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Chapter 1

Introduction

The classification of (super)conformal field theories in two-dimensions plays important role both in compactification of (super)string theory and in statistical mechanical system. In particular, the solvable (super)conformal field theories, i.e. the ones in which spectrum and correlation functions may be exactly computed, are of special importance since they give rise to computable physical predictions. For example, the minimal conformal field theories [3], and their supersymmetric generalization [4, 33], WZW theories [39, 49, 70], Z_k parafermionic models [37, 38, 73], and orbifolds [21, 22] etc. String theory is a generalised quantum field theory in which the basic objects are not point particles but one dimensional string. The distinctive feature of such a quantum field theory is that all the states have nonnegative norm, i.e. the state space is a Hilbert space. In other words these theories are unitary. Statistical mechanical models can be unitary or nonunitary. For example Ising model [3] corresponds to unitary minimal conformal field theory with central charge $c = \frac{1}{2}$. The Lee-Yang edge singularity [12] corresponds to the nonunitary minimal conformal field theory with central charge $c = -\frac{22}{5}$.
The complete classification of the unitary (super)conformal field theories in two-dimensions is accomplished by classifying the unitary highest weight representations of the (super)conformal algebras, and the classifications of their modular invariant partition functions. The unitarity combined with conformal invariance, strongly constraints on the allowed physical system. In two-dimensional unitary (super)conformal field theories the scaling dimensions of fields are determined by the unitary representations of the Virasoro(conformal) algebra. By classifying all possible unitary representations of the (super)conformal algebra, one is able to find all possible conformal dimension in two-dimensions.
The aim of this thesis is to make contribution to the classification of unitary conformal and $N = 2$ superconformal field theories in two-dimensions with central charge $c = 2$ and $c = 3$, respectively. This will widen the basis for compactification of (super)string theory.
This paper is organized as follows : in remaining part of this chapter we briefly review features of the $N = 0, 1, 2$ superconformal algebra and superconformal field theories that are relevant to our studies. In chapter 2, We study the moduli space $\mathcal{C}^2$ of unitary two–dimensional conformal field theories with central charge $c = 2$. We construct all the 28 nonexceptional nonisolated irreducible components of $\mathcal{C}^2$ that may be obtained by an orbifold procedure from toroidal theories. The parameter spaces and partition functions are calculated explicitly. In chapter 3, we determine multicritical points and lines and show that all but four of the 28 irreducible components of $\mathcal{C}^2$ corresponding to nonexceptional orbifolds are directly or indirectly connected to the moduli space of toroidal theories in $\mathcal{C}^2$. In chapter 4, we construct $\mathbb{Z}_M$,$M = 2, 3, 4, 6$ orbifold models of

the $N = 2$ superconformal field theories with central charge $c = 3$ [23]. Then we check the description of the $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds by the $N = 2$ superconformal Landau-Ginzburg models with $c = 3$, by comparing the spectrum of chiral fields, in particular the Witten index $Tr(-1)^F$. In chapter 5.1 we review modular group, Dedekind eta function and jacobi theta functions.

1.1 Representation of the Virasoro algebra

The Virasoro algebra [69] is infinite dimensional. It is generated by the holomorphic(left moving) traceless stress-energy tensor $T(z)$ with conformal dimension h equal to 2. It has Laurent expansion

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2}, \tag{1.1}$$

Similar relation holds for antiholomorphic (right moving) stress-energy tensor. In the following we only concentrate on the holomorphic sector. The operator product expansion (OPE) is

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{1.2}$$

where we ignored all non-singular terms in this expansion. The OPE (1.2) equivalent to the following commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \ , \quad m, n \in \mathbb{Z}. \tag{1.3}$$

where c is the central charge and commutes with all the other generators L_n. Classically, $c = 0$ but as we shall see, it plays a crucial role in any quantum mechanical applications. If representation of the Virasoro algebra is irreducible c will have to be a real number. The unitarity of the representation of Virasoro algebra means that it contains no negative-norm states or the hermiticity relation

$$L_n^+ = L_{-n} \tag{1.4}$$

holds with respect to some positive definite inner product on the representation space. Highest weight means that all the states in the representation space can be generated from a unique highest-weight state $|h>$ satisfying

$$L_0|h\rangle = h|h\rangle \ , \quad L_n|h\rangle = 0 \ , \quad n > 0. \tag{1.5}$$

An irreducible highest weight representation is characterized by the pair of numbers (c, h). Using a formula of Kac [46] for the determinants of the matrix of inner products of vectors with a given eigenvalue of L_0, Friedan, Qiu and Shenker [34] established the following condition for the existence of unitary minimal highest weight representation of the Virasoro algebra either

$$c \geq 0 \ , \quad h \geq 0, \tag{1.6}$$

or

$$c = 1 - \frac{6}{m(m+1)} \ , \quad m = 3, 4 \cdots, \tag{1.7}$$

and

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}, \tag{1.8}$$

where $p = 1, 2, \cdots, m-1$; $q = 1, 2, \cdots, p$. An explicitly unitary construction of the discrete series representations of the Virasoro algebra out of representations of affine Kac-Moody algebra was presented by Goddard-Kent-Olive (GKO) in [42, 43]. Their method produces not only all the minimal models but also many other rational conformal field theories (RCFT). Therefore, the conditions (1.6) and (1.7) of Friedan, Qiu and Shenker are necessary and sufficient. It was shown by J. L. Cardy [11] that the unitary minimal models are the only consistent conformal field theories with a finite number of primary fields at $c \leq 1$. Their chiral algebra is just Virasoro algebra.

We would like to give some remarks on the non-unitary minimal conformal field theories [45]. Belavin, Polyakov and Zamolodchikov [3] have shown that there exist minimal degenerate representation of the Virasoro algebra admitting a finite number of primary conformal fields with central charge

$$c = 1 - \frac{6(r-r')^2}{rr'},$$

and with conformal dimensions

$$h_{p,q} = \frac{(rp - r'q)^2 - (r-r')^2}{4rr'}$$

with

$$p = 1, \cdots, r'-1, \quad q = 1, \cdots, r, \tag{1.9}$$

where (r, r') a pair of coprime positive integers. For (r, r') coprimes, there always exist two integers (p_0, q_0) satisfying (1.9) such that $rp_0 - r'q_0 = 1$, $(p_0, q_0) \neq 1$. The corresponding dimension of the conformal primary field

$$h_{p_0,q_0} = \frac{1-(r-r')^2}{4rr'}$$

is always negative, except if $|r - r'| = 1$, in which case it vanishes and results in unitary conformal field theories with central charge (1.7) and dimension (1.8). If negative dimension primary fields contribute to the partition function

$$Z(\sigma) = trq^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}},$$

then its small $q = e^{2\pi i\sigma}$ behavior is no longer given by $(q\overline{q})^{-\frac{c}{24}}$, but by $(q\overline{q})^{-\frac{c_{eff}}{24}}$, where $c_{eff} = c - 24h_{min}$, h_{min} denotes the conformal weight of the minimal highest weight state, c is the central charge of the unitary theories. One finds $c_{eff} = 1 - \frac{6}{rr'}$. Indeed, the minimal models are just those with $c_{eff} < 1$.

The moduli space of all conformal field theories with $c_{eff} = 1$ and its generalization to the extended supersymmetric conformal algebras were investigated by M. Flohr [29, 30].

To summarise, for $0 < c < 1$, there exists a complete classification of the unitary highest-weight representations of the Virasoro algebra in the form of the discrete unitary series, as well as a classification of the modular invariant partition functions [8, 9, 10, 11, 12, 34, 35]. On the other hand the conformal field theories with $c \geq 1$ have a quite different structure. Namely, besides a set of isolated models, there exists continuum of inquivalent theories and one can consider smooth deformations of the conformal field theory by the exactly marginal operators. The general features of the moduli space of two-dimensional unitary conformal field theories with central charge $c \geq 1$ have been studied by R. Dijkgraaf et al [14]. The complete descriptions of the two-dimensional unitary conformal field theories with $c = 1$ has been studied in [15, 40, 41]. We will recall them in the following section as they are closely related to our studies.

1.2 Unitary conformal field theories with central charge $c = 1$

The complete classification of modular invariant partition functions for $c = 1$ unitary conformal theories consists of the toroidal, $\mathbb{Z}_2$ orbifold, and three isolated models. The toroidal model describes a single scalar field φ compactified on the circle $S^1 = R/2\pi r\mathbb{Z}$ with the action

$$S = \frac{1}{2\pi}\int d^2z\partial\varphi\overline{\partial}\varphi. \tag{1.10}$$

Corresponding partition function is

$$\begin{aligned} Z^{c=1}(r) &= trq^{L_0-\frac{1}{24}}\overline{q}^{\overline{L}_0-\frac{1}{24}} = \frac{1}{|\eta(\sigma)|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{p^2}{2}}\overline{q}^{\frac{\overline{p}}{2}} \\ &= \frac{1}{|\eta(\sigma)|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{4}(\frac{n}{r}+mr)^2}\overline{q}^{\frac{1}{4}(\frac{n}{r}-mr)^2}, \end{aligned} \tag{1.11}$$

where $q = e^{2\pi i\sigma}$, $\sigma = \sigma_1 + i\sigma_2$ parametrizes the world sheet torus and $\eta(\sigma)$ is the Dedekind eta function defined as

$$\eta(\sigma) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$

The Virasoro zero mode operator in (1.11) for a single boson is given by

$$L_0 = \sum_{m>0}\alpha_{-m}\alpha_m + \frac{p^2}{2}, \quad \overline{L}_0 = \sum_{m>0}\overline{\alpha}_{-m}\overline{\alpha}_m + \frac{\overline{p}^2}{2}$$

with $\alpha_0 \equiv p = (\frac{n}{r} + mr)$, and $\overline{\alpha}_0 \equiv \overline{p} = (\frac{n}{r} - mr)$. At the self dual point radius $r = 1$, for $m = n = \pm 1$ we find two $(1, 0)$ states, and for $m = -n = \pm 1$ two $(0, 1)$ states. In operator language these states are created by the operators

$$j^{\pm}(z) = e^{\pm i\sqrt{2}\varphi(z)}, \quad \overline{j}^{\pm}(\overline{z}) = e^{\pm i\sqrt{2}\overline{\varphi}(\overline{z})}$$

with conformal weights $(h, \overline{h}) = (1, 0)$ and $(0, 1)$. At arbitrary radius r, we always have $(1, 0)$ and $(0, 1)$ oscillators states $\alpha_{-1}|0, 0\rangle$ and $\overline{\alpha}_{-1}|0, 0\rangle$, created by the operators

$$j(z) = i\partial\varphi(z), \quad \overline{j}(\overline{z}) = i\partial\overline{\varphi}_1(\overline{z}).$$

The operators j,$j^{\pm}$ and $\overline{j}$, $\overline{j}^{\pm}$ generate an $SU(2)_L \times SU(2)_R$ Kac-Moody algebra. Thus we see that the circle theory at radius $r = 1$ has an $SU(2)_L \times SU(2)_R$
The $\mathbb{Z}_2$ *orbifold* models are obtained from the corresponding toroidal model (1.11) by identifying φ with $-\varphi$. Its partition function has the form

$$\begin{aligned} Z^{c=1}_{orb}(r) &= \frac{1}{2}\left(Z^{c=1}(r) + 2\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|\right) \\ &= \frac{1}{2}\left(Z^{c=1}(r) + 2Z^{c=1}(2) - Z^{c=1}(1)\right). \end{aligned} \tag{1.12}$$

