UNIVERSITÄT BONN
Physikalisches Institut

Polarization-free generators for the Belavin model

by
Tom-Dieter Albert

Employing a change of basis, the so-called factorizing Drinfel'd twist, we construct polarization-free and completely symmetric creation operators for a face type model equivalent to the Belavin model. A resolution of the nested structure of the Bethevectors is achieved.

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Contents

1 Introduction **1**

2 Quantum Inverse Scattering Method **4**

2.1 Integrability 4

2.2 Algebraic Bethe ansatz 6

2.3 Solution of the inverse problem and thermodynamical limit 14

3 Functional Bethe ansatz **16**

3.1 Spectral analysis 17

3.2 Seperation of variables 20

4 The Belavin model **21**

4.1 Definition and properties 21

4.2 The corresponding IRF model 22

5 The F-basis **23**

5.1 Motivation 23

5.2 Factorizing F-matrix 24

5.3 F-transformed creation operators 27

6 Bethe wavevectors **32**

7 Conclusion **37**

A Notation **38**

B Spin-$\frac{1}{2}$ Heisenberg model **39**

C Theta functions **43**

References **45**

Index **49**

1 Introduction

In the quest for understanding the world surrounding us, nature and its presumably basic constituents have been modeled over the entire history of mankind. These models range from Empedocles' theory of 4 elements to the intricate picture modern string theory projects: a world consisting of tiny string-like objects and its (mem)brane generalizations, whose vibrations manifest itself in the known elementary particles which in turn provide the building blocks of matter and forces. During this development the notion of symmetry has proven extremely helpful. On the one hand side it allows to unify seemingly different aspects of nature. This idea is probably most apparent in the concept of gauge theory, which allows to incorporate all known forces besides gravity into the standard model whose predictions have been checked with an accuracy unparalleled in the history of physics. On the other hand side it serves as a means of simplifying complex aspects of phenomena under consideration. Although our world with its huge variety of features certainly is not symmetric in a stringent sense, such considerations lead us to gain insight into certain aspects. Such simplified models are used to test ideas and methods underlying more realistic albeit more involved theories.

In this sense the area of integrable quantum systems in one space and one time dimension and their two dimensional counterparts in statistical physics, has led over the last seventy years to a remarkable success in both providing exactly solvable models, and in relating different areas of physics and mathematics. The first step on this way was taken by Bethe in 1931 with his solution of the isotropic Heisenberg model [1] by means of a method now called (coordinate) Bethe ansatz, which has been applied to many other models since. With the solution of the square Ising model by Onsager [2] in 1944 (providing the first example of a phase transition at non-vanishing temperature), the importance of the star-triangle relation as a condition on the Boltzmann weights in order to make the model exactly solvable was revealed. Subsequently this relation was used to obtain solutions to the anisotropic generalizations of the Heisenberg models by Yang and Yang for the XXZ case [3] and by Baxter for the XYZ case [4]. In the modern form this relation is now refered to as Yang-Baxter equation and encodes the dynamical symmetry of the system, thus providing an algebraic explanation of the integrability of some models. These ideas led to the reformulation of the Bethe ansatz in an algebraic version, the Quantum Inverse Scattering Method (QISM) by Faddeev and Takhtadzhan [5] and the discovery of underlying structures by Sklyanin [6]. This method stimulated the investigation of various areas in mathematical physics, such as quantum groups, braid groups, theory of knots etc. These efforts resulted in a plethora of models soluble by means of the QISM,

insofar that eigenvalues and eigenvectors of the Hamiltonian were found explicitely. Despite the indisputable achievements of the QISM and the rather simple action of the inverse problem operators, which can be interpreted as creation and annihilation operators for quasiparticles and as generating functions for the conserved quantities respectively, the study of correlation functions and formfactors has proven to be rather intricate. This is partly due to the fact that the solution of the inverse problem (expressing the original microscopic operators by means of the operators figuring in the algebraic Bethe ansatz) has only been achieved recently [7, 8, 9] for some particular models tractable by the algebraic Bethe ansatz. An important contribution has been the application of factorizing Drinfel'd twists by Maillet and Sanchez de Santos to inhomogeneous spin chain models [10] for which the algebraic Bethe ansatz is available. It was shown that the similarity transformation provided by the factorizing twists gives rise to a completely symmetric representation of the monodromy matrix. They used as paradigmata for their argumentation the rational XXX and the trigonometric XXZ model with underlying group $sl(2)$. A striking aspect of the results in [10] is related to the fact that no polarization clouds are attached to quasiparticle creation and annihilation operators in the basis in which the monodromy matrix is completely symmetric. This means in terms of a particle notation that no virtual particle–antiparticle pairs are present in the wave vectors generated by the action of the creation operators to the ground state (the reference state of the Bethe ansatz), or in spin chain terminology that the creation and annihilation operators are exclusively built from local spin raising and spin lowering operators respectively (that is, there are no compensating pairs of local raising and lowering spin operators). It was noted in [10] that this latter feature underscores the neat connection between the quantum spin chain models and their respective quasiclassical limits, which are Gaudin magnets [11], insofar as the appearance of the quasi particle operators of the quantum models in the particular basis differ from the corresponding operators in the quasiclassical limit models only by a "diagonal dressing" (see below). A generalized transformation has subsequently been used to resolve the nested hierarchy in the Bethevectors of the $sl(n)$ XXX model [12].
In this thesis we want to report the generalization of this concept to multicomponent variants of the eight-vertex model.

This thesis is organized as follows: In Chapter 2 we give a concise introduction into the Quantum Inverse Scattering Method concentrating on the ideas underlying the construction of eigenvectors and eigenvalues of the transfer matrix. We recount various aspects of the algebraic Bethe ansatz and of its generalization to models with higher rank groups or internal degrees of freedom, the nested Bethe ansatz. Furthermore we present the solution of the inverse problem and remark on the thermodynamical limit. Chapter 3 presents an alternative to the traditional Bethe ansatz, Sklyanin's funcional Bethe ansatz, which permits one to treat models in which the pseudovacuum for the Lax operator does not exist. This method bears some resemblence with the ideas of factorizing twists as devised by Maillet and Sanchez de Santos. The fourth Chapter contains a short survey of the the Belavin model as a possible n-state generalization of the eight vertex model and its corresponding reformulation as a face-type model is introduced. This is intended to set grounds for the next two chapters in which our main results are presented. Chapter 5 is concerned with the construction of a factorizing twist for the face model and the calculation of the creation operators in this new basis. In Chapter 6 we discuss the resolution of the nested Bethe hierarchy for the wave vectors. Chapter 7 contains our conclusions, while three appendices contain our notations and some technical details concerning the Heisenberg model and theta functions. For the convenience of the reader we included an index.

This thesis is based on the previously published papers:

Albert, T.-D., Boos, H., Flume, R., Ruhlig, K.: Resolution of the nested hierarchy for rational sl(n) models, *J. Phys.* **A33**, (2000), 4963-4980, nlin-SI/0002027.

Albert, T.-D., Boos, H., Flume, R., Poghossian, R.H., Ruhlig, K.: An F-twisted XYZ model, nlin-SI/0005023.

Albert, T.-D., Ruhlig, K.: Polarization free generators for the Belavin model, nlin-SI/0007020.

2 Quantum Inverse Scattering Method

2.1 Integrability

The Liouville concept of integrability in classical mechanics [13] states that a Hamiltionian system $H[p(x,t), q(x,t)]$ given by the nonlinear equations

$$\dot{p} = -\frac{\partial H}{\partial q} , \quad \dot{q} = \frac{\partial H}{\partial p} \tag{2.1}$$

is integrable if it possesses a number (half of the dimension of the phase space) of independent quantities, which are mutually in involution with respect to the Poisson bracket (called integrals of motion)

$$\{I_i, I_k\} = 0 .$$

Integrability implies the existence of action-angle variables, which means that there is a canonical transformation $(p(x,t), q(x,t)) \to (a(\lambda), b(\lambda,t))$, such that the new Hamiltonian depends only on the action variables, i.e. $H[a(\lambda)]$.
In this case the dynamical equations $\dot{a} = -\frac{\partial H}{\partial b} = 0, \quad \dot{b} = \frac{\partial H}{\partial a} = \omega$ can be solved trivially and furthermore $a(\lambda)$ is the generator of the conserved quantities ln $a(\lambda) = \sum_j I_j \lambda^j$, with $\dot{I}_j = 0$.
An important development in the theory of integrable systems is the inverse scattering method (ISM).
The important feature of this method is that, instead of attacking the nonlinear equation (2.1) directly, one constructs a corresponding linear problem

$$\frac{\partial \mathcal{T}(x,\lambda)}{\partial x} = L(q(x,t), p(x,t), \lambda)\ \mathcal{T}(x,\lambda),$$

where the Lax operator $L(q,p,\lambda)$ with the fields q, p contains all information about the original nonlinear system[1]. The asymptotic form of the linear problem can be interpreted as a scattering problem, with $\mathcal{T}(x = \pm\infty)$ playing the role of the Jost function. The parameter λ, known as the spectral parameter, acts like the rapidity of the scattering waves, while the original field q becomes the scattering potential.
The function $a(\lambda)$ being related to the transition and $b(\lambda,t)$ to the reflection coefficient of the scattering process are linked with the action and the angle variables, respectively [14].

[1]The Lax operator can be interpreted as a connection on a vector bundle with fiber $\mathbb{C}^2$ over the base $\mathbb{R}^2$ (2 dimensional spacetime) realizing parallel transport along constant-time contours.

The aim of the ISM is to find the canonical mapping between the original field and the action-angle variables and using it to construct the exact solutions for the original field. The Hamiltonian is identified from the set of conserved quantities $\{I_j\}$, generated by $\ln a(\lambda)$. The Lax operator $L(q,p,\lambda)$ therefore can be seen as the representative of an integrable system.

To obtain a quantum counterpart of the above definition, we will replace Poisson brackets with commutators, and say that a system is quantum integrable if there exist the same number of commuting operators as the number of degrees of freedom[2]

$$[I_i, I_k] = 0 \ . \tag{2.2}$$

We will give a concise description of the main ideas underlying the Quantum Inverse Scattering Method (QISM) which enables one to determine eigenfuctions and eigenvalues of the Hamiltionian explicitely [16, 17]. The basic idea of the QISM is purely algebraic and resembles the determination of the spectrum of the quantum harmonic oscillator or the hydrogen atom (Coulomb problem) by algebraic methods. In both cases the idea is to include the conserved quantities (forming a commuting algebra) into some bigger algebra $\mathcal{A}$.
In the case of the harmonic oscillater the number operator N forms together with the creation and annihilation operators a, $a^\dagger$ and the identity the Heisenberg Lie algebra, while for the hydrogen atom the components of the angular momentum and of the Runge-Lenz vector (together with the relation that the energy is conserved) form the Lie algebra $so(4)$. The QISM consists in using an algebra which describes the dynamical symmetry of quantum integrable systems.
The analogues of the action-angle variables become in the quantum case operators with commutation relation

$$A(\lambda)B(\mu) = f(\lambda-\mu)B(\mu)A(\lambda) \ . \tag{2.3}$$

The involution of the conserved quantities (2.2) is guaranteed by a family of commuting transfer matrices $[\mathcal{F}(\lambda), \mathcal{F}(\mu)] = 0$, since the transfer matrix $\mathcal{F}(\lambda)$ generates the conserved operators $\ln \mathcal{F}(\lambda) = \sum_j I_j \lambda^j$. The existence of commuting transfer matrices in turn is assured by a matrix relation known as the quantum Yang–Baxter equation (YBE)

$$R(\lambda,\mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda,\mu) \tag{2.4}$$

involving the quantum $R(\lambda,\mu)$-matrix and the monodromy matrix $T(\lambda) = \prod_{i=1}^{N} L_i(\lambda)$, where $L_i(\lambda)$ is the quantum Lax operator of the corresponding lattice model at point i and $\mathcal{F}(\lambda) =$

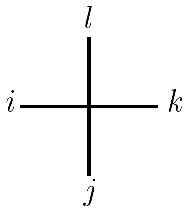
[2]We will disregard all problems arising from correctly defining functional independence of integrals of motion in the quantum case, for a rigorous discussion of this point see ref. [15].

$trT(\lambda)$.
The YBE (2.4) associated with the quantum integrable systems and ensuring the commutativity of action variables for different λ, defines a quasi-triangular Hopf algebra [18] and represents in fact a set of commutation relations like (2.3) between generalized action and angle variables, thus describing the dynamical symmetry of the system.

In this chapter we will proceed to give an elementary introduction to some features of the QISM, concentrating on the construction of eigenvectors and eigenvalues of the Hamiltonian, and we will not dwell on the underlying algebraic structures (cf. Appendix B for a concise discussion of some of those aspects for the Heisenberg model).

2.2 Algebraic Bethe ansatz

In the first step we will reformulate 2d statistical systems as 1d quantum systems.
Let us consider a quadratic, two-dimensional lattice with spin variables $j \in \mathbb{Z}_n$ placed on the bonds. Denote with $R_{ij}^{kl}(\lambda)$ the Boltzmann weight for a single vertex



$$\equiv R_{ij}^{lk}$$

Taking $\{e_i\}$ as the standard orthonormal basis of $V \cong \mathbb{C}^n$, these Boltzmann weights give a representation of the map $R : V_1 \otimes V_2 \to V_1 \otimes V_2$ on the basis $e_i \otimes e_j$

$$R(e_i \otimes e_j) = \sum_{k,l} (e_k \otimes e_l) R_{ij}^{kl} \ . \tag{2.5}$$

We call R the *R-matrix* of the model. It depends on two spectral parameters λ_1 and λ_2 associated to these two vector spaces. In what follows we denote it by $R_{12}(\lambda_1, \lambda_2)$.
Identifying one of the two linear spaces of this R-matrix with the Hilbert space $\mathcal{H}_i$ of the spin corresponding to site i of a linear chain, we construct the quantum *Lax operator*

$$L_i(\lambda, z_i) = R_{0i}(\lambda, z_i) \tag{2.6}$$

where z_i is an arbitrary inhomogeneity parameter depending on site i. The Lax operator acts in the tensorproduct $V_0 \otimes \mathcal{H}_i$, where the auxilliary space V_0 is isomorphic to $\mathbb{C}^n$.

The quantum *monodromy matrix* is obtained as an ordered product of Lax operators[3]

$$T(\lambda, \{z_1, \ldots, z_N\}) = L_N(\lambda, z_N)\, L_{N-1}(\lambda, z_{N-1}) \ldots L_1(\lambda, z_1)\ . \tag{2.7}$$

It can be represented in the auxilliary space as a $n \times n$ matrix whose elements are linear (nonlocal, i.e. acting on all sites $i = 1, \ldots, N$) operators acting in the quantum space of states of the chain[4] $\mathcal{H} = \otimes_{i=1}^{N} \mathcal{H}_i$

$$T(\lambda) = \begin{pmatrix} A_{11} & \ldots & A_{1\,n-1} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{n-1\,1} & \ldots & A_{n-1\,n-1} & B_{n-1} \\ C_1 & \ldots & C_{n-1} & D \end{pmatrix} . \tag{2.8}$$

By taking the trace in the auxilliary space we obtain the *transfer matrix*

$$\mathcal{F}(\lambda) = tr_0 T(\lambda) = \sum_{i=1}^{N-1} A_{ii}(\lambda) + D(\lambda) \tag{2.9}$$

which can be interpreted as a discrete evolution operator for one time step, being related to the Hamiltonian of the quantum chain by

$$H = i \frac{d\, \ln \mathcal{F}(\lambda)}{d\, \lambda}\Big|_{\lambda=0} . \tag{2.10}$$

The partition function of the original vertex model being given as the sum over all spin configurations $\mathcal{C}$ weighted by the energy of the respective configuration can thus be reformulated in terms of the transfer matrix

$$\mathcal{Z} = \sum_{\mathcal{C}} \exp\{E(\mathcal{C})\} = Tr_{\mathcal{H}}(\mathcal{F}^N)$$

where the trace $Tr_{\mathcal{H}}$ is taken in the whole Hilbert space of states [19]. This reduces the problem of calculating the partition function to the diagonalization of an operator in $\mathcal{H}$.

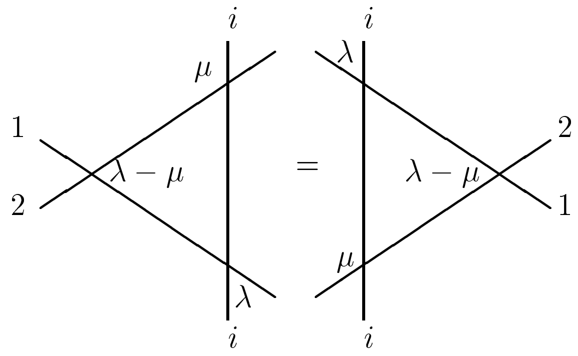
Our discussion up to now holds irrespectively of the integrability of the model. The integrability of a model is based on the fact that a local commutation relation for the Lax operators exists

$$R_{12}(\lambda - \mu)\, L_i(\lambda, z_i) \otimes L_i(\mu, z_i) = L_i(\mu, z_i) \otimes L_i(\lambda, z_i)\, R_{12}(\lambda - \mu) \tag{2.11}$$

where the R-matrix acts in the tensorproduct of two auxilliary spaces denoted 1 and 2.

