ln \frac{H_k}{k}\right)^{1-\frac{m^2}{M^2}} \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} & \text{for } \; a_i H_i < k < me^{-\frac{m^2}{3M^2}}. \end{cases} \tag{9.43}$$

[9]As it should be, at $k = k_t = me^{-\frac{m^2}{3M^2}}$ the two expressions are equal: in the upper expression the square root in the denominator is dominated by m^2, in the lower expression by H_k. At $k = k_t$ one has $H_k \sim H_t \sim m$.

[10]Usually we always neglected all numerical prefactors as they were of the order of unity. However $\left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}}$ will be very small for already moderate values of $\frac{m}{M} > 1$, so we cannot throw it away.

For $m < M$ this reduces to

$$|\delta\sigma_k|^2 k^3\Big|_f \sim M^2\Big(\ln\frac{H_k}{k}\Big)^{1-\frac{m^2}{M^2}}, \tag{9.44}$$

knowing that $(2/3e)^{\frac{m^2}{M^2}} \sim \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \sim \mathcal{O}(1)$ for $\frac{m}{M} \leq 1$. For the other extreme, $m > H_i$, one gets

$$|\delta\sigma_k|^2 k^3\Big|_f \sim \frac{k^3}{m}. \tag{9.45}$$

9.4 The energy density and the power spectrum for $m < M$

First we shall treat the easiest case with the scalar being heavy, where the spectrum looks similar to the one for $m > H$ in de Sitter space, see (8.12). So one gets

$$\langle\varrho_{\delta\sigma}\rangle_f \sim mM^3 \quad \text{and} \quad \mathcal{P}^2_{\delta\sigma}(k)\Big|_f \sim \left(\frac{k}{M}\right)^3\left(1-\frac{k}{M}\right)^3. \tag{9.46}$$

From (5.5) we find the constraint

$$m \ll \frac{1}{M}, \tag{9.47}$$

so that m can have values up to the Planck scale. For the power spectrum we get

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f \sim \left(\frac{k}{M}\right)^{\frac{3}{2}} \quad \text{for} \quad a_iH_i \ll k \ll M. \tag{9.48}$$

Such a spectrum would change by over 7 orders of magnitude from the galactic to the CMBR scale and is hence not acceptable, so our model with the scalar being heavy can be considered to be ruled out.

Now we consider light scalar fields. The energy density is found with

$$\begin{aligned}\langle\varrho_{\delta\sigma}\rangle_f &\sim m^2 \int\limits_{a_iH_i}^{H_f} M^2\Big(\ln\frac{H_k}{k}\Big)^{1-\frac{m^2}{M^2}}\frac{dk}{k} \\ &\sim m^2M^2(-1)^{1-\frac{m^2}{M^2}}\int\limits_{\varepsilon}^{L}\frac{(\ln l')^{1-\frac{m^2}{M^2}}}{l'}dl'.\end{aligned} \tag{9.49}$$

Unlike the energy densities we calculated so far, *we now need the lower cut-off to be $\neq 0$ for the integral not to diverge*:[11]

$$\begin{aligned}\langle \varrho_{\delta\sigma}\rangle|_f &\sim \frac{m^2M^2}{2-\frac{m^2}{M^2}}(-1)^{1-\frac{m^2}{M^2}}\left[(\ln L)^{2-\frac{m^2}{M^2}}-(\ln\varepsilon)^{2-\frac{m^2}{M^2}}\right]\\ &\sim \frac{m^2M^2}{2-\frac{m^2}{M^2}}\left[\left(\ln\frac{1}{\varepsilon}\right)^{2-\frac{m^2}{M^2}}-\left(\ln\frac{1}{L}\right)^{2-\frac{m^2}{M^2}}\right].\end{aligned} \tag{9.50}$$

Now $\frac{1}{\varepsilon}\gg\frac{1}{L} \quad\Rightarrow\quad \ln\frac{1}{\varepsilon}>\ln\frac{1}{L}$, so with the exponent $2-\frac{m^2}{M^2}$ being bigger than zero, one can neglect the second integral. Furthermore taking $H_k\approx H_i$ one gets

$$\langle\varrho_{\delta\sigma}\rangle_f\sim m^2M^2\left(\ln\frac{a_f}{a_i}\right)^{2-\frac{m^2}{M^2}}. \tag{9.51}$$

Compared to (8.30), in this expression for the gravitationally produced particles one finds a direct dependence on the mass of the scalar field and also the number of e-folds of inflation. We shall use this expression in the next chapter. So from (5.5) we get for $m\ll M$ that

$$m\ll\frac{1}{\ln\frac{a_f}{a_i}}\quad\Rightarrow\quad m\ll\varphi_i^{-2} \tag{9.52}$$

and for $m\approx M$ that

$$m,M\ll\varphi_i^{-1}. \tag{9.53}$$

The last two conditions are fulfilled, because one wants to stay below the self-reproduction scale, see (4.50). From (5.46) one easily gets (again *not neglecting* the lower cut-off) that

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f\sim\frac{\left(\ln\frac{H_k}{k}\right)^{\frac{1}{2}-\frac{m^2}{2M^2}}}{\left(\ln\frac{a_f}{a_i}\right)^{1-\frac{m^2}{2M^2}}}, \tag{9.54}$$

(again the number of e-folds appears in the equation). So one finds an almost scale invariant spectrum with a slight logarithmic deviation, similar to the one we found in Ch.4, except that now we're dealing with nongaussian isocurvature perturbations. The spectrum is slightly red, loosing its redness the higher one takes m to be. So we make the educated guess that for

---

[11] The expression for the energy density during inflation looks like

$$\langle\varrho_{\delta\sigma}\rangle\sim\frac{m^2M^2}{2-\frac{m^2}{M^2}}\left(\frac{H}{M}\right)^{2\frac{m^2}{M^2}}\left[\left(\ln\frac{1}{\varepsilon}\right)^{2-\frac{m^2}{M^2}}-\left(\ln\frac{H_k}{aH}\right)^{2-\frac{m^2}{M^2}}\right],$$

so that again one finds that one reaches a "plateau" fairly soon, see (8.19).

$M < m$ we'll find a blue spectrum. We'll find that (9.54) will hardly change in the radiation dominated period following inflation. For $m \ll M$ one gets

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f \sim \frac{\left(\ln \frac{H_k}{k}\right)^{\frac{1}{2}}}{\ln \frac{a_f}{a_i}} \tag{9.55}$$

and for $m = M$ one has

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f \sim \frac{1}{\left(\ln \frac{a_f}{a_i}\right)^{\frac{1}{2}}}, \tag{9.56}$$

a Harrison-Zel'dovich spectrum.

9.5 The energy density and the power spectrum for $m > M$

We'll now calculate the energy density for $M < m < H_i$. With (9.43) and (5.26) one gets

$$\begin{aligned}
\langle \varrho_{\delta\sigma} \rangle_f \;&\sim\; m^2 M^2 \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \int\limits_{a_i H_i}^{k_t} \left(\ln \frac{H_k}{k}\right)^{1-\frac{m^2}{M^2}} \frac{dk}{k} \;+\; m^2 \int\limits_{k_t}^{M} \frac{k^2}{m} dk \\
&\sim\; m^2 M^2 \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \Big(-1\Big)^{1-\frac{m^2}{M^2}} \int\limits_{\frac{a_i H_i}{H_k}}^{\frac{k_t}{H_k}} \frac{\left(\ln l'\right)^{1-\frac{m^2}{M^2}}}{l'} dl' \;+\; \frac{m}{3}(M^3 - k_t^3) \\
&\sim\; m^2 M^2 \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \frac{(-1)}{2 - \frac{m^2}{M^2}} \left[\left(\ln \frac{H_k}{k_t}\right)^{2-\frac{m^2}{M^2}} - \left(\ln \frac{H_k}{a_i H_i}\right)^{2-\frac{m^2}{M^2}}\right] + \frac{m}{3}(M^3 - k_t^3).
\end{aligned} \tag{9.57}$$

For $m > \sqrt{2}M$ we make the approximation[12] that $2-\frac{m^2}{M^2} \approx -\frac{m^2}{M^2}$ and get, *this time neglecting the term containing* $\ln \frac{H_k}{a_i H_i}$,[13] (because its exponent is now negative)

$$\langle \varrho_{\delta\sigma} \rangle_f \sim M^4 \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \left(\ln \frac{H_k}{k_t}\right)^{2-\frac{m^2}{M^2}} + \frac{m}{3}(M^3 - k_t^3). \tag{9.58}$$

With $H_k \sim M$ and $k_t \sim e^{-\frac{m^2}{3M^2}}$ one gets

$$\left(\ln \frac{H_k}{k_t}\right)^{2-\frac{m^2}{M^2}} \sim \left(\ln \frac{M}{m} + \frac{m^2}{3M^2}\right)^{2-\frac{m^2}{M^2}} \sim \left(\frac{m^2}{3M^2}\right)^{2-\frac{m^2}{M^2}} \tag{9.59}$$

so that

$$\langle \varrho_{\delta\sigma} \rangle_f \sim \left(\frac{2}{e}\right)^{\frac{m^2}{M^2}} \frac{m^4}{9} + \frac{m}{3}\left(M^3 - m^3\, e^{-\frac{m^2}{M^2}}\right) \sim \frac{m^4}{3}\left[\left(\frac{2^{\frac{m^2}{M^2}}}{3} - 1\right)e^{-\frac{m^2}{M^2}} + \frac{M^3}{m^3}\right]. \tag{9.60}$$

The first term in brackets rapidly converges to zero for $\frac{m}{M} > 1$ and increasing. So already for moderate values of $\frac{m}{M}$ one gets

$$\langle \varrho_{\delta\sigma} \rangle_f \sim m M^3, \tag{9.61}$$

which is the same formula for the energy density as for heavy scalars. Hence one also gets the constraint (9.47). To calculate the power spectrum we use (5.35) and (5.38) to get

$$\begin{aligned} \frac{\langle \varrho_{\delta\sigma} \rangle_f^2 \, \mathcal{P}_{\delta\sigma}^2(k)\Big|_f}{m^4 k^2} \quad &\sim \quad \frac{S^2}{H_k^2} \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma(l')}{l'^2}\left[\Sigma(l+l') - \Sigma(l-l')\right]\, dl' \\ &+ \quad \frac{S^2}{H_k^2} \int\limits_{l+\varepsilon}^{L-l} \frac{\varsigma(l')}{l'^2}\left[\Sigma(l+l') - \Sigma(l'-l)\right]\, dl'. \end{aligned} \tag{9.62}$$

[12]Naturally the following question comes up: how does $\langle \varrho_{\delta\sigma} \rangle|_f$ look like for $m = \sqrt{2}M$? With

$$\lim_{\frac{m^2}{M^2}\to 2} \left\{ \frac{-1}{2-\frac{m^2}{M^2}} \left(\left(\ln \frac{H_k}{k_t}\right)^{2-\frac{m^2}{M^2}} - \left(\ln \frac{H_k}{a_i H_i}\right)^{2-\frac{m^2}{M^2}} \right) \right\} = \ln\ln \frac{H_k}{a_i H_i} - \ln\ln \frac{H_k}{k_t} \sim \ln\ln \frac{H_k}{a_i H_i}$$

we get that

$$\langle \varrho_{\delta\sigma} \rangle \sim M^4.$$

One can see that the two expressions for the energy density, (9.51) and (9.61) converge to the same point at $m = \sqrt{2}$, as it should be.

[13]Note that since the upper cut-off is more important than the lower cut-off, the SWA-at-crossing range (which k's are rather low) is not important at all, giving another good reason for having neglected it.