Here and in the following $\vartheta_i(z, \sigma)$, $i \in \{1, \cdots, 4\}$ denote the classical Jacobi theta functions, and $\vartheta_i(\sigma) := \vartheta_i(0, \sigma)$. Both circle and orbifold partition functions are invariant under the duality transformation $r \to \frac{1}{r}$ in which the roles of winding number m and

momentum number n are simply interchanged. The circle theory at radius $r = 2$ is equivalent to the orbifold theory at $r = 1$

$$Z^{c=1}(2) = Z^{c=1}_{orb}(1).$$

Because each of the theories can be constructed by $\mathbb{Z}_2$ twist, either by a reflection or by a shift by a half period, and that the two $\mathbb{Z}_2$ construction are equivalent due to the $SU(2)_L \times SU(2)_R$ symmetry.
Here, one can construct further models by modding out the finite subgroups of $SO(3) = SU(2)/\mathbb{Z}_2$. These are the cyclic group with l elements C_l; the non abelian dihedral group with 2n elements D_l; the tetrahedron T, octahedron O, and Icosahedron I groups, with respective number of elements 12, 24, and 60. Modding out (1.11) by the cyclic group of rotation of order l gives points on the circle line, but at a radius decreased to r/l, namely

$$Z_l \equiv Z^{c=1}(r/l).$$

Modding out by the dihedral group D_l gives points on the orbifold line. For odd l, the mutually commuting elements of the group D_l are one cyclic subgroup C_l of order l, and l cyclic subgroups C_2 of order two. For even l, the mutually commuting elements of D_l are one cyclic subgroup C_l and $n/2$ dihedral subgroups D_2. By perfoming the orbifold procedure for both l odd and l even cases, one obtains the following partition function

$$Z^{c=1}_{D_l} = \frac{1}{2l}\left(Z_l + l(2Z_2 - Z_1)\right) = \frac{1}{2}\left(Z_l + 2Z_2 - Z_1\right) = Z^{c=1}_{\mathbb{Z}_2-orb}(r/l),$$

where we subtracted of $Z_1 = Z^{c=1}(r)$ from the second term to avoid overcounting the contribution from the identity element, which is commen to all the abelian subgroups we summed over. The T, O and I orbifold models live neither on the circle line nor on the orbifold, but they form isolated points in the moduli space for $c = 1$ conformally invariant theories. The mutually commuting elements of T fall into four cyclic subgroups C_3 of order three, and three cyclic subgroups C_2 of order two. The mutually commuting elements of O fall into three cyclic subgroups of order four, four cyclic subgroups of order three, and six cyclic subgroups of order two. The mutually commuting elements of I fall into six cyclic subgroups of order five, ten cyclic subgroups of order three, and fifteen cyclic subgroups of oder two. These lead to the following partition function

$$\begin{aligned}
Z^{c=1}_T &= \frac{1}{12}\left(4(3Z_3 - Z_1) + 3(2Z_2 - Z_1) + Z_1\right) \\
&= \frac{1}{2}\left(2Z_3 + Z_2 - Z_1\right) \\
Z^{c=1}_O &= \frac{1}{24}\left(3(4Z_4 - Z_1) + 4(3Z_3 - Z_1) + 6(2Z_2 - Z_1) + Z_1\right) \\
&= \frac{1}{2}\left(Z_4 + Z_3 + Z_2 - Z_1\right) \\
Z^{c=1}_I &= \frac{1}{60}\left(6(5Z_5 - Z_1) + 10(3Z_3 - Z_1) + 15(2Z_2 - Z_1) + Z_1\right) \\
&= \frac{1}{2}(Z_5 + Z_3 + Z_2 - Z_1). \qquad (1.13)
\end{aligned}$$

1.3 $N = 1$ superconformal algebra

The $N = 1$ superconformal algebra [57, 58, 63] is a linear extension of Virasoro algebra (1.2) or (1.3). It is defined by (1.2) together with

$$T(z)G(w) = \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} \tag{1.14}$$

$$G(z)G(w) = \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2T(w)}{z-w},$$

where $G(z)$ is the supercharge with conformal dimension $h = 3/2$. It is a function of z only and has the following Laurent expansion

$$G(z) = \sum_{r=-\infty}^{+\infty} G_r z^{-r-\frac{3}{2}}. \tag{1.15}$$

The OPE (1.14) is equivalent to the (anti)commutation relations

$$[L_n, G_r] = (\frac{n}{2} - r)G_{n+r}, \tag{1.16}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{3}c(r^2 - \frac{1}{4})\delta_{r+s}, \tag{1.17}$$

where $r, s \in \mathbb{Z}$ in the Ramond (R) algebra or $r, s \in \mathbb{Z} + 1/2$ in the Neveu-Schwarz (NS) algebra. Because of unitarity, G_r satisfies hermiticity condition

$$G_r^+ = G_{-r}. \tag{1.18}$$

Highest-weight state $|h\rangle$ of the Neveu-Schwarz (NS) and Ramond (R) algebras satisfy

$$G_r|h\rangle = 0 \ , \quad r > 0. \tag{1.19}$$

as well as (1.5). Unitarity again restricts the possible values of central charge and conformal dimension. Eq. (1.17) implies that a highest weight state has eigenvaule $h - \frac{c}{16}$ under G_0^2 in the Ramond sector. For $c > \frac{3}{2}$, the unitary representations exist for all values of (c,h) with $h \geq 0$ in the NS sector and $h \geq \frac{c}{16}$ in the R sector. On the other hand, for $c < \frac{3}{2}$, the unitary representations of the $N = 1$ superconformal algebra can only occur at the discrete values of the central charge, which corresponds to the $N = 1$ unitary minimal models [4, 33, 56, 62]

$$c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right) \ , \quad m = 2, 3, \cdots, \tag{1.20}$$

and

$$h_{p,q} = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} + \epsilon \ , \tag{1.21}$$

where $p = 1, 2, \cdots, m-1$ and $q = 1, 2, \cdots, m+1$. $\epsilon = \frac{1}{32}(1 - (-1)^{p-q})$ is 0 and $p - q$ even in the Neveu-Schwarz sector, while ϵ is $\frac{1}{16}$ and $p - q$ odd in the Ramond sector. A method for constructing representations of the $N = 1$ superconformal algebras out of representations of affine Kac-Moody algebras was presented by Goddard-Kent-Olive (GKO) in [44] and shown to yield not only the minimal discrete series of irreducible highest weight representation but also many other rational conformal fiels theories. A

complete classification of the modular invariant partition function was discussed in [7, 47]. The tricritical Ising model [33, 62] in two-dimensions is an example of the simplest $N = 1$ superconformal field theories with Virasoro central charge $c = \frac{7}{10}$.
The complete description of the moduli space of two-dimensional unitary $N = 1$ superconformal field theories with $c = \frac{3}{2}$ has been studied by Dixon, Ginsparg and Harvey [19]. We will briefly review them in the next section.

1.4 $N = 1$ Unitary superconformal field theories with central charge $c = 3/2$

The complete classification of modular invariant partition functions of $c = 3/2$ theories consists of the circle partition function $Z^{c=\frac{3}{2}}(r)$, $\mathbb{Z}_2(R)$ orbifold $Z^{c=\frac{3}{2}}_{\mathbb{Z}_2-orb}(r)$, superaffine, superorbifold , and orbifold-prime partition function as well as six isolated models. The $N = 1$ superconformal models with $c = \frac{3}{2}$ are realized by a single superfield, consisting of a free boson and a free Majorana(real) fermion. Therefore partition function for the $N = 1$ superconformal cirlce line is constructed by tensoring the theory of a single free boson compactified on a circle of radius r, together with the theory of a single Majorana fermion with left right components $\psi(z)$ and $\overline{\psi}(\overline{z})$. The partition function for a single boson is given in (1.11). For a real single fermion, partition function [40] is obtained by summing over equal spin structures for left and right components

$$Z_{Ising} = \frac{1}{2}\left(\left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right| + \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right| + \left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right|\right). \tag{1.22}$$

Thus the $N = 1$ superconformal circle partition function takes the form

$$Z^{c=\frac{3}{2}}_{circ}(r) = Z^{c=1}(r) Z_{Ising}. \tag{1.23}$$

The other four theories in the $c = 3/2$ moduli space can be constructed from (1.23) by modding out the $\mathbb{Z}_2$ symmetries generated by

$$R, \quad (-1)^{F_s}, \quad S_\delta \equiv (-1)^{F_s} t_\delta, \quad S_R \equiv (-1)^{F_s} R$$

that exist for all values of the radius and leave the world sheet supersymmetry generator invariant. Where R is a reflection, $(-1)^{F_s}$ is the $\mathbb{Z}_2$ symmetry of any superconformal field theory. It acts by $+1$ on states in the Neveu-Schwarz sector and by -1 on states in the Ramond sector of the theory. $t_\delta = e^{2\pi i p \frac{\delta}{\sqrt{2}}}$ is a translation symmetry wich acts on the bosonic $c = 1$ theory. Let us briefly discuss the four orbifold theories.
An $N = 1$ $\mathbb{Z}_2(R)$ *orbifold* line is obtained from circle line (1.23) by identifying φ with $-\varphi$ and ψ with $-\psi$. The partition function $Z^{c=\frac{3}{2}}_{orb}(r)$ is equal to $c = 1$ circle orbifold (1.12) times the Ising partition function (1.22), since the sum over the spin structures for the fermion is invariant under $\psi \to -\psi$. Thus the $\mathbb{Z}_2(R)$ orbifold partition function has the form

$$Z^{c=\frac{3}{2}}_{orb}(r) = \frac{1}{2}\left(Z^{c=1}(r) + 2\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right| + 2\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|\right) Z_{Ising}. \tag{1.24}$$

By the same argument as $c = 1$ circle theory, the orbifold theory at $r = 1$ is equivalent to the circle theory at $r = 2$

$$Z^{c=\frac{3}{2}}_{orb}(1) = Z^{c=\frac{3}{2}}_{circ}(2). \tag{1.25}$$

due to the $SU(2)_L \times SU(2)_R$ symmetry at radius $r = 1$ of the circle line.
The *superaffine* line $Z^{c=\frac{3}{2}}_{s-a}(r)$ is obtained by twisting the circle line (1.23) by the $\mathbb{Z}_2(S_\delta)$ group generated by $S_\delta \equiv (-1)^{F_s} t_\delta$. The partition function for this theory is

$$\begin{aligned} Z^{c=\frac{3}{2}}_{s-a}(r) &= \frac{1}{2}\left(\left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right| + \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right|\right) Z^{c=1}_{(+)}(r) + \frac{1}{2}\left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right| Z^{c=1}_{(-)}(r) \\ &+ \frac{1}{2}\left(\left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right| - \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right|\right) Z^{c=1}_{\delta(+)}(r) + \frac{1}{2}\left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right| Z^{c=1}_{\delta(-)}(r), \end{aligned} \tag{1.26}$$

where $Z^{c=1}_{(+)}(r)$ and $Z^{c=1}_{(-)}(r)$ are defined as is $Z^{c=1}(r)$ but the summation for n restricted to $n \in 2\mathbb{Z}$, $n \in 2\mathbb{Z}+1$ respectively; $Z^{c=1}_{\delta(\pm)}(r)$ also are defined as is $Z^{c=1}(r)$ but now the lattice shifted by δ, i.e. summation for m restricted to $m \in \mathbb{Z} + \frac{1}{2}$.
The *superorbifold* line may be constructed twisting superaffine line (1.26) by the $\mathbb{Z}_2(R)$ group generated by R. Or it can also be constructed by twisting the circle theory (1.23) by the $D_2 = \mathbb{Z}_2(R) \times \mathbb{Z}_2(S_\delta)$ group generated by R and S_δ. The Partition function for this theory is

$$\begin{aligned} Z^{c=\frac{3}{2}}_{s-orb}(r) &= \frac{1}{2}\left(Z^{c=\frac{3}{2}}_{s-a}(r) + \left|\frac{\vartheta_3(\sigma)}{\vartheta_2(\sigma)}\right| + \left|\frac{\vartheta_3(\sigma)}{\vartheta_4(\sigma)}\right| + \left|\frac{\vartheta_4(\sigma)}{\vartheta_2(\sigma)}\right| \right. \\ &\left. + \left|\frac{\vartheta_4(\sigma)}{\vartheta_3(\sigma)}\right| + \left|\frac{\vartheta_2(\sigma)}{\vartheta_3(\sigma)}\right| + \left|\frac{\vartheta_2(\sigma)}{\vartheta_4(\sigma)}\right|\right). \end{aligned} \tag{1.27}$$