[3]Sometimes we just omit the dependence on the local inhomogeneities $\{z_i\}$, and write $T(\lambda)$.
[4]If all spaces V_i and the auxilliary space V_0 are isomorphic, the model is denotes fundamental.

The relation can be visualized as follows:



The local relation (2.11) induces a global relation for the monodromy matrix[5]

$$R_{12}(\lambda-\mu)\, T(\lambda,\{z_i\}) \otimes T(\mu,\{z_i\}) = T(\mu,\{z_i\}) \otimes T(\lambda,\{z_i\})\, R_{12}(\lambda-\mu)\ . \tag{2.12}$$

Taking the trace in the auxilliary spaces and using the cyclic property of the trace we obtain for the transfer matrix (2.9) ($tr A \otimes B = tr A\, tr B$)

$$[\mathcal{F}(\lambda), \mathcal{F}(\mu)] = 0 \tag{2.13}$$

i.e. we have a one parameter family of commuting transfer matrices. It follows that the model is integrable in the above sense.
It can be shown that the conserved quantities [20, 21]

$$I_k = i^k \frac{d^k\, \ln \mathcal{F}(\lambda)}{d\,\lambda^k}\Big|_{\lambda=0} \tag{2.14}$$

couple $(k+1)$ nearest neighbors on the chain, with I_1 being the Hamiltonian (cf. (2.10)) with nearest neighbor interaction.
The consistency of (2.12) (i.e. the associativity of a product of three operators T) implies the sufficient relation to be fulfilled by the R-matrix

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)R_{13}(\lambda_1-\lambda_3)R_{12}(\lambda_1-\lambda_2) \tag{2.15}$$

on the tensor product of three auxilliary spaces. R_{ij} is acting nontrivially only on the space V_i and V_j.
This relation is commonly denoted as *Yang-Baxter equation* [6]. It can be considered as a hallmark of integrability, and has applications in a wide variety of fields, such as elastic scattering

[5]Note that the difference property of the R-matrix stipulates the distribution of the local inhomogeneities $\{z_i\}$ to be identical..
[6]The classical limit of this equation yields the Jacobi identity: $R = \mathbb{1} + \hbar r + \mathcal{O}(\hbar^2)$, the first order in $\hbar$ is trivially fulfilled, the second order is equivalent to the Jacobi identity for r.

in (1+1) dimensions, quantum groups, braid groups. For an overview cf. [22].
As can be seen from (2.12), for $\lambda = \mu$, the R-matrix reduces to the permutation operator Π acting on the auxilliary spaces $V_1 \otimes V_2$

$$R_{12}(0) = \Pi_{12} \ . \tag{2.16}$$

It is furthermore unitary:

$$R_{ij}(\lambda_i - \lambda_j) R_{ji}(\lambda_j - \lambda_i) = \mathbb{1} \ . \tag{2.17}$$

With the notation $T_{0,23} = R_{03} R_{02}, \quad R_{0i} \equiv R_{0i}(z_i)$, where the index 0 refers to an auxilliary space $\mathbb{C}^n_{(0)}$, one may rewrite Eq. (2.15) in the form

$$R_{23}^{\sigma_{23}} T_{0,23} = T_{0,32} R_{23}^{\sigma_{23}}$$

with σ_{23} the transposition of space labels $(2,3)$.
It is straightforward to generalize Eq. (2.18) to a N-fold tensor product of spaces:
With the definition $T_{0,1\ldots N} = R_{0N} \ldots R_{01}$ the generalization reads

$$R^{\sigma}_{1\ldots N} T_{0,1\ldots N} = T_{0,\sigma(1)\ldots\sigma(N)} R^{\sigma}_{1\ldots N} \tag{2.18}$$

where σ is now an element of the symmetric group $\mathcal{S}_N$ and $R^{\sigma}_{1\ldots N}$ denotes a product of R-matrices occuring in (2.18), the product corresponding to a decomposition of σ into elementary transpositions.
The order of the upper matrix indices α_i of the R^σ reads according to the above prescription as follows:

$$\left(R^{\sigma}_{1\ldots N}\right)^{\alpha_{\sigma(N)}\ldots\alpha_{\sigma(1)}}_{\beta_N\ldots\beta_1} \ .$$

Eq. (2.18) implies the composition law

$$R^{\sigma'\sigma}_{1\ldots N} = R^{\sigma}_{\sigma'(1)\ldots\sigma'(N)} R^{\sigma'}_{1\ldots N} \tag{2.19}$$

for a product of two elements in $\mathcal{S}_N$. The factor $R^{\sigma'}_{\sigma(1)\ldots\sigma(N)}$ on the r.h.s.of Eq. (2.19) satisfies for itself the relation

$$R^{\sigma'}_{\sigma(1)\ldots\sigma(N)} T_{0,\sigma(1)\ldots\sigma(N)} = T_{0,\sigma\sigma'(1)\ldots\sigma\sigma'(N)} R^{\sigma'}_{\sigma(1)\ldots\sigma(N)}. \tag{2.20}$$

The next step in the program of the QISM consists in constructing eigenfunctions and eigenvalues of the transfer matrix (2.9). For reasons of clarity we will demonstrate this method with

the example of the $\mathbb{Z}_2$ (or spin-$\frac{1}{2}$) six-vertex model (cf. Appendix B).
The R-matrix of this model is

$$R(\lambda-\mu)=\left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & b(\lambda-\mu) & c(\lambda-\mu) & 0 \\ 0 & c(\lambda-\mu) & b(\lambda-\mu) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{2.21}$$

with

$$\begin{aligned} b(\lambda-\mu) &= \frac{\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+\eta)} \\ c(\lambda-\mu) &= \frac{\sinh(\eta)}{\sinh(\lambda-\mu+\eta)} \end{aligned}$$

where η is a complex parameter.
The monodromy matrix is

$$T(\lambda)=\left(\begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array}\right) \tag{2.22}$$

with the transfer matrix $\mathcal{F}(\lambda)=tr_0\, T(\lambda)=A(\lambda)+D(\lambda)$.
The associated Hamiltonian is that of the XXZ model

$$H=\sum_{i=1}^{N}\left\{\sigma_i^x\,\sigma_{i+1}^x+\sigma_i^y\,\sigma_{i+1}^y+\triangle\sigma_i^z\,\sigma_{i+1}^z\right\}$$

with periodic boundary conditions $\sigma_{N+1}^i=\sigma_1^i$ (the σ^i are the familiar Pauli matrices).
The diagonalization of $\mathcal{F}(\lambda)$ for all values of λ starts with the existence of a pseudovacuum Ω[7] for which

$$\begin{aligned} A(\lambda)\Omega &= a(\lambda)\Omega, \quad D(\lambda)\Omega=d(\lambda)\Omega \\ B(\lambda)\Omega &= 0, \qquad\quad C(\lambda)\Omega\neq 0\,. \end{aligned} \tag{2.23}$$

In our case it is given by the tensorproduct of local lowest weight states $\Omega=\otimes_{i=1}^N v_i;\ v_i=\binom{0}{1}$ and the eigenvalues are $d(\lambda)=1,\ a(\lambda)=\prod_{i=1}^N b(\lambda,z_i)$.
Other eigenstates Ψ are obtained by choosing an ansatz (called the Bethe vector)

$$\Psi\equiv\Psi(\lambda_1,\ldots,\lambda_n)=\prod_{j=1}^{n}C(\lambda_j)\Omega\,. \tag{2.24}$$

Employing commutation relations between the operators A, D and C which are given by the relation (2.12) to commute the operators A, D through to the vacuum for which the eigenvalues

[7]This pseudovacuum coincides with the physical vacuum only for the ferromagnetic case $\triangle\geq 1$, which is why it is also denoted reference state.

of the transfer matrix are known, we can obtain eigenvalues for the Bethe vectors (2.24). Of the alltogether 16 relations of (2.12) we need in particular

$$\begin{aligned}\Big[C(\lambda),\, C(\mu)\Big] &= 0 &(2.25)\\ A(\lambda)C(\mu) &= \frac{1}{b(\mu,\lambda)}C(\mu)A(\lambda)+\frac{b(\lambda,\mu)}{c(\lambda,\mu)}C(\lambda)A(\mu)\\ D(\lambda)C(\mu) &= \frac{1}{b(\lambda,\mu)}C(\mu)D(\lambda)+\frac{b(\mu,\lambda)}{c(\mu,\lambda)}C(\lambda)D(\mu)\,. &(2.26)\end{aligned}$$

While the first term in the commutation relations (2.26) preserves the arguments in the operators, the second does not, and its contribution must vanish in order to obtain eigenvectors. This is achieved by imposing that the coefficient of the unwanted contribution vanishes. The requirement results in the so called Bethe ansatz equations (BAE)

$$\frac{a(\lambda_j)}{d(\lambda_j)}=\prod_{\substack{k=1\\k\neq j}}^{n}\frac{b(\lambda_k,\lambda_j)}{b(\lambda_j,\lambda_k)}\;;\quad j=1,\ldots,n \tag{2.27}$$

These equations state that the vector (2.24) is an eigenvector of the transfer matrix only if the spectral parameters $\{\lambda_j\}$ fulfill the equations (2.27) [1].
The eigenvalues $f(\mu,\{z_i\})$ of the transfer matrix with respect to the Bethe vector are then given by[8]

$$f(\mu,\{\lambda_i\})=a(\mu)\prod_{j=1}^{n}\frac{1}{b(\lambda_j,\mu)}+d(\mu)\prod_{j=1}^{n}\frac{1}{b(\mu,\lambda_j)} \tag{2.28}$$

which in turn yields the eigenvalues of all integrals of motion via (2.14). We have thus achieved our goal to construct eigenvalues and eigenfunctions of the transfer matrix.
The operators $C(\lambda)$ can be intepreted as creation operators for pseudoparticles of rapidity λ, which are called magnons (the real solutions of the BAE are refered to as elementary magnons while the complex solutions correspond to bound states of elementary magnons, so called strings). The operators $B(\lambda)$ correspondingly annihilate magnons of rapidity λ. Due to the commutativity of the creation operators it is clear that the Bethe vector is symmetric under the exchange of rapidities.

[8]The form of the eigenvalues allows an elegant derivation of the BAE: The general eigenvalue $f(\lambda)$ should be analytical in λ as it is to generate the eigenvalues of the conserved quantities through (2.14). Consequently the residue at singular points is to vanish $\mathrm{res}_{\lambda=\lambda_j} f(\lambda)=0$. This yields exactly the Eq.'s (2.27).

It is possible to apply the method of the ABA to a generalized Heisenberg model with spin variables $j \in \mathbb{Z}_n$ [23]. The R-matrix is given as an analogue of (2.21) in the form [24]

$$\begin{aligned} R(\lambda)^{ij}_{kl} &= b(\lambda)\delta_{ik}\delta_{jl}\, e^{\lambda \mathrm{sign}(i-j)} + c(\lambda)\delta_{il}\delta_{kj}\,; \quad k \neq l \\ R(\lambda)^{ii}_{kk} &= \delta_{ik}\,. \end{aligned} \tag{2.29}$$

This model has $n(2n-1)$ non-vanishing vertices, and for $n = 2$ we recover the six-vertex or XXZ-model (2.21). Proceeding as in the case of spin-$\frac{1}{2}$, we look for a pseudovacuum $\Omega_N^{(n)}$ which is annihilated by the elements C_i of the monodromy matrix (2.8)

$$\begin{aligned} B_i(\lambda)\Omega_N^{(n)} &= 0 \\ C_i(\lambda)\Omega_N^{(n)} &\neq 0 \\ A_{ij}(\lambda)\Omega_N^{(n)} &= \delta_{ij}a(\lambda)\Omega_N^{(n)} \\ D(\lambda)\Omega_N^{(n)} &= \Omega_N^{(n)}\,. \end{aligned}$$

The reference state (pseudovacuum) $\Omega_N^{(n)}$ is constituted as a N-fold tensor product of lowest weight states $v_n^{(i)} = {}^t(0,0,\ldots,0,1)$ in $\mathbb{C}^n_{(i)}$. The properties of the reference state suggest a block decomposition of the R-matrix in the following form

$$R(\lambda) = \begin{pmatrix} \tilde{R}^{ij}_{kl}(\lambda) & 0 & 0 & 0 \\ 0 & \delta_{ik}\, b(\lambda)\, e^{-\lambda} & \delta_{ik}\, c(\lambda) & 0 \\ 0 & \delta_{ik}\, c(\lambda) & \delta_{ik}\, b(\lambda)\, e^{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.30}$$

with $i, j, k, l = 1, \ldots, n-1$, and the matrix $\tilde{R}^{ij}_{kl}$ is the R-matrix of the $\mathbb{Z}_{n-1}$ spin problem acting on $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$.

From the global commutation relation (2.12) we obtain

$$\begin{aligned} A_{ij}(\lambda)C_k(\mu) &= C_l(\mu)A_{im}(\lambda)\frac{\tilde{R}^{ml}_{jk}(\lambda-\mu)}{c(\lambda-\mu)} - \frac{b(\lambda-\mu)\, e^{(\lambda-\mu)}}{c(\lambda-\mu)}C_j(\lambda)A_{ik}(\mu) \\ D(\lambda)C_i(\mu) &= C_i(\mu)D(\lambda)\frac{1}{c(\mu-\lambda)} - \frac{b(\mu-\lambda)\, e^{-(\mu-\lambda)}}{c(\mu-\lambda)}C_i(\lambda)D(\mu)\,. \end{aligned} \tag{2.31}$$

The creation operators C_i themselves fulfill the relation

$$C_i(\lambda)C_j(\mu) = C_k(\mu)C_l(\lambda)\tilde{R}(\lambda-\mu)^{kl}_{ij}\,. \tag{2.32}$$

The appearance of the "reduced" (in the sense that it refers to the $\mathbb{Z}_{n-1}$ spin problem) R-matrix $\tilde{R}$ in the commutator relations (2.31) and (2.32) constitutes the main difference as compared to the spin-$\frac{1}{2}$ XXZ model.

In order to obtain eigenstates to the transfer matrix (2.9) we will consider a vector which consists of linaer superpositions of products of C_i (with $card\, C_i$ fixed)

$$\Psi^{(n)} = \sum_{\alpha_1,\dots,\alpha_p} \Phi^{(n)}_{\alpha_1,\dots,\alpha_p} C_{\alpha_1}(\lambda_1)\dots C_{\alpha_p}(\lambda_p)\,\Omega^{(n)}_N \tag{2.33}$$

with c-number coefficients $\Phi^{(n)}_{\alpha_1,\dots,\alpha_p}$ still to be determined.
By inspection of (2.31) we notice that again only the first term on the r.h.s. can produce eigenvectors, while the contribution of the second which exchanges the arguments must vanish, giving rise to equations to be fulfilled by the parameters $\{\lambda_i\}$, the Bethe ansatz equations.
Commuting $\sum_{i=1}^{n-1} A_{ii}(\lambda)$ through the $C_j(\lambda_j)$, the first term of the sum in (2.31) however gives rise to an expression of the form ($A_{ik}\Omega = \delta_{ik} a\Omega$)

$$\sum_{i=1}^{n-1} A_{ii}(\lambda)\Psi^{(n)} = \frac{a(\lambda)}{\prod_{l=1}^{p} c(\lambda-\lambda_l)} \sum_{\beta_1,\dots,\beta_p} \prod_{l=1}^{p} C_{\beta_l}(\lambda_l)\,\mathcal{F}^{(n-1)}(\lambda;\{\lambda_p\})^{\beta_1,\dots,\beta_p}_{\alpha_1,\dots,\alpha_p}\,\Phi^{(n)}_{\alpha_1,\dots,\alpha_p}\,\Omega^{(n)}_N$$

with

$$\begin{aligned}\mathcal{F}^{(n-1)}(\lambda;\{\lambda_p\}) &= \sum_{i=1}^{n-1} T^{(n-1)}_{ii}(\lambda;\{\lambda_p\})\\ &= \sum_{i=1}^{n-1} \tilde{R}^{k_1\,\beta_1}_{i\,\alpha_1}(\lambda-\lambda_1)\tilde{R}^{k_2\,\beta_2}_{k_1\,\alpha_2}(\lambda-\lambda_1)\dots\tilde{R}^{i\quad\beta_p}_{k_{p-1}\,\alpha_p}(\lambda-\lambda_p)\end{aligned}$$

refering to a transfer matrix of an inhomogeneous $\mathbb{Z}_{n-1}$ spin chain of length p. This fact inspires us to choose the coefficient

$$\Phi^{(n)}_{\alpha_1,\dots,\alpha_p} = \left(\sum_{\beta_1,\dots,\beta_{p_1}} \Phi^{(n-1)}_{\beta_1,\dots,\beta_{p_1}} \tilde{C}_{\beta_1}(\mu_1,\{\lambda_p\})\dots\tilde{C}_{\beta_{p_1}}(\mu_{p_1},\{\lambda_p\})\,\Omega^{(n-1)}_p\right)_{\alpha_1,\dots,\alpha_p} \tag{2.34}$$

as a component of the "reduced" Bethe vector, where the operators $\tilde{C}_i(\mu,\{\lambda_p\})$ are creation operators from the monodromy matrix of the $\mathbb{Z}_{n-1}$ problem for an inhomogeneous chain of length p. We have thus reduced the diagonalization of the transfermatrix for a $\mathbb{Z}_n$ problem to that of a $\mathbb{Z}_{n-1}$ problem.
This *nesting procedure* can be repeated $n-2$ times until we arrive at a spin-$\frac{1}{2}$ problem, which can be solved by the algebraic Bethe ansatz descibed above.
Summarizing, Ψ_n in the form (2.33) is eigenvector of the transfer matrix $t(\lambda) = \sum_i T_{ii}(\lambda)$ if

i) the parameters $\lambda_1,\dots,\lambda_p$ satisfy a certain system of equations, the famous Bethe ansatz equations

and if

ii) the c-number coefficients are chosen s.t. they constitute the components of a inhomogeneous $\mathbb{Z}_{n-1}$ transfer matrix, with the inhomogeneities given by the spectral parameters of the $\mathbb{Z}_n$ Bethe vector.
The Bethe ansatz equation become very intricate as the spectral parameters of the reduced problem have to fulfill similar equations, down to the final spin-$\frac{1}{2}$ problem, thus yielding a set of coupled algebraic equations determining the set of numbers $\lambda_{j_l}; \quad j_l \leq p_l; \; i = 1, \ldots, n-1$. They can be derived again by imposing that the residue of the eigenvalues vanishes, and are found in [24].
Although it seems from (2.32) that the Bethe vector is not symmetric under the exchange of two spectral parameters, a closer inspection along with the explicit form of the coefficients $\Phi_{\alpha_1,\ldots,\alpha_p}$ (2.34) shows that the Bethe vector is nevertheless symmetric [25, 26]. The "reduced" R-matrix arising from (2.32) can be absorbed into the coefficient (2.34), thus yielding a symmetric function of the λ_i.