With (9.61) and *considering modes with $a_i H_i \ll k \ll k_t$* this gives, with $H_k \sim M$,

$$
\begin{aligned}
\frac{M^6 \ \mathcal{P}^2_{\delta\sigma}(k)\Big|_f}{m^2 \ k^2} \ \sim \ & \frac{S^2_{SR\to osc}}{H_k^2} \int\limits_{\varepsilon}^{l-\varepsilon} \frac{\varsigma_{SR\to osc}(l')}{l'^2} \Big[\Sigma_{SR\to osc}(l+l') - \Sigma_{SR\to osc}(l-l')\Big] \ dl' \\
+ \ & \frac{S^2_{SR\to osc}}{H_k^2} \int\limits_{l+\varepsilon}^{l_t} \frac{\varsigma_{SR\to osc}(l')}{l'^2} \Big[\Sigma_{SR\to osc}(l+l') - \Sigma_{SR\to osc}(l'-l)\Big] \ dl' \\
+ \ & \frac{S^2_{osc}}{H_k^2} \int\limits_{l_t}^{L-l} \frac{\varsigma_{osc}(l')}{l'^2} \Big[\Sigma_{osc}(l+l') - \Sigma_{osc}(l'-l)\Big] \ dl'.
\end{aligned} \tag{9.63}
$$

The integrands of the first two terms have singularities for $l' = 0$ and $l = l'$, so that they can be approximated the way explained in section 5.4. The last term can easily be calculated exactly. One gets

$$
\begin{aligned}
\frac{M^6 \ \mathcal{P}^2_{\delta\sigma}(k)\Big|_f}{m^2 \ k^2} \ \sim \ & M^4 \left[\left(\frac{m^2}{M^2} \right)^{\frac{m^2}{M^2}} \left(\frac{2}{3e} \right)^{\frac{m^2}{M^2}} \right]^2 \frac{\left(\ln \frac{M}{k} \right)^{1-\frac{m^2}{M^2}}}{\frac{k^2}{M^2}} \ \frac{\left(\ln \frac{M}{k} \right)^{2-\frac{m^2}{M^2}}}{\frac{m^2}{M^2}} \\
+ \ & \frac{2k}{m^2} \int\limits_{k_t}^{M-k} k'^2 dk'
\end{aligned} \tag{9.64}
$$

so that

$$
\mathcal{P}^2_{\delta\sigma}(k)\Big|_f \sim \left[\left(\frac{m^2}{M^2} \right)^{\frac{m^2}{M^2}} \left(\frac{2}{3e} \right)^{\frac{m^2}{M^2}} \right]^2 \left(\ln \frac{M}{k} \right)^{3-2\frac{m^2}{M^2}} + \frac{2}{3} \left(\frac{k}{M} \right)^3. \tag{9.65}
$$

As we deal with $k \ll k_t$, it's easy to check that one can neglect the second term so that

$$
\mathcal{P}_{\delta\sigma}(k)\Big|_f \sim \ \left(\frac{2}{3e} \right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2} \right)^{\frac{m^2}{M^2}} \left(\ln \frac{M}{k} \right)^{\frac{3}{2}-\frac{m^2}{M^2}}. \tag{9.66}
$$

Note that in the case with $m < M$ the logarithmic dependence on scale gave an amplitude *increasing* for larger scales, whereas for $m > \sqrt{2}M$ one finds the amplitude *decreasing*, due to the negative exponent in the logarithm, the spectrum is blue! (9.66) is going to remain unaltered throughout the radiation dominated period.

Chapter 10

Quintessential inflation

This chapter lies somewhat outside the main scope of this text, the following chapters will not refer to this one. We follow an idea stated by Peebles and Vilenkin in [7], but use a more accurate expression for the energy density of the gravitationally produced particles (found in the last chapter), and consider a different potential.

10.1 The model without radiation

In the scenario presented in Ch.2 the φ-field decays after inflation to produce the radiation of the Universe. As was pointed out already in section 2.5, the couplings between the decay products and the inflaton must neither be too large nor too weak. But "models satisfying these criteria do not arise naturally in particle physics; they have to be constructed ad hoc.", [7]. To preserve the mechanism of chaotic inflation but circumvent the problem stated above, Peebles and Vilenkin proposed a potential that is like a quartic chaotic potential for negative φ (with self-coupling constant $\lambda = 10^{-14}$, see footnote 15 in Ch.4), which then gradually vanishes for positive φ, the effective mass approaching zero. The inflaton is supposed not to couple to radiation or other fields (or at least the coupling is so weak that its decay can be neglected). *Instead the reheating comes from gravitational particle production*[1]. For reasons that will become obvious later on we shall consider the quadratic chaotic potential used so far, modified to look like[2]

$$V(\varphi) = \frac{M^2}{2} \times \begin{cases} \varphi^2 + \varphi_*^2 & \text{for } \varphi \leq 0 \\ \frac{\varphi_*^4}{\varphi^2+\varphi_*^2} & \text{for } \varphi > 0, \end{cases} \tag{10.1}$$

[1]See the chapters about the σ-field in (quasi) de Sitter space.

[2]Quintessential inflation works also with other potentials in (10.1), one can have other chaotic potentials and there's also a variety of decaying potentials for the right branch of the curve.

with $\varphi_* \ll 1$ and $M \sim 10^{-6}$ (this comes from the assumption that *the inflaton's fluctuations cause the CMBR-anisotropy*[3]). So one has a shifted parabola for negative φ, giving chaotic inflation for $\varphi \ll -\varphi_*$. As we'll see, the branch with positive φ is a quintessence form for $\varphi \gg \varphi_*$, a quintessence being a field that acts like an effective cosmological constant (which can have survived till today). This is where the name for this model stems from.

We shall first *not* take the radiation of the gravitationally produced particles into account, to clarify the dynamics of the φ-field. When inflation ends, the potential energy is of the same order of magnitude as the kinetic energy. Unlike the original chaotic model we do not have a minimum to oscillate around (with the average α being zero), but instead the field continues its roll onto the branch for positive φ. At $\varphi = 0$ one has $\varrho_{kin} \gg V$, so that for positive φ one encounters *a period where the kinetic energy density (its value approximated by the potential energy at the end of inflation) dominates over the potential energy*. Since the results we derived in section 2.2 are potential-independent, we can take them over straight away (see (2.13) to (2.16) and footnote 7 in Ch.2):

$$\ddot{\varphi} + 3H\dot{\varphi} = 0 \quad \Rightarrow \quad \dot{\varphi} \propto a^{-3}. \tag{10.2}$$

Hence one gets

$$\varrho_{kin} \sim \frac{M^2}{a^6} \ \text{ and } \ \varphi \sim \ln a \tag{10.3}$$

and thus

$$H \sim \frac{M}{a^3} \ \text{ and hence } \ t = \frac{a^3 - 1}{M} \approx \frac{a^3}{M} \ \text{ for } a \gg 1, \tag{10.4}$$

with $t_f := 0$. From (10.2) we get that with φ increasing one has $\varrho_{kin}(\varphi)$ exponentially decreasing, while the potential just drops $\propto \varphi^{-2}$ (see (10.1)). Consequently after a while one cannot avoid to enter a *second period where the potential dominates over the kinetic term*, the moment of entrance denoted by the subscript$_q$. There one has

$$\begin{aligned}
\varrho_{kin}(a_q) &\sim V(a_q) \\
\Rightarrow \quad \frac{M^2}{a_q^6} &\sim M^2 \frac{\varphi_*^4}{\varphi_q^2} \sim M^2 \frac{\varphi_*^4}{(\ln a_q)^2} \\
\Rightarrow \quad a_q &\sim \left(\frac{\ln a_q}{\varphi_*^2}\right)^{\frac{1}{3}} \\
&\sim \left(\frac{1}{\varphi_*^2}\right)^{\frac{1}{3}} \ \text{ to } 0^{th} \text{ order} \\
\Rightarrow \quad a_q &\sim \left(\frac{-\ln \varphi_*}{\varphi_*^2}\right)^{\frac{1}{3}} \ \text{ to } 1^{st} \text{ order,}
\end{aligned} \tag{10.5}$$

[3] A very recent paper by Felder, Kofman and Linde criticises exactly this proposition, see the end of the second section of this chapter.

so that

$$t_q \sim \frac{-\ln \varphi_*}{M\varphi_*^2}, \tag{10.6}$$

similar to the corresponding results in [7]. With $\varphi_* \ll 1$ one has $a_q \gg a_f$. To describe the evolution for $a > a_q$, one starts with (2.6) and (2.47), using $\varrho_{kin} \ll V$:

$$H = \frac{M\varphi_*^2}{\sqrt{3}\varphi} \quad \text{and} \quad \ddot{\varphi} + \sqrt{3}M\varphi_*^2\frac{\dot{\varphi}}{\varphi} = M^2\frac{\varphi_0^4}{\varphi^3} \quad \Rightarrow \quad \dot{\varphi} = \frac{M}{\sqrt{3}}\frac{\varphi_*^2}{\varphi^2}, \tag{10.7}$$

the last step possible if we assume slow roll. From the last equation one gets

$$\frac{\varrho_{kin}}{V} \sim \frac{1}{\varphi^2}, \tag{10.8}$$

so if φ increases, the dominance of the potential also increases. One also gets

$$\ddot{\varphi} = -2\frac{M}{\sqrt{3}}\frac{\varphi_*^2}{\varphi^3}\dot{\varphi} \quad \Rightarrow \quad \left|\frac{\ddot{\varphi}}{H\dot{\varphi}}\right| = \frac{1}{\varphi^2} \ll 1, \tag{10.9}$$

proving the validity of SR, *which gets even better with φ increasing*. For the time dependence of φ one finds from (10.7)

$$\varphi \sim \left(\sqrt{3}M\varphi_*^2(t - t_q) + \varphi_q^3\right)^{\frac{1}{3}}. \tag{10.10}$$

So the field increases, the potential energy thereby decreases. Using $\frac{d}{dt} = \dot{\varphi}\,\frac{d}{d\varphi}$ one gets from the first equation in (10.7) that

$$a \sim a_q\ e^{\frac{1}{2}(\varphi^2 - \varphi_q^2)}, \tag{10.11}$$

which gives an accelerated expansion, so that "the expansion then enters inflation from which it never recovers", [7].

10.2 The model with radiation

But one assumes to have gravitationally produced particles after inflation, too. From this the radiation and matter in our present Universe was formed. *Only if the energy density of radiation dominates over the potential energy at $\varphi = \varphi_q$ one has a chance to postpone the second inflationary phase, with today's Universe developing before, an expansion like in (10.11) yet to come.* So the model can only work if "the energy density of radiation at the end of inflation is large enough to dominate the inflaton energy density at $a_r < a_q$", [7], the subscript$_r$ denoting the moment when the ultrahard kinetic energy density drops below the energy density of the radiation produced by the gravitationally produced particles. On the other hand we also have to demand that the energy density of the gravitationally produced particles, ϱ_{gpp} is much smaller than the energy density of the φ-field during inflation, so that

$$\varrho_{gpp} \ll V_\varphi\Big|_f \sim M^2. \tag{10.12}$$

The energy $\mathcal{E}$ of a particle is given by

$$\mathcal{E} = \sqrt{\frac{k^2}{a^2} + m^2}. \tag{10.13}$$

If the modes with k near H_f are most important for the integration that leads to $\langle\varrho\rangle$ (as in the case of de Sitter space, and hence in [7]), one can say that a typical energy is given by $\mathcal{E} \sim \frac{H_f}{a}$ (if $m \ll H$ and a not to large), giving the usual redshift for ultrarelativistic particles. But in our case the modes that crossed the horizon early are the most important ones, so we can neglect the first term inside the square root so that

$$\mathcal{E} \sim m. \tag{10.14}$$

If the gravitationally produced particles decay to radiation soon after inflation, its energy density goes like

$$\varrho_{rad} \sim \frac{\varrho_{gpp}}{a^4}. \tag{10.15}$$

One of the by-products in the last chapter is a formula (see (9.51)) for the energy density of the fluctuations of a scalar field with $m \ll M$ in quasi de Sitter space of a quadratic chaotic potential so that, assuming the existence of a lot of light scalar fields with mass $m \ll M$,

$$\varrho_{gpp} = \sum_{i=1}^{N_{tot}} \langle \varrho_{\delta\sigma_i} \rangle \sim M^2 \mathcal{N}^2 \sum_{i=1}^{N_{tot}} m_i^2, \tag{10.16}$$

with N_{tot} the total number of light scalar fields. For the typical energy of a particle we shall use (10.14), and postulate that the dust of the graviationally produced particles decays to radiation. $\langle\varrho_{\delta\sigma}\rangle$ strongly depends on the mass of the scalar considered, so if one considers many scalar fields with many different masses only the scalar fields with the highest masses will contribute substantially to the *total* energy density of the gravitationally produced particles. So we may write

$$\varrho_{gpp} \sim N_s\, m^2 M^2 \mathcal{N}^2, \tag{10.17}$$

with N_s *denoting just the number of scalar fields with similar masses being much higher than the rest of the scalar fields' masses (we assume* $N_s \leq 10$*).* We assume m to be high enough so that one has $H < m$ fairly quickly so that the fields of the gravitationally produced particles do not stay constant, so that the energy density decays $\propto a^{-3}$ till the particles thermalize to become radiation. N_s was used for the total number of scalar fields in [7][4]:

[4] In [7] also the constraints on N_s not to have an overproduction of gravitational waves were treated: N_s mustn't be too small for the energy density of the gravitationally produced particles to exceed the energy density of gravitational waves; we shall not pursue this issue further.