The circle and superorbifold theories are equivalent at $r = \sqrt{2}$

$$Z^{c=\frac{3}{2}}_{s-orb}(\sqrt{2}) = Z^{c=\frac{3}{2}}(\sqrt{2}). \tag{1.28}$$

The *orbifold-prime* theories is constructed by twisting the circle theory (1.23) by the $\mathbb{Z}_2(S_R)$ group generated by S_R. Or it may also be constructed by twisting the orbifold line (1.24) by the $\mathbb{Z}_2((-1)^{F_s})$ group generated by $(-1)^{F_s}$. The partition function of this theory is given by

$$Z^{c=\frac{3}{2}}_{orb'}(r) = Z^{c=\frac{3}{2}}_{orb}(r) - 3. \tag{1.29}$$

where $Z^{c=\frac{3}{2}}_{orb}(r)$ is given in (1.24). The orbifold-prime and superaffine line, the superorbifold and orbifold-prime line intersect at

$$Z^{c=\frac{3}{2}}_{orb'}(1) = Z^{c=\frac{3}{2}}_{s-a}(2) \tag{1.30}$$

$$Z^{c=\frac{3}{2}}_{s-orb}(2) = Z^{c=\frac{3}{2}}_{orb'}(2). \tag{1.31}$$

For more details and the constructions of isolated components of the $c = \frac{3}{2}$ moduli space read reference [19].
As in the cases of $N = 0$ conformal field theory and $N = 1$ superconformal field theory, the $N = 2$ superconformal field theory is classified by the representation of the $N = 2$ superconformal algebra. One notes that the classification of the $N = 2$ superconformal field theory with central charge $c < 3$ is analogous to the $N = 0$ class for $c < 1$, and the $N = 1$ class for $c < 3/2$. The case $N = 2$ for $c = 3$ analogous to $N = 0$ for $c = 1$ and $N = 1$ for $c = 3/2$.

1.5 $N = 2$ superconformal algebra

The $N = 2$ superconformal algebra has aplications in supersymmetric critical phenomina in two diemsional statistical mechanical systems and in the compactification of superstring theory, where it is well known that in order to get an $N = 1$ space-time supersymmetric string theory one needs an $N = 2$ supersymmetric conformal field theory on the world-sheet.
The $N = 2$ superconformal algebras are generated by the stress-energy tensor $T(z)$ and its super partners (two fermionic generators) $G^\alpha(z)$, $\alpha = 1, 2$ ($G^\pm(z) = G^1(z) \pm iG^2(z)$) and $U(1)$ current $J(z)$ with conformal dimensions h equal to 2, 3/2, and 1 respectively. They are functions of z only and have the Laurent expansions (1.1), (1.15) and

$$J(z) = \sum_{n=-\infty}^{+\infty} J_n z^{-n-1}. \tag{1.32}$$

All of the generators L_n, G^α_r and J_n satisfy the hermicity conditions (1.4), (1.18) and

$$J^+_n = J_{-n}. \tag{1.33}$$

All the charecteristic properties of the $N = 2$ superconformal invariant theory can be expressed by the operator product expansions (OPE) of T, G and J. The operator product algebra $T(z)T(w)$ and $T(z)G^\alpha(w)$ are given in (1.2) and (1.14), respectively. The remaining of the $N = 2$ superconformal operator product algebras are given as

$$\begin{aligned}
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)} \\
G^\alpha(z)G^\beta(w) &= \left[\frac{2}{(z-w)^3} + \frac{2T(z)}{(z-w)}\right]\delta^{\alpha\beta} + i\left[\frac{2J(w)}{(z-w)^2} + \frac{\partial J(w)}{(z-w)}\right]\varepsilon^{\alpha\beta} \\
J(z)G^\alpha(w) &= \frac{i}{(z-w)}\varepsilon^{\alpha\beta}G^\beta(w) \\
J(z)J(w) &= \frac{c/3}{(z-w)^2},
\end{aligned} \tag{1.34}$$

where we ignored all non-singular terms in the expansion. The OPE (1.34) are equivalent to the following (anti)commutation relations

$$\begin{aligned}
&[L_n, J_m] = -mJ_{m+n}\ , \quad \{G^\pm_r, G^\pm_s\} = 0 \\
&\{G^+_r, G^-_s\} = 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}
\end{aligned} \tag{1.35}$$

and

$$[J_n, G^\pm_r] = \pm G^\pm_{n+r}\ , \quad [J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}.$$

There are three different $N = 2$ superconformal algebras, namely Ramond (R) (or periodic P), Nevue-Schwarz (NS) (or antiperiodic A) and twisted (T) algebras which correspond to different ways of choosing baundary conditions on the cylinder. Whatever baundary condition we choose the Virasoro generators L_n are always integrally moded, because the bosonic stress-energy tensor is always periodic on the cylinder. For the Ramond (R) algebra J_n and G^α_r are integrally moded i.e n and r run over integral values. For the Neveu-Schwarz (NS) algebra J_n are integrally moded, G^α_r are half integrally moded i.e r run over half integral values. The twisted (T) algebra has integer modes for G^1_r, half integer modes for J_n and G^2_r.

A highest weight state(hws.) of $N = 2$ superconformal algebra is a state $|h, Q>$ which satisfies the following relation

$$L_n|h,Q\rangle = 0 \ , \quad G_r^{\pm}|h,Q\rangle = 0 \ , \quad J_m|h,Q\rangle = 0 \quad \text{for n, r, m positive .} \tag{1.36}$$

Operators L_n, J_m, G_r^{α} and L_{-n}, J_{-m}, G_{-r}^{α} are called annihilation and creation operators for positive n, m, r, respectively. In the NS sector there are only two zero mode operators L_0 and J_0 which satisfy

$$L_0|h,Q\rangle = h|h,Q\rangle \ , \qquad J_0|h,Q\rangle = Q|h,Q\rangle, \tag{1.37}$$

where h is the conformal dimension and Q is the $U(1)$ charge of the state. In the Ramond sector we also have to include $G_0^{\pm}$. If a state in the Ramond sector of the theory is annihilated by both $G_0^{\pm}$, then it is called Ramond ground state.
The determinant formula for the Ramond, Neveu-Schwarz and twisted $N = 2$ superconformal algebra, and a classification of the unitary highest weight representations of these algebra were presented [5, 16, 17, 56]. For the Ramond and Neveu-Schwarz algebras, it is no longer the case that all $c \geq 3$, $h \geq 0$ representations are unitary. The unitary highest weight representations of the $N = 2$ superconformal algebra having the central charge $c < 3$ form a discrete series

$$c = \frac{3m}{m+2}, \quad m = 1, 2, \cdots. \tag{1.38}$$

The primary fields of the $N = 2$ superconformal algebra are labeled by two integeres p and q where p, takes its value in the range $0, \cdots, k$. q labeles the $U(1)$ charge of the field, and is defined modulo $2(m+2)$. In addition, fields are labeled by their sector, either NS or R. In the NS sector the conformal dimensions and $U(1)$ charges are given as

$$h_{p,q} = \frac{p(p+2) - q^2}{4(m+2)} \ , \qquad Q = \frac{q}{m+2} \ , \tag{1.39}$$

for $p = 0, \cdots, m$ and $q = -p, -p+2, \cdots, p$. In the Ramond sector the dimensions are

$$h_{p,q} = \frac{p(p+2) - (q \pm 1)^2}{4(m+2)} + \frac{1}{8} \ , \qquad Q = \frac{q \pm 1}{m+2} \mp \frac{1}{2}. \tag{1.40}$$

The unitary $N = 2$ superconformal field theories with $c < 3$ are the theories with finite number of superconformal fields. They were constructed in [36, 60, 61, 73, 74].
We remark that the unitary highest weight representations of the $N = 3$ and $N = 4$ superconformal algebras exist at the discrete values of central charge [26, 66].

$$\begin{aligned} N = 3: \quad & c = \frac{3k}{2}, \quad k = 1, 2, \cdots, \\ N = 4: \quad & c = 6k, \quad k = 1, 2, \cdots, \end{aligned}$$

and unlike $N = 0$, $N = 1$ and $N = 2$, here no continuum vaules of central charge are allowed by unitarity.
The moduli space of $N = 4$ unitary superconformal field theories with $c = 6$ has been studied by W. Nahm and K. Wendland [53], P. S. Aspinwall and D. R. Morrison [2], as well as N. Seiberg [65], and references therein.

1.6 Chiral primary fields

Left chiral and left antichiral fields in the NS Hilbert space defined as

$$G^+_{-\frac{1}{2}}|\phi\rangle = 0, \qquad G^-_{-\frac{1}{2}}|\phi\rangle = 0 \tag{1.41}$$

and similarly right chiral and right antichiral states are defined by replacing G with $\overline{G}$. Primary chiral states satisfy in addition to (1.41), the condition

$$G^-_{n+\frac{1}{2}}|\phi\rangle \;=\; G^+_{n+\frac{1}{2}}|\phi\rangle \;= 0 \qquad for \quad n \geq 0. \tag{1.42}$$

From (1.35) one writes

$$\{G^-_{\frac{1}{2}}, G^+_{-\frac{1}{2}}\} = 2L_0 - J_0,$$

where $\{G^-_{\frac{1}{2}}, G^+_{-\frac{1}{2}}\}$ is the positive operator since we are dealing with the unitary theory. By taking expectation value of this positive operator for any state in the NS Hilbert space one can get the inequality $h_\chi \geq \frac{Q_\chi}{2}$ and equality occurs precisely when χ is primary chiral field. Therefore conformal dimension of the holomorphic primary chiral fields is equal to the one-half of its charge $h_\phi = \frac{Q_\phi}{2}$. Similarly for a antichiral field one can deduce that $h_\phi = -\frac{Q_\phi}{2}$. One can also build another inequality for the dimension of a primary chiral and antichiral fields of the $N = 2$ superconformal theories by using the $N = 2$ superconformal algebra relation (1.35) in the particular case

$$\{G^-_{\frac{3}{2}}, G^+_{-\frac{3}{2}}\} = 2L_0 - 3J_0 + 2\frac{c}{3}. \tag{1.43}$$

If one takes the expectation value of this positive operator for any chiral primary state , and use $h_\phi = \frac{Q_\phi}{2}$ for such a state, one obtains

$$h \leq \frac{c}{6}. \tag{1.44}$$

Thus, there are finite number of chiral primary fields for a unitary nondegenerate $N = 2$ superconformal field theories since the conformal dimension of primary chiral field is less than or equal to $c/6$.