The above described method of the Bethe ansatz has been applied to a plethora of models in (1+1) dimensions, such as for instance the non-linear Schrödinger model, the non-linear sigma model, the Liouville model, the Gross-Neveu model or the Hubbard model. The procedure relies on the existence of a pseudovacuum and allows to construct eigenvectors of the transfer matrix in the form of Bethe vectors of the type (2.24) due to commutation relations of the form (2.12).

2.3 Solution of the inverse problem and thermodynamical limit

Another important issue is the calculation of correlation functions for the microscopic local spin variables σ^i with respect to the Bethe vectors of the form (2.24). For this purpose we need an expression of these operators in terms of operators of the monodromy matrix, i.e. we have to solve the inverse problem [7].
Note that from (2.22) it follows for the R-matrix (2.21) that

$$R_{ij}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Pi_{ij} \tag{2.35}$$

where $\Pi_{ij}$is the permutation operator interchanging the i-th and the j-th space. Furthermore the R-matrix has an inverse due to the unitatity relation, $R_{12}(\lambda_1, \lambda_2) R_{21}(\lambda_2, \lambda_1) = \mathbb{1}$.

Using the definition of the monodromy matrix (2.7) and the identity $\Pi_{jk}\Pi_{jl} = \Pi_{kl}\Pi_{kj}$ for permutation operators we get

$$T(0) = \Pi_{0N}\Pi_{0\,N-1}\dots\Pi_{01} = \Pi_{01}\Pi_{1N}\Pi_{1\,N-1}\dots\Pi_{12} = \Pi_{01}U$$

with $U = \Pi_{1N}\Pi_{1\,N-1}\dots\Pi_{12} = \Pi_{12}\Pi_{23}\dots\Pi_{N-1\,N}$ being the shift operator in the Hilbert space (U acts on a local operator x_n as $U\,x_n\,U^{-1} = x_{n+1}$).
Writing this equation in matrix form

$$\begin{pmatrix} A(0) & B(0) \\ C(0) & D(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\mathbb{1}_1 + \sigma_1^z)U & \sigma_1^- U \\ \sigma_1^+ U & \frac{1}{2}(\mathbb{1}_1 - \sigma_1^z)D(0) \end{pmatrix} .$$

Comparing the matrix elements we obtain for the spin variables acting on the first site of the chain

$$\begin{aligned} \sigma_1^+ &= C(0)U^{-1} \\ \sigma_1^- &= B(0)U^{-1} \\ \sigma_1^z &= \left(A(0) - D(0)\right)U^{-1} \end{aligned}$$

and for the shift operator U

$$U = A(0) + D(0) = \mathcal{F}(0) .$$

In order to obtain the expression for a spin variable a site n we can use the shift operator $U^{n-1}\sigma_1^j U^{1-n} = \sigma_n^j$ and the fact that $U^N = \mathbb{1}$. We finally obtain

$$\sigma_n^+ = U^{n-1}C(0)U^{N-n} \tag{2.36}$$

$$\sigma_n^- = U^{n-1}B(0)U^{N-n} \tag{2.37}$$

$$\sigma_n^z = U^{n-1}\left(A(0) - D(0)\right)U^{N-n} . \tag{2.38}$$

As the spin operators are expressed entirely in terms of the operators of the monodromy matrix, the inverse problem is solved [8, 9].

The last issue we want to address in our review of the QISM is that of the thermodynamical limit. We introduce suitably normalized operators in order to deal with finite norms of states and finite eigenvalues in the thermodynamical (TD) limit [27][9]:

$$\begin{aligned} \tilde{A}(\lambda) = a^{-1}(\lambda)A(\lambda) \quad &, \quad \tilde{D}(\lambda) = d^{-1}(\lambda)D(\lambda) \\ \tilde{B}(\lambda) = a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)B(\lambda) \quad &, \quad \tilde{C}(\lambda) = a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)C(\lambda) . \end{aligned} \tag{2.39}$$

[9] Our prescription differs from the one given in [27] in an inessential way.

Thus the operators $\tilde{A}(\lambda)$ and $\tilde{D}(\lambda)$ have correspondingly the eigenvalue 1. In the sense of generalized functions we obtain in the TD limit the simplified relations

$$\begin{aligned} \tilde{A}(\lambda)\tilde{B}(\mu) &= \frac{1}{b(\lambda,\mu)}\tilde{B}(\mu)\tilde{A}(\lambda) \\ \tilde{C}(\lambda)\tilde{D}(\mu) &= \frac{1}{b(\lambda,\mu)}\tilde{D}(\mu)\tilde{C}(\lambda) \, . \end{aligned} \tag{2.40}$$

One notes, comparing with the commutation relations arising from (2.12), that the exchange terms have dropped out. The BAE can therefore be disregarded in the TD limit (as long as one restricts the attention to the sector of elementary magnons). Normalized asymptotic scattering states are generated by acting with creation operators $Z^\dagger(\lambda) = \tilde{C}(\lambda)\tilde{D}^{-1}(\lambda)$ on the vacuum (the lowest weight state) and are annihilated by operators $Z(\lambda) = -\tilde{A}^{-1}(\lambda)\tilde{B}(\lambda)$ [28]. The action of the operators $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$ are easily deduced from the relations (2.40) and from the fact that the vacuum is an eigenstate with unit eigenvalue of $\tilde{A}(\lambda)$ and $\tilde{D}(\lambda)$ and therefore also of $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$.
An incoming scattering state is given by

$$Z^\dagger(\lambda_1)\ldots Z^\dagger(\lambda_n)|0\rangle \tag{2.41}$$

if the rapidities are ordered in such a way that $\lambda_1 > \ldots > \lambda_n$, and represents an outgoing state for $\lambda_1 < \ldots < \lambda_n$.
To relate the incoming to the outgoing states use has to be made of the relation

$$Z^\dagger(\lambda)\, Z^\dagger(\mu) = S(\lambda,\mu) Z^\dagger(\mu)\, Z^\dagger(\lambda) \tag{2.42}$$

with $S(\lambda,\mu) = \frac{b(\lambda,\mu)}{b(\mu,\lambda)}$ the two-body S-matrix. It is easily seen that the n-magnon S-matrix is given as a product of 2-magnon S-matrices.
The wavefunctions (2.41) are normalized to delta functions with a unit prefactor.

3 Functional Bethe ansatz

The Functional Bethe Ansatz (FBA) method of Sklyanin [29, 30] represents an alternative way to obtain eigenvalues and eigenvectors of the transfer matrix without being restricted to lowest weight representations (i.e. the existence of a pseudovacuum, which is absent in models like the Toda model or the XXX-model with non-compact symmetry groups [31] describing

QCD in the high energy limit [32]) and also circumventing the problem of the completeness of Bethe states, that is the question whether the Bethe states (2.24) cover all eigenfunctions of the transfer matrix [33] .
We will content ourselves with describing the method of FBA in the case of the rational $sl(2)$ XXX model (cf. Appendix B) in the finite-dimensional representation V_{2l_i} ($\dim V_{2l_i} = 2l_i + 1$) of $sl(2)$.

3.1 Spectral analysis

Assume that the monodromy matrix $T(\lambda)$ is a polynomial in the spectral parameter λ of degree N:

$$T(\lambda) = T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \sum_{i=0}^{N} \lambda^i \begin{pmatrix} \mathcal{A}_i(\lambda) & \mathcal{B}_i(\lambda) \\ \mathcal{C}_i(\lambda) & \mathcal{D}_i(\lambda) \; . \end{pmatrix} \tag{3.43}$$

There exists a central element in the algebra defined by (2.12), denoted as quantum determinant [34]

$$\Delta(\lambda) \equiv det_q T(\lambda) = A(\lambda - \frac{\eta}{2}) D(\lambda + \frac{\eta}{2}) - B(\lambda - \frac{\eta}{2}) C(\lambda - \frac{\eta}{2}) \tag{3.44}$$

which commutes with all elements of the monodromy matrix

$$[det_q T(\lambda) \, , \, T(\mu)] = 0 \; .$$

The idea of the Functional Bethe Ansatz is to define the operator valued roots $\hat{x}_n$, $1 \leq n \leq N$ of the polynomial $B(\lambda)$ of degree N:

$$B(\lambda) = \mathcal{B}_N \prod_{n=1}^{N} (\lambda - \hat{x}_n), \tag{3.45}$$

by constructing a map between Fun$\mathbb{X}$ and the space of states $\mathcal{H}$ of the chain. Here Fun$\mathbb{X}$ is the space of functions on a finite set $\mathbb{X} \subset \mathbb{C}^N$.
$\mathbb{X}$ is the common spectrum of $\{\hat{x}_1, \ldots, \hat{x}_N\}$, where the $\hat{x}_n$ are realized on Fun$\mathbb{X}$ as the operators of multiplication by the corresponding coordinates (the n-th coordinate) in $\mathbb{C}^N$:

$$[\hat{x}_n f](\mathbf{x}) = x_n \, f(\mathbf{x}), \quad \forall f \in \text{Fun}\mathbb{X}, \quad \forall \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{X} \; . \tag{3.46}$$

The idea of operator roots needs some clarification (for a rigorous discussion cf. [30]):
Assume that the coefficient of λ^N of $T(\lambda)$ is a number matrix and that the quantum determinant

is a c-numbered function along with the condition that both $B(\lambda)$ and $\Delta(\lambda)$ are polynomials of maximal degree. Then define symmetric polynomials $\hat{b}_n$ of $B(\lambda)$ by $\hat{b}_n \equiv (-1)^n \mathcal{B}_{N-n}/\mathcal{B}_N$, and express $B(\lambda)$ as

$$B(\lambda) = \mathcal{B}_N \left(\lambda^N - \hat{b}_1 \lambda^{N-1} + \hat{b}_2 \lambda^{N-2} - \dots\right) .$$

Due to the fact that the $B(\lambda)$ commute among themselves, the $\left\{\hat{b}_i\right\}$ also commute. Denote by $\mathbb{B}$ their common spectrum. This is assumed to be simple (this is the most restrictive condition, which nevertheless holds for most models), that is the commuting $\left\{\hat{b}_i\right\}$ have a common set of eigenfunctions and to every point $\mathbf{b} \in \mathbb{B}$ there exists only one eigenfunction. The $\left\{\hat{b}_i\right\}$ are symmetric polynomials of the roots of $B(\lambda)$, which still have to be determined. For this purpose consider the mapping

$$\Theta: \quad \mathbf{x} \to \mathbf{b}$$

given by

$$\hat{b}_n(\mathbf{x}) = \sigma_n(\mathbf{x})$$

where σ_n is the elementary symmetric polynomial of degree n of N variables. Then $\mathbb{X}$ as the common spectrum of $\{\hat{x}_i\}$ is identified with the pre-image $\Theta^{-1}(\mathbb{B})$, which is completely symmetric under permutations of the coordinates $\{\hat{x}_i\}$.

Operators $\hat{X}_n^+$, $\hat{X}_n^-$ are then defined from polynomials $A(\lambda)$ and $D(\lambda)$ by substitution "from the left" (i.e. with operator ordering) of λ by $\hat{x}_n$:

$$\hat{X}_n^- = \sum_{p=0}^{N} \hat{x}_n^p \mathcal{A}_p \equiv [A(\lambda)]_{\lambda=\hat{x}_n} , \qquad \hat{X}_n^+ = \sum_{p=0}^{N} \hat{x}_n^p \mathcal{D}_p \equiv [D(\lambda)]_{\lambda=\hat{x}_n} . \tag{3.47}$$

The commutation relations for $\hat{x}_n$, $\hat{X}_m^\pm$ follow from those of A, B, C, and D (2.12):

$$[\hat{x}_m, \hat{x}_n] = [\hat{X}_m^\pm, \hat{X}_n^\pm] = 0, \quad \forall m,n, \qquad [\hat{X}_m^+, \hat{X}_n^-] = 0, \quad \forall m,n,\ m \neq n, \tag{3.48}$$

$$\hat{X}_m^\pm \hat{x}_n = (\hat{x}_n \pm \eta \delta_{mn}) \hat{X}_m^\pm, \quad \forall m,n, \qquad \hat{X}_n^\pm \hat{X}_n^\mp = \Delta(\hat{x}_n \pm \frac{\eta}{2}), \quad \forall n . \tag{3.49}$$

The operators $A(\lambda)$ and $D(\lambda)$ can be reconstructed in terms of the $\hat{x}_n$, $\hat{X}_m^\pm$ by means of polynomial interpolation:

$$\begin{aligned}
B(\lambda) &= \mathcal{B}_N \prod_{n=1}^{N} (\lambda - \hat{x}_n), \\
A(\lambda) &= \sum_{n=1}^{N} \Big\{ \prod_{i \neq n} \frac{\lambda - \hat{x}_i}{\hat{x}_n - \hat{x}_i} \Big\} \hat{X}_n^-, \qquad D(\lambda) = \sum_{n=1}^{N} \Big\{ \prod_{i \neq n} \frac{\lambda - \hat{x}_i}{\hat{x}_n - \hat{x}_i} \Big\} \hat{X}_n^+, \\
C(\lambda) &= B^{-1}(\lambda + \eta)[A(\lambda + \eta) D(\lambda) - \Delta(\lambda + \frac{\eta}{2})] .
\end{aligned} \tag{3.50}$$

which, with (3.45) and the fact that C can be obtained in terms of B, A, D and the quantum determinant Δ, provides expressions of the matrix elements of the monodromy matrix only in terms of operators $\hat{x}_n$ and $\hat{X}_n^{\pm}$.