Peebles and Vilenkin based their calculation on (8.30) instead of (9.51), thus approximating *the quartic-potential quasi de Sitter space by de Sitter space with $H = H_f \sim \sqrt{\lambda}$ (hence furthermore the energy density goes like a^{-4} so that one automatically has ultrarelativistic material right from the start)*. But then neither the mass of the scalar fields nor the number of e-folds of inflation appears in the expression of the energy density of the gravitationally produced particles! Hence they had two unknown parameters less to consider, which is why in their treatment they could find exact numbers for φ_*, a_r, $\mathcal{T}_{th}$, etc. (see below), *unlike the results we are going to get.*[5] But replacing (8.30) by (9.51) requires to use a *quadratic* quintessential potential, which is why we use (10.1) instead of the potential used by Peebles & Vilenkin. So condition (10.12) poses a constraint on m and $\mathcal{N}$, which should be fulfilled with $m \ll 10^{-6}$, $\mathcal{N} \leq 10^6$. [6]

Before finding a_r we first have to find the moment when the dust of the gravitationally produced particles has decayed to thermalized ultrarelativistic material (indicated by the subscript$_{th}$) by interaction via the exchange of gauge bosons. Thermalization is not possible if the expansion rate of the Universe is larger than the mutual interaction rate $\mathcal{I}$ among the gravitationally produced particles, so that we need

$$\mathcal{I}(a_{th}) > H(a_{th}). \tag{10.18}$$

$\mathcal{I}$ is the product of the particles' number density and their cross section (see e.g. [40]) so that

$$\mathcal{I} \sim n \;\times\; \frac{G^2}{\mathcal{E}^2}, \tag{10.19}$$

with G being the gauge coupling constant $\mathcal{O}(10^{-2} - 10^{-1})$.The number density is found with

$$n = \frac{\frac{\varrho_{gpp}}{a^3}}{\mathcal{E}}. \tag{10.20}$$

Collecting all these equations gives

$$1 < \frac{\varrho_{gpp} G^2}{M m^3} \sim N_s \mathcal{N}^2 G^2 \frac{M}{m}. \tag{10.21}$$

As the a_{th}'s cancel, one basically has the dust decayed to thermalized radiation right from the start of the post-inflationary period if

$$\frac{N_s\, \mathcal{N}^2}{m} > (MG^2)^{-1} \sim 10^{10}, \tag{10.22}$$

[5](10.12) is fulfilled naturally in their treatment, because one has $10^{-2} N_{tot} \lambda^2 \ll \lambda \;\Rightarrow\; N_{tot} \ll 10^{16}$ (see footnote 15 in Ch.4.)

[6]Of course we could have repeated the procedure that lead to (9.51) for a quartic potential, but the integrals involved in this calculation are far more complicated.

a condition that is fulfilled even with the lowest possible $\mathcal{N}$ and the highest possible m, so that no new constraints on m and $\mathcal{N}$ arise. To calculate the temperature at that moment one uses the Stefan-Boltzmann law,

$$\mathcal{T} \sim \varrho_{rad}^{\frac{1}{4}}, \tag{10.23}$$

so that

$$\mathcal{T}_{th} \sim \frac{m^3 M}{\varrho_{gpp}^{3/4} G^2} \sim \frac{m^{3/2}}{N_s^{3/4} M^{1/2} \mathcal{N}^{3/2} G^2}. \tag{10.24}$$

To find a_r one considers

$$\begin{aligned} \varrho(a_r) &\sim \varrho_{kin}(a_r) \\ \Rightarrow \quad \frac{\varrho_{gpp}}{a_r^4} \sim \frac{N_s m^2 M^2 \mathcal{N}^2}{a_r^4} &\sim \frac{M^2}{a_r^6} \\ \Rightarrow \quad a_r \sim \frac{M}{\sqrt{\varrho_{gpp}}} &\sim \left(\sqrt{N_s}\; m\; \mathcal{N}\right)^{-1} \\ \Rightarrow \quad t_r \sim \frac{M^2}{\varrho_{gpp}^{3/2}} \sim \frac{a_r^3}{M} &\sim \left(N_s^{3/2}\; m^3\; \mathcal{N}^3\; M\right)^{-1}. \end{aligned} \tag{10.25}$$

From this one also gets with (10.12) that

$$1 \ll \left(\sqrt{N_s}\; m\; \mathcal{N}\right)^{-1} \sim a_r. \tag{10.26}$$

The temperature at the moment of the radiation dominated period is given by

$$\mathcal{T}_r \sim \frac{\varrho_{gpp}^{3/4}}{M} \sim N_s^{3/4}\; m^{3/2}\; \mathcal{N}^{3/2} M^{1/2}. \tag{10.27}$$

As $\mathcal{T}_r$ mustn't be too low or too high due to several reasons (e.g. we must have GUT baryogenesis) one has a further constraint on the parameters m and $\mathcal{N}$. But this constraint depends on the particular model of nuclear synthesis, so that we shall not pursue in this direction any further.

Let's now follow the evolution of φ for $a > a_r$, after the regime of ultrahard matter has been left so that one has radiation-dominance, $\varrho_{rad} > \varrho_{kin} > V$. The equation of motion still hardly depends on the potential gradient, so that we can still use (10.2), but now with $a \propto \sqrt{t}$ (instead of $t^{1/3}$ as before) so that

$$\varphi \sim C_1 - \frac{C_2}{\sqrt{t}}. \tag{10.28}$$

The initial conditions are given by φ_r and $\dot{\varphi}_r$ so that

$$\varphi \sim \varphi_r + 2\dot{\varphi}_r t_r \left(1 - \sqrt{\frac{t_r}{t}}\right). \tag{10.29}$$

With (10.3) and (10.25) one finds

$$\begin{aligned}\varphi_r &\sim \ln\Big(\frac{1}{\sqrt{N_s}\, m\, \mathcal{N}}\Big)\\ \text{and}\quad \dot{\varphi}_r &\sim \frac{\dot{a}}{a}\Big|_r = H_r = \frac{1}{2}\,\frac{1}{t_r},\end{aligned} \tag{10.30}$$

so that

$$\varphi \sim \varphi_r + \left(1 - \sqrt{\frac{t_r}{t}}\right) \quad \text{and} \quad \dot{\varphi} \sim \frac{\sqrt{t_r}}{2t^{3/2}}. \tag{10.31}$$

(10.31) gives how $\varphi(t)$ evolves in time if the energy density of the radiation dominates and if the potential gradient is negligible. Due to (10.26) one can surely assume that $\varphi_r > 1$. If φ_r is an order of magnitude bigger than unity[7] then φ changes by at most 10% while (10.31) is still valid. *We thus have a scalar field that roughly stays constant.* The kinetic energy density quickly decreases so that one sooner or later gets $\varrho_{rad} > V > \varrho_{kin}$. This happens at $a = a_q > a_r$:

$$\begin{aligned}\frac{M^2}{a_q^6} &\sim \frac{M^2\varphi_*^4}{\varphi_r^2}\\ \Rightarrow \quad a_q &\sim \frac{\varphi_r^{1/3}}{\varphi_*^{2/3}}.\end{aligned} \tag{10.32}$$

To find when neglecting the potential gradient is not valid anymore, we follow the same pattern as in [7]: We drop the second derivative in the equation of motion, and use H in a radiation dominated background, to get the stable increasing solution.

$$\frac{3\dot{\varphi}}{2t} - \frac{M^2\varphi_*^4}{\varphi^3} = 0 \Rightarrow \quad \varphi \sim M^{1/2}\varphi_* t^{1/2}. \tag{10.33}$$

Skipping the potential gradient is no longer valid from the moment on when solution (10.31) (approximately constant$=\varphi_r$) becomes comparable to (10.33), which happens at

$$t_a \sim \frac{\varphi_r^2}{M\varphi_*^2}. \tag{10.34}$$

To identify the cosmological constant with the nearly *constant* quintessence, it is necessary to demand that t_a would be somewhere in the *future*,

$$t_a \overset{!}{>} t_0 \sim 10^{59}. \tag{10.35}$$

[7]It's probably safe to assume $\varphi_r \ll 100$, remembering that m represents the masses of the *heaviest* light scalar fields.

In other words, to follow the further evolution from $a = a_r$ onwards one assumes that the parameters of the model are such that (I) the kinetic energy shortly afterwards became less than the potential energy ($a_q < a_0$), (II) the potential energy "just" started dominating over radiation/matter[8] and (III) that till that same moment the potential gradient in the Klein-Gordon equation was negligible so that *we have a potential that is nearly constant, dominating the energy density of the Universe*. If we hence identify the cosmological constant with $V(\varphi_r)$, one gets

$$\frac{M^2\varphi_*^4}{\varphi_r^2} \sim \Lambda \sim 10^{-120}. \tag{10.36}$$

With $M \sim 10^{-6}$ (and assuming $\varphi_r > 1$) this gives

$$\begin{aligned} &\varphi_*^4 < 10^{-108} \\ \Rightarrow \quad &\varphi_* < 10^{-29}. \end{aligned} \tag{10.37}$$

We can see from this result that (10.32) gives

$$a_q < 10^{20}. \tag{10.38}$$

Furthermore (10.35) is fulfilled with

$$t_a > 10^{65}. \tag{10.39}$$

From today on the field's energy dominates, so we get similar results as for (10.9), (10.10) and (10.11):

$$\varphi \sim \left(\sqrt{3}M\varphi_*^2(t-t_0) + \varphi_r^3\right)^{1/3} \quad \text{and} \quad a = a_0 \; e^{\frac{1}{2}(\varphi^2-\varphi_r^2)}. \tag{10.40}$$

So up to a few constraints we get the same results as in [7], the more refined expression for the energy density of the gravitationally produced particles does not force the general idea of the model to be changed. However it should be mentioned that a very recent paper, see [38], pointed out that, among other flaws[9] *the temperature anisotropies we observe today should not be supposed to come from the inflaton's fluctuation, but from the inhomogeneities of the gravitationally produced particles*, since their energy density later dominates the Universe. They propose a new model, still with a non-oscillatory potential, but the radiation coming from instant preheating due to a coupling of the inflaton to the light scalar field.

[8] Λ started to dominate only fairly recently at redshift $z = 0.14 \;\cong\; 8 \cdot 10^9 yrs$ ago. Since that time the scale factor has increased by a mere factor of $\mathcal{O}(1)$.

[9] They furthermore argue that there are certain constraints on the initial conditions; furthermore the cosmological production of gravitinos and moduli fields causes problems, too.

Chapter 11

The $\delta\sigma$-field in quasi de Sitter space, strongly coupled to the inflaton

Now we'll treat the case with the two fields coupled.[1] An attractive feature of this scenario is that as long as initially $|\varphi| > |\sigma|$, the σ-field rolls exponentially rapidly (see next section) into the valley near $\langle\sigma\rangle = 0$ and stays there, so that *one needs high temperature effects only to a small degree*. One automatically has to work with $\delta\sigma \gg \langle\sigma\rangle$. We'll focus on

$$m^2 \ll \mathsf{min}\Big\{g^2\varphi^2, M^2\Big\}, \tag{11.1}$$

so that the Klein-Gordon equation approximately looks like

$$\delta\ddot{\sigma}_k + 3H\delta\dot{\sigma}_k + \left(\frac{k^2}{a^2} + \alpha H^2\right)\delta\sigma_k = 0. \tag{11.2}$$

To have a feeling what to expect we get from (8.35)[2], identifying m^2 with αH^2, that

$$n \approx 1 + \frac{4\alpha}{3}. \tag{11.3}$$

11.1 The k-range of LWA

We'll proceed the same way as in the last chapter. At horizon crossing one has SWA if (see (7.1))

$$H_k^2 > |H_f^2 + (\alpha - 2)H_k^2|. \tag{11.4}$$

To get a feeling for this inequality, we shall quickly consider several cases:

[1] This case was first treated in [41].