The left-right chiral primary fields denoted as (c, c); left antichiral and right chiral primary fields denoted as (a, c) and their complex conjugates are (a, a) and (c, a). Thus, in general, for $N = 2$ superconformal models there are four types of rings arising from the various combinations of left-right chiral and left-right antichiral fields. We denote these rings by (c, c), (a, a), (a, c), (c, a). They are pairwise conjugate. In $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifold models of $N = 2$ superconformal field theories with $c = 3$, and $N = 2$ Landau-Ginzburg models, one obtains only (c, c) and its conjugate (a, a) ring. For such models the (a, c) and (c, a) rings are trivial and consist only of the identity operator.We shall discuss this point in detail in chapter four.

Let us now consider the operator product (algebra) of two chiral primary fields. By dimensional analysis one can write

$$\phi_1(z)\phi_2(w) = \sum_i (z - w)^{h_{\psi_i} + h_{\phi_1} + h_{\phi_2}} \psi_i(w), \tag{1.45}$$

where h_{ψ_i} is the conformal dimension of the field ψ_i. The $U(1)$ charge of which is $Q_{\psi_i} = Q_{\phi_1} + Q_{\phi_2}$ and hence $\psi_i \geq h_{\phi_1} + h_{\phi_2}$. Now we can see that there are no singular terms in the operator product of two chiral primary fields. If $z \to w$ the only terms wich

survive on the right hand side of (1.45) are those for which ψ itself is chiral primary. Therefore the NS chiral primary fields yield a nonsingular and closed ring algebra. we will assume that the left right moving $U(1)$ charges $(Q, \overline{Q})$ of all NS and R states satisfy the following relation

$$Q - \overline{Q} \in \mathbb{Z}. \tag{1.46}$$

Let us consider now some aspects of the Ramond sector. As in the NS sector case, if one takes the expectation value of the anticommutator $\{G_0^-, G_0^+\}$ for any statte in the Ramond sector, one would get $h \geq \frac{c}{24}$. Equality holds only for states which are annihilated by both G_0^- and G_0^+. These are pricisely ground states of the Ramond sector which also contribute to the Witten index $Tr(-1)^F$ [71]. The operator $(-1)^F$, where $F = F_L + F_R$, and F_L, F_R are left-right moving fermion numbers, defined to anticommute with all the fermionic operators $(-1)^F \psi(z) = -\psi(z)(-1)^F$, and to commute with all the bosonic operators $(-1)^F \varphi(z) = \varphi(z)(-1)^F$, as well as to satisfy $((-1)^F)^2 = 1$. For $N = 2$ superconformal field theories which satisfy (1.46) the opertor $(-1)^F$, can be defined in terms of the $U(1)$ currents as

$$(-1)^F = e^{i\pi(J_0 - \overline{J}_0)}. \tag{1.47}$$

1.7 Spectral flow

It is well known that one can connect the Neveu-Schwarz sector to the Ramond sector by spectral flow [66] operation. It is the continuous transformation and has the following form

$$\begin{aligned} L_n^\theta &= L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0}, \\ J_n^\theta &= J_n + \frac{c}{3}\theta \delta_{n,0}, \\ G_r^{\pm\theta} &= G_{r\pm\theta}^{\pm}. \end{aligned} \tag{1.48}$$

The θ twisted operators L_n^θ, $G_r^{\pm\theta}$ and J_n^θ still satisfy the $N = 2$ superconformal algebra for an arbitrary value of the parameter θ. In particular, the zero mode eigenvalues h of L_0 and Q of J_0 are changed by spectral flow as

$$h_\theta = h + \theta Q + \theta^2 \frac{c}{6}\,, \qquad Q_\theta = Q + \theta \frac{c}{3}. \tag{1.49}$$

Under the spectral flow by $\theta = \frac{1}{2}$, the left-right ground states of the Ramond sector flow to the left-right chiral primary fields of the Neveu-Schwarz sector. The flow between the NS and NS as well as R and R can be obtained by the flow parameter $\theta = 1$. Besides, under the left-right symmetric spectral flow, $Q - \overline{Q}$ does not change. Thus the Witten's index [51] is

$$\begin{aligned} Tr(-1)^F &= Tr_R \left[(-1)^{J_0 - \overline{J}_0} q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right] \\ &= Tr_{\mathcal{H}_\theta} \left[(-1)^{J_0^\theta - \overline{J}_0^\theta} q^{L_0^\theta - \frac{c}{24}} \overline{q}^{\overline{L}_0^\theta - \frac{c}{24}} \right] \\ &= Tr_{NS} \left[(-1)^{J_0 - \overline{J}_0} q^{L_0 + \frac{1}{2}J_0} \overline{q}^{\overline{L}_0 + \frac{1}{2}\overline{J}_0} \right] = \sum_{\mathcal{R}} e^{i\pi(Q - \overline{Q})}, \end{aligned} \tag{1.50}$$

where the $\mathcal{H}_\theta$ in the second line is the Hilbert space of states which is twisted by the parameter θ. The $\mathcal{R}$ in the last line denotes the chiral ring. First line implies that the ground state of the Ramond sector gives nonvanishing contribution to the Witten index.

The secend line is obtained by applying the spectral flow to the first line. By setting $\theta = \frac{1}{2}$ one can flow from Ramond sector to the NS. (Note that $J_0^\theta - \overline{J}_0^\theta = J_0 - \overline{J}_0$). Thus the Witten index receives contributions from either the ground states of the Ramond sector or the chiral primary states in the NS sector. The only difference between the charges of the NS chiral primary states and that of the Ramond ground states is $\frac{c}{6}$. Therefore the relation between the $U(1)$ character valued degeneracy of the Ramond ground states and character valued sum over the ring $\mathcal{R}$ in the Neveu-Schwarz sector is

$$Tr_R \left[t^{J_0} \overline{t}^{\overline{J}_0} \right]_{G_0^\pm = \overline{G}_0^\pm = 0} = (t\overline{t})^{-\frac{c}{6}} Tr_{NS} \left[t^{J_0} \overline{t}^{\overline{J}_0} \right]_{\mathcal{R}} = (t\overline{t})^{-\frac{c}{6}} P(t, \overline{t}), \tag{1.51}$$

where t and $\overline{t}$ are independent variables. If we set $t = \overline{t}^{-1} = e^{i\pi}$ into (1.51) we get (1.50). Due to charge conjugation invariance of the ground state of the Ramond sector, (1.51) implies that Poincaré polinomial $P(t, \overline{t})$ of the conformal field theory

$$P(t, \overline{t}) = Tr_{\mathcal{R}} t^{J_0} \overline{t}^{\overline{J}_0} \tag{1.52}$$

satisfies the following duality relation

$$P(t, \overline{t}) = (t\overline{t})^{\frac{c}{3}} P(\frac{1}{t}, \frac{1}{\overline{t}}). \tag{1.53}$$

The duality property of the Poincaré polinomial implies that there exists an unique primary chiral field with highest left-right charges $Q = \overline{Q} = \frac{c}{3}$ and dimensions $h = \overline{h} = \frac{c}{6}$.

Chapter 2

Unitary Conformal Field Theories with Central Charge $c = 2$

In this chapter we study the moduli space $\mathcal{C}^2$ of unitary two–dimensional conformal field theories with central charge $c = 2$. The component $\mathcal{T}^2$ of the moduli space corresponding to compactification on a two–dimensional torus is well understood [13, 54]. One can conjecture that every theory in $\mathcal{C}^2$ either corresponds to compactification on a torus or on an orbifold thereof. It was stated in [14] that it is not difficult to classify all possible types of $c = 2$ (symmetric) orbifold models which can be obtained by modding out an automorphism group of a theory in $\mathcal{T}^2$. However, to our knowledge this analysis has not been carried out explicitly up to now.
This chapter is organized as follows: In section 2.1 we briefly review the features of $\mathcal{T}^2$ relevant to our studies. To find all nonisolated components of $\mathcal{C}^2$ we use the standard classification of crystallographic groups in two dimensions, which is discussed in section 2.3. In chapter 2.2 we will discuss symmetric points in $\mathcal{T}^2$. In section 2.4 we will give the prescription for the construction of orbifold theories. Section 2.5 contains a case by case study of all the 28 irreducible components of $\mathcal{C}^2$ obtained from $\mathcal{T}^2$ by modding out crystallographic groups. All consistent choices of the B–field on the original toroidal theory and the effect of discrete torsion are discussed, which also leads to some insight into the role of the B–field in a conformal field theory. We explicitly calculate the corresponding partition functions and determine the parameter space for each component.
Unitary two–dimensional field theories can be described as Minkowskian theories on the circle or equivalently as Euclidean theories on the torus with parameter σ in the upper half plane. The world sheet coordinates are called ξ_0, ξ_1, and we frequently use $z = e^{\xi_0 + \sigma \xi_1}, z \in Z$ to parametrize the worldsheet on an annulus $Z \subset \mathbb{C}^*$.

2.1 The moduli space $\mathcal{T}^2$ of toroidal theories

Let us briefly recall the structure of the moduli space $\mathcal{T}^2$ of theories corresponding to toroidal compactification in two dimensions (see also [14]). Consider a torus $T^2 = \mathbb{R}^2/\Lambda$, where $\Lambda \subset \mathbb{R}^2$ is a nondegenerate lattice with generators $\lambda_1, \lambda_2 \in \Lambda$. The nonlinear σ–model on T^2 describes two real massless scalar fields $\Phi^\mu : Z \to T^2, \mu \in \{1, 2\}$, governed by the action

$$S = \frac{1}{2\pi} \int_Z d^2 z \, (G_{\mu\nu} + B_{\mu\nu}) \partial \Phi^\mu(z, \overline{z}) \overline{\partial} \Phi^\nu(z, \overline{z}), \tag{2.1}$$

where we have set $\alpha' = 1$ by choosing a unit of length. Action (2.1) depends on four real parameters or moduli, the constant symmetric tensor $G_{\mu\nu} = \langle \lambda_\mu, \lambda_\nu \rangle$ defines the

metric on T^2 and the antisymmetric tensor $B_{\mu\nu} = -B_{\nu\mu}$ is known as B-field. The four dimensional parameter space of the background fields is called moduli space of the conformal quantum field theory with central charge $c = 2$. Each Φ^μ in (2.1) decomposes into a left- and a rightmoving part $\Phi^\mu(z,\overline{z}) = \frac{1}{2}(\varphi^\mu(z) + \overline{\varphi}^\mu(\overline{z}))$. The fields $j_\mu = i\partial\varphi^\mu$ are the two generic abelian $U(1)$ currents of the theory which generate translations along the coordinate axes of T^2. The energy momentum tensor is given in the Sugawara form

$$T = \frac{1}{2}(:j_1 j_1: + :j_2 j_2:), \quad \overline{T} = \frac{1}{2}(:\overline{j}_1\overline{j}_1: + :\overline{j}_2\overline{j}_2:). \tag{2.2}$$