Therefore, if we compute explicitly the action of $\hat{X}_n^{\pm}$ in a basis which diagonalizes simultaneously all $\hat{x}_n$, we will obtain explicit expressions for A, B, C, and D in a basis in which B is diagonal.
This can be done by considering the actions of $\hat{x}_n$, $\hat{X}_m^{\pm}$ in an explicit basis of Fun$\mathbb{X}$ (which is isomorphic to $\mathcal{H}$). The point is thus to determine the action of $\hat{X}_n^{\pm}$ on Fun$\mathbb{X}$.
Following Sklyanin, let us define the function $\Delta_n^{\pm} = \hat{X}_n^{\pm}\omega$, where ω is the constant function $\omega \equiv 1$ of Fun$\mathbb{X}$. The action of $\hat{x}_n^{\pm}$ on an arbitrary function $f \in$ Fun$\mathbb{X}$ is then given in terms of $\Delta_n^{\pm}$:

$$[\hat{X}_n^{\pm} f](\mathbf{x}) = \Delta_n^{\pm}(\mathbf{x})\, f(E_n^{\pm}\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{X}, \tag{3.51}$$

where $E_n^{\pm}$ are the shift operators in $\mathbb{C}^N$:

$$E_n^{\pm} : (x_1, \ldots, x_n, \ldots, x_N) \longrightarrow (x_1, \ldots, x_n \pm \eta, \ldots, x_N).$$

It is thus sufficient for our purpose to compute $\Delta_n^{\pm}$. Commutation relations for $\hat{x}_n$, $\hat{X}_m^{\pm}$ impose conditions on $\Delta_n^{\pm}$, which determine $\Delta_n^{\pm}$ up to multiplication by an arbitrary function (here set equal to one which corresponds to fixing the isomorphism between $\mathcal{H}$ and Fun$\mathbb{X}$): In the basis

$$\{f_{1\,k_1} \times f_{2\,k_2} \times \ldots \times f_{N\,k_N} \,|\, f_{i\,k_i} \in \mathrm{Fun}\Lambda_i,\ k_i = l_i, l_i - 1, \ldots, -l_i\} \tag{3.52}$$

of Fun$\mathbb{X}$ defined by

$$f_{n\,k_n}(z_n + \eta p_n) = \xi_+^{-l_n + k_n}\, \delta_{k_n, p_n} \quad \forall p_n \in \{-l_n, \ldots, l_n\}, \tag{3.53}$$

we have

$$\Delta_n^{\pm}(\mathbf{x}) = \xi_{\pm}\Delta^{\pm}(x_n) \quad \text{with} \quad \Delta^{\pm}(x_n) = \prod_{i=1}^{N}(x_n - z_i \pm \eta l_i), \tag{3.54}$$

with ξ_+, $\xi_- \in \mathbb{C}$ are such that $\xi_+\xi_- = 1$. Thus $\hat{x}_n$, $\hat{X}_n^{\pm}$ are given by

$$\hat{x}_n = z_n + \eta S_n^z, \qquad \hat{X}_n^{\pm} = \eta \prod_{\substack{i=1 \\ i \neq n}}^{N}(z_n - z_i + \eta S_n^z \pm \eta l_i)\, S_n^{\pm}, \tag{3.55}$$

where S_n^z, $S_n^{\pm}$ are matrix representations (dimension $2l_n + 1$) of the spin operators.

In the case of the XXX-model, the spectrum $\mathbb{Y}$ can be determined directly from the diagonal elements of the lower triangular matrix $B(\lambda)$

$$\mathrm{Diag}B(\lambda) = \prod_{i=1}^{N}(\lambda - z_i - \eta S_i^z),$$

so that the associated spectrum is

$$\mathbb{X} = \Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_N, \quad \Lambda_i = \{z_i + \eta k \,|\, k = -l_i, -l_i + 1, \ldots, l_i\}. \tag{3.56}$$

where $\Delta(\lambda) = \prod_{n=1}^{N}(\lambda - z_n - l_n\eta - \frac{\eta}{2})(\lambda - z_n + l_n\eta + \frac{\eta}{2})$ is the quantum determinant of the monodromy matrix $T_{XXX}(\lambda)$.

3.2 Seperation of variables

We are now in the position to reach the final goal of the FBA, to determine the spectrum of the transfer matrix (2.9).
Let $\Psi \in \mathrm{Fun}(\mathbb{X})$ be a common eigenfunction of the transfer matrix $\mathcal{F}(\lambda)$ with eigenvalue $f(\lambda)$

$$(D(\lambda) + A(\lambda))\,\Psi = f(\lambda)\Psi\ . \tag{3.57}$$

Substitute into (3.57) consecutively $\lambda = \hat{x}_n;\quad n = 1, \ldots, N$ and use the relations (3.47, 3.51) and (3.54) to obtain

$$\Delta_-(x_n)f(E_n^-\mathbf{x}) + \Delta_+(x_n)f(E_n^+\mathbf{x}) = f(x_n)\Psi(\mathbf{x})\ . \tag{3.58}$$

This set of equations allow the seperation of variables. Let

$$\Psi(\mathbf{x}) = \prod_{n=1}^{N}\phi_n(x_n)\ . \tag{3.59}$$

Then the functions $\phi_n \in \mathrm{Fun}(\Lambda_n)$ of one variable x_n satisfy the multiparameter spectral problem

$$\Delta_-(x_n)\phi_n(x_n - \eta) + \Delta_+(x_n)\phi_n(x_n + \eta) = f(x_n)\phi_n(x_n)\ . \tag{3.60}$$

Due to the one-to-one correspondence between the eigenfunctions Ψ of the original multidimensional spectral problem and those of the related one-dimensional ones there is no completeness problem.

This method was applied for instance to the Toda chain and to various Gaudin models. Its generalization to models with nested structure remains yet to be clarified.
The expressions (3.50) (when all spins $l_i = 1/2$) coincide with the expressions for the operators B, C, D, A in the F-basis obtained by J.M. Maillet and J. Sanchez de Santos in [10] (there the operator D was diagonalized). The new basis obtained by Functional Bethe Ansatz technique appears to be a generalization of the F-basis [35].

4 The Belavin model

4.1 Definition and properties

The Belavin model [36] represents a possible n-state generalization of the eight-vertex model [4]. The Boltzmann weights S^{ij}_{kl} for a single vertex with state values of $\mathbb{Z}_n$-spin fulfill the conditions (so called $\mathbb{Z}_n \otimes \mathbb{Z}_n$ invariance)

$$\begin{aligned} S^{ij}_{kl} &= 0, \ \text{ unless } i+j = k+l, \text{ mod } n \\ S^{i+p,j+p}_{k+p,l+p} &= S^{ij}_{kl} \ \text{ for every } i,j,k,l \text{ and } p \in \mathbb{Z}_n\,. \end{aligned} \tag{4.1}$$

They solve the Yang–Baxter equation [37], [38]

$$S(z_1-z_2)^{i_1,i_2}_{k_1,k_2}\, S(z_1-z_3)^{k_1,i_3}_{j_1,k_3}\, S(z_2-z_3)^{k_2,k_3}_{j_2,j_3} = S(z_2-z_3)^{i_2,i_3}_{k_2,k_3}\, S(z_1-z_3)^{i_1,k_3}_{k_1,j_3}\, S(z_1-z_2)^{k_1,k_2}_{j_1,j_2} \tag{4.2}$$

Here and subsequently double indices k mean summation over $k = 0, \ldots, n-1$.
The elements can be parametrized in the following way [39][10]

$$S(z)^{k,l}_{i,j} = \frac{h(w)h(z)}{h(z+w)} \frac{\theta^{(i-j)}(z+w)}{\theta^{(i-k)}(w)\theta^{(k-j)}(z)} \delta^{k+l}_{i+j} \tag{4.3}$$

with

$$h(z) = \prod_{i=0}^{n-1} \theta^{(i)}(z) \left(\prod_{i=1}^{n-1} \theta^{(i)}(0) \right)^{-1} \quad ; \quad h(0) = 0 \tag{4.4}$$

where $\theta^{(i)}(z) = \theta \left[\begin{array}{c} \frac{1}{2} - \frac{i}{n} \\ \frac{1}{2} \end{array} \right](z)$ represents the theta function of rational characteristics $1/2 - i/n, 1/2$ (cf. appendix A).

[10] The slight difference compared to [39] originates from a different normalization of the S-matrix.

The matrix (4.3) is unitary, $S(z)S(-z) = \mathbb{1}$ and obeys the initial condition $S(0) = \Pi$ [40], where Π is the permutation operator.

4.2 The corresponding IRF model

The non-conservation of spin in the Belavin model is responsible for the absence of a local vacuum for the Lax operator associated to the R-matrx (4.3).
To avoid the problems arising from this fact we exploit a vertex-face map which transforms the Belavin model into a face type model [41], which in turn can be thought of as a multicomponent generalization of the six-vertex model.
The correspondence is given by

$$S(z_1 - z_2) M_a^\mu(z_1) \otimes M_{a+\hat{\mu}}^\nu(z_2) = M_{a+\hat{\nu}'}^{\mu'}(z_1) \otimes M_a^{\nu'}(z_2) R(z_1 - z_2|a)_{\mu',\nu'}^{\mu,\nu} \tag{4.5}$$

where $R(z|a)$ has the form [42]

$$\begin{aligned} R(z|a)_{\mu',\nu'}^{\mu,\nu} &= b^{\mu,\nu}(z|a)\delta_{\mu'\mu}\delta_{\nu'\nu} + c^{\mu,\nu}(z|a)\delta_{\mu'\nu}\delta_{\nu'\mu} && (4.6) \\ b^{\mu,\nu}(z|a) &= \frac{h(z)}{h(z+w)} \frac{h(s^\nu - s^\mu + w(a-\hat{\nu})^{\nu,\mu})}{h(s^\nu - s^\mu + w(a-\hat{\nu})^{\nu,\mu} + w)} \prod_{i=0}^{n-1} \frac{g_{i\nu}(a)}{g_{i\nu}(a+\hat{\mu})} \prod_{i=0}^{n-1} \frac{g_{i\mu}(a+\hat{\nu})}{g_{i\mu}(a)} \\ c^{\mu,\nu}(z|a) &= \frac{h(z+w)}{h(z)} \frac{h(s^\nu - s^\mu + w(a-\hat{\nu})^{\nu,\mu} + w + z)}{h(s^\nu - s^\mu + w(a-\hat{\nu})^{\nu,\mu} + w)} && (4.7) \end{aligned}$$

$$g_{i\mu}(a) = \begin{cases} 1 & \text{if } i \geq \mu \\ h(s^0 - s^\mu + wa^{0,\mu}) & \text{if } i = 0 \\ h(s^i - s^\mu + wa^{i,\mu} - w) & \text{if } 0 < i < \mu \end{cases} . \tag{4.8}$$

The s^μ are arbitrary complex numbers, $a \in \sum_{\mu=0}^{n-1} \mathbb{C}\Lambda_\mu$ with Λ_μ weights of the affine Lie algebra $A_{n-1}^{(1)}$, $\hat{\mu} = \Lambda_{\mu+1} - \Lambda_\mu$, $\mu = 0, 1, \ldots, n-1$ and $\Lambda_0 = \Lambda_n$. The $a^{\mu,\nu}$ is given in [41] and obeys $(a - \hat{\nu})^{\mu,\mu} = 0$.
The intertwining vector $M_a^\mu(z)$ is given by

$$\begin{aligned} M_a^\mu(z) &= {}^t(\phi_a^\mu(z)_0, \ldots, \phi_a^\mu(z)_{n-1}) \prod_{i=0}^{n-1} g_{i\mu}^{-1}(a) && (4.9) \\ \phi_a^\mu(z)_j &= \theta^{(j)}(z - nwa^{\mu,0}). && (4.10) \end{aligned}$$

Applying a sum of threefold tensorproducts of functions (4.9) to (4.2) using the vertex-face map (4.5) and exploiting the explicit index structure of (4.6) we obtain a modified Yang-Baxter equation

$$
\begin{aligned}
& R(z_1-z_2|a+\hat{\alpha}_3)^{\alpha_1,\alpha_2}_{\mu_1,\mu_2} R(z_1-z_3|a)^{\mu_1,\alpha_3}_{\beta_1,\mu_3} R(z_2-z_3|a+\hat{\beta}_1)^{\mu_2,\mu_3}_{\beta_2,\beta_3} \\
= \; & R(z_2-z_3|a)^{\alpha_2,\alpha_3}_{\mu_2,\mu_3} R(z_1-z_3|a+\hat{\mu}_2)^{\alpha_1,\mu_3}_{\mu_1,\beta_3} R(z_1-z_2|a)^{\mu_1,\mu_2}_{\beta_1,\beta_2} \, .
\end{aligned} \tag{4.11}
$$

This can be written symbolically as (we set $R(z_i - z_j|a) = R_{ij}(a)$)

$$
R_{12}(a+\hat{\sigma}_3) R_{13}(a) R_{23}(a+\hat{\sigma}_1) = R_{23}(a) R_{13}(a+\hat{\sigma}_2) R_{12}(a) \, .
$$

The monodromy matrix for a chain with N sites associated with the R-matrix (4.6) is

$$
T_{0,1\ldots N}(\lambda|a) = R_{0N}(a+\hat{\sigma}_1+\ldots+\hat{\sigma}_{N-1}) \ldots R_{02}(a+\hat{\sigma}_1) R_{01}(a) \tag{4.12}
$$

where 0 denotes the horizontal auxiliary space with spectral parameter λ; and $1,\ldots,N$ label the vertical quantum spaces (with local inhomogeneities $\{z_i\}$), whose tensorproduct constitute the physical Hilbertspace $\mathcal{H}_N$. An algebraic construction of eigenvalues of the monodromy matrix associated to the matrix (4.3) was performed in [42] and parallels the procedure for the eight-vertex model in [5] (One has to take into account that our construction of the monodromy matrix (4.12) differs from the monodromy matrix in [42] by an additional transformation in the physical space).

5 The F-basis

5.1 Motivation

Despite the achievements of the QISM, the treatmant of correlation functions proves to be rather intricate [43]. The reason for this lies in the fact that the apparantly simple action of the quasiparticle creation and annihilation operators which figure in the construction of the eigenfunctions (Bethe wavevectors) is obscured on the level of the microscopic variables (spin raising and lowering operators) by nonlocal effects arising from polarization clouds (compensating pairs of local raising and lowering spin operators).

This property can already be seen in the example of the spin-$\frac{1}{2}$ XXZ model, where the creation operators have the following form when expressed in the original microscopic spin variables

$$
C(\lambda) \;=\; \sum_{i=1}^{N} \sigma_i^+ \Omega_i + \sum_{i\neq j\neq k} \sigma_i^+ (\sigma_j^- \sigma_k^+) \Omega_{ijk} + \text{terms of higher order} \tag{5.1}
$$

where higher order means more compensating exchange terms $(\sigma_j^- \sigma_k^+)$, and the Ω_i, Ω_{ijk} etc. are diagonal and act on all site but i; i, j, k and so on.
In a seminal paper Maillet and Sanchez de Santos [10] revealed an application of so called factorizing Drinfel'd twists F defined by $R_{ij} = F_{ij} F_{ji}^{-1}$ to obtain a basis for the $sl(2)$ XXX and XXZ model which allows to express the creation and annihilation operators in a completely symmetric way with the further advantage of being polarization free, that is being built from the respective quasiclassical Gaudin operators dressed diagonally, thus minimizing quantum effects for these operators.
In the new basis the operators $\tilde{C}(\lambda) \equiv F(\lambda)\, C(\lambda) F^{-1}(\lambda)$ consist of only N terms (as compared to 2^N in the original basis) of the form

$$\tilde{C}(\lambda) = \sum_{i=1}^{N} \sigma_i^+ \otimes_{j \neq i} \begin{pmatrix} b(\lambda - z_j) b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]} . \tag{5.2}$$

These results allowed a simple derivation of formulas (previously known, but involving so called auxilliary fields [44]) for the scalar products of Bethe vectors and correlation functions [7, 45].
In this chapter we will concentrate on the construction of a similar basis for multicomponent generalizations of the anisotropic Heisenberg model, the Belavin model in its face type reformulation (4.7).

5.2 Factorizing F-matrix

The factorizing F-matrix for N sites (N quantum spaces) is defined by the relation $R^{\sigma}_{1\ldots N}(a) = F^{-1}_{\sigma(1\ldots N)}(a) F_{1\ldots N}(a)$ [10].
For the face model (4.7) it turns out to be given by formally the same expression as found in [12] for the $sl(n)$ XXX-model

$$F_{1\ldots N}(a) \quad = \quad \sum_{\sigma \in \mathcal{S}_N} \sum_{\alpha} {}^{*} \prod_{i=1}^{N} P_{\sigma(i)}^{\alpha_{\sigma(i)}} R^{\sigma_\alpha}_{1\ldots N}(z_1, \ldots, z_N | a) \tag{5.3}$$

where $(P_i^{\alpha_i})_{kl} = \delta_{k,\alpha_i} \delta_{l,\alpha_i}$ projects on the α_i-th component in the i-th space and the labels $\alpha_{\sigma(i)}$ satisfy the conditions

$$\begin{aligned} \alpha_{\sigma(i+1)} \quad &\geq \quad \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) > \sigma(i) \\ \alpha_{\sigma(i+1)} \quad &> \quad \alpha_{\sigma(i)} \quad \text{if} \quad \sigma(i+1) < \sigma(i) \; . \end{aligned} \tag{5.4}$$

This form of the F-matrix ensures down-triangularity and as the diagonal elements are non-vanishing, also non-degeneracy.