[2] One can compare these two cases, because both times we get a power law (see section 10.2) for the spectrum and hence the final results in both cases depend on the upper cut-off.

If $\alpha = 0$, one has

$$H_k^2 > 2H_k^2 - H_f^2 \quad \Rightarrow \quad H_k^2 < H_f^2, \tag{11.5}$$

a condition that is never fulfilled, so that *one has no SWA at crossing at all.*

If $\alpha = 1$, one has

$$H_k^2 > H_k^2 - H_f^2 \quad \Rightarrow \quad H_f^2 > 0, \tag{11.6}$$

which is always fulfilled, so that *one has no LWA at crossing at all.* Now let's consider the general case with $\alpha \leq 1$. Since $H_f^2 + \alpha H_k^2 < 2H_k^2$, one gets from (11.4)

$$H_k^2 > 2H_k^2 - H_f^2 - \alpha H_k^2 \quad \Rightarrow \quad H_k^2 < \frac{H_f^2}{1-\alpha} \;\Leftrightarrow\; a_k > e^{\frac{1}{2} - \frac{1}{2(1-\alpha)}}. \tag{11.7}$$

From this one can find that one has LWA at crossing for the lower part of the range. Up to $\alpha \approx 0.7$ this part is big so that *one may neglect any SWA effects* (for $\alpha > 0.7$ one also has take into account the appearance of a "dip" again).

If $\alpha = 2$, one has

$$H_k^2 > H_f^2, \tag{11.8}$$

which is always fulfilled, *so one has no LWA at crossing at all.* From $\alpha = 2$ onwards the range with SWA at horizon crossing gradually gets smaller: the LWA is possible in the upper part of the k-range.

If $\alpha \geq 3$, one has

$$H_k^2 > (\alpha - 2)H_k^2 + H_f^2 \quad \Rightarrow \quad (3-\alpha)H_k^2 > H_f^2, \tag{11.9}$$

which is never fulfilled, so *one has no SWA at all*, so for $\alpha \geq 3$ one can use the LWA again. *However, here we shall only treat the case with $g < M$, namely $\alpha < 0.7$.*

11.2 The spectrum of the $\delta\sigma$-field

So within the LWA we get

$$\ddot{\delta\sigma}_k + 3H\dot{\delta\sigma}_k + \alpha H^2 \delta\sigma_k = 0. \tag{11.10}$$

Assuming SR this reads

$$3H\dot{\delta\sigma}_k + \alpha H^2 \delta\sigma_k = 0 \quad \Rightarrow \quad 3aH^2 \delta\sigma_k' + \alpha H^2 \delta\sigma_k = 0, \tag{11.11}$$

having used (2.22). *The same equation is also valid for $\langle\sigma\rangle$.* From the first equation we get, if $H(t)$ changes only little in time,

$$\frac{\dot{\langle\sigma\rangle}}{\langle\sigma\rangle} = -\frac{\alpha H}{3} \quad \Rightarrow \quad \langle\sigma\rangle \propto e^{-\frac{\alpha H}{3}t}, \tag{11.12}$$

getting the exponential plunge of the $\langle\sigma\rangle$-field into the valley, as mentioned in the beginning. From the second equation we get

$$\frac{d\,(\delta\sigma_k)}{\delta\sigma_k} = -\frac{\alpha}{3}\frac{da}{a}\ . \tag{11.13}$$

Hence

$$\delta\sigma_k^{sr} \propto a^{-\frac{\alpha}{3}} = a^{-\frac{g^2}{M^2}}. \tag{11.14}$$

Now we first have to find the condition for SR to be valid:

$$\begin{aligned}
1 \overset{!}{\ll} \left|\frac{\ddot{\delta\sigma}_k^{sr}}{3H\dot{\delta\sigma}_k^{sr}}\right| &= \left|\frac{-\frac{g^2}{M^2}\Big(-\frac{g^2}{M^2}-1\Big)a^{-\frac{g^2}{M^2}-2}\dot{a}^2 - \frac{g^2}{M^2}a^{-\frac{g^2}{M^2}-1}\ddot{a}}{-3H\frac{g^2}{M^2}a^{-\frac{g^2}{M^2}-1}\dot{a}}\right| \\
&= \frac{1}{3}\left|\frac{\Big(-\frac{g^2}{M^2}-1\Big)\frac{\dot{a}^2}{a^2}+\frac{\ddot{a}}{a}}{3H\frac{\dot{a}^2}{a^2}}\right| = \frac{1}{3}\left|-\frac{g^2}{M^2}-1+\frac{\frac{\ddot{a}}{a}}{H^2}\right| \\
&= \frac{1}{3}\left|-\frac{g^2}{M^2}-1+\frac{H^2+\dot{H}}{H^2}\right| \approx \frac{g^2}{3M^2}.
\end{aligned} \tag{11.15}$$

So we get

$$3M^2 \gg g^2 \ \text{ or } \ \alpha \ll 9 \tag{11.16}$$

as the necessary condition for slow roll. Since we consider $\alpha < 0.7$, this condition is well-fulfilled. To find the decaying mode, we consult Liouville's formula, getting for the integral

$$\int a^{-4+\frac{2\alpha}{3}}(1-2\ln a)^{-\frac{1}{2}}\ da \sim H^{-1}\int a^{-4+\frac{2\alpha}{3}}\ da \sim H^{-1}a^{-3-\frac{2\alpha}{3}}. \tag{11.17}$$

This confirms the second solution to be decaying much faster than the slow roll solution. Fixing the constants of integration as usual gives

$$|\delta\sigma_k|^2k^3 \sim H_k^2\Big(\frac{a_k}{a}\Big)^{\frac{2\alpha}{3}}. \tag{11.18}$$

With (5.13) this gives

$$|\delta\sigma_k|^2k^3\Big|_f \sim M^2\Big(\ln\frac{H_k}{k}\Big)^{1-\frac{\alpha}{3}}\ \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}. \tag{11.19}$$

11.3 The energy density and the power spectrum

With the last result and (5.26) one gets (notice that we don't need the lower cut-off for this integration: the power law dominates over the logarithm)

$$\langle\varrho_{\delta\sigma}\rangle_f \ \sim\ m^2M^2\int_0^M \Big(\ln\frac{H_k}{k}\Big)^{1-\frac{\alpha}{3}}\left(\frac{k}{M}\right)^{\frac{2\alpha}{3}} dk\ \ \Bigg|\ \ 1-\frac{\alpha}{3}\approx 1\ \text{ for the logarithm}$$

$$\sim \quad -m^2 M^2 \left(\frac{H_k}{M}\right)^{\frac{2\alpha}{3}} \int\limits_0^{\frac{M}{H_k}} \ln l' \quad l'^{\frac{2\alpha}{3}-1} dl' \tag{11.20}$$

so that we get with $H_k \sim M$

$$\langle \varrho_{\delta\sigma} \rangle_f \sim \frac{m^2 M^2}{\alpha^2}. \tag{11.21}$$

So one gets from (5.5) the constraint that $\alpha \gg m$ or $g \gg \sqrt{m}M$. To calculate the power spectrum we use (5.46) and get

$$\begin{aligned} \mathcal{P}^2_{\delta\sigma}(k)\Big|_f \quad &\sim \quad \alpha^2 \left(\ln \frac{M}{k}\right)^{1-\frac{\alpha}{3}} \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}} \int\limits_0^{\frac{k}{M}} \left(\ln \frac{M}{k}\right)^{1-\frac{\alpha}{3}} \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}-1} d\frac{k}{M} \\ &\sim \quad \alpha \left(\ln \frac{M}{k}\right)^{1-\frac{\alpha}{3}} \left(\frac{k}{M}\right)^{\frac{4\alpha}{3}} \left[\left(\ln \frac{M}{k}\right)^{1-\frac{\alpha}{3}} + \frac{3}{2\alpha}\right]. \end{aligned} \tag{11.22}$$

So for $k \ll M \ e^{-\frac{3}{2\alpha}}$ one gets

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f \ \sim \ \sqrt{\alpha} \left(\ln \frac{M}{k}\right)^{1-\frac{\alpha}{3}} \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}. \tag{11.23}$$

So one finds $n \sim 1 + \frac{4\alpha}{3}$, exactly as the rough approximation in the beginning of this chapter predicted. We find a blue spectrum as in the last chapter for $m > M$, but this time even in form of a power law and not just a logarithm. Note that the power spectrum of the fluctuations of the inflaton depends directly on the mass of the inflaton, whereas the power spectra for the σ-field depend on the ratio $\frac{m}{M}$ or $\frac{g}{M}$. Unlike the two power spectra we found for the fields uncoupled, this power spectrum will change throughout the radiation dominated period due to its dependence on high-k modes. This introduces a dependence of the power spectrum on $\frac{m}{M}$.

Chapter 12

The $\delta\sigma$-field in a radiation-dominated background

12.1 The evolution of the field

After inflation has ended, we assume that the inflaton field (which energy density still dominates the total energy density) rapidly decays (e.g. via parametric resonance) into radiation or other ultrarelativistic material with $\alpha = \frac{1}{3}$, so that we have virtually no period of dust (with $H \propto a^{-3/2}$) between inflation and the radiation-dominated Universe. *We assume furthermore that the σ-field does not couple to the radiation and use (6.7) to describe its evolution in time.* From (1.5) and (1.10) one gets

$$H(a) = \frac{H_f}{a^2} \approx \frac{M}{a^2} \tag{12.1}$$

and hence

$$u_k'' + \left(\frac{m^2}{M^2}a^2 + \frac{k^2}{M^2}\right)u_k = 0. \tag{12.2}$$

We shall work out an approximate solution to this equation.[1] If $a > \frac{k}{m}$ right from the beginning of the radiation dominated period at $a = a_f = 1$ (if $\frac{m}{M} > 1$ this is always fulfilled after inflation), we can make the LWA and neglect the k-dependence in (12.2), so that (see [35])

$$u_k'' + \frac{m^2}{M^2}\; a^2\; u_k = 0 \quad \Rightarrow \quad u_k = \; \mathcal{C}_1\; \sqrt{a}\; J_{-\frac{1}{4}}\Big(\frac{m}{M}\; a^2\Big) + \; \mathcal{C}_2\; \sqrt{a}\; J_{\frac{1}{4}}\Big(\frac{m}{M}\; a^2\Big), \tag{12.3}$$

[1] Using the transformation

$$a = \sqrt{\frac{M}{2im}} \;\; b$$

one can transform (12.2) into Weber's equation (see (6.49)) for $u(b)$. But the result looks rather awkward, with complex coefficients and arguments.

which gives hence

$$\delta\sigma_k = \frac{1}{\sqrt{a}}\Bigg(C_1\ J_{-\frac{1}{4}}\Big(\frac{m}{M}\ a^2\Big) +\ C_2\ J_{\frac{1}{4}}\Big(\frac{m}{M}\ a^2\Big)\Bigg). \tag{12.4}$$

If one has furthermore

$$a > \sqrt{\frac{M}{m}} \quad \Rightarrow \quad k > \sqrt{mM}, \tag{12.5}$$

we get with (8.4) that

$$\delta\sigma_k = \frac{C_5}{a^{3/2}}\ \times \text{ oscillating function.} \tag{12.6}$$

When this formula is valid we say to be in the regime of *oscillating Bessel modes*.[2] For $\frac{m}{M} > 1$ the LWA gives an oscillating Bessel modes for all k right from the start, so in this case the evolution of $\delta\sigma_k$ in a radiation dominated Universe is rather simple, with the field obeying (12.6) from the beginning, hence

$$|\delta\sigma_k|^2 k^3\Big|_{a>1}^{m>M} \sim \frac{|\delta\sigma_k|^2 k^3\Big|_f}{a^3}. \tag{12.7}$$