In the following, we will work with φ^μ and $\overline{\varphi}^\mu$ separately, but the left–right transformed analogue of some statement will often not be mentioned explicitly in order to avoid tedious repetitions. The Virasoro mode operators for the bosons are given by

$$L_n = \sum_{m=-\infty}^{+\infty} \alpha^\mu_{n-m}\alpha^\mu_m \ , \quad \overline{L_n} = \sum_{m=-\infty}^{+\infty} \overline{\alpha}^\mu_{n-m}\overline{\alpha}^\mu_m, \tag{2.3}$$

where the oscillator operators satisfy the following commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta^{\mu\nu}\delta_{m+n}, \quad [\alpha^\mu_m, \overline{\alpha}^\nu_n] = 0, \quad {\alpha^\mu_m}^+ = \alpha^\mu_{-m}.$$

The Virasoro zero mode operators are given by

$$L_0 = \sum_{n>0} \alpha^\mu_{-n}\alpha^\mu_n + \frac{1}{2}p^2(\lambda,\mu), \quad \overline{L}_0 = \sum_{n>0} \overline{\alpha}^\mu_{-n}\overline{\alpha}^\mu_n + \frac{1}{2}\overline{p}^2(\lambda,\mu). \tag{2.4}$$

The Hilbert space $\mathcal{H}$ of our theory decomposes into an infinite number of sectors according to different winding and momentum numbers of the ground state. We label ground states with winding mode $\lambda = m_2\lambda_1 + m_1\lambda_2 \in \Lambda$ and momentum mode $\mu = n_2\mu_1 + n_1\mu_2 \in \Lambda^*$ by $|m_1, m_2, n_1, n_2\rangle$, where (μ_1, μ_2) is the basis dual to (λ_1, λ_2). The Hilbert space $\mathcal{H}$ is composed of all linear combinations of states of the form

$$\alpha^\mu_{-n_1} \cdots \alpha^\mu_{-n_l}\overline{\alpha}^\nu_{-n_{l+1}} \cdots \overline{\alpha}^\nu_{-n_k}|m_1, m_2, n_1, n_2>, \tag{2.5}$$

where $\alpha^\mu_n|m_1, m_2, n_1, n_2\rangle = 0$ and $\overline{\alpha}^\nu_m|m_1, m_2, n_1, n_2\rangle = 0$ for $n > 0$, $m > 0$. So the total number of states is given by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2(1-\overline{q}^n)^2}.$$

The left-right moving zero mode momentum p and $\overline{p}$ in (2.4) are defined as

$$(p(\lambda,\mu), \overline{p}(\lambda,\mu)) := \frac{1}{\sqrt{2}}(\mu - B\lambda + \lambda, \mu - B\lambda - \lambda) \tag{2.6}$$

The vertex operator corresponding to $|m_1, m_2, n_1, n_2\rangle$ is defined as

$$V_{\lambda,\mu} := c_{\lambda,\mu} : \exp[ip(\lambda,\mu)\cdot\varphi(z) + i\overline{p}(\lambda,\mu)\cdot\overline{\varphi}(\overline{z})] : \ . \tag{2.7}$$

where $c_{\lambda,\mu}$ is a cocycle factors In particular, $V_{\lambda,\mu}$ has charge $(p(\lambda,\mu), \overline{p}(\lambda,\mu))$ with respect to $(j_1, j_2, \overline{j}_1, \overline{j}_2)$, and by (2.4) the action of the zero modes L_0 and $\overline{L}_0$ on the ground states $|m_1, m_2, n_1, n_2\rangle$ are given by

$$\begin{aligned} L_0|m_1, m_2, n_1, n_2\rangle &= \frac{1}{2}p^2(\lambda,\mu)|m_1, m_2, n_1, n_2\rangle \\ \overline{L}_0|m_1, m_2, n_1, n_2\rangle &= \frac{1}{2}\overline{p}^2(\lambda,\mu)|m_1, m_2, n_1, n_2\rangle \end{aligned} \tag{2.8}$$

Hence our theory has partition function

$$Z_{\Lambda,B}(\sigma) = tr_{\mathcal{H}} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24} = \frac{1}{\eta^2 \overline{\eta}^2} \sum_{\lambda \in \Lambda, \mu \in \Lambda^*} q^{\frac{1}{2}(p(\lambda,\mu))^2} \overline{q}^{\frac{1}{2}(\overline{p}(\lambda,\mu))^2}, \tag{2.9}$$

By [13, 54] toroidal theories are determined uniquely by their charge lattice

$$\Gamma = \Gamma(\Lambda, B) := \{(p(\lambda,\mu), \overline{p}(\lambda,\mu)) \mid (\lambda,\mu) \in \Lambda \oplus \Lambda^*\}. \tag{2.10}$$

This is an even unimodular lattice in $\mathbb{R}^{2,2} = \mathbb{R}^2 \times \mathbb{R}^2$ which is equipped with the scalar product

$$(p, \overline{p}) \cdot (p', \overline{p}') := p \cdot p' - \overline{p} \cdot \overline{p}'.$$

The moduli space of toroidal conformal field theories with central charge $c = 2$ is

$$\mathcal{T}^2 = O(2) \times O(2) \backslash O(2,2;\mathbb{R}) / O(2,2;\mathbb{Z}) \tag{2.11}$$

[13, 54]. In the two–dimensional case it is convenient to group the four real parameters $G_{\mu\nu}, B_{\mu\nu}$ of the theory in terms of two parameters τ and ρ in the upper complex half plane as follows

$$\tau = \tau_1 + i\tau_2 := \frac{G_{12}}{G_{22}} + i\frac{\sqrt{\det(G_{\mu\nu})}}{G_{22}}, \quad \rho = \rho_1 + i\rho_2 := B_{12} + i\sqrt{\det(G_{\mu\nu})}. \tag{2.12}$$

Here τ represents the complex structure of the target space torus T^2, and ρ is its complexified Kähler structure; both take values on the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$. Now the generators $\lambda_1, \lambda_2 \in \Lambda$ are given by

$$\lambda_1 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \text{ and } B = \frac{\rho_1}{\rho_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

By (2.12) for $\lambda = m_2\lambda_1 + m_1\lambda_2 \in \Lambda$ and $\mu = n_2\mu_1 + n_1\mu_2 \in \Lambda^*$ as above (2.15) reads

$$\overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}} \left\{ \begin{pmatrix} n_2\tau_2 \\ -n_2\tau_1 + n_1 \end{pmatrix} + \rho_1 \begin{pmatrix} m_1\tau_2 \\ -m_2 - m_1\tau_1 \end{pmatrix} \overset{+}{_{(-)}} \rho_2 \begin{pmatrix} m_2 + m_1\tau_1 \\ m_1\tau_2 \end{pmatrix} \right\}. \tag{2.14}$$

We may also write (2.14) in the following form

$$\begin{aligned} p^2 &= \frac{1}{2\tau_2\rho_2} |n_1 - \tau n_2 - \rho(m_2 + \tau m_1)|^2 \\ \overline{p}^2 &= \frac{1}{2\tau_2\rho_2} |n_1 - \tau n_2 - \overline{\rho}(m_2 + \tau m_1)|^2. \end{aligned} \tag{2.15}$$

If (Λ, B) are related to (τ, ρ) by (2.12), for the partition function (2.9) we write

$$Z(\tau_1, \tau_2, \rho_1, \rho_2) := Z_{\Lambda,B}(\sigma) = \frac{1}{\eta^2 \overline{\eta}^2} \sum_{\lambda \in \Lambda, \mu \in \Lambda^*} q^{\frac{1}{2}(p(\lambda,\mu))^2} \overline{q}^{\frac{1}{2}(\overline{p}(\lambda,\mu))^2}. \tag{2.16}$$

Note that if $\tau_1 = \rho_1 = 0$, then the torus theory is a tensor product of two theories with $c = 1$ corresponding to compactification of single real bosons on circles of radii $r = \sqrt{G_{22}} = \sqrt{\rho_2/\tau_2}$ and $r' = \sqrt{G_{11}} = \sqrt{\rho_2\tau_2}$. The partition function (2.16) factorizes correspondingly:

$$Z(0, \tau_2, 0, \rho_2) = Z^{c=1}(\sqrt{\rho_2/\tau_2}) Z^{c=1}(\sqrt{\rho_2\tau_2}). \tag{2.17}$$

In terms of the new parameters (τ, ρ) the duality group $O(2,2;\mathbb{Z})$ in (2.11) translates into the group generated by $PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z})$. Its generators are

$$S : \rho \to \rho\ ,\ \ \tau \to -\frac{1}{\tau} \quad T : \rho \to \rho\ ,\ \ \tau \to \tau + 1$$

and

$$S' : \rho \to -\frac{1}{\rho}\ ,\ \ \tau \to \tau \quad T' : \rho \to \rho + 1\ ,\ \ \tau \to \tau.$$

The first two transormation S and T represents that the target space is a two dimensional torus, and thus unmodified by modular transformations of its complex structure. For $B = 0$ the third transformation S' takes the volume in target space into its invers. 2.15 exposes three more symmetries,

$$U(\tau, \rho) := (\rho, \tau),\ V(\tau, \rho) := (-\overline{\tau}, -\overline{\rho}). \tag{2.18}$$

The transformation U exchanges the complex and kähler structure of the target space torus T^2. The symmetry V changes the sign of G_{12} and B_{12}, while leaving G_{11} and G_{22}. The last symmetry is the world sheet parity ($\sigma \to -\sigma$) which interchanges p and $\overline{p}$. It is given by $B \to -B$ or $(\tau, \rho) \to (\tau, -\overline{\rho})$. Since this symmetry interchanges p with $\overline{p}$, it changes the sign of the Lorentzian norm $p^2 - \overline{p}^2$ and thus is not a duality symmetry. In terms of the parameters (τ, ρ) the moduli space (2.11) therefore is

$$\mathcal{T}^2 = (\mathbb{H}/PSL(2,\mathbb{Z}) \times \mathbb{H}/PSL(2,\mathbb{Z}))\,/(\mathbb{Z}_2 \times \mathbb{Z}_2). \tag{2.19}$$

2.2 Points of enhanced continous symmetry in $\mathcal{T}^2$

Classically, tori cannot have continnous symmetries beyond the translations, but for certain parameters quantum effects can enhance the symmetry. It is well known that for a particular value of the antisymmetric tensor field $B_{\mu\nu}$ and constant background metric $G_{\mu\nu}$ the four dimensional physical modular space contains points with enhanced gauge symmetry groups $SU(2)_L^2 \times SU(2)_R^2$ and $SU(3)_L \times SU(3)_R$. Let us discuss those special points somewhat in detail.
Torus has $SU(2)_L^2 \times SU(2)_R^2$ symmetry at $(\tau, \rho) = (i, i)$. Where we can read of from (2.8) the eigenvalues of L_0 and $\overline{L}_0$ for the $|m_1, m_2, n_1, n_2\rangle$ states as $\frac{1}{4}\left((m_1 \pm n_1)^2 + (m_2 \pm n_2)^2\right)$. For $(m_1, n_1, m_2, n_2) = (\pm 1, \pm 1, 0, 0)$ and $(0, 0, \pm 1, \pm 1)$, we find four $(1,0)$ states, and for $(\pm 1, \mp 1, 0, 0)$, $(0, 0, \pm 1, \mp 1)$ four $(0,1)$ states. In operator language these states are created by the operators

$$j_\mu^{\pm}(z) = e^{\pm i\sqrt{2}\varphi_\mu(z)}, \quad \overline{j}_\mu^{\pm}(\overline{z}) = e^{\pm i\sqrt{2}\overline{\varphi}_\mu(\overline{z})}, \quad \mu \in \{1, 2\}.$$

with conformal weights $(h, \overline{h}) = (1,0)$ and $(0,1)$. At arbitrary values of τ and ρ, we always have two $(1,0)$ and two $(0,1)$ oscillators states $\alpha_{-1}^\mu|0,0,0,0\rangle$ and $\overline{\alpha}_{-1}^\mu|0,0,0,0\rangle$, created by the operators

$$j_\mu(z) = i\partial\varphi_\mu(z), \quad \overline{j}_\mu(\overline{z}) = i\partial\overline{\varphi}_1(\overline{z}). \tag{2.20}$$