The modification of the Yang-Baxter equation (4.11) requires a particular rule for the handling of the parameter a in the formation of the intertwining matrix $R^\sigma(a)$ (related to the permutation σ), which can be read off from the modified composition law

$$\begin{aligned} R^{\sigma\sigma_i}(a) &= R_{\sigma(i),\sigma(i+1)}(\tilde{a}_i)R^\sigma(a) \\ \tilde{a}_i &= a+\hat{\sigma}_{\sigma(1)}+\ldots+\hat{\sigma}_{\sigma(i-1)} \end{aligned} \tag{5.5}$$

where σ_i is the transposition of $i, i+1$, and σ an arbitrary permutation.
$R^\sigma(a)$ has the intertwining property

$$R^\sigma(a)T_{0,1\ldots N}(a) = T_{0,\sigma(1)\ldots\sigma(N)}(a)R^\sigma(a+\hat{\sigma}_h) \tag{5.6}$$

where σ_h is associated with the matrix indices in the space 0.
The matrix $F_{1\ldots N}(a)$ satisfies the factorizing equation

$$R^\sigma_{1\ldots N}(a) = F^{-1}_{\sigma(1\ldots N)}(a)F_{1\ldots N}(a) \ . \tag{5.7}$$

An argument for the proof of (5.7) emphasizing the geometric meaning (loosely speaking a pure gauge field representation of R^σ on the symmetric group $\mathcal{S}_N$ as base space) may be sketched as follows:
First we note that (5.7) can be decomposed stepwise into elementary transpositions:

$$\begin{aligned} F_{\sigma(1)\ldots\sigma(N)}\ R^\sigma_{1\ldots N} &= F_{\sigma_1\sigma_2\ldots\sigma_k(1,\ldots,N)}\ R^{\sigma_k}_{\sigma_1\sigma_2\ldots\sigma_{k-1}(1,\ldots,N)}\ R^{\sigma_{k-1}}_{\sigma_1\sigma_2\ldots\sigma_{k-2}(1,\ldots,N)}\cdots R^{\sigma_1}_{1\ldots N} \\ &= F_{\sigma_1\sigma_2\ldots\sigma_{k-1}(1,\ldots,N)}\ R^{\sigma_{k-1}}_{\sigma_1\sigma_2\ldots\sigma_{k-2}(1,\ldots,N)}\ R^{\sigma_{k-2}}_{\sigma_1\sigma_2\ldots\sigma_{k-3}(1,\ldots,N)}\cdots R^{\sigma_1}_{1\ldots N} \\ &= \ldots\ldots\ldots = F_{\sigma_1(1,\ldots,N)}\ R^{\sigma_1}_{1\ldots N}\ = \ F_{1\ldots N} \end{aligned}$$

where the composition law (5.5) was used. Thus it is sufficient to prove Eq. (5.7) for arbitrary elementary transpositions $\sigma = \sigma_i;\quad \sigma_i:\ (i,i+1)\to(i+1,i)$,

$$R^{\sigma_i}_{1\ldots N} = F^{-1}_{\sigma_i(1\ldots N)}F_{1\ldots N} \ . \tag{5.8}$$

A special solution $\hat{F}^i$ of (5.8) is the two site matrix $F_{i\,i+1}$, embedded into the N-fold tensor product

$$\hat{F}^i = \mathbb{1}\otimes\ldots\otimes F_{i\,i+1}\otimes\ldots\otimes\mathbb{1}$$

and the most general solution of (5.8) is given by

$$F_{1\ldots N} = X(1,\ldots,\{i,i+1\},\ldots,N)\hat{F}^i \tag{5.9}$$

where $X(1,\ldots,\{i,i+1\},\ldots,N)$ denotes a non-singular matrix in the quantum space $\otimes^N \mathbb{C}^n$ which is symmetric with respect to an exchange of the space labels i and $i+1$ and of the respective local inhomogeneity parameters. The two site matrix $F_{i\,i+1}$ is given by

$$F_{i\,i+1} = \sum_{n-1\geq\alpha_{i+1}\geq\alpha_i} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}}\, \mathbb{1}_{i\,i+1} + \sum_{n-1\geq\alpha_i>\alpha_{i+1}} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}} R_{i\,i+1}^{\sigma_i\,i+1}\,. \tag{5.10}$$

That this matrix fulfills the factorization property can be checked directly, using the unitarity of the R-matrix and the fact that

$$P_i^\alpha P_j^\alpha R_{ij} = P_i^\alpha P_j^\alpha \mathbb{1}_{ij} \tag{5.11}$$

which allows to shift the equality sign in the summation in Eq. (5.10).

So we note that the proof of Eq. (5.7) is complete if we can show that

$$F_{1\ldots N}\left(\hat{F}^i\right)^{-1} = X^{(i)}_{(1,\ldots,N)} \tag{5.12}$$

is symmetric in the labels i and $i+1$. For this purpose it is convenient to consider the permutation group element σ_α of Eq. (5.12) as being decomposed into products of elementary transpositions, which also leads to a factorization of R^{σ_α} into a product of elementary (two site) matrices R^{σ_x}, and to subdivide the sum $\sum_\alpha$ in Eq. (5.3) into terms with the transposition σ_i and into terms without. It is furthermore possible to arrange the order of the transpositions such that R^{σ_i} does occur at most once and then on the very right end of the respective product of matrices R^{σ_x}. Schematically we obtain

$$\sum_\alpha P^\alpha R^{\sigma_\alpha} = \sum_{\alpha'} P^{\alpha'} R^{\sigma_{\alpha'}} R^{\sigma_i} + \sum_{\alpha''} P_{\alpha''} R^{\sigma_{\alpha''}}\,. \tag{5.13}$$

Each factor $P^{\alpha'} R^{\sigma_{\alpha'}}$ in the first sum has a counterpart $P^{\alpha''} R^{\sigma_{\alpha''}}$ in the second sum related by an interchange of the labels i and $i+1$.
Keeping this in mind one verifies straightforwardly by inspection that the multiplication of Eq. (5.13) by

$$\hat{F}_{\sigma_i}^{-1} = \left(\sum_{n-1\geq\alpha_i>\alpha_{i+1}} \mathbb{1}_{i,i+1} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}} + \sum_{n-1\geq\alpha_{i+1}\geq\alpha_i} R^{\sigma_i} P_i^{\alpha_i} P_{i+1}^{\alpha_{i+1}}\right) \Delta^{-1}_{i\,i+1}$$

with the diagonal matrix

$$[\Delta_{ij}]^{\beta_i,\beta_j}_{\alpha_i,\alpha_j} = \delta_{\alpha_i\beta_i}\delta_{\alpha_j\beta_j}\begin{cases} 1 & \text{if } \alpha_i = \alpha_j, \\ b_{ij} & \text{if } \alpha_i > \alpha_j, \\ b_{ji} & \text{if } \alpha_j > \alpha_i \end{cases}$$

from the right side produces an expression which is symmetric in i and $i+1$.

5.3 F-transformed creation operators

We will proceed by computing elements of the monodromy matrix (those acting as creation operators) in the new basis provided by the F-matrix. The transformation law $\tilde{T}_{0,1\ldots N}(a) = F_{1\ldots N}(a)T_{0,1\ldots N}(a)F^{-1}_{1\ldots N}(a+\hat{\sigma}_h)$ is enforced by the requirement that the resulting operator is symmetric, i.e. $\tilde{T}_{0,1\ldots N}(a) = \tilde{T}_{0,\sigma(1)\ldots\sigma(N)}(a)$, which then follows from (5.6).

We start with illustrating the method for the simplest case of a diagonal element. In order to computate the diagonal element $T_{n-1\,n-1}(\lambda|a)$ let us consider the action of the matrix $F(a)$ on $T_{n-1\,n-1}(a)$:

$$\begin{aligned} F_{1\ldots N}(a)T_{n-1\,n-1}(a) &= \sum_{\sigma\in\mathcal{S}_N}\ \sideset{}{^*}\sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} R^{\sigma}_{1\ldots N}(a)P^{n-1}_0 T_{0,1\ldots N}(a)P^{n-1}_0 \\ &= \sum_{\sigma\in\mathcal{S}_N}\ \sideset{}{^*}\sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{i=1}^{N} P^{\alpha_{\sigma(i)}}_{\sigma(i)} P^{n-1}_0 T_{0,\sigma(1)\ldots\sigma(N)}(a)P^{n-1}_0 R^{\sigma}_{1\ldots N}(a+\sigma_h) \end{aligned} \tag{5.14}$$

The specialization to the entry $(n-1,n-1)$ of the auxiliary space here is achieved by the projectors P^{n-1}_0. For the second equality in (5.14) we have used relation (5.6) and the obvious fact that P^{n-1}_0 commutes with $R^{\sigma}_{1\ldots N}(a)$. To simplify the following argument we distinguish in the sum $\sum^*$ cases of various multiplicities of the occurrence of the group index $n-1$:

$$\begin{aligned} F_{1\ldots N}(a)T_{n-1\,n-1}(a) &= \sum_{\sigma\in\mathcal{S}_N}\sum_{k=0}^{N}\ \sideset{}{^{*'}}\sum_{\alpha_{\sigma(1)}\ldots\alpha_{\sigma(N)}}\prod_{j=N-k+1}^{N} P^{n-1}_{\sigma(j)}\delta_{\alpha_{\sigma(j)},n-1}\prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)} \\ &\quad \times P^{n-1}_0 T_{0,\sigma(1)\ldots\sigma(N)}(a)P^{n-1}_0 R^{\sigma}_{1\ldots N}(a+\sigma_h)\,. \end{aligned} \tag{5.15}$$

Let us consider the prefactor of $R^{\sigma}_{1\ldots N}(a+\sigma_h)$ on the r.h.s. of Eq. (5.15) more closely. Using specific features of the R-matrices we can rewrite it as:

$$\prod_{j=1}^{N-k} P^{\alpha_{\sigma(j)}}_{\sigma(j)}\prod_{j=N-k+1}^{N} P^{n-1}_{\sigma(j)}P^{n-1}_0\, T_{0,\sigma(1)\ldots\sigma(N)}(a)\, P^{n-1}_0$$

$$
\begin{aligned}
&= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \left(R_{0,\sigma(N)}(a+\sum_{m=1}^{N-1}\hat{\sigma}_{\sigma(m)}) \right)_{n-1n-1}^{n-1n-1} \dots \left(R_{0,\sigma(N-k+1)}(a)\right)_{n-1n-1}^{n-1n-1} \\
&\quad \times P_0^{n-1}\, T_{0,\sigma(1)\dots\sigma(N-k)}(a)\, P_0^{n-1} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} \\
&= \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{n-1}\, T_{0,\sigma(1)\dots\sigma(N-k)}(a)\, P_0^{n-1} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} \\
&= \prod_{i=1}^{N-k} \left(R_{0\sigma(i)}(a+\sum_{m=1}^{i-1}\hat{\sigma}_{\sigma(m)}) \right)_{n-1,\alpha_{\sigma(i)}}^{n-1,\alpha_{\sigma(i)}} \prod_{j=1}^{N-k} P_{\sigma(j)}^{\alpha_{\sigma(j)}} \prod_{j=N-k+1}^{N} P_{\sigma(j)}^{n-1} P_0^{n-1}
\end{aligned}
\tag{5.16}
$$

Inserting the r.h.s. of (5.16) into Eq. (5.15) one sees that the product $\prod_i \left(R_{0\sigma(i)}(a+\sum_{m=1}^{i-1}\hat{\sigma}_{\sigma(m)}) \right)_{n-1,\alpha_{\sigma(i)}}^{n-1,\alpha_{\sigma(i)}}$ creates a diagonal dressing factor for $\tilde{T}_{n-1\,n-1}(a)$ and the product of projectors applied to $R^{\sigma}(a+\sigma_h)$ gives $F_{1\dots N}(a+\sigma_h)$.
One obtains

$$
\tilde{T}_{n-1\,n-1}(\lambda|a) = Y_{n-1}(a) \otimes_{i=1}^{N} \operatorname{diag}\{b(\lambda-z_i),\dots,b(\lambda-z_i),1\}\,. \tag{5.17}
$$

with the abbreviations

$$
\begin{aligned}
Y_j(a) &= \prod_{m=0}^{j-1} \left\{ \prod_{i=0}^{n-1} \frac{g_{ij}(\widetilde{a_m}+k_m\hat{m})}{g_{ij}(\widetilde{a_m})} \prod_{j_m=1}^{k_m} \left[\prod_{i=0}^{n-1} \frac{g_{im}(\widetilde{a_m}+(j_m-1)\hat{m})}{g_{im}(\widetilde{a_m}+(j_m-1)\hat{m}+\hat{j})} \right. \right. \times \\
&\quad \left. \left. \times \frac{h(s^m-s^j+w[\widetilde{a_m}+(j_m-2)\hat{m}]^{m,j})}{h(s^m-s^j+w[\widetilde{a_m}+(j_m-2)\hat{m}]^{m,j}+w)} \right] \right\} \\
\widetilde{a_m} &= a+\sum_{i=0}^{m-1} k_i\,\hat{i} \\
b(\lambda) &= \frac{h(\lambda)}{h(\lambda+w)}
\end{aligned}
\tag{5.18}
$$

and k_m gives the multiplicity of the upper matrix labels $\alpha_i = m$.
Note that this element is completely diagonal.
To compute $T_{n-1\,n-2}(\lambda|a)$ one has to distinguish in the sum $\sum^*$ cases of various multiplicities k_{n-1} and k_{n-2} of the occurrence of group indices $n-1$ and $n-2$:

$$
\begin{aligned}
F_{1\dots N}(a)T_{n-1\,n-2}(a) &= \sum_{\sigma\in\mathcal{S}_N} \sum_{k_{n-1}=0}^{N} \sum_{k_{n-2}=0}^{N-k_{n-1}} \sum_{\alpha_{\sigma(1)}\dots\alpha_{\sigma(N)}}^{*''} \prod_{j_{n-1}=N-k_{n-1}+1}^{N} P_{\sigma(j_{n-1})}^{n-1} \prod_{j_{n-2}=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} P_{\sigma(j_{n-2})}^{n-2} \\
&\quad \times \prod_{j=1}^{N-k_{n-1}-k_{n-2}} P_{\sigma(j)}^{\alpha_{\sigma(j)}} P_0^{n-1} T_{0,\sigma(1)\dots\sigma(N)}(a) P_0^{n-2} R_{1\dots N}^{\sigma}(a+\sigma_h)\,.
\end{aligned}
\tag{5.19}
$$

Evaluating the matrix product in $T_{0,\sigma(1)\ldots\sigma(N)}(a)$ leads to

$$
\begin{aligned}
& \prod_{j_{n-1}=N-k_{n-1}+1}^{N} P^{n-1}_{\sigma(j_{n-1})} \prod_{j_{n-2}=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} P^{n-2}_{\sigma(j_{n-2})} \prod_{j=1}^{N-k_{n-1}-k_{n-2}} P^{\alpha_{\sigma(j)}}_{\sigma(j)} P^{n-1}_0 T_{0,\sigma(1)\ldots\sigma(N)}(a) P^{n-2}_0 \\
= & \sum_{i=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} \left(R_{0\widetilde{N-k_{n-1}}}(\hat{a}_{N-k_{n-1}})\right)^{n-1\,n-2}_{n-1\,n-2} \ldots \left(R_{0\widetilde{i+1}}(\hat{a}_{i+1})\right)^{n-1\,n-2}_{n-1\,n-2} \\
& \times \left(R_{0\tilde{i}}(\hat{a}_i)\right)^{n-1\,n-2}_{n-2\,n-1} \left(R_{0\widetilde{i-1}}(\hat{a}_{i-1})\right)^{n-2\,n-2}_{n-2\,n-2} \ldots \left(R_{0\widetilde{N-k_{n-1}-k_{n-2}+1}}(\hat{a}_{N-k_{n-1}-k_{n-2}+1})\right)^{n-2\,n-2}_{n-2\,n-2} \\
& \times \prod_{k=1}^{N-k_{n-1}-k_{n-2}} \left(R_{0\tilde{k}}(\hat{a}_k)\right)^{n-2\,\alpha_{\tilde{k}}}_{n-1\,\alpha_{\tilde{k}}} E^{\tilde{i}}_{n-2,n-1} \prod_{j\neq i} P^{\alpha_{\tilde{j}}}_{\tilde{j}} P^{n-1}_{\tilde{i}} E^0_{n-1,n-2} \\
= & \sum_{i=N-k_{n-1}-k_{n-2}+1}^{N-k_{n-1}} \prod_{l=i+1}^{N-k_{n-1}} b^{n-1,n-2}(\lambda - z_{\tilde{l}}|\hat{a}_l) c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\hat{a}_i) \\
& \times \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E^{\tilde{i}}_{n-2,n-1} \prod_{j\neq i} P^{\alpha_{\tilde{j}}}_{\tilde{j}} P^{n-1}_{\tilde{i}} E^0_{n-1,n-2}
\end{aligned}
\tag{5.20}
$$

with the abbreviations

$$
\begin{aligned}
\hat{a}_i &= a + \sum_{m=1}^{i-1} \hat{\sigma}_{\tilde{m}} \\
\tilde{i} &= \sigma(i) \\
\text{and} \quad \left(E^i_{a,b}\right)^{\alpha_i}_{\beta_i} &= \delta_{\alpha_i,a}\delta_{\beta_i,b} \quad \text{denoting the root operators of } sl(n).
\end{aligned}
$$

One notes that in the calculation the index $\alpha_{\tilde{i}}$ has changed from $n-2$ to $n-1$. As the distribution of α's is therefore no longer consistent with the conditions (5.4) in the sum $\sum^*$ one has to correct this fact by commuting the site $\tilde{i}$ through all higher sites $\tilde{j}$ with $\alpha_{\tilde{j}} = n-2$. So, taking into account (5.5), one has to insert an additional factor

$$
R_{\tilde{i}\,\widetilde{N-k_{n-1}}}(\hat{a}_{\widetilde{N-k_{n-1}}} + \sigma_h) \ldots R_{\tilde{i}\,\widetilde{i+1}}(\hat{a}_{\widetilde{i+1}} + \sigma_h) \tag{5.21}
$$

between the projectors and $R^{\sigma}_{1\ldots N}(a+\sigma_h)$ in (5.19). Because of Eq. (5.11) no further corrections are neccessary. For the following calculation two equations are needed

$$
P^{n-1}_i P^{n-2}_j \mathbb{1}_{ij} = P^{n-1}_i P^{n-2}_j \left\{ b^{n-1,n-2}(z_i - z_j|a)^{-1} R_{i\,j}(a) - \frac{c^{n-2,n-1}(z_i - z_j|a)}{b^{n-1,n-2}(z_i - z_j|a)} \Pi_{i\,j} \right\} \tag{5.22}
$$

and

$$
E^i_{n-1,n} P^{n-1}_j \Pi_{i\,j} = E^j_{n-1,n} P^{n-1}_i \tag{5.23}
$$

with $(\Pi_{ij})^{\alpha_i\,\alpha_j}_{\beta_i\,\beta_j} = \delta_{\alpha_i,\beta_j}\delta_{\alpha_j,\beta_i}$ the permutation operator in space i and j.