For the case of $m \ll M$ (the rest of this section will deal with this case) one can also have (12.6) for certain a and k, but with a different value for C_5. But we also have to consider the case $\frac{k}{m} < a < \sqrt{\frac{M}{m}}$ and $a < \frac{k}{m}$, too (ranges II and III in Fig.12.1, respectively). For the first case (12.4) is still valid, but the Bessel function must be approximated according to (8.6), so that one gets

$$\delta\sigma_k = C_1\left(\frac{m}{M}\right)^{\frac{1}{4}} + C_2\left(\frac{M}{m}\right)^{\frac{1}{4}} a^{-1}. \tag{12.8}$$

Modes evolving to this equation we say to be in the regime of the *power law Bessel modes*. For $a < \frac{k}{m}$ (12.2) gives

$$u'' + \frac{k^2}{M^2}\ u = 0 \ \Rightarrow\ u = \mathcal{C}_3 \cos\Big(\frac{k}{M}\ a\Big) + \mathcal{C}_4 \sin\Big(\frac{k}{M}\ a\Big), \tag{12.9}$$

so that

$$\delta\sigma_k = \frac{1}{a}\Bigg(C_3 \cos\Big(\frac{k}{M}\ a\Big) + C_4 \sin\Big(\frac{k}{M}\ a\Big)\Bigg). \tag{12.10}$$

We'll call a mode like this a *trigonometric mode*. How a certain mode with a certain $\frac{m}{M} < 1$ evolves at some moment a is sketched in Fig.12.1. One can see three k-ranges for the

[2]Notice that

$$a = \sqrt{\frac{M}{m}} \ \Leftrightarrow\ H = m,$$

which is equivalent to the moment when the solutions of (6.49) start to oscillate, too. So one has oscillations for $k >\max\Big\{ma,\ Ha\Big\}$.

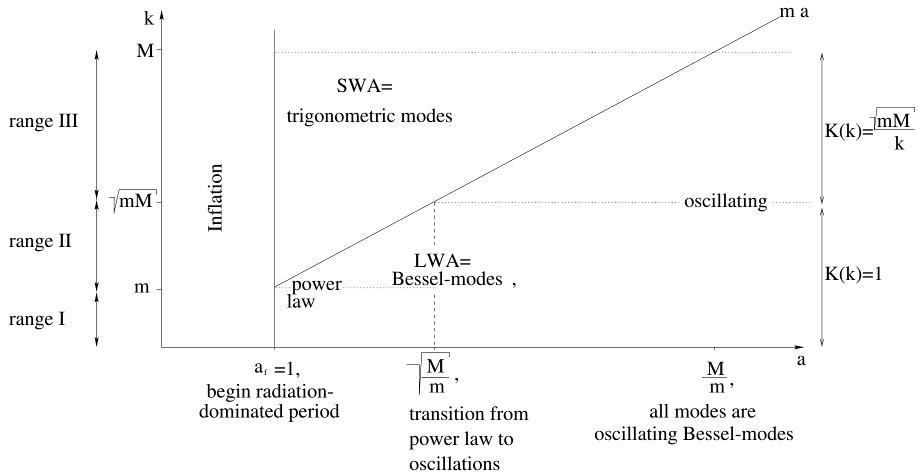

Figure 12.1: The evolution of the modes after inflation for $\frac{m}{M} < 1$.

evolution of modes up to $a = \frac{M}{m}$, from where finally all modes evolve in the oscillating Bessel regime:

$$\begin{array}{rll} (I) & a_i H_i \le k < m & \text{Inflation} \to \text{power law Bessel} \to \text{oscillating Bessel} \\ (II) & m \le k < \sqrt{mM} & \text{Inflation} \to \text{trigonometric} \to \text{power law Bessel} \to \text{oscillating Bessel} \\ (III) & \sqrt{mM} \le k \le M & \text{Inflation} \to \text{trigonometric} \to \text{oscillating Bessel.} \end{array} \tag{12.11}$$

To see how $|\delta\sigma_k|^2 k^3$ evolves from $|\delta\sigma_k|^2 k^3\Big|_f$ at $a = a_f$ to the time when $a > \frac{M}{m}$ (when all the modes are oscillating Bessel modes), we follow the evolution through all three k-ranges, gluing together the different solutions (12.6), (12.8) and (12.10) at the moments of transition from one regime to another. We start with (I). First we'll determine C_1 and C_2. From (12.8) we get that at $a = a_f$ we have

$$\begin{aligned} \delta\sigma_k(a_f) &= C_1 \left(\frac{m}{M}\right)^{\frac{1}{4}} + C_2 \left(\frac{M}{m}\right)^{\frac{1}{4}} \\ \delta\sigma'_k(a_f) &= \qquad\qquad - C_2 \left(\frac{M}{m}\right)^{\frac{1}{4}}. \end{aligned} \tag{12.12}$$

From (9.43) we get

$$\delta\sigma'_k \sim \frac{m^2}{aH^2} \delta\sigma_k, \tag{12.13}$$

so that[3]

$$\delta\sigma_k'|_f \sim \frac{m^2}{M^2}\delta\sigma_k'|_f \ll \delta\sigma_k, \tag{12.14}$$

which is easily understood since we had SR for the σ-field with $m < M$. Adding the two equations (12.12) gives

$$\delta\sigma_k|_f \left(1 - \frac{m^2}{M^2}\right) = C_1\left(\frac{m}{M}\right)^{\frac{1}{4}} \quad\Rightarrow\quad C_1 \approx \delta\sigma_k|_f\left(\frac{M}{m}\right)^{\frac{1}{4}}, \tag{12.15}$$

so that[4]

$$C_2 \approx 0, \tag{12.16}$$

the decaying particular solution $J_{-\frac{1}{4}}$ not playing an important role. So for the power law Bessel modes one gets with (12.8)

$$\delta\sigma_k(a) \sim \delta\sigma_k|_f = constant. \tag{12.17}$$

At $a = \sqrt{\frac{M}{m}}$ the power law Bessel modes become oscillating Bessel modes, so gluing the two regimes together at $a = \sqrt{\frac{M}{m}}$ gives, see (12.6),

$$\delta\sigma_k|_f = \frac{C_5}{\sqrt{\frac{M}{m}}^3}, \tag{12.18}$$

and hence

$$|\delta\sigma_k|^2k^3\Big|_{a>\sqrt{\frac{M}{m}}}^{range\ I} \sim \left(\frac{M}{m}\right)^{\frac{3}{2}} \frac{|\delta\sigma_k|^2k^3\Big|_f}{a^3}. \tag{12.19}$$

For (II) we get from (12.10) for $a \approx 1$ and $k \leq \sqrt{mM} \ll M$ that

$$\delta\sigma_k = \frac{C_3}{a} + C_4\frac{k}{M} \tag{12.20}$$

and hence

$$\begin{aligned} \delta\sigma_k|_f &= C_3 + C_4\frac{k}{M} \\ \text{and}\ \ \delta\sigma_k'|_f &= -C_3. \end{aligned} \tag{12.21}$$

[3]One finds a similar result, namely that $\frac{\delta\sigma_k'|_f}{\delta\sigma_k|_f} \ll 1$ for φ and σ strongly coupled, so all following results, like all previous results in this section, are equally valid for that case, too.

[4]Note that it is possible to find an exact expression for C_1 and C_2 in terms of a lot of Bessel functions, but it does not clarify the physics at all.

This gives

$$C_4 \approx \frac{M}{k}\delta\sigma_k|_f \quad \text{and} \quad C_3 \approx 0. \tag{12.22}$$

So (12.10) reads

$$\delta\sigma_k \sim \delta\sigma_k|_f \frac{M}{k} \frac{\sin\left(\frac{k}{M}a\right)}{a}. \tag{12.23}$$

The argument of the sine at the moment of transition to power law Bessel modes, $a = \frac{k}{m}$, is given by

$$\frac{k^2}{mM} < 1, \tag{12.24}$$

so that the expansion for small arguments is still valid; *but this means that the trigonometric modes in this k-range stay roughly as constant as the power law Bessel modes, so that (I) and (II) exhibit roughly the same behaviour.*[5] We get

$$|\delta\sigma_k|^2 k^3\Big|_{a>\sqrt{\frac{M}{m}}}^{range\ I+II} \sim \left(\frac{M}{m}\right)^{\frac{3}{2}} \frac{|\delta\sigma_k|^2 k^3\Big|_f}{a^3}. \tag{12.25}$$

To find $|\delta\sigma_k|^2 k^3$ for (III), we note that (12.22) is also valid for $k > \sqrt{mM}$, with k still being of the same order of magnitude as $\sqrt{mM}$. So at the moment of transition ($a = \frac{k}{m}$) from the trigonometric modes to the oscillating Bessel modes we get

$$|\delta\sigma_k|^2 k^3\Big|_f \frac{M^2}{k^2} \frac{\sin^2\left(\frac{k^2}{mM}\right)}{\frac{k^2}{m^2}} = \frac{|C_3(k)|^2}{\frac{k^3}{m^3}} \sin^2\left(\frac{k^2}{mM}\right). \tag{12.26}$$

Hence

$$|C_3(k)|^2 \sim \frac{M^2}{mk} = \left(\frac{M}{m}\right)^{\frac{3}{2}} \frac{\sqrt{mM}}{k}. \tag{12.27}$$

To see how this changes for $k \gg \sqrt{mM}$ we note that for $k = M$ one has at $a = \frac{M}{m}$ (matching (12.26) with (12.27) to (12.6)

$$\begin{aligned} \frac{m^2}{M^2} \quad &\times \quad |\delta\sigma_k|^2 k^3\Big|_f \ \times \ \text{oscillating function of } \left(\frac{M}{m}\right) \\ &= \ \frac{m^3}{M^3} \ \times \ |C_5|^2 \ \times \ \text{oscillating function of } \left(\frac{M}{m}\right) \end{aligned} \tag{12.28}$$

so that

$$|C_5|^2 = \frac{M}{m}|\delta\sigma_k|^2 k^3\Big|_f = |C_3(M)|^2. \tag{12.29}$$

[5] Always keeping in mind that we do just order-of-magnitude calculations; this of course is also true for the following treatment of range III.

We hence take (12.27) to be valid approximately for the entire range III. So we finally get

$$|\delta\sigma_k|^2 k^3\Big|_{a>\frac{M}{m}} \sim \left(\frac{M}{m}\right)^{\frac{3}{2}} \frac{|\delta\sigma_k|^2 k^3\Big|_f}{a^3} \times \begin{cases} \frac{\sqrt{mM}}{k} & \text{for } \sqrt{mM} \le k \le M \\ 1 & \text{for } a_i H_i \le k < \sqrt{mM} \end{cases} \tag{12.30}$$

$$|\delta\sigma_k|^2 k^3\Big|_{a>\frac{M}{m}} \sim \left(\frac{M}{m}\right)^{\frac{3}{2}} \frac{|\delta\sigma_k|^2 k^3\Big|_f}{a^3} \times K(k) \tag{12.31}$$

with $K(k)$ being a transfer function (see Fig.12.1 and footnote 6 in Ch.3) given by

$$K(k) = \left[1 + \left(\frac{\sqrt{mM}}{k} - 1\right) \times \Theta\left(\sqrt{mM} - k\right)\right], \tag{12.32}$$

Θ being Heaviside's step function. This result applies also to the spectrum at the end of inflation with σ and φ being coupled. The $a^{-3/2}$ in (12.31) and (12.7) leads to $\varrho \propto a^{-3}$, cold $\delta\sigma$-dust, *thereby giving a $\delta\sigma$-field which energy density decays slowlier than the energy density of radiation.* Hence later on the power spectrum of the $\delta\sigma$-field will be more important than the inflaton's power spectrum, so that the CMBR anisotropy we observe today might be caused not by the fluctuations of the inflaton but by the σ-field's fluctuations, see section 5.1.