Each triple j_μ,$j_\mu^\pm$ generates an $SU(2)_1$ Kac-Moody algebra. At this $SU(2)_L^2 \times SU(2)_R^2$ symmetric point there exist sixteen operators with conformal dimension $(\frac{1}{2}, \frac{1}{2})$, eight of them have $\mathbb{Z}_2$ symmetry; there are thirtysix operators of dimension $(1,1)$ marginal operators, corresponding to combinations of the $SU(2)_L^2 \times SU(2)_R^2$ currents $j_\mu^i \overline{j}^l_\mu$, $i, l = 1, 2, 3$. But these are all related by $SU(2)_L^2 \times SU(2)_R^2$ to the four exactly marginal

operators $\partial\varphi^\mu(z)\overline{\partial}\overline{\varphi}^\mu(\overline{z})$. Twenty of them are invariant under $\mathbb{Z}_2$ transformation. At this special point, the torus partition function (2.17) takes the form

$$\begin{aligned} Z(0,1,0,1) &= \frac{1}{\eta\overline{\eta}}\sum_{n_1,m_1} q^{\frac{1}{4}(n_1+m_1)^2}\overline{q}^{\frac{1}{4}(n_1-m_1)^2}\frac{1}{\eta\overline{\eta}}\sum_{n_2,m_2} q^{\frac{1}{4}(n_2+m_2)^2}\overline{q}^{\frac{1}{4}}(n_2-m_2)^2 \\ &= Z^{c=1}(1)Z^{c=1}(1). \end{aligned} \tag{2.21}$$

We may also express $Z(\tau=\rho=i)$ in terms of theta functions. To do that, first we write $Z(1)$ in the form

$$\begin{aligned} Z(1) &= \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{4}(n+m)^2}\overline{q}^{\frac{1}{4}(n-m)^2} \\ &= \frac{1}{|\eta|^2}\sum_{\substack{n+m\\ n-m}\in 2\mathbb{Z}} q^{\frac{1}{4}(n+m)^2}\overline{q}^{\frac{1}{4}(n-m)^2} + \frac{1}{|\eta|^2}\sum_{\substack{n+m\\ n-m}\in 2\mathbb{Z}+1} q^{\frac{1}{4}(n+m)^2}\overline{q}^{\frac{1}{4}(n-m)^2} \\ &= \frac{1}{|\eta|^2}\sum_{\mu,\nu\in\mathbb{Z}} q^{\mu}\overline{q}^{\nu} + \frac{1}{|\eta|^2}\sum_{\mu,\nu\in\mathbb{Z}} q^{\mu+\frac{1}{2}}\overline{q}^{\nu+\frac{1}{2}} = \left(\left|\frac{\vartheta_3(2\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_2(2\sigma)}{\eta(\sigma)}\right|^2\right). \end{aligned}$$

By using $\vartheta_2(2\sigma)=\sqrt{\vartheta_3^2(\sigma)-\vartheta_4^2(\sigma)}$ and $\vartheta_3(2\sigma)=\sqrt{\vartheta_3^2(\sigma)+\vartheta_4^2(\sigma)}$, one writes

$$Z(0,1,0,1) = \frac{1}{2}\left(\left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right|^4 + \left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right|^4 + \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right|^4\right). \tag{2.22}$$

Torus line has an $SU(3)_L\times SU(3)_R$ symmetry at $(\tau,\rho)=(\frac{1}{2}+i\frac{\sqrt{3}}{2},\frac{1}{2}+i\frac{\sqrt{3}}{2})$. For $(m_1,n_1,m_2,n_2)=(\pm1,\pm1,0,0)$, $(0,\pm1,\pm1,\pm1)$, and $(\mp1,0,\pm1,\pm1)$, there are six dimension $(1,0)$ operators; for $(m_1,n_1,m_2,n_2)=(0,0,\pm1,\pm1)$, $(\mp1,\pm1,\pm1,0)$ and $(\mp1,\pm1,0,\pm1)$, there are six $(0,1)$ operators. Generically we always have two $(1,0)$ and two $(0,1)$ operators in (2.20).
Partition function (2.17) has simple form at $(\tau,\rho)=(i,2i)$

$$\begin{aligned} Z(0,1,0,2) &= \frac{1}{|\eta|^2}\sum_{n_1,m_1} q^{\frac{1}{2}(\frac{n_1}{2}+m_1)^2}\overline{q}^{\frac{1}{2}(\frac{n_1}{2}-m_1)^2} \\ &= \frac{1}{|\eta|^2}\sum_{n_2,m_2} q^{\frac{1}{2}(\frac{n_2}{2}+m_2)^2}\overline{q}^{\frac{1}{2}(\frac{n_2}{2}-m_2)^2} \\ &= Z^{c=1}(\sqrt{2})Z^{c=1}(\sqrt{2}). \end{aligned} \tag{2.23}$$

Since

$$\begin{aligned} Z^{c=1}(\sqrt{2}) &= \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{8}(n+2m)^2}\overline{q}^{\frac{1}{8}(n-2m)^2} \\ &= \frac{1}{|\eta|^2}\sum_{\substack{n\in 2\mathbb{Z}\\ m\in\mathbb{Z}}} q^{\frac{1}{8}(n+2m)^2}\overline{q}^{\frac{1}{8}(n-2m)^2} + \sum_{\substack{n\in 2\mathbb{Z}+1\\ m\in\mathbb{Z}}} q^{\frac{1}{8}(n+2m)^2}\overline{q}^{\frac{1}{8}(n-2m)^2} \\ &= \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}}\left(q^{\frac{1}{2}(n+m)^2}\overline{q}^{\frac{1}{2}(n-m)^2} + q^{\frac{1}{2}(n+\frac{1}{2}+m)^2}\overline{q}^{\frac{1}{2}(n+\frac{1}{2}-m)^2}\right) \\ &= \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}}\left(q^{\frac{1}{2}n^2}\overline{q}^{\frac{1}{2}m^2} + q^{\frac{1}{2}(n+\frac{1}{2})^2}\overline{q}^{\frac{1}{2}(m+\frac{1}{2})^2}\right)\frac{1+(-1)^{n+m}}{2} \\ &= \frac{1}{2}\left(\left|\frac{\vartheta_3(\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_4(\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_2(\sigma)}{\eta(\sigma)}\right|^2\right) = Z_{Dirac}. \end{aligned} \tag{2.24}$$

Thus the torus partition function (2.23) take the form

$$Z(0,1,0,2) = (Z^{c=1}(\sqrt{2}))^2 = (Z_{Dirac})^2. \tag{2.25}$$

At this point, there are eight operators of dimension $(\frac{1}{2},\frac{1}{2})$; four of them are $\mathbb{Z}_2$ invariant; twenty operators of dimension $(1,1)$, twelf of them are $\mathbb{Z}_2$ invariant. For $\tau_1 = \rho_1 = 0$, and $\tau_2 \neq \rho_2$ there is no $(1,0)$ and $(0,1)$ fields.
If $\tau_1 = \rho_1 = \frac{1}{2}$, $\tau_2 = \rho_2$, there are two $(1,0)$ states for $(m_1,n_1,m_2,n_2) = (0,\pm 1,\pm 1,\pm 1)$ and two $(0,1)$ states for $(0,0,-1,\pm 1)$. If $\tau_1 = \rho_1 = 0$, $\tau_2 = \rho_2$ there are two $(1,0)$ states for $(0,0,\pm 1,\pm 1)$ and two $(0,1)$ for $(0,0,-1,\pm 1)$. This implies that there is $SU(2)_L \times SU(2)_R$ symmetry.
If $\tau = \rho$ and $|\tau| = |\rho| \neq 1$, for $(0,0,-1,\pm 1)$ one can find two $(0,1)$ states. They yield $SU(2)_R$ symmetry. If $\tau = -\overline{\rho}$ and $|\tau| = |\rho| \neq 1$, we find two $(1,0)$ states for $(0,0,\pm 1,\pm 1)$. They yield $SU(2)_L$ symmetry.
Orbifolds by finite subgroups of the enhanced symmetry groups would merit seperate study. The following discussion concentrates on geometric symmetries.

2.3 Symmetries of the two–dimensional torus

To find nonisolated orbifold components of the moduli space $\mathcal{C}^2$ we must employ the orbifold procedure for all possible discrete symmetry groups of the torus. In two dimensions, there are seventeen inequivalent crystallographic space groups [59], i.e. discrete subgroups $G \subset O(2) \ltimes \mathbb{R}^2$ that leave invariant some lattice Λ' and therefore act on a torus $T^2 = \mathbb{R}^2/\Lambda$, where $\Lambda \subset \Lambda'$. Figure 2.1 shows all these symmetry groups by depicting the orbit of some symbol ▶ under G. Each lattice Λ' in figure 2.1 is formed by fixed combinations of the symbol ▶, which we call motive, in various orientations. Then $\Lambda \subset \Lambda'$ is given by those motives which have the same orientation. The space group G is a semi-direct product of a finite point group $P \subset O(2)$ (point group P is the discrete group of rotations) and a "translationary" group $\triangle \subset O(2) \ltimes \mathbb{R}^2$ of elements which do not fix the origin. In figure 2.1 the group Δ is the minimal subgroup of G which acts transitively on motives. The finite group P is determined by inspection of the particular motives which comprise the orbit of the symbol ▶ under P each.
By the above, P is an automorphism group of the two–dimensional lattice Λ, and if $(S,\delta) \in \Delta$, then there is some $N \in \mathbb{Z}$ such that $N\delta \in \Lambda$. Therefore if $A \in P$ has order M then $M \in \{2,3,4,6\}$. The values $M = 3$ or $M = 6$ require Λ to be a hexagonal lattice ($\tau = e^{2\pi i/3}$); $M = 4$ requires a square lattice ($\tau = i$). As to symmetry groups of order $M = 2$, $\mathbb{Z}_2$ acts by $x \mapsto -x$ as automorphism on every lattice Λ. Moreover, the reflection symmetry group $\mathbb{Z}_2(R)$ is an automorphism group of lattices with $\tau_1 \in \{0,\frac{1}{2}\}$, where R acts on the coordinates of T^2 by

$$R = R_1 : (x^1,x^2) \mapsto (x^1,-x^2) \text{ or } R = R_2 : (x^1,x^2) \mapsto (-x^1,x^2). \tag{2.26}$$

Translations $T_\delta = e^{2\pi i p \cdot \frac{\delta}{\sqrt{2}}}$ by $\delta \in \Lambda$ are the basic symmetries of the torus $T^2 = \mathbb{R}^2/\Lambda$. The result of modding out any torus by a translation symmetry T_δ, $N\delta \in \Lambda$, $N \in \mathbb{N}$ minimal with this property, gives another torus with lattice generated by Λ and δ. To produce a surface different from the torus (and later on non–toroidal conformal field theories), we must combine the translation with the reflection symmetry which we denote $T_R := Re^{2\pi i p \cdot \frac{\delta}{\sqrt{2}}}$. More precisely, we will need this symmetry only in the case $\tau_1 = 0$ and