Let us now concentrate on the term with $i = N - k_{n-1} - k_{n-2} + 1$ in (5.20) and use (5.22) to create the needed factor (5.21). Because of (5.23) the second term in (5.22) gives rise to an $E^{\tilde{j}}_{n-2,n-1}$ with $\tilde{j} \neq \tilde{i}$. So the only possibility to get an $E^{\tilde{i}}_{n-2,n-1}$ is to use the first term in (5.22). Corrections in the other terms with $j > N - k_{n-1} - k_{n-2} + 1$ cannot lead to an expression with $E^{\tilde{i}}_{n-2,n-1}$ as $\tilde{j}$ has not to be commuted with the site $\tilde{i}$. So the only term that contains $E^{\tilde{i}}_{n-2,n-1}$ after the corrections for that special $R^{\sigma}_{1\ldots N}$ in (5.20) is

$$\begin{aligned} &\prod_{l=i+1}^{N-k_{n-1}} \frac{b^{n-1,n-2}(\lambda - z_{\tilde{l}}|\hat{a}_l)}{b^{n-1,n-2}(z_{\tilde{i}} - z_{\tilde{l}}|\hat{a}_l)} c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\hat{a}_i) \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E^{\tilde{i}}_{n-2,n-1} \\ = &\prod_{l=i+1}^{N-k_{n-1}} \frac{b(\lambda - z_{\tilde{l}})}{b(z_{\tilde{i}} - z_{\tilde{l}})} c^{n-2,n-1}(\lambda - z_{\tilde{i}}|\tilde{a}_{n-2}) \prod_{k=1}^{N-k_{n-1}-k_{n-2}} b^{n-2,\alpha_{\tilde{k}}}(\lambda - z_{\tilde{k}}|\hat{a}_k) E^{\tilde{i}}_{n-2,n-1}. \end{aligned} \tag{5.24}$$

Because of the symmetry of $\tilde{T}_{0,1\ldots N}(\lambda|a)$ all other terms have to be of the same form as (5.24). Taking into account the action of the projectors the resulting expression is

$$\begin{aligned} \tilde{T}_{n-1\,n-2}(\lambda|a) &= Y_{n-2}(a) \sum_{i=1}^{N} c^{n-2,n-1}(\lambda - z_i|\tilde{a}_{n-2}) E^{i}_{n-2,n-1} \\ &\quad \otimes^{N}_{j\neq i} \operatorname{diag}\{b(\lambda - z_j), \ldots, b(\lambda - z_j), b(\lambda - z_j) b^{-1}(z_i - z_j), 1\}. \end{aligned} \tag{5.25}$$

For the calculation of $\tilde{T}_{n-1\,n-3}(\lambda|a)$ one has to distinguish the cases $n-1$, $n-2$ and $n-3$ in the sum $\sum^{*}$. The only difference compared to $\tilde{T}_{n-1\,n-2}(\lambda|a)$ is a term containing a product $E^{i}_{n-3,n-2}\,E^{j}_{n-2,n-1}$ now showing up in the matrix product in $T_{0,\sigma(1)\ldots\sigma N}(a)$. One again has to correct the distribution of $\alpha's$ with the analog of the equations (5.22) and (5.23)

$$P^{\alpha_i}_i\,P^{\alpha_j}_j\,\mathbb{I}_{ij} = P^{\alpha_i}_i\,P^{\alpha_j}_j \left\{ b^{\alpha_i,\alpha_j}(z_i - z_j|a)^{-1} R_{ij}(a) - \frac{c^{\alpha_j,\alpha_i}(z_i - z_j|a)}{b^{\alpha_i,\alpha_j}(z_i - z_j|a)} \Pi_{ij} \right\} \tag{5.26}$$

$$E^{i}_{a,b}\,P^{a}_{j}\,\Pi_{ij} = E^{j}_{a,b}\,P^{a}_{i} \tag{5.27}$$

and also with a new relation which has to be taken into account when dealing with the term containing the product $E^{i}_{n-3,n-2}\,E^{j}_{n-2,n-1}$:

$$E^{i}_{n-3,n-1}\,P^{n-2}_{j}\,P_{ij} = E^{i}_{n-3,n-2} E^{j}_{n-2,n-1}. \tag{5.28}$$

This reasoning leads to ($b_{i\,k} = b(z_i - z_k)$):

$$
\begin{aligned}
\tilde{T}_{n-1\,n-3}(\lambda|a) &= Y_{n-3}(a) \sum_{i=1}^{N} c^{n-3,n-1}(\lambda - z_i|\tilde{a}_{n-3}) E^{i}_{n-3,n-1} \\
&\quad \otimes^{N}_{k\neq i} \mathrm{diag}\{b(\lambda - z_k), \ldots, b(\lambda - z_k), b(\lambda - z_k) b^{-1}_{i\,k}, b(\lambda - z_k) b^{-1}_{i\,k}, 1\} \\
&\quad + Y_{n-3}(a) \sum_{i\neq j}^{N} f_2(n-3, n-2; n-1) E^{i}_{n-3,n-2}\, E^{j}_{n-2,n-1} \\
&\quad \otimes^{N}_{k\neq i,j} \mathrm{diag}\{b(\lambda - z_k), \ldots, b(\lambda - z_k), b(\lambda - z_k) b^{-1}_{i\,k}, b(\lambda - z_k) b^{-1}_{j\,k}, 1\}
\end{aligned} \tag{5.29}
$$

with

$$
\begin{aligned}
f_2(n-3, n-2; n-1) &= c^{n-2,n-1}(\lambda - z_j|\tilde{a}_{n-2}) c^{n-3,n-2}(\lambda - z_i|\tilde{a}_{n-3}) \\
&\quad - \frac{c^{n-2,n-1}(z_i - z_j|\tilde{a}_{n-2})}{b^{n-1,n-2}(z_i - z_j|\tilde{a}_{n-2})} b^{n-1,n-2}(\lambda - z_j|\tilde{a}_{n-2}) c^{n-3,n-1}(\lambda - z_i|\tilde{a}_{n-3})
\end{aligned} \tag{5.30}
$$

where the second term in (5.30) has its origin in the second term in (5.26) and in (5.28).

Proceeding in an analogous manner one obtains in the general case

$$
\begin{aligned}
\tilde{T}_{n-1\,\alpha}(\lambda|a) &= \sum_{k=1}^{\alpha} \sum_{i_1\neq\ldots\neq i_k} \sum_{n-1-\alpha<n_1<\ldots n_k<n-1} \\
&\quad E^{(i_1)}_{n_1,n_2} \otimes E^{(i_2)}_{n_2,n_3} \otimes \ldots \otimes E^{(i_k)}_{n_k,n-1} \;\; f_k(n_1, n_2, \ldots, n_k; n-1)\; Y_{n-1-\alpha}(a)
\end{aligned}
$$

$$
\otimes_{j\neq\{i_k\}} \mathrm{diag}\left\{ b(\lambda - z_j), \ldots, \underbrace{b(\lambda - z_j) b^{-1}_{i_1 j}, \ldots, b(\lambda - z_j) b^{-1}_{i_1 j}}_{n_2 - n_1}, \ldots, \underbrace{b(\lambda - z_j) b^{-1}_{i_k j}, \ldots, b(\lambda - z_j) b^{-1}_{i_k j}}_{n-1-n_k}, 1 \right\} \tag{5.31}
$$

with $f_k(n_1, n_2, \ldots, n_k; n_0)$ being defined recursively by[11]

$$
f_k(n_1, n_2, \ldots, n_k; n_0) = -\frac{c^{n_k,n_0}_{i_{k-1},i_k}}{b^{n_0,n_k}_{i_{k-1},i_k}} b^{n_0,n_k}_{0,i_k} f_{k-1}(n_1, n_2, \ldots, n_{k-1}; n_0) + c^{n_k,n_0}_{0,i_k} f_{k-1}(n_1, n_2, \ldots, n_{k-1}; n_k) \tag{5.32}
$$

where

$$
f_1(n_1; n_0) = c^{n_1,n_0}_{0,i_{n_1}} \,.
$$

[11] In the rational $sl(n)$ case this recursion relation leads directly to the result in Eq. (42) in [12].

We have thus acheived to compute the creation operators in the new basis. They can be regarded as the corresponding Gaudin operators (defined as the semiclassical limit of the quantum monodromy matrix) being dressed diagonally.

Remark: As the above procedure to obtain the operators of the monodromy matrix in the F-basis does not rely on the invariance of the monodromy matrix under combined $sl(n)$ rotations in the auxilliary and quantum space, the $\mathbb{Z}_n$ XXZ-model can be treated in the same way, and we obtain expressions for the $\tilde{T}_{n-1\,\alpha}$ similar to that in (5.31), with the diffenence that the functional dependence on a vanishes, and the parametrization of the elements of the R-matrix is trigonometric instead of elliptic ($c(\lambda) = \frac{sinh(w)}{sinh(\lambda+w)}$, $b(\lambda) = \frac{sinh(\lambda)}{sinh(\lambda+w)}$).

6 Bethe wavevectors

The operators $T_{n-1\,\alpha}$ $(\alpha < n-1)$ act as quasiparticle creation operators and satisfy the commutation relation

$$T_{n-1\,\alpha}(\lambda|a)T_{n-1\,\beta}(\mu|a+\hat{\alpha}) = T_{n-1\,\gamma}(\mu|a)T_{n-1\,\delta}(\lambda|a+\hat{\gamma})R(\lambda-\mu|a)^{\delta\,\gamma}_{\alpha\,\beta} \tag{6.1}$$

or in components of the R-matrix

$$\begin{aligned} [T_{n-1\,\alpha}(\lambda|a),\, T_{n-1\,\alpha}(\mu|a+\hat{\alpha})] &= 0 && (6.2)\\ T_{n-1\,\alpha}(\lambda|a)T_{n-1\,\beta}(\mu|a+\hat{\alpha}) &= \frac{1}{b^{\beta\alpha}(\mu-\lambda|a)}T_{n-1\,\beta}(\mu|a)T_{n-1\,\alpha}(\lambda|a+\hat{\beta}) \\ &- \frac{c^{\beta\alpha}(\mu-\lambda|a)}{b^{\beta\alpha}(\mu-\lambda|a)}T_{n-1\,\beta}(\lambda|a)T_{n-1\,\alpha}(\mu|a+\hat{\beta})\,. && (6.3) \end{aligned}$$

An ansatz for a Bethe vector Ψ_n is given in terms of a linear superposition of products of operators $T_{n-1\,\alpha}$ acting on a reference state $\Omega_N^{(n)}$ [23]:

$$\Psi_n(N;\lambda_1,\ldots,\lambda_p|a) = \sum_{\alpha_1,\ldots,\alpha_p} \Phi_{\alpha_1,\ldots,\alpha_p} T_{n-1\,\alpha_1}(\lambda_1|a)T_{n-1\,\alpha_2}(\lambda_2|a+\hat{\alpha}_1)\ldots T_{n-1\,\alpha_p}(\lambda_p|a+\sum_{i=1}^{n-1}\hat{\alpha}_i)\,\Omega_N^{(n)} \tag{6.4}$$

where the reference state $\Omega_N^{(n)}$ is constituted as a N-fold tensor product of lowest weight states $v_n^{(i)} = {}^t(0,\ldots,0,1)$ in $\mathbb{C}_n^{(i)}$

$$\Omega_N = \otimes_{i=1}^N v_n^{(i)}$$

and the $\Phi_{\alpha_1,\dots,\alpha_p}$ denote some c-number coefficients, which themselves have to be chosen s.t. they are components of a $\mathbb{Z}_{n-1}$ wavevector, leading to a nested structure finally giving a $\mathbb{Z}_2$ eigenvalue problem.
It is important to note that the reference state is invariant under the F-transformation

$$F\,\Omega_N^{(n)} = \Omega_N^{(n)}$$

due to the special form of the R-matrix.
We will not impose the Bethe ansatz equations for the spectral parameters $\{\lambda_i\}$ which turns the vector (6.4) into an eigenvector of the transfer matrix, that is we consider the Bethe wavevector being "off-shell" [46].
In what follows we want to determine the functional form of such vectors, using the explicit form of the operators relevant for the Bethe wavevectors.
In order to clarify the arguments employed in the course of the computation we will present the case of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in quite a detail. The generalization to the general case of $\mathbb{Z}_n$, $n > 3$ will then be rather straightforward.
For the $\mathbb{Z}_2$ case we have from (5.25)

$$\tilde{T}_{10}(\lambda|a) = \sum_{i=1}^{N} c^{0,1}(\lambda - z_i|a)\sigma_+^{(i)} \otimes_{j\neq i} \begin{pmatrix} b(\lambda - z_j)b_{ij}^{-1} & 0 \\ 0 & 1 \end{pmatrix}_{[j]} \tag{6.5}$$

$$\begin{aligned}\tilde{\Psi}_2(N;\lambda_1,\dots,\lambda_p|a) &= \tilde{T}_{10}(\lambda_1|a)\tilde{T}_{10}(\lambda_2|a+\hat{0})\dots\tilde{T}_{10}(\lambda_p|a+(p-1)\cdot\hat{0})\,\Omega_N^{(2)} \\ &= \sum_{i_1\neq\dots\neq i_p} B_p^{(2)}(\lambda_1,\dots,\lambda_p|z_{i_1},\dots,z_{i_p}|a)\sigma_+^{(i_1)}\dots\sigma_+^{(i_p)}\,\Omega_N^{(2)}\,.\end{aligned} \tag{6.6}$$

The c-number coefficient $B^{(2)}(\{\lambda_i\}|\{z_i\}|a)$ is, due to the "diagonally dressed" spin raising operators σ_+^i in (6.5), of the form

$$B_p^{(2)}(\lambda_1,\dots,\lambda_p|z_1,\dots,z_p) = \sum_{\sigma\in\mathcal{S}_p}\prod_{m=1}^{p} c^{0,1}(\lambda_m - z_{\sigma(m)}|a+(m-1)\cdot\hat{0})\prod_{l=m+1}^{p}\frac{b(\lambda_m - z_{\sigma(l)})}{b(z_{\sigma(m)} - z_{\sigma(l)})}. \tag{6.7}$$

We now turn to the $\mathbb{Z}_3$ case. The strategy will rely on the symmetry of the wavevector under the exchange of arbitrary spectral parameters (the verification of this fact follows the same lines as the one in [25] using (4.11) and (6.1)) which enables us to concentrate on a particularily simple term in the sum (6.4), and the repeated use of the commutation relations (6.3).
These ideas lead us to propose the following form for the Bethe wavevector

$$\tilde{\Psi}_3(N,\lambda_1,\dots,\lambda_{p_0};\lambda_{p_0+1},\dots,\lambda_{p_0+p_1}|a)$$

$$
\begin{aligned}
= & \sum_{\sigma\in\mathcal{S}_{p_0}} B^{(2)}_{p_1}(\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p_1)}|a) \\
\times & \prod_{k=1}^{p_1}\prod_{l=p_1+1}^{p_0} b^{1,0}(\lambda_{\sigma(k)}-\lambda_{\sigma(l)}|a+(k-1)\cdot\hat{0}+(l-p_1-1)\cdot\hat{1})^{-1} \\
\times & \tilde{T}_{21}(\lambda_{\sigma(p_1+1)}|a)\ldots\tilde{T}_{21}(\lambda_{\sigma(p_0)}|a+(p_0-p_1-1)\cdot\hat{1}) \\
\times & \tilde{T}_{20}(\lambda_{\sigma(1)}|a+(p_0-p_1)\cdot\hat{1})\ldots\tilde{T}_{20}(\lambda_{\sigma(p_1)}|a+(p_0-p_1)\cdot\hat{1}+(p_1-1)\cdot\hat{0})\,\Omega^{(3)}_N\,. \qquad (6.8)
\end{aligned}
$$