12.2 The energy density and the power spectrum

Before we perform the necessary integrations, we note again that the k-range is set by inflation, the only period when quantum fluctuations can be stretched to cross the horizon to become classical. *So we 'll use (4.4) also for $a \ge 1$ (after inflation) for all integrations over k.* For the energy density we get the general expression

$$\langle \varrho_{\delta\sigma} \rangle\Big|_{a>\frac{M}{m}}^{m<M} \sim \frac{m^2}{a^3}\left(\frac{M}{m}\right)^{\frac{3}{2}} \left[\int\limits_{a_i H_i}^{\sqrt{mM}} |\delta\sigma_k|^2 k^3\Big|_f \frac{dk}{k} \; + \; \sqrt{mM} \int\limits_{\sqrt{mM}}^{M} |\delta\sigma_k|^2 k^3\Big|_f \frac{dk}{k^2}\right]. \tag{12.33}$$

Let's first treat the strongly coupled case. As the lower integration bound can be set to zero we choose $H_k \sim M$. So with (11.19) one gets

$$\langle \varrho_{\delta\sigma} \rangle\Big|_{a>\frac{M}{m}}^{m<min\{g\varphi,\ M\}} \sim \frac{m^{1/2} M^{7/2}}{a^3} \left[\int\limits_{0}^{\sqrt{\frac{m}{M}}} \ln\frac{M}{k} \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}} \frac{dk}{k} + \sqrt{\frac{m}{M}} \int\limits_{\sqrt{\frac{m}{M}}}^{1} \ln\frac{M}{k} \left(\frac{k}{M}\right)^{\frac{2\alpha}{3}} \frac{dk}{k^2}\right]. \tag{12.34}$$

The first integral will have a prefactor of $1/\alpha$, the second one $\sqrt{\frac{m}{M}}/(\alpha-1)$ instead, so that one can safely ignore the second integral. Consequently only the modes with k up to $\sqrt{mM}$ are important for the integral. This is a direct consequence of the transfer function. One gets

$$\left\langle \varrho_{\delta\sigma}\right\rangle\Big|_{a>\frac{M}{m}}^{m<min\{g\varphi,\ M\}} \sim \frac{m^{1/2}M^{7/2}}{a^3\alpha}\left(\frac{m}{M}\right)^{\frac{\alpha}{3}}\left[\ln\frac{M}{m}+\frac{3}{2\alpha}\right]. \tag{12.35}$$

To calculate the power spectrum we'll focus only on large scales with $k \ll \sqrt{mM}$. Performing the analogous calculation as in (11.22) with (5.46) and the result above one gets that

$$\mathcal{P}^2_{\delta\sigma}(k)\Big|_{a>\frac{M}{m}}^{m<min\{g\varphi,\ M\}} \sim \frac{\alpha^2\left(\frac{M}{m}\right)^{\frac{2\alpha}{3}}}{\left[\ln\frac{M}{m}+\frac{3}{2\alpha}\right]^2}\left(\ln\frac{M}{k}\right)^{1-\frac{\alpha}{3}}\left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}\int\limits_0^{\frac{k}{M}}\left(\ln\frac{M}{k}\right)^{1-\frac{\alpha}{3}}\left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}\frac{dk}{k}. \tag{12.36}$$

The integration introduces a factor $1/\alpha$. So for $k \ll$min$\left\{\sqrt{mM}, M\ e^{-\frac{3}{2\alpha}}\right\}$ this gives

$$\mathcal{P}_{\delta\sigma}(k)\Big|_{a>\frac{M}{m}}^{m<min\{g\varphi,\ M\}} \sim \frac{\sqrt{\alpha}\left(\frac{M}{m}\right)^{\frac{\alpha}{3}}}{\ln\frac{M}{m}+\frac{3}{2\alpha}}\left(\ln\frac{M}{k}\right)^{1-\frac{\alpha}{3}}\left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}. \tag{12.37}$$

Now we consider the case with no coupling between σ and φ and $m < M$. As we saw in Ch.9, one has all relevant integrals determined by the modes with k near the lower cut-off. So the energy density looks like

$$\left\langle \varrho_{\delta\sigma}\right\rangle\Big|_{a>\frac{M}{m}}^{g\varphi<m\leq M} \sim \frac{m^{1/2}M^{7/2}}{a^3}\left(\ln\frac{a_f}{a_i}\right)^{2-\frac{m^2}{M^2}}. \tag{12.38}$$

To determine the power spectrum we use (5.46). Compared to the end of inflation, one hardly finds any changes, due to the dependence on modes near a_iH_i:

$$\mathcal{P}_{\delta\sigma}(k)\Big|_{a>1}^{g\varphi<m\leq M} \sim \frac{\left(\ln\frac{H_k}{k}\right)^{\frac{1}{2}-\frac{m^2}{2M^2}}}{\left(\ln\frac{a_f}{a_i}\right)^{1-\frac{m^2}{2M^2}}}. \tag{12.39}$$

So how comes that there is a change from (11.23) to (12.37), but hardly any change from (9.54) to (12.39)? The power spectrum of the fields strongly coupled depends on the upper cut-off, *so the modes that left the horizon during the late stages are most important*. But for modes that are close to the horizon the approximation that everything is frozen is not very good, so microphysics still does influence the power spectrum during the radiation dominated period. If the fields are not coupled, the power spectrum depends on the lower cut-off, hence it depends on the modes that left the horizon in the very beginning and those modes are consequently far outside the horizon, truly frozen, thus the power spectrum remains unaltered.[6] On the other hand, (9.66) remains unaltered, too,

$$\begin{aligned}\langle \varrho_{\delta\sigma} \rangle \Big|_{a>1}^{g\varphi<\sqrt{2}M<m} &\sim mM^3 \\ \text{and } \mathcal{P}_{\delta\sigma}(k)\Big|_{a>1}^{g\varphi<\sqrt{2}M<m} &\sim \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}} \left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}} \left(\ln\frac{M}{k}\right)^{\frac{3}{2}-\frac{m^2}{M^2}}, \end{aligned} \tag{12.40}$$

although it depends on the upper cut-off: for the mass of the σ-field being larger than the inflaton's mass one has (12.7). m is too heavy for $\delta\sigma_k$ to change a lot, once it crossed the horizon.

[6]So also a dust dominated period between inflation and the radiation dominated period would not alter the result in this case.

Chapter 13

Applications

13.1 Constraints on the model

We shall now compare the CMBR-anisotropy (3.1) with the quasi de Sitter power spectra in the radiation dominated period. First we consider the case with no coupling and $m < M$. One gets from (12.39)

$$\frac{\left(\ln \frac{H_k}{k}\right)^{\frac{1}{2}-\frac{m^2}{2M^2}}}{\left(\ln \frac{a_f}{a_i}\right)^{1-\frac{m^2}{2M^2}}} \sim 10^{-5}. \tag{13.1}$$

With $\ln \frac{H_k}{k} \sim 100$ one finds

$$\left(\ln \frac{a_f}{a_i}\right)^{1-\frac{m^2}{2M^2}} \sim 10^{6-\frac{m^2}{M^2}}. \tag{13.2}$$

For our model to be consistent for $m \ll M$ as a necessary prerequisite one needs at least 10^6 e-folds of inflation, or $\varphi_i > 1000$. So not to be near the self reproduction scale (see section 4.4) one hence needs furthermore $M < 10^{-6}$. These constraints work nicely with the one we obtained already in Ch.9, see (9.52). For $m \sim M$ one needs even 10^{10} e-folds, or $\varphi_i \sim 10^5$, so that one has to have $M < 10^{-10}$.

Next we shall consider the case with the fields strongly coupled, (12.37). As k appears not just logarithmically in this equation (as in the previous case), but in a power-law, we need to choose explicitly a scale to work with. As explained in section 4.3, the galactic scale during inflation is given by

$$k \sim 10^{-25} M, \tag{13.3}$$

so that we get

$$\left[\frac{\sqrt{\alpha}\left(\frac{M}{m}\right)^{\frac{\alpha}{3}}}{\ln \frac{M}{m} + \frac{3}{2\alpha}}\, 50^{-\frac{\alpha}{3}}\right] 10^{-\frac{50\alpha}{3}} \sim 10^{-7}. \tag{13.4}$$

With $\alpha < 0.7$ but not too small (as it will turn out) the term in brackets is of the order of unity for a big variety of $\frac{M}{m}$, so that one gets[1]

$$\frac{50\alpha}{3} \sim 7 \quad \Rightarrow \quad \alpha \sim 0.4 \quad \Rightarrow \quad \frac{g}{M} \sim 0.3. \tag{13.5}$$

From this one can see immediately that the perturbations change by a factor of 10 if one goes from galactic scales to CMBR-scales, fluctuations are suppressed on large scales.

Now we compare (12.40) with (3.1). With the logarithm of $\mathcal{O}(100)$ (the final result does not depend strongly on the exact number) one finds numerically that

$$m \sim 2M. \tag{13.6}$$

Of course this result is only a crude approximation, for $m = 2M$ the k-range of SWA-at-crossing modes is still quite big, and our calculation leading to (9.66) contained the assumption $\frac{m^2}{M^2} \gg 2$. But nevertheless, as an order-of-magnitude estimate it should be sufficient, so that we have

$$m > M \quad \text{but} \quad m = \mathcal{O}(M). \tag{13.7}$$

So we arrive at a scenario where
- due to its mass being higher than the inflaton's mass, the second field automatically rolls down into the valley so that one can get easily $\langle\sigma\rangle \ll \delta\sigma$
- the resulting power spectrum is blue
- there is no other parameter the power spectrum depends on
- matching the power spectrum with observational constraints fixes the mass of the σ-field, *all* parameters are fixed relative to each other.[2]

13.2 Remarks to *Inflation and Large Internal Dimensions*

Recently N.Kaloper and A.Linde discussed the constraints on the inflaton mass if our world is embedded in a higher dimensional space. This idea is motivated by one of the the big flaws of the standard model of elementary particles: "Since the natural cut-off of any quantum field theory is the Planck mass ..., the renormalization effects should drive all scalar masses [and hence the unification scale μ] up to the Planck scale [$\sim \mathcal{O}(1)$]. As a result, all interacting particles should be very heavy.", [42]. But experimental data give an upper limit on the mass

[1] Note that if we had ignored the distortion of the power spectrum from the end of inflation to the point with $a > \frac{M}{m}$, i.e. if we had compared (11.23) with (3.1), we also would have found $\alpha \sim 10^{-14}$, a totally different result!

[2] The last point is due to the fact that the modes that crossed late are most important so that the number of e-folds does not appear in the final result (like in the previous case), and furthermore one does not have to consider the coupling constant as an extra parameter.

of the scalar Higgs field of $\sim 10^{-15}$. This is the so-called *hierarchy problem*.[3] But the Planck mass that one compares with experimental results refers to a value derived from $\hbar$, c and G. The latter is obtained from Newton's law, only valid in three dimensions, whereas the Planck mass providing the cut-off in QFT does not refer to any dimension. This suggests an alternative way out of the hierarchy problem: *we live in a world with more than three spatial dimensions, but only three are not compactified and hence visible; there exists a* fundamental *Planck mass which equals the unification scale μ, and an* effective *Planck mass (~ 1 in our units) which depends on the number of dimensions*. From [42] one finds the corresponding relation[4]

$$\mu \sim R^{\frac{3-\mathcal{D}}{\mathcal{D}-1}}, \tag{13.8}$$

with R being the radius of the compactified dimension (assuming that all compactified dimensions have a similar radius) and $\mathcal{D}$ the total number of spatial dimensions. If either $\mathcal{D} = 3$ or $R = l_p = 1$ one has $\mu = 1$. But if $\mathcal{D} > 3$ and $R \gg 1$ (this is why [42] speaks of *large* internal dimensions)[5], we get a unification scale being the natural cut-off for QFT that can be many orders of magnitude smaller than unity, thereby resolving the hierarchy problem, or "the hierarchy problem then becomes a problem of choosing the radius of stabilization of the internal space.", [42]. If one wants to work with the simple scenario that compactification and stabilization happened before inflation, so that inflation occurs only in three dimensions, one needs to fulfil the condition (see [42])

$$M \le H < \mu^2. \tag{13.9}$$

Assuming the CMBR-anisotropy to come from the fluctuations of the inflaton (see 4.36), one has $M \sim 10^{-6}$ so that one needs to have

$$10^{-3} < \mu. \tag{13.10}$$

So the idea of large internal dimensions would be ruled out for $\mu < 10^{-3}$. On the other hand, if one assumes a second scalar field to be the cause of the inhomogeneities (uncoupled to the inflaton and the mass being smaller or equal to the inflaton's mass) one gets that – as long as there was a high number of e-folds of inflation – with any scale of unification that is *smaller* than $\sqrt{10^{-6}}$ the model explained above could have been realized. For the scenario with two fields strongly coupled one can have *any* unification mass as long as $m \ll M$ and $\alpha \sim 0.4$. Similarly, if $m > M$ but nevertheless $m = \mathcal{O}(M)$ (no coupling) one can have any unification scale, too. So the scenario of large internal dimensions in combination with the conventional theory of the origin of the initial inhomogeneities poses stronger constraints on the unification scale than the results of the model proposed in this text, allowing a bigger mass range.