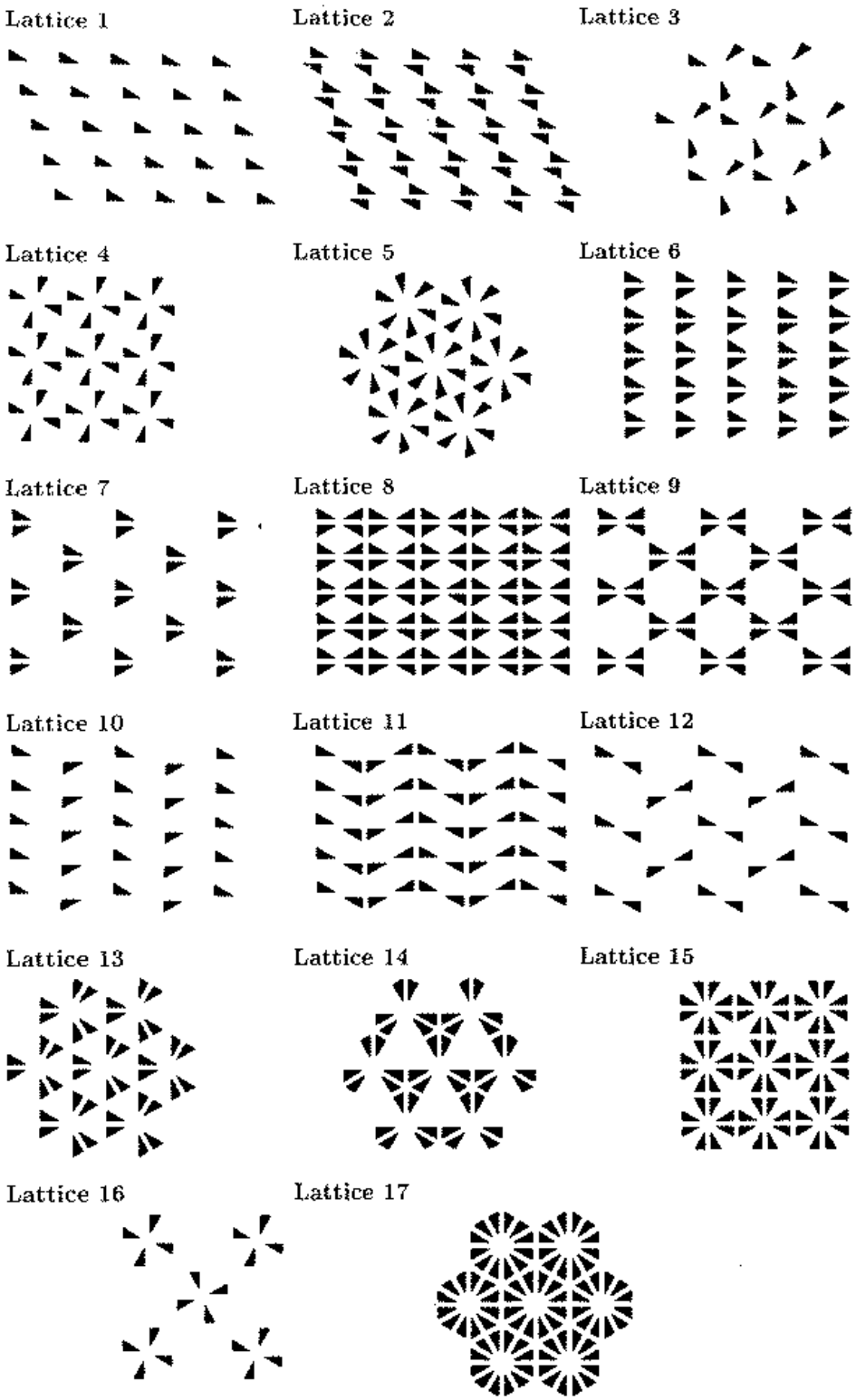

Figure 2.1: The seventeen inequivalent crystallographic space groups [6].

$N = 2$, and we set

$$\begin{array}{rclcrclcrcl} \delta_1 & := & \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}1/2\\0\end{pmatrix}, & & \delta_2 & := & \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}0\\\tau_2/2\end{pmatrix}, & & \delta' & := & \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix}1/2\\\tau_2/2\end{pmatrix}; \\ T_{R_\mu} & := & R_\mu e^{2\pi i p\cdot\frac{\delta_\mu}{\sqrt{2}}} & & T'_{R_\mu} & := & R_\mu e^{2\pi i p\cdot\frac{\delta'}{\sqrt{2}}}, & & \widehat{T}_{R_2} & := & R_2 e^{2\pi i p\cdot\frac{\delta_1}{\sqrt{2}}}. \end{array} \tag{2.27}$$

where $\mu \in \{1,2\}$. The groups of type $\mathbb{Z}_2$ generated by T_R or T'_R are denoted $\mathbb{Z}_2(T_R)$ or $\mathbb{Z}_2(T'_R)$, respectively, where either $R = R_1$ or $R = R_2$. We denote by $A(\theta) \in \mathbb{Z}_M$ the rotation by an angle of θ. Then $R_2 = A(\pi)R_1$, $T^{(\prime)}_{R_2} = A(\pi)T^{(\prime)}_{R_1}$, and we have the noncyclic crystallographic groups

$$\begin{array}{rcl} D_2 & := & \{1, A(\pi), R_1, R_2\}, \\ D_3(R) & := & \mathbb{Z}_3 \cup R\mathbb{Z}_3, \\ D_4 & := & \mathbb{Z}_4 \cup R_1\mathbb{Z}_4 = \mathbb{Z}_4 \cup R_2\mathbb{Z}_4, \\ D_6 & := & \mathbb{Z}_6 \cup R_1\mathbb{Z}_6 = \mathbb{Z}_6 \cup R_2\mathbb{Z}_6, \\ D_2(T_R) & := & \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}, \\ D_2(T'_R) & := & \{1, A(\pi), T'_{R_1}, T'_{R_2}\}, \\ D_4(T'_R) & := & \mathbb{Z}_4 \cup T'_{R_1}\mathbb{Z}_4 = \mathbb{Z}_4 \cup T'_{R_2}\mathbb{Z}_4. \end{array}$$

The symmetries that correspond to the lattices in figure 2.1 are

Lattice	1	2	3	4	5	6	7	8	9
Symmetries	$\{T_\delta, \delta \in \Lambda\}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_2(R)$	$\mathbb{Z}_2(R)$	D_2	D_2

Lattice	10	11	12	13	14	15	16	17
Symmetries	$\mathbb{Z}_2(T_R)$	$D_2(T_R)$	$D_2(T'_R)$	$D_3(R_1)$	$D_3(R_2)$	D_4	$D_4(T'_R)$	D_6

. (2.28)

2.4 General orbifold prescription

The notion of *orbifold* arises when one considers a manifold that has a discrete symmetry group G. One may consider a new manifold M/G which is obtained from old one by modding out the discrete symmetry group. If the group G acts freely on M in other words if M has no fixed points under the group G action then M/G is a smooth manifold. If M has fixed points under the group G action, then M/G is no longer a smooth manifold. For example, consider a real line $\mathbb{R} \equiv M$, coordinatised by x, with group action $G = \mathbb{Z}_2$: $\mathbb{R} \to \mathbb{R}$ defined by the generator $g :\ x \to -x$. This group action has fixed points at $x = 0$. The orbifold $\mathbb{R}/\mathbb{Z}_2$ is the half line with an orbifold point at the boundary $x = 0$. The real line has another discrete infinite symmetry group, namely, translation T: $x \to x + 2\pi r$. This symmetry is freely acting on $\mathbb{R}$ i.e. the real line has no fixed points under the translation, and the resulting orbifold is a smooth manifold, namely a circle of radius r. Our aim is to construct modular invariant 2-dimensional orbifold theory from a given modular invariant theory, whose Hilbert space admits a discrete symmetry group G. To do that we now give the general procedure for the construction of orbifold theories by modding out a symmetry group G of a conformal field theory with central charge c by following the articles [18, 20, 21, 22].

Let $\mathcal{H}$ be the Hilbert space of an orbifold theory. It has two sectors, namely untwisted and twisted sector, i.e, $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_t$. Let us consider first the untwisted sector of the orbifold theory. The untwisted Hilbert space $\mathcal{H}_u$ will be a subspace of the Hilbert space for the conformal field theories with c. In the path integral for the partition function

this means that the fields obey periodic boundary conditions along the space direction of the torus and twisted periodic boundary conditions in time. So on an orbifold, the untwisted sector boundary conditions on the field are given as

$$\begin{aligned} \varphi(1) &= \varphi(0) + 2\pi\Lambda \\ \varphi(\sigma) &= g\varphi(0) + 2\pi\Lambda \end{aligned} \tag{2.29}$$

where $g \in G$. The untwisted Hilbert space $\mathcal{H}_u$ decomposes into G invariant and non-invariant space of states. In oder to construct consistent models, we must project out the group noninvariant space of states. In the Hamiltonian formalism, group invariant states are obtained by insertion of the projection operator $P = \frac{1}{|G|}\sum_{g\in G} g$ into the trace over states. Here $|G|$ is the number of elements in G and the sum $\sum g$ runs over all elements in G. Thus the untwisted sector partition function is

$$Z_u = tr_{\mathcal{H}_u} P q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \tag{2.30}$$

Here $tr_{\mathcal{H}_u}$ denote the trace in the untwisted Hilbert space sectors . In the path integral formalism, projection on the group invariant states in the untwisted sector is represented as

$$Z_u = \frac{1}{|\mathbb{Z}_M|}\sum_{g\in\mathbb{Z}_M} g\begin{array}{|c|}\hline\ \ \\ \hline 1 \\ \hline\end{array},$$

where we sum over all possible twistings in the time direction of the torus. $g\begin{array}{|c|}\hline\ \ \\ \hline 1 \\ \hline\end{array}$ represents boundary conditions on any generic fields in the theory twisted by g in the time direction of the torus. The partition function of the original model is simply given by $Z = 1\begin{array}{|c|}\hline\ \ \\ \hline 1 \\ \hline\end{array}$.