Consider a special term in the sum (6.4) of the form (which is motivated by the fact that the associated coefficient Φ is especially simple to compute, see below)

$$
\begin{aligned}
& \tilde{T}_{20}(\lambda_1|a)\tilde{T}_{20}(\lambda_1|a+\hat{0})\ldots\tilde{T}_{20}(\lambda_{p_1}|a+(p_1-1)\cdot\hat{0}) \\
\times & \tilde{T}_{21}(\lambda_{p_1+1}|a+p_1\cdot\hat{0})\tilde{T}_{21}(\lambda_{p_1+2}|a+p_1\cdot\hat{0}+\hat{1})\ldots\tilde{T}_{21}(\lambda_{p_0}|a+p_1\cdot\hat{0}+(p_0-p_1-1)\cdot\hat{1})\,\Omega^{(3)}_N\,.
\end{aligned}
\qquad (6.9)
$$

Commuting all $\tilde{T}_{20}(\lambda_1|a)$ to the right using the first term in (6.3) yields an additional factor

$$
\prod_{x=1}^{p_1}\prod_{y=p_1+1}^{p_0}\left\{b^{1,0}(\lambda_y-\lambda_x|a+(x-1)\cdot\hat{0}+(y-p_1-1)\cdot\hat{1})\right\}^{-1}\,.
$$

It has to be noted that the associated c-number coefficient $\Phi^{(2)}_{0\ldots01\ldots1}$ in (6.9) is not evaluated in the $\mathbb{Z}_3$ F basis. It can however be expressed in the form (6.7) as it is invariant under the action of the $\mathbb{Z}_2$ F-matrix. This is due to the fact that it constitutes a component of the $\mathbb{Z}_2$ vector whose labels (a non-decreasing series of α_i with respect to the original ordering of sites i) correspond via (5.4) to the identity permutation in the definition of the F-matrix (5.3). Invoking the exchange symmetry we arrive thus at the formula (6.8).
The creation operators with respect to the lowest weight state are of the form (cf. (5.31))

$$
\begin{aligned}
\tilde{T}_{21}(\lambda|a) &= Y_1(a)\sum_{i=1}^{N} c^{1,2}(\lambda-z_i|\widetilde{a_1})E^{(i)}_{1,2}\otimes_{j\neq i}\mathrm{diag}\{b(\lambda-z_j),b(\lambda-z_j)b^{-1}_{ij},1\}_{[j]} \qquad (6.10)\\
\tilde{T}_{20}(\lambda|a) &= \sum_{i=1}^{N} c^{0,2}(\lambda-z_i|\widetilde{a_0})E^{(i)}_{0,2}\otimes_{j\neq i}\mathrm{diag}\{b(\lambda-z_j)b^{-1}_{ij},b(\lambda-z_j)b^{-1}_{ij},1\}_{[j]} \\
&+ \text{ terms involving } E^{(i)}_{1,2}\otimes E^{(j)}_{0,1}\,. \qquad (6.11)
\end{aligned}
$$

The second term in (6.11) annihilates the vacuum $\Omega^{(3)}_N$, which is why we did not cite the explicit form of the prefactors accompanying these roots. The form of the creation operators permits us to further simplify the wavevector (6.8). Taking into account the action of the roots on the respective dressing as well as the fact that both groups of creation operators generate a factor similar to the $\mathbb{Z}_2$ problem, we obtain

$$
\begin{aligned}
&\tilde{\Psi}_3(N,\lambda_1,\ldots,\lambda_{p_0};\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|a)\\
=&\sum_{\sigma\in\mathcal{S}_{p_0}}\sum_{i_1\neq\ldots\neq i_{p_0}}\prod_{k=1}^{p_1}\prod_{l=p_1+1}^{p_0} b^{1,0}(\lambda_{\sigma(l)}-\lambda_{\sigma(k)}|a+(k-1)\cdot\hat{0}+(l-p_1-1)\cdot\hat{1})^{-1}\; b(\lambda_{\sigma(l)}-z_{i_k})\\
\times&\; B^{1,2}_{p_0-p_1}(\lambda_{\sigma(p_1+1)},\ldots,\lambda_{\sigma(p_0)}|z_{i_{p_1+1}},\ldots,z_{i_{p_0}}|a)\; B^{0,1}_{p_1}(\lambda_{p_0+1},\ldots,\lambda_{p_0+p_1}|\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p_1)}|a)\\
\times&\; B^{0,2}_{p_1}(\lambda_{\sigma(1)},\ldots,\lambda_{\sigma(p_1)}|z_{i_1},\ldots,z_{i_{p_1}}|a+(p_0-p_1)\cdot\hat{1})\; E^{(i_{p_1+1})}_{1,2}\ldots E^{(i_{p_0})}_{1,2}E^{(i_1)}_{0,2}\ldots E^{(i_{p_1})}_{0,2}\Omega^{(3)}_N
\end{aligned}
\tag{6.12}
$$

where we defined as a generalization of (6.7)

$$
\begin{aligned}
&B^{\alpha,\beta}_p(\lambda_1,\ldots,\lambda_p|z_1,\ldots,z_p|a)\\
=&\sum_{\sigma\in\mathcal{S}_p}\prod_{m=1}^{p}\left\{c^{\alpha,\beta}(\lambda_m-z_{\sigma(m)}|\widetilde{a_\alpha}+(m-1)\cdot\hat{\alpha})\,Y_\alpha(a+(m-1)\cdot\hat{\alpha})\right\}\prod_{l=m+1}^{p}\frac{b(\lambda_m-z_{\sigma(l)})}{b(z_{\sigma(m)}-z_{\sigma(l)})}.
\end{aligned}
\tag{6.13}
$$

We are now in the position to proceed to the general $\mathbb{Z}_n$ case.
We start with a special term of (6.4), where all operators $\tilde{T}_{n-1\,\alpha}$ are to the left of operators $\tilde{T}_{n-1\,\beta}$ if $\alpha<\beta$. The associated coefficient Φ again remains invariant under the action of F, which enables us to express it in a manner similar to (6.7). We then use the commutation relations (6.3) to reverse the order of all operators. Once again only the term containing one $E^i_{a,b}$ in the expression (5.31) of the respective operators $\tilde{T}_{n-1\,\alpha}$ contributes in this special ordering.
The wavevector can be expressed in a form similar to (6.12) ($\bar{\alpha}\equiv n-2-\alpha$)

$$
\tilde{\Psi}_n(N,p_0,p_1,\ldots,p_{n-2})=\sum_{i_1\neq\ldots\neq i_{p_0}} B^{(n)}_{p_0,p_1,\ldots,p_{n-2}}(\lambda_1,\ldots,\lambda_{p_0+\ldots p_{n-2}}|z_{i_1},\ldots,z_{i_{p_0}}|a)\prod_{\alpha=0}^{n-2}\prod_{j=p_{\bar{\alpha}+1}+1}^{p_{\bar{\alpha}}} E^{(i_j)}_{\alpha\;n-1}\Omega^{(n)}_N
\tag{6.14}
$$

with the recursion relation for $B^{(n)}$

$$
\begin{aligned}
&B^{(n)}_{p_0\,p_1\,\ldots\,p_{n-2}}(\lambda_1,\ldots,\lambda_{p_0+p_1+\ldots p_{n-2}}|z_1,\ldots,z_{p_0}|a)\\
=&\sum_{\sigma\in\mathcal{S}_{p_0}}\prod_{\alpha=0}^{n-3}\prod_{\beta=\alpha+1}^{n-2}\quad\prod_{k_\alpha=p_{\bar{\alpha}+1}+1}^{p_{\bar{\alpha}}}\quad\prod_{l_\beta=p_{\bar{\beta}+1}+1}^{p_{\bar{\beta}}} b(\lambda_{\sigma(l_\beta)}-z_{k_\alpha})\\
\times&\; b^{\beta,\alpha}\left(\lambda_{\sigma(l_\beta)}-\lambda_{\sigma(k_\alpha)}|a+\sum_{m=0}^{\alpha-1}(p_{\bar{m}}-p_{\bar{m}+1})\cdot\hat{m}+(k_\alpha-p_{\bar{\alpha}+1}-1)\cdot\hat{\alpha}+(l_\beta-p_{\bar{\beta}+1}-1)\cdot\hat{\beta}\right)^{-1}
\end{aligned}
$$

$$\times \prod_{\gamma=0}^{n-2} B^{\gamma,n-1}_{p_{\bar\gamma}-p_{\bar\gamma+1}}(\lambda_{\sigma(p_{\bar\gamma+1}+1)}\dots\lambda_{\sigma(p_{\bar\gamma})}|z_{p_{\bar\gamma+1}+1}\dots z_{p_{\bar\gamma}}|a+\sum_{m=\gamma+1}^{n-2}(p_{\bar m}-p_{\bar m+1})\cdot\hat m)$$

$$\times \quad B^{(n-1)}_{p_1\dots p_{n-2}}(\lambda_{p_0+1}\dots\lambda_{p_0+p_1+\dots+p_{n-2}}|\lambda_{\sigma(1)}\dots\lambda_{\sigma(p_1)}|a) \tag{6.15}$$

which can be solved explicitely to yield

$$\begin{aligned}
&B^{(n)}_{p_0\,p_1\,\dots\,p_{n-2}}(\lambda_1,\dots,\lambda_{p_0+p_1+\dots+p_{n-2}}|z_1,\dots,z_{p_0}|a)\\
&=\sum_{\sigma_0\in\mathcal{S}_{p_0}}\sum_{\sigma_1\in\mathcal{S}_{p_1}}\cdots\sum_{\sigma_{n-3}\in\mathcal{S}_{p_{n-3}}}\prod_{i=0}^{n-3}\prod_{\alpha_i=0}^{n-3-i}\prod_{\beta_i=\alpha_i+1}^{n-2-i}\prod_{k_{\alpha_i}=p_{\bar\alpha_i+1}+1}^{p_{\bar\alpha_i}}\prod_{l_{\beta_i}=p_{\bar\beta_i+1}+1}^{p_{\bar\beta_i}} b(\lambda_{q_{i-1}+\sigma_i(l_{\beta_i})}-\lambda_{q_{i-2}+\sigma_{i-1}(k_{\alpha_i})})\\
&\times b^{\beta_i,\alpha_i}\left(\lambda_{q_{i-1}+\sigma_i(l_{\beta_i})}-\lambda_{q_{i-1}+\sigma_i(k_{\alpha_i})}|a+\sum_{m=0}^{\alpha_i-1}(p_{\bar m}-p_{\bar m+1})\cdot\hat m+(k_{\alpha_i}-p_{\bar\alpha_i+1}-1)\cdot\hat\alpha_i+(l_{\beta_i}-p_{\bar\beta_i+1}-1)\cdot\hat\beta_i\right)^{-1}\\
&\times\prod_{\gamma_i=0}^{n-2-i} B^{\gamma_i,n-1}_{p_{\bar\gamma_i}-p_{\bar\gamma_i+1}}\left(\lambda_{q_{i-1}+\sigma_i(p_{\bar\gamma_i+1}+1)}\dots\lambda_{q_{i-1}+\sigma_i(p_{\bar\gamma_i})}|\lambda_{q_{i-2}+\sigma_{i-1}(p_{\bar\gamma_i+1}+1)}\dots\lambda_{q_{i-2}+\sigma_{i-1}(p_{\bar\gamma_i})}|a+\sum_{m=\gamma_i+1}^{n-2-i}(p_{\bar m}-p_{\bar m+1})\cdot\hat m\right)\\
&\times\quad B^{0,1}_{p_{n-2}}\left(\lambda_{q_{n-3}+1}\dots\lambda_{q_{n-3}+p_{n-2}}|\lambda_{q_{n-4}+\sigma_{n-3}(1)}\dots\lambda_{q_{n-4}+\sigma_{n-3}(p_{n-2})}|a\right)
\end{aligned}\tag{6.16}$$

where we defined

$$q_i=\sum_{j=0}^{i}p_j;\quad q_{-1}=0$$

and

$$\lambda_{\sigma_{-1}(k)}=z_k\,.$$

By expressing the $\mathbb{Z}_n$ wavevector (6.14) with the help of $\mathbb{Z}_2$ building blocks (6.13) we have achieved a resolution of the hierarchy.

<u>Remark</u>: Once again the arguments in this section apply to the $\mathbb{Z}_n$ XXZ-model as well, leading to a wavevector for this model which displays the same features as (6.14) in connection with (6.16), without the functional dependence on a and with a trigonometric parametrization.

7 Conclusion

We accomplished the construction of a factorizing F-matrix for the Belavin model enabling one to construct completely symmetric creation operators which moreover are devoid of non-local effects from polarization clouds. These operators were used to resolve the intricacies of the nested structure of Bethe wavevectors.

In contrast to the $sl(n)$ XXX-model the above method does not rely on an $sl(n)$ invariance of the monodromy matrix, which renders the extraction of the elements in the grid of the monodromy matrix much more cumbersome. The only ingredients needed in our computation are the form of the R-matrix (4.6), i.e. its structure, unitarity and the fact that it constitutes a representation of the permutation group, and the property (5.11). Thus our findings directly yield the corresponding expression for the generalized $\mathbb{Z}_n$ XXZ-model.

Led by the similarities between the generalized rational, trigonometric and elliptic model it is tempting to ask whether such a distinguished basis exists for all integrable models in two dimensions. In [47] it was shown that in every integrable two-dimensional quantum field theory there exist semi-local polarization-free generators which are localized in wedge-shaped regions of Minkowski space and which generate one particle states from the vacuum. It is conceivable that there is a relation between these operators and polarization-free operators in lattice spin models.

For this purpose the generalization of the above procedure for the thermodynamical limit would be desireable.

We hope that these results might prove useful for the construction of formfactors starting from the microscopic level.

In view of the similarities between the results obtained by the functional Bethe ansatz and those in the F-basis for the $sl(2)$ XXX and XXZ model, it seems plausible that some version of the functional Bethe ansatz should be feasible both for the ellipic case (XYZ model) as well as for the nested form.