[3] One proposal to avoid this problem is by including SUSY: "There are powerful nonrenormalization theorems ... that show that higher order interactions do not renormalize the mass scale." [43].

[4] This relation comes from the integration over the compactified dimensions in the higher dimensional effective action.

[5] Nevertheless R has to be still small enough that we do not notice the extra dimensions in our every day life.

Chapter 14

Summary and conclusion

Classical cosmology (p.13-17) is faced with the horizon and the flatness problem (p.17-19). A solution to both of them is given if one has a period of accelerated expansion – called inflationary period – during the early stages of the Universe, washing away all inhomogeneities and expanding space so much that one can hardly detect any curvature (p.19-20). Chaotic inflation [the easiest and most natural model among the numerous inflationary models (p.30-32)] comes up almost inevitably as one goes backwards in time to high densities where matter has to be described in terms of fields: a scalar field (the φ-field, called the inflaton, with mass M) dominated by its potential term provides an effective cosmological constant, causing quasi de Sitter space; the field only slowly decays, ending inflation when it drops below unity. The inflaton is then supposed to decay into ultrarelativistic material to produce the entropy of the Universe, (p.21-30).

An extra bonus of the theory of inflation is a possible mechanism for the generation of the seeds of today's observable structure in the cosmic microwave background radiation from quantum fluctuations during the inflationary period: small scale inhomogeneities grow faster than the Hubble radius, they become stretched to cross the horizon where microphysics cannot affect their evolution anymore, the perturbations "freeze"; no other period but an inflationary one can provide this effect. After inflation the horizon grows faster than the perturbations, the frozen microphysics of the inflationary period re-enter (p.39-40). Since in the linear regime the modes in Fourier space evolve independently (p.34), it is best to decompose the operator related to the fluctuations in terms of modes and creation/annihilation operators (p.41-42). A measure for the relative energy density perturbation on a certain length scale is the power spectrum (p.34-36) (we always calculated the power spectrum for large scales). If the average of the field is much larger than the fluctuations itself, the power spectrum is easily related to the solutions of the equations of motion of the field and the metric's perturbation (p.42-44). Assuming the fluctuations of the inflaton to be the main source of today's inhomogeneity, one finds a power spectrum of Gaussian curvature perturbations that is logarithmically increasing for large scales. The temperature anisotropies of the cosmic microwave background radiation fix M to be of the order of 10^{-6} (p.44-46).

But since modern theories of elementary particles give rise to many scalar fields, this text treats fluctuations occuring if one assumes a second scalar field, the σ-field with mass m, to be the source of the inhomogeneities [the final results are given in Ch.12 (p.110-112) and Ch.13 (p.113-115)]: the σ-field is taken to be energetically insignificant during inflation so that the inflaton still drives inflation. Hence one deals with isocurvature perturbations[1], which makes solving the Klein-Gordon equation much easier than in the previous case, because one does not have to take into account the backreaction of the metric's perturbations. Assuming the σ-field not to decay after inflation, its energy density decays less rapidly than the energy density of the radiation produced by the inflaton; so after a while the σ-field becomes dominant, hence the inflaton's fluctuations could be of prime importance for the large scale structure of the Universe (p.49-52). Furthermore the average of the σ-field is supposed to vanish due to a high (effective) mass and/or high-temperature effects. But if $\delta\sigma \gg \sigma$ then the perturbations are no longer Gaussian (p.52-54), because density and field fluctuations are no longer proportional (so naturally one observational constraint that has to be fulfilled for the three scenarios investigated in this text is the nongaussianity of the temperature fluctuations); furthermore, unlike the case with fluctuations that are small compared to the average, the expression for the power spectrum now involves a tricky integration which has to be treated approximately (p.54-58).

Several transformations to solve the σ-field's Klein-Gordon equation were treated (p.59-65), and also some approximations (p.67-69). Although there are modes outside the horizon for which one may not neglect the spatial derivatives (see Fig.9.1 on p.81), it turns out that for all $\frac{m}{M}$, with the coupling not too large (p.80-82, 101-102) – though not being obvious – these modes play an important role neither for the calculation of the energy density of the σ-field nor its power spectrum .

After exercising with de Sitter space as a simplified toy-model (p.71-78), getting

$$\mathcal{P}_{\delta\sigma}(k)\Big|_f^{m\ll H} \sim \frac{m}{H}\left(\frac{k}{H}\right)^{\frac{2m^2}{3H^2}},$$

three cases were considered: the σ-field coupled to the inflaton, with the coupling term larger than m (p.101-104), and the two fields uncoupled [surprisingly one is able to find an exact analytic solution to the Klein-Gordon equation (p.64-65, 82)], with $m < M$ and $m > M$ (p.79-92):

– The power spectrum at the end of inflation with the two fields coupled is influenced strongly by modes that left the horizon near the end of inflation (p.104, 110-111). Thus it

[1]As was already emphasised in [32], "one should note that from the point of view of large scale structure formation, the difference between adiabatic and isothermal density perturbations with a flat spectrum is not very large. Such isothermal perturbations finally give rise to adiabatic perturbations with nearly flat spectrum ... Therefore the potential importance of the isothermal perturbations stems not from their isothermal origin, but from the possibility of obtaining practically any spectrum of perturbations one wants by a proper choice of the underlying theory of elementary particles."

changes considerably during the radiation dominated period, as the modes near the horizon are still influenced by microphysics (p.105-110). One gets

$$\mathcal{P}_{\delta\sigma}(k)\Big|_{a>\frac{M}{m}}^{m<min\{g\varphi,\ M\}} \sim \frac{\sqrt{\alpha}\,\left(\frac{M}{m}\right)^{\frac{\alpha}{3}}}{\ln\frac{M}{m}+\frac{3}{\alpha}}\Big(\ln\frac{M}{k}\Big)^{1-\frac{\alpha}{3}}\left(\frac{k}{M}\right)^{\frac{2\alpha}{3}}.$$

Comparing this to the temperature anisotropies of the cosmic microwave background radiation fixes the coupling constant relative to the inflaton while allowing a big range of $\frac{M}{m}$ (p.113-114). The perturbations have a spectral index slightly bigger than unity, so that cosmic microwave background anisotropy is suppressed, compared to the anisotropy on smaller scales.

– In the case of no coupling and $m < M$ a power spectrum with a logarithmic scale dependence increasing for larger scales was obtained (p.88-90). Since it is determined mainly by the modes that left the horizon early during inflation, it hardly changes during the radiation dominated period. One has

$$\mathcal{P}_{\delta\sigma}(k)\Big|_{a\geq 1}^{g\varphi<m\leq M} \sim \frac{\left(\ln\frac{H_k}{k}\right)^{\frac{1}{2}-\frac{m^2}{2M^2}}}{\left(\ln\frac{a_f}{a_i}\right)^{1-\frac{m^2}{2M^2}}}.$$

Comparison with the temperature anisotropy requires a huge number of e-folds of inflation for the model to be consistent, but on the other hand the possible value for $\frac{m}{M}$ is fairly arbitrary (p.113).

– For $m > H_i$ one finds a spectrum $\propto k^{3/2}$ (p.88); this suppresses the large scale inhomogeneity way beyond observational bounds and is thus ruled out.

– The result for $M < m < H_i$ is the most interesting one. For this case the k-range of interest is split into two subranges: for the upper range the modes oscillate from the moment of horizon crossing on, while for the lower range one first there is slow roll; by the time the Hubble parameter drops below m the modes start to oscillate, too (p.85-87; this behaviour is quite general and can be found for a radiation dominated background, too (p.106)). The resulting power spectrum is logarithmically scale dependent, *decreasing* for large scales. Evolution in the radiation dominated period does not significantly alter the power spectrum due to the largeness of m (p.112),

$$\mathcal{P}_{\delta\sigma}(k)\Big|_{a\geq 1}^{g\varphi<\sqrt{2}M<m} \sim \left(\frac{2}{3e}\right)^{\frac{m^2}{M^2}}\left(\frac{m^2}{M^2}\right)^{\frac{m^2}{M^2}}\left(\ln\frac{M}{k}\right)^{\frac{3}{2}-\frac{m^2}{M^2}}.$$

The temperature anisotropy then fixes m to be of the same order of magnitude as M (p.114); no further requirements (unlike in the previous case with the need for a large number of e-folds of inflation) arise. *We prefer the last scenario the most, with two scalar fields with masses of the same order of magnitude, the lighter field driving inflation (and decaying to radiation afterwards) and the heavier one having rolled quickly into the potential valley so that its average vanishes, causing a nongaussian, isothermal power spectrum decreasing on large scales.*

These results were applied to a recent paper by Kaloper and Linde that treats the implications of large internal dimensions (as a solution to the hierarchy problem) on the mass of the inflaton. While the model with the perturbations coming from the inflaton is compatible with the proposal of large internal dimensions only if the unification scale is above 10^{-3}, the three power spectra presented in this text allow for a big mass range for the unification scale (p.114-115). The expression for the energy density of gravitationally produced particles in quasi de Sitter space was applied to a recent paper by Peebles and Vilenkin, where only a rough approximation was used instead. With a few constraints one finds that the general idea of the model can still be maintained (p.93-100).

Hopefully the considerations and estimates given in this text spark off some more refined calculations based on a concrete particle model (maybe using numerical methods).

Appendix A

Tables of symbols, abbreviations etc.

Greek character	**meaning**
α	proportionality constant between p and ϱ; $\frac{3g^2}{M^2}$
$\alpha(x)$	coefficient of $f''(x)$
$\beta(x)$	coefficient of $f'(x)$
$\gamma(x)$	coefficient of $f(x)$
$\Gamma(x)$	gamma function
$\Gamma_\nu(x)$	incomplete gamma function
δ	density contrast
$\delta\ldots$	inhomogeneous component of a spatial quantity
$\delta(k)$	Dirac's distribution in k-space
δ_{ab}	Kronecker symbol
Δ	$\lvert\vec{l}-\vec{l}'\rvert$
$\Delta_{sr}\varphi$	change via slow roll during characteristic time
Δt	duration of inflation
ϵ	slow roll parameter
ε	$a_i H_i/H_k$
ζ	$\sqrt{\frac{9}{4}-\frac{m^2}{H^2}}$
η	conformal time
$\eta_{\mu\nu}$	component of the Minkowski metric
ϑ	polar angle
$\Theta(k)$	Heaviside step function in k-space

Greek character	meaning (cont.)
$\vec{\kappa}$	wavenumber in Minkowski space
λ	coupling constant of a general chaotic potential; wavelength
Λ	cosmological constant
μ	unification scale
ν	power law exponent; coefficient of mathematical functions
$\xi(\vec{r})$	correlation function of the energy density's fluctuations
π	3.141592
ϱ	energy density
σ	second scalar field
ς	containing all terms of $\lvert\delta\sigma_k\rvert^2 k^3$ that depend on k
$\Sigma(\Delta)$	$\int \frac{\varsigma(\Delta)}{\Delta^2} d\Delta$
τ	$\frac{2m^2}{3H^2}$
ϕ	azimuthal angle; arbitrary function of space and time in the perturbed metric
φ	scalar field driving inflation = inflaton;
φ_*	constant in the quintessential potential
Φ	gauge invariant potential
χ	$\sqrt{\frac{9}{4} - \frac{m^2+H_f^2}{H_k^2} - \frac{g^2}{H_f^2}}$
ψ	angular coordinate in $\varphi/\dot{\varphi}$-phase space; arbitrary function of space and time in the perturbed metric
Ψ	another gauge invariant potential
ω	angular frequency
Ω	density parameter

Latin character	meaning
a	scale factor
$\hat{a}$	annihilation operator
$A(a)$	auxiliary function to simplify a second order differential equation
b	variable to solve the equation of motion in a radiation dominated Universe
B	arbitrary function of space and time in the perturbed metric
c	speed of light in vacuum; arbitrary constant
$c(x)$	auxiliary function
$C,\ \mathcal{C}$	constants of integration

Latin character	**meaning (cont.)**
d	total derivative
$D_\nu(x)$	Whittaker's representation of the parabolic cylindrical function
$\mathcal{D}$	total number of spatial dimensions
∂	partial derivative
ds^2	line element
e	Euler's constant
E	energy;
	arbitrary function of space and time in the perturbed metric
$E(\eta)$	auxiliary function to simplify a second order differential equation
$\mathcal{E}$	energy of a gravitationally produced particle
$f(x)$	auxiliary function of x
g	absolute value of the metric's determinant;
	coupling constant between σ and φ
$g_{\mu\nu}$	component of the metric tensor
$\mathbf{g}$	metric tensor
G	Newton's constant
	gauge coupling constant
$\mathbf{G}$	Einstein tensor
h	$\sqrt{3}\varphi$
$\hbar$	Planck's constant
H	Hubble parameter;
	constant characterising de Sitter space
$\mathcal{H}(h)$	auxiliary function to simplify a second order differential equation
$\mathbf{H}_\nu(x)$	Hermite polynomial
i	$\sqrt{-1}$
I	$\varsigma(l')/l'^2 \ \cdot\ \Big[\Sigma(l+l') - \Sigma(\lvert l-l'\rvert)\Big]$
$J_\nu(x)$	Bessel function
$k,\ \vec{k}$	comoving wavenumber
k_B	Boltzmann's constant
$K(k)$	transfer function of the spectrum for the radiation dominated period
$\mathcal{K}$	sign of the the Universe's curvature
l	length
$l,\ \vec{l}$	k/H_k, $\vec{k}/H_k$
L	H_f/H_k
$\mathcal{L}$	Lagrangian
m	mass of the σ-field
M	mass of the φ-field

Latin character	**meaning (cont.)**
$\mathcal{M}$	mass on a certain length scale
n	spectral index; exponent of φ in a chaotic potential; particle density
N	particle number
$\mathcal{N}$	number of e-folds of inflation
p	pressure
p_b	component of the generalised momentum
P	probability
$\mathcal{P}$	power spectrum
q_a	component of the generalised coordinate
r	radial coordinate in a Robertson-Walker spacetime; radial coordinate in $\varphi/\dot{\varphi}$-phase space
$\vec{r}$	difference between two comoving coordinates
R	radius of compactified dimension
$\mathcal{R}$	length scale
$\Re(\ldots)$	real part of ...
s_1	known solution to second order differential equation
s_2	unknown solution to second order differential equation
S	containing all terms of $\lvert\delta\sigma_k\rvert^2 k^3$ that do not depend on k
t	(cosmic) time
$T_\mu^{\ \nu}$	component of the energy momentum tensor
$\mathbf{T}$	energy momentum tensor
$\mathcal{T}$	absolute temperature
u_k	$A\ \delta\sigma_k$
$U_\nu(x)$	parabolic cylindrical function
v_k	$E\ \delta\sigma_k$
V	potential term; volume
$V_\nu(x)$	parabolic cylindrical function
$\mathcal{V}$	v/Z
w_k	$\delta\sigma_k/\mathcal{H}$
W	Wronskian
x	variable
$\vec{x}$	comoving coordinate
$\vec{y}$	another comoving coordinate
z	$\ln\frac{aH}{k}$
z	redshift
$Z(z)$	auxiliary function to simplify a second order differential equation

subscript	meaning
b	where a certain mode re-enters the LWA regime
cc	causally connected
com	comoving
$crossing$	at the moment of horizon crossing
$dust$	dust
$\delta\sigma$	of the σ-field with $\delta\sigma \gg \langle\sigma\rangle$
eq	equal energy densities of radiation and dust
φ	of the inflaton with $\varphi \gg \delta\varphi$
gpp	gravitationally produced particles
H	Hubble
hom	homogeneous
f	at the end of inflation
i	at the beginning of inflation
$\mathcal{I}$	at the beginning of kination
k	Fourier mode of the comoving wavenumber k; at the moment of horizon crossing
$\vec{k}$	Fourier mode of the comoving wavenumber $\vec{k}$
$\vec{\kappa}$	Fourier mode of the Minkowski wavenumber $\vec{\kappa}$
kin	kinetic; kination
l	where a certain mode leaves the regime of LWA
L	lowest mode contributing to the energy density
Λ	cosmological constant
max	where $a\sqrt{\lvert m^2 + H_f^2 - 2H^2\rvert}$ has its local maximum
osc	oscillation
p	Planck
$phys$	physical
q	begin of the period in the quintessential model with $V > \varrho_{kin}$
r	radiation dominated period
rad	radiation
rec	moment of recombination
$rest$	rest
rms	root mean square
$root$	where $a\sqrt{\lvert m^2 + H_f^2 - 2H^2\rvert}$ has its minimum
th	thermalization

subscript	meaning (cont.)
eff	effective
SR	with SR at the moment of horizon crossing
$SR \to osc$	with SR at the moment of horizon crossing, later oscillating decay
σ	of the σ-field
$\langle\sigma\rangle$	of the σ-field with $\langle\sigma\rangle \gg \delta\sigma$
t	moment when slow roll solution goes over to exponential decay
$today$	today
tot	total
U	Universe
$zero$	where the potential is zero
0	today

superscript	meaning
d	decaying solution
sr	slow roll solution

abbreviation	meaning
CDM	cold dark matter
CMBR	cosmic microwave background radiation
GR	general relativity
GUT	grand unified theory
HDM	hot dark matter
LWA	long wavelengths approximation
QFT	quantum field theory
SR	slow roll approximation
SUSY	supersymmetry
SUGRA	supergravity
SWA	short wavelengths approximation

other notations	meaning
$\dot{}$	derivative with respect to time
$'$	derivative with respect to a;
	derivative with respect to x in section 7.2
$\star$	derivative with respect to conformal time
$\tilde{}$	derivative with respect to $\ln \frac{aH}{k}$
$\diamond$	derivative with respect to h
$\vert$	spatial covariant derivative

Bibliography

[1] S. Perlmutter et al., astro-ph/9812133

[2] Riess, A.G. et al., 1998, astro-ph/9805201

[3] M. Turner, *Cosmology solved? - Quite possible!*, astro-ph/9811364

[4] Alan H. Guth, *The inflationary Universe* (Addison Wesley, USA, 1997)

[5] Ch. Misner, K. Thorne, J. Wheeler, *Gravitation* (W.H. Freeman and Company, USA, 1997)

[6] V.A. Belinski, L.P. Grishchuk, Ya. B. Zel'dovich, I.M. Khalatnikov, Sov. Phys. JETP 62, 195 (1985)

[7] P.J.E. Peebles, A. Vilenkin, *Quintessential Inflation*, Phys. Rev. D59 (1999) 063505, astro-ph/9810509

[8] Robert Brandenberger, *Inflation and the theory of cosmological perturbations*, astro-ph/9711106

[9] L. Kofman, A. Linde, A. Starobinsky, hep-th/9405187, Phys. Rev. Lett. 73 (1994) 3195-3198

[10] A.H. Guth, Phys. Rev. D 23 (1982), 347

[11] A.D. Linde, Phys. Lett. 108B (1982), 389

[12] A. Albrecht, P.J. Steinhardt, Phys. Rev. Lett. 48 (1982), 1220

[13] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, *Theory of Cosmological Perturbations*, Phys. Rept. 215.203 (1992)

[14] Lucchin, F. and Matarese, S., Phys. Rev. D 32 (1985), 1316 (17)

[15] B. Spokoiny, Phys. Lett. B 315, 40 (1993)

[16] P.J.E. Peebles, B. Ratra, Astrophys. J. 352, L17 (1988)

[17] Andrei Linde, *Chaotic Inflation*, Physics Letters, vol. 129B (1983)

[18] Joseph Silk, *The Big Bang* (W.H. Freeman and Company, USA, 1997)

[19] P. Coles, F. Lucchin, *Cosmology*, (John Wiley & Sons Ltd, England, 1997)

[20] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Germany, 1990)

[21] P.J.E Peebles, *The Large-Scale Structure of the Universe* (Princeton University Press, USA, 1980)

[22] T. Padmanabhan, *Structure formation in the universe* (Cambridge University Press, Great Britain, 1995)

[23] Sachs, R. K. and Wolfe, A.M., 1967, *Perturbations of a cosmological model and angular variations of the microwave background*, Astrophys. J. 147,1,73

[24] J.Lidsey, A. Liddle, E. Kolb, E. Copeland, astro-ph/9508078, Rev. Mod. Phys. 69 (1997) 373-410

[25] Paul Davies, "Particles do not exist", *The Quantum Theory of Gravity* (edited by S.M. Christensen, Bristol A. Hilger, 1984)

[26] Steven Weinberg, *The Quantum Theory of Fields*, Vol. I (Press Syndicate of the University of Cambridge, USA, 1995)

[27] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, USA, 1993)

[28] V.F. Mukhanov, P.J. Steinhardt, *Density Perturbations in Multifield Inflationary Models*, astro-ph/9710038, Phys. Lett. B422 (1998) 52-60

[29] C. Kiefer, D. Polarski, A.A. Starobinsky, *Quantum-to-classical transition for fluctuations in the early Universe*, gr-qc/9802003, Int. J. Mod. Phys. D7 (1998) 455-462

[30] A. Linde, *The Eternally Existing, Self-Reproducing Inflationary Universe*, proceedings of the Nobel symposium on *Unification of Fundamental Interactions* (World Scientific, Singapore, 1987)

[31] A. Linde, Phys.Lett. 185B, 375 (1985)

[32] L.A. Kofman, A.D. Linde, Nuc.Phys. B282 (1987) 555-588

[33] L.Wang, V.F.Mukhanov, P.J.Steinhardt, Phys.Lett. B414 (1997) 18-27, astro-ph/9709032

[34] P.J.E. Peebles, astro-ph/9805194

[35] E. Kamke, *Differentialgleichungen - Lösungsmethoden und Lösungen I* (Akademische Verlagsgesellschaft Geest & Portig K.G., Leipzig, Germany, 1961)

[36] J. Spanier and K. Oldham, *An Atlas of Functions* (Hemisphere Publishing Corporation, Washington, 1987)

[37] L.H. Ford, Phys. Rev. D35, 2955 (1987)

[38] G. Felder, L. Kofman, A. Linde, hep-ph/9903350

[39] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover Publ., New York, 1965)

[40] E. Kolb, M. Turner, *The Early Universe* (Addison-Wesley Publishing Company, USA, 1990)

[41] A. Linde, V. Mukhanov, astro-ph/9610219, Phys. Rev. D56 (1997) 535-539

[42] N. Kaloper, A. Linde, hep-th/9811141

[43] M. Kaku, *Quantum Field Theory* (Oxford University Press, New York, USA, 1993)

Acknowledgements

I am deeply indebted to my supervisor in Munich, *Prof.Dr. Slava Mukhanov*, for looking after me throughout my diploma year. He helped and supported me a lot.

Many thanks also to *Prof.Dr. Hans-Peter Nilles* who made my outward diploma thesis possible by being my co-corrector in Bonn.

Furthermore I would like to thank *Prof.Dr. Wolfgang Priester* and *Dr. Carsten van de Bruck* for their efforts on mediating me from Bonn to Munich.

I had many helpful discussions with the "Chet Baker of k-Inflation", *Christian Armendáriz-Picón*, my office-mate in Munich. *Andreij Neronov*'s mathematical insight has given me some good hints, too.

Last, but not least, I would like to thank my parents for having such a good influence on me being the person I am today.