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A Notation

We use the following notation:

<u>Tensor product:</u> The tensor product of two $n \times n$ matrices X, Y (yielding a $n^2 \times n^2$ matrix $X \otimes Y$) is defined as

$$(X \otimes Y)^{ij}_{kl} = X_{ik}\, Y_{jl} \quad (i, j, k, l = 1, \ldots, n)\ . \tag{A.1}$$

The matrix elements of the $n^2 \times n^2$ matrix are labled by block indices i, k and by intrinsic indices j, l as is indicated in the following example of a 4×4 matrix R

$$R = \begin{pmatrix} R^{11}_{11} & R^{11}_{12} & R^{11}_{21} & R^{11}_{22} \\ R^{12}_{11} & R^{12}_{12} & R^{12}_{21} & R^{12}_{22} \\ R^{21}_{11} & R^{21}_{12} & R^{21}_{21} & R^{21}_{22} \\ R^{22}_{11} & R^{22}_{12} & R^{22}_{21} & R^{22}_{22} \end{pmatrix}\ .$$

The matrix product of two $n^2 \times n^2$ matrices S, T in this notation is given by

$$(ST)^{ij}_{kl} = S^{ij}_{mn} T^{mn}_{kl}\ . \tag{A.2}$$

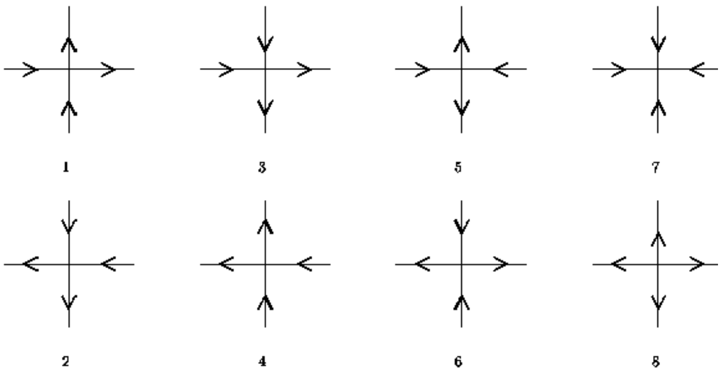
The unit matrix $E = \mathbb{1} \otimes \mathbb{1}$ and the permutation matrix Π are given respectively

$$E^{ij}_{kl} = \delta_{ik}\delta_{jl}, \qquad \Pi^{ij}_{kl} = \delta_{il}\delta_{jk} \tag{A.3}$$

$$\Pi^2 = E\ . \tag{A.4}$$

B Spin-$\frac{1}{2}$ Heisenberg model

In this appendix we collect the basic definitions of the XYZ, XXZ and XXX-Heisenberg model in the terminology of the QISM.
Spin-$\frac{1}{2}$ results in 16 Boltzmann weights R_{ij}^{kl}.
We impose the condition $(i+j=k+l;\ \bmod 2)$ on the vertex weights R_{ij}^{kl}, leaving eight of the originally 16 vertices non-zero, which in the absence of an external magnetic field exhibit a $\mathbb{Z}_2$ symmetry (simultaneous reversal of all spins), thus giving four independent vertices, which have to be parametrized in a way consisient with the YBE (4.11)



$$
\begin{aligned}
a(\lambda) &= exp(-\beta\epsilon_1) = exp(-\beta\epsilon_2)\\
b(\lambda) &= exp(-\beta\epsilon_3) = exp(-\beta\epsilon_4)\\
c(\lambda) &= exp(-\beta\epsilon_5) = exp(-\beta\epsilon_6)\\
d(\lambda) &= exp(-\beta\epsilon_7) = exp(-\beta\epsilon_8)\ .
\end{aligned}
$$

The weight d is responsible for for non-conservation of spin at a vertex.
The associated R-matrix is given by

$$
R_{XYZ}(\lambda-\mu) = \begin{pmatrix} 1 & 0 & 0 & d(\lambda-\mu) \\ 0 & b(\lambda-\mu) & c(\lambda-\mu) & 0 \\ 0 & c(\lambda-\mu) & b(\lambda-\mu) & 0 \\ d(\lambda-\mu) & 0 & 0 & 1 \end{pmatrix} \tag{B.1}
$$

with (cf. Appendix C for some facts about elliptic functions)

$$
\begin{aligned}
b(\lambda-\mu) &= \frac{\mathrm{sn}(\lambda-\mu)}{\mathrm{sn}(\lambda-\mu+\eta)}\\
c(\lambda-\mu) &= \frac{\mathrm{sn}(\eta)}{\mathrm{sn}(\lambda-\mu+\eta)}\\
d(\lambda-\mu) &= k\frac{\mathrm{sn}(\eta)}{\mathrm{sn}(\lambda-\mu+\eta)}\ .
\end{aligned} \tag{B.2}
$$

The associated Hamiltonian is

$$H_{XYZ} = \sum_{i=1}^{N} \left\{ J_x \sigma_i^x \, \sigma_{i+1}^x + J_y \sigma_i^y \, \sigma_{i+1}^y + J_z \sigma_i^z \, \sigma_{i+1}^z \right\} \tag{B.3}$$

with the couplings given by

$$J_x = 1 + k\,\mathrm{sn}^2\eta; \quad J_y = 1 - k\,\mathrm{sn}^2\eta; \quad J_z = \mathrm{cn}\eta\,\mathrm{dn}\eta \;.$$

The trigonometric limit $k \to 0$ of the XYZ model yields the six-vertex model, which conserves spin, and whose R-matrix is

$$R_{XXZ}(\lambda-\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda-\mu) & c(\lambda-\mu) & 0 \\ 0 & c(\lambda-\mu) & b(\lambda-\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.4}$$

with

$$\begin{aligned} b(\lambda-\mu) &= \frac{\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+\eta)} \\ c(\lambda-\mu) &= \frac{\sinh(\eta)}{\sinh(\lambda-\mu+\eta)} \;. \end{aligned} \tag{B.5}$$

The associated Hamiltonian is

$$H_{XXZ} = \sum_{i=1}^{N} \left\{ \sigma_i^x \, \sigma_{i+1}^x + \sigma_i^y \, \sigma_{i+1}^y + \cos i\eta \, \sigma_i^z \, \sigma_{i+1}^z \right\} \;. \tag{B.6}$$

The isotropic limit (the so called XXX-model) is obtained by rescaling $\lambda \to \epsilon\lambda; \quad \eta \to \epsilon\eta$ and letting $\epsilon \to 0$:

$$R_{XXX}(\lambda-\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda-\mu) & c(\lambda-\mu) & 0 \\ 0 & c(\lambda-\mu) & b(\lambda-\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.7}$$

with

$$\begin{aligned} b(\lambda-\mu) &= \frac{\lambda-\mu}{\lambda-\mu+\eta} \\ c(\lambda-\mu) &= \frac{\eta}{\lambda-\mu+\eta} \;. \end{aligned} \tag{B.8}$$

The associated Hamiltonian is

$$H_{XXX} = \sum_{i=1}^{N} \left\{ \sigma_i^x \, \sigma_{i+1}^x + \sigma_i^y \, \sigma_{i+1}^y + \sigma_i^z \, \sigma_{i+1}^z \right\} \;. \tag{B.9}$$

Let us briefly consider the algebra generated by the local commutation relation (2.11)

$$R_{12}(\lambda-\mu)\, L_i(\lambda,z_i)\otimes L_i(\mu,z_i) = L_i(\mu,z_i)\otimes L_i(\lambda,z_i)\, R_{12}(\lambda-\mu)\ . \tag{B.10}$$

Inserting the above R_{XYZ}-matrix and the Lax operator into (2.11), we arrive at a set of algebraic relations for S^i; $i=0,1,2,3$

$$[S^0,S^\alpha] = iJ_{\beta\gamma}\left\{S^\beta,S^\gamma\right\}, \quad [S^\alpha,S^\beta] = i\left\{S^0,S^\gamma\right\} \tag{B.11}$$

involving commutator [,] and anticommutator { , } along with the Casimir operators

$$\bar{C}_1 = \Sigma_{i=0}^3 (S^i)^2, \quad \bar{C}_2 = \Sigma_{\alpha=1}^3 J_\alpha (S^a)^2. \tag{B.12}$$

The triplet of indices (α,β,γ) represents any cyclic permutation of $(1,2,3)$ and the structure constants $J_{\alpha\beta} = \frac{-J_\alpha+J_\beta}{J_\gamma}$ obey the constraint $J_{12}+J_{23}+J_{31}+\ J_{12}J_{23}J_{31}=0$.
They are given by

$$J_1 = \frac{\vartheta_2(2\eta)\vartheta_2(0)}{\vartheta_2^2(\eta)}, \quad J_2 = \frac{\vartheta_3(2\eta)\vartheta_3(0)}{\vartheta_3^2(\eta)}, \quad J_3 = \frac{\vartheta_4(2\eta)\vartheta_4(0)}{\vartheta_4^2(\eta)}\ .$$

Note that in contrast to a Lie algebra, (B.11) is a quadratic algebra, which depends on the parameters η, k. This quadratic algebra, discovered in [6] is known as the *Sklyanin algebra.*

The corresponding trigonometric limit of the Sklyanin algebra may be obtained from the limiting values of the parameters giving the structure constants

$$J_{12}=0,\ J_{23}=-J_{31}=\tan^2\frac{\eta}{2}\ . \tag{B.13}$$

This reduces (B.11) to

$$\begin{gathered}[S^0,S^3]=0, \quad [S^3,S^\pm]=\pm\left\{S^3,S^\pm\right\}, \\ [S^0,S^\pm]=\mp\tan^2\frac{\eta}{2}\left\{S^3,S^\pm]\right\}, \quad [S^+,S^-]=4S^0S^3\ .\end{gathered} \tag{B.14}$$

The Casimir operators (B.12) reduce to

$$\bar{C}_1\to C_1=\Sigma_{i=0}^3(S^i)^2, \quad \bar{C}_1-\bar{C}_2\to C_2=\cos^2\frac{\eta}{2}\ (S^0)^2+\sin^2\frac{\eta}{2}\ (S^3)^2\ . \tag{B.15}$$

This trigonometric Sklyanin algebra is also quadratic and depends on a single parameter η. Let us notice first that the Casimir operator C_2 given in the form of an ellipse suggests a simple realization (putting $C_2=\frac{1}{4}\mathbb{1}$)

$$S^0=\frac{\cos(\eta s^3)}{2\cos\frac{\eta}{2}}, \quad S^3=\frac{\sin(\eta s^3)}{2\sin\frac{\eta}{2}} \tag{B.16}$$

through a single operator s^3.
This realization together with the same $S^\pm$, reduces the quadratic algebra (B.14) to

$$[s^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = [2s^3]_q. \tag{B.17}$$

The remaining Casimir operator in (B.12) reduces to

$$[j]_q[j+1]_q = [s^3]_q[s^3-1]_q + S^+S^-. \tag{B.18}$$

where the $q-bracket$ is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin(\eta x)}{\sin \eta}, \quad q = e^{i\eta}.$$

This is the *quantum algebra* $U_q(sl(2))$ [18]. At the limit $q \to 1$ (or equivalently at $\eta \to 0$) , when

$$[2s^3]_q \approx 2s^3 + \frac{\eta^3}{3}(2s^3)^3 + \cdots,$$

the deformed algebra (B.17) goes to the familiar $su(2)$ spin algebra

$$[s^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2s^3. \tag{B.19}$$

The related Casimir operator reduces to $j(j+1) = s^3(s^3-1) + S^+S^-$.
We should remember that the quantum algebra, sometimes called the *quantum group*, is in fact neither quantum nor a group. Nevertheless, it has got some analogy with the quantization procedure (the deformation parameter q can be interpreted as $q = e^{i\hbar\kappa}$ with $\hbar$ the Planck constant and κ the coupling of the model, see for example [48]) as well as with group properties.

C Theta functions

For a complex number τ in the upper half plane, let $\Lambda_\tau \equiv \mathbb{Z} + \mathbb{Z}\tau$ be the lattice generated by 1 and τ and $\mathbb{T}_\tau = \mathbb{C}/\Lambda_\tau$ the associated complex torus.
The Jacobi *theta function* is defined by the Fourier series [49]

$$\theta(z,\tau) = \sum_{m\in\mathbb{Z}} \exp\left(\pi\, i\tau m^2 z + 2\pi\, imz\right) \tag{C.1}$$

with

$$\theta(z,\tau) = 0 \;\text{ at }\; z = \frac{1}{2} + \frac{\tau}{2}, \quad \mathrm{mod}\Lambda_\tau. \tag{C.2}$$

For every holomorphic function $f(z)$ and $a, b \in \mathbb{R}$ let

$$\begin{aligned} (T_a f)(z) &= f(a + z\tau)\exp\left\{2\pi\, i(az + a^2\tau/2)\right\} \\ (S_b f)(z) &= f(b+z)\,. \end{aligned}$$

The theta function of a complex variable z with arbitrary *characteristics* $a, b \in \mathbb{R}$ (a, b fixed) is defind by

$$\begin{aligned} \theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) &\equiv (S_b T_a \theta)(z,\tau) \\ &= \sum_{m\in\mathbb{Z}} \exp\left(\pi\, i\tau(m+a)^2 z + 2\pi\, i(m+a)(z+b)\right) \end{aligned} \tag{C.3}$$

and has zeros at $z = (\frac{1}{2} - b) + (\frac{1}{2} - a)\tau$.
The action of the Heisenberg group is given by

$$\begin{aligned} \left(\mathcal{S}_p \theta\begin{bmatrix} a \\ b \end{bmatrix}\right)(z,\tau) &= \theta\begin{bmatrix} a \\ b+p \end{bmatrix}(z,\tau) \\ \left(T_q \theta\begin{bmatrix} a \\ b \end{bmatrix}\right)(z,\tau) &= \exp(-2\pi\, i\, bq)\theta\begin{bmatrix} a+q \\ b \end{bmatrix}(z,\tau) \\ \theta\begin{bmatrix} a+k \\ b+l \end{bmatrix}(z,\tau) &= \exp(2\pi\, i\, al)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) \end{aligned} \tag{C.4}$$

for $p, q \in \mathbb{R}$ and $k, l \in \mathbb{Z}$. In terms of transformations with respect to z (quasiperiodicity) these relation can be rewritten as

$$\begin{aligned} \theta\begin{bmatrix} a \\ b \end{bmatrix}(z+p+q\tau,\tau) &= \exp\left(-\pi\, iq(q\tau + 2p + 2b + 2z)\right)\theta\begin{bmatrix} a+q \\ b+p \end{bmatrix}(z,\tau) \\ \theta\begin{bmatrix} a \\ b \end{bmatrix}(z+\mu+\nu\tau,\tau) &= \exp\left(-\pi\, i\nu(\nu\tau + 2z) + 2\pi\, i(\mu a - \nu b)\right)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) \end{aligned} \tag{C.5}$$

with $p, q \in \mathbb{R}$ and $\mu, \nu \in \mathbb{Z}$.
We use the abbreviation $\theta^{(j)}(z)$ to denote the theta function of rational characteristics $1/2 - j/n$, $1/2$

$$\theta^{(j)}(z) = \theta \left[\begin{array}{c} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{array} \right] (z, n\tau) . \tag{C.6}$$

A characteristics $\left[\begin{array}{c} a \\ b \end{array} \right]$ is called half-periods, if a, b are 0 or $\frac{1}{2}$. A half-periods is said to be even if $4\,ab \equiv 0 \pmod 2$ and odd otherwise.
The function $\theta \left[\begin{array}{c} a \\ b \end{array} \right] (z, \tau)$ is even or odd, according as $\left[\begin{array}{c} a \\ b \end{array} \right]$ is an even or odd half-period.

To make contact with the notation used in [5] we set

$$\begin{aligned} i\,\vartheta_1(z,\tau) = \theta \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z, n\tau), \quad \vartheta_2(z,\tau) &= \theta \left[\begin{array}{c} 1/2 \\ 0 \end{array} \right] (z, n\tau), \\ \vartheta_3(z,\tau) = \theta \left[\begin{array}{c} 0 \\ 0 \end{array} \right] (z, n\tau), \quad \vartheta_4(z,\tau) &= \theta \left[\begin{array}{c} 0 \\ 1/2 \end{array} \right] (z, n\tau) . \end{aligned} \tag{C.7}$$

We put

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}, \quad k' = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)} \quad K = \frac{1}{2}\pi\vartheta_3^2(0) . \tag{C.8}$$

The doubly periodic Jacobi *elliptic functions* are

$$\mathrm{sn}(u) = \frac{1}{\sqrt{k}} \frac{\vartheta_1(v)}{\vartheta_4(v)}, \quad \mathrm{cn}(u) = \frac{\sqrt{k'}}{\sqrt{k}} \frac{\vartheta_2(v)}{\vartheta_4(v)}, \quad \mathrm{dn}(u) = \sqrt{k'} \frac{\vartheta_3(v)}{\vartheta_4(v)} \tag{C.9}$$

with $v = u/(2K)$.
The limit $\tau \to i\infty$ (or $k \to 0$) of the elliptic functions yields trigonometric functions

$$\mathrm{sn}u \to \sin u, \quad \mathrm{cn}u \to \cos u, \quad \mathrm{dn}u \to 1 . \tag{C.10}$$

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Index

Bethe ansatz
  algebraic, 2, 6, 13
  equations, 10, 12, 13, 32
  functional, 2, 16, 19
  nested, 2, 3, 32, 36
Bethe wavevector, 10, 11, 13, 14, 22, 23, 31, 32, 36

elliptic function, 38, 43

F-matrix, 20, 23, 24, 26, 33, 36
factorizing twist, 3, 23

inverse problem, 2, 14, 15

Lax operator, 3–7, 40
limit
  classical, 8, 41
  semiclassical, 31
  thermodynamical, 2, 14, 15, 36
  trigonometric, 39, 40, 43

model
  Belavin, 3, 20, 21, 23, 36
  eight-vertex, 3, 20, 38
  face model, 3, 21–23
  Heisenberg
    XXX, 23, 36, 38, 39
    XXZ, 10–12, 22, 23, 31, 35, 36, 38
    XYZ, 38–40
  integrable, 4, 5, 7, 8, 36
  six-vertex, 9, 11, 21, 39
  XXX, 16, 19
monodromy matrix, 5, 6, 8, 9, 11, 13–16, 18, 22, 26, 31, 36

pseudovacuum (see also reference state), 3, 10, 11, 13, 16

quantum determinant, 16–18
quantum group, 8, 41
Quantum inverse scattering method (QISM), 2, 5, 6, 9, 15, 22, 38

R-matrix, 6–9, 11–14, 22, 31, 32, 36, 38, 39
reference state, 2, 11, 31, 32

Sklyanin algebra, 40, 41
space
  auxilliary, 6–8, 22, 26, 31
  Hilbert, 6, 7, 14, 22

theta function, 21, 42, 43
  characterisitics, 21, 42, 43
  even, 43
  half-periods, 43
  odd, 43
transfer matrix, 5, 7–13, 16, 19, 32

vertex, 6, 7, 11, 20, 38
vertex-face map, 21

Yang-Baxter equation (YBE), 5, 8, 20, 22, 24