The untwisted sector partition function is not modular invariant. To gain modular invariant partition function, we therefore need to consider the contributions of twisted sector Hilbert space of states. For G abelian, the twisted Hilbert space decomposes into a set of twisted sectors labeled by $h \in G$, and in each twisted sector there is a projection onto G invariant states. If G is not abelian, the twisted Hilbert space decomposes into a set of twisted sectors labeled by conjugacy classes $\{h\}$ of G. In the path integral description the a generic field obey the following twisted boundary conditions

$$\begin{aligned} \varphi(1) &= h\varphi(0) + 2\pi\Lambda \\ \varphi(\sigma) &= g\varphi(0) + 2\pi\Lambda. \end{aligned} \tag{2.31}$$

where h and g are twists on the fields in the space and time direction of the torus. The twisted Hilbert space $\mathcal{H}_t$ decomposes into G invariant and noninvariant space of states. To construct consistent models, we again have to project onto group invariant states. In the Hamiltonian formalism, group invariant states are obtained by insertion of the projection operator $P_h := \frac{1}{|G|}\sum_{g\in G:\,[g,h]=0} g$ into the trace over states. The prefactor is adjusted correctly in order to take care of the multiplicities in each twisted sector. Namely, it takes care of overcounting if later on we sum over all $h \in G, h \neq 1$ instead of conjugacy classes $\{h\}$ which actually label twisted sectors by the above (see [20]). In the path integral formalism, projection onto group invariant states in the twisted sector is representes as

$$Z_t = \frac{1}{|G|}\sum_{\substack{g,h\in G,\\ h\neq 1,[g,h]=0}} g\begin{array}{|c|}\hline\ \ \\ \hline h \\ \hline\end{array}.$$

Thus the twisted sector partition function has the form

$$Z_t = \sum_{h\in G, h\neq 1} tr_{\mathcal{H}_h} P_h q^{L_0-\frac{c}{24}} \overline{q}^{\overline{L}_0-\frac{c}{24}} = \frac{1}{|G|} \sum_{\substack{g,h\in G,\\ h\neq 1,[g,h]=0}} g\underset{h}{\square}. \tag{2.32}$$

The total modular invariant orbifold partition function is

$$Z_{G-orb} = \sum_{f\in G} tr_{\mathcal{H}_h} P_h q^{L_0-\frac{c}{24}} \overline{q}^{\overline{L}_0-\frac{c}{24}} = \frac{1}{|G|} \sum_{\substack{g,h\in G,\\ [g,h]=0}} g\underset{h}{\square}, \tag{2.33}$$

where we set $\mathcal{H}_1 := \mathcal{H}$ and $P_1 := P$. For general $h, g \in G$ the contribution $g\underset{h}{\square}$ can also be calculated by using modular transformations:

$$g\underset{h}{\square}(-\frac{1}{\sigma}) = h\underset{g}{\square}(\sigma)\ , \quad g\underset{h}{\square}(\sigma+1) = h\circ g\underset{h}{\square}(\sigma). \tag{2.34}$$

In summary, in order to construct an orbifold model, one first formulates the Hilbert space of states on the torus, then one projects onto the group invariant states, finally one includes twisted sector contributions. For more details see ref.[21] [22] [18].

2.5 Sixteen orbifolds of the torus theories with $c = 2$

In (2.28) we have listed all the seventeen possible symmetry groups G of a two–dimensional torus $T^2 = \mathbb{R}^2/\Lambda$. Because the first of them, corresponding to lattice 1, is the translation group $G \cong \Lambda$ which acts trivially on T^2, this implies that we can construct at most sixteen different types of orbifold theories corresponding to different compactifications on T^2/G. To do so, we must show that these symmetries can be continued to symmetries of the corresponding two-dimensional conformal field theories. Since the action of $g \in G$ on the abelian currents j_μ, which generate translations along the coordinate axes of T^2, is determined by the action on T^2, this amounts to continuing every $g \in G$ to a symmetry of the charge lattice (2.10). By (2.14) it is easy to see that this is possible iff $B = g^T B g$. In particular, any of the symmetries listed in (2.28) which corresponds to a lattice characterized by parameters $\tau \in \mathbb{H}$ and $\rho_2 \in \mathbb{R}^+$ immediately gives a symmetry of the toroidal conformal field theory with parameters $(\tau, 0, \rho_2)$, i.e. $B = 0$. But nonzero values for ρ_1 might be possible, too. Note in particular that ρ_1, as parameter in $\mathcal{T}^2$, is only defined modulo $\mathbb{Z}$. In other words, g can be continued to a symmetry of the toroidal conformal field theory iff

$$B = g^T B g + \frac{n}{\rho_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ n \in \mathbb{Z}. \tag{2.35}$$

Below, we will discuss all possible B–field values for each of the symmetry groups listed in (2.28).

Lattices 2 to 5: $\mathbb{Z}_M$ Orbifold theories

In the following, let γ be a generator of $\mathbb{Z}_M$ and assume $T^2 = \mathbb{R}^2/\Lambda$ to be a torus with $\mathbb{Z}_M$ symmetry, where Λ is characterized by specific values of τ and ρ_2 as explained in section 2.1. By the discussion in section 2.3 this means that $\tau = e^{2\pi i/M}$ if $M \in \{3, 4, 6\}$, and $\rho_2 \in \mathbb{R}^+$ arbitrary, whereas $\mathbb{Z}_2$ is a symmetry for every torus. Because by (2.13) γ

commutes with B for any value of ρ_1, from (2.35) we know that every toroidal conformal field theory with parameters $(\tau,\rho) \in \mathbb{H}\times\mathbb{H}$, $\tau = e^{2\pi i/M}$ for $M \in \{3,4,6\}$, has $\mathbb{Z}_M$ symmetry. The action of the rotation group $\mathbb{Z}_M$ on the bosonic fields and on its modes are

$$(\gamma^k\varphi^{\pm})(z) = e^{\pm\frac{2\pi ik}{M}}\varphi^{\pm}(z)\ , \quad \gamma^k\alpha_n^{\pm}\gamma^{-k} = e^{\pm\frac{2\pi ik}{M}}\alpha_n^{\pm}, \quad k\in\{1,2,\cdots,M-1\}. \tag{2.36}$$

The mode expansion of the bosonic field which satisfies the boundary conditions (2.31) is

$$\varphi^+(z) = q_f^+ + i\sum_{n\in\mathbb{Z}+k/M}\frac{1}{n}\alpha_n^+ z^{-n}. \tag{2.37}$$

One can not have nonzero momentum or winding number here, since they are not consistent with the twisted boundary conditions. In this mode expansion q_f^+ denote the fixed points of T^2 under the $h\in G$. The index f labels these fixed points. So the corresponding bosonic twisted ground state has conformal dimensions

$$h_t = \overline{h}_t = \frac{1}{2}\frac{1}{M}\left(1-\frac{1}{M}\right) \tag{2.38}$$

(see also [20]). The action of $\mathbb{Z}_M$ on the charge lattice (2.10) is given by

$$\gamma\in\mathbb{Z}_M\ , \quad \gamma : (p,\overline{p}) \mapsto (\gamma p, \gamma\overline{p}). \tag{2.39}$$

It follows that the $\mathbb{Z}_M$ action commutes with Möbius transformations on ρ. The $\mathbb{Z}_2$ action commutes with the entire $PSL(2,\mathbb{Z})^2\times\mathbb{Z}^2$ of (2.19), so for the families of $\mathbb{Z}_M$ orbifold conformal field theories with $c=2$ we get the following irreducible components of $\mathcal{C}^2$:

$$\begin{aligned} & & \mathcal{C}_{\mathbb{Z}_2-orb} &\cong \mathcal{T}^2, \\ \text{for } M\in\{3,4,6\}: & & \mathcal{C}_{\mathbb{Z}_M-orb} &= \left\{(\tau,\rho)\mid \tau = e^{2\pi i/M}, \rho\in\mathbb{H}/PSL(2,\mathbb{Z})\right\} \\ & & &\cong \mathbb{H}/PSL(2,\mathbb{Z}). \end{aligned} \tag{2.40}$$

As will see, there is no discrete torsion for the $\mathbb{Z}_M$ orbifolds, since all boxes ${}_g\square_h$ are related by modular tranformations to a box of type ${}_g\square_1$. Mathematically, the discrete torsion for a discrete group G is obtained from the cohomology $H_2(G)$, which vanishes for $G=\mathbb{Z}_M$ [68].

Lattice 2: The $\mathbb{Z}_2$ orbifold

By (2.40) lattice 2 depicts an arbitrary lattice. We may now produce another family of theories, i.e. the $G=\mathbb{Z}_2$ orbifold conformal field theories with the same set of moduli as the torus theories by following the general orbifold prescription introduced in previous section. The action of $\gamma\in\mathbb{Z}_2$ on the bosonic Hilbert space sectors $|m_1,m_2,n_1,n_2\rangle$ is given by

$$\gamma|m_1,m_2,n_1,n_2\rangle = |-m_1,-m_2,-n_1,-n_2\rangle. \tag{2.41}$$

Under the $\mathbb{Z}_2$ symmetry the untwisted bosonic Hilbert spaces $\mathcal{H}_u$ decomposes into $\gamma=\pm1$ eigenspaces $\mathcal{H}_u = \mathcal{H}_u^+\oplus\mathcal{H}_u^-$ as

$$\mathcal{H}_u^+ \ = \ \{\alpha_{-k_1}^+\cdots\alpha_{-k_l}^+\overline{\alpha}_{-k_{l+1}}^+\cdots\overline{\alpha}_{-k_{2j}}^+(1+\gamma)|m_1,m_2,n_1,n_2\rangle\}$$

$$
\begin{aligned}
& + \{\alpha^+_{-k_1}\cdots\alpha^+_{-k_l}\overline{\alpha}^+_{-k_{l+1}}\cdots\overline{\alpha}^+_{-k_{2j+1}}(1-\gamma)|m_1,m_2,n_1,n_2\rangle\} \\
\mathcal{H}^-_u &= \{\alpha^+_{-k_1}\cdots\alpha^+_{-k_l}\overline{\alpha}^+_{-k_{l+1}}\cdots\overline{\alpha}^+_{-k_{2j+1}}(1+\gamma)|m_1,m_2,n_1,n_2\rangle\} \\
& +\{\alpha^+_{-k_1}\cdots\alpha^+_{-k_l}\overline{\alpha}^+_{-k_{l+1}}\cdots\overline{\alpha}^+_{-k_{2j}}(1-\gamma)|m_1,m_2,n_1,n_2\rangle\},
\end{aligned}
$$

where k_i takes positive integer values. By (2.30), untwisted $\mathbb{Z}_2$ orbifold partition function is

$$
Z_u = (q\overline{q})^{-\frac{1}{8}} tr_{\mathcal{H}_u}\frac{1}{2}(1+\gamma)q^{L_0}\overline{q}^{\overline{L}_0}.
$$

The first term in the trace is equal to the partition function in (2.16) since there is no twist along the two cycles of the torus; the second term in the trace with γ inserted receives only contribution from the sector $m_1 = m_2 = n_1 = n_2 = 0$ because each state obtained by acting on $(1+\gamma)|m_1,m_2,n_1,n_2\rangle$ with creation operators has a counter part with the same L_0 eigenvalue obtained by acting on $(1-\gamma)|m_1,m_2,n_1,n_2\rangle$ with the same creation operators; however, these two states have opposite eigenvalues under $\gamma \in \mathbb{Z}_2$, and their contributions cancel in the trace. Thus, only the states obtained by acting creation operators α^+_{-k} or $\overline{\alpha}^+_{-k}$ on the vacuum $|0,0,0,0\rangle$ will contribute. Therefore the overall untwisted sector partition function is

$$
Z_u = \frac{1}{2}\left(Z(\tau,\rho) + \frac{(q\overline{q})^{-\frac{1}{12}}}{\prod_{n=1}^{\infty}(1+q^n)^2(1+\overline{q}^n)^2}\right) = \frac{1}{2}\left(Z(\tau,\rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right).
$$

Under the symmetry action $\gamma : \varphi^+ \to -\varphi^+$ the torus has four fixed points. By (2.38), there are four twisted ground states with conformal dimension $h = \overline{h} = 1/8$. So one may build four distinct Hilbert space sectors. However, these sectors lead to isomorphic physics, as they are related by translation symmetry of the torus. Denote the four twisted sector ground states by $|\frac{1}{8},\frac{1}{8}\rangle_f$, where $f = 1,2,3,4,$. As untwisted bosonic Hilbert space sector, the twisted bosonic Hilbert space decomposes into $\gamma = \pm 1$ eigenspaces $\mathcal{H}_t = \mathcal{H}^+_t \oplus \mathcal{H}^-_t$ as

$$
\begin{aligned}
\mathcal{H}^+_t &= \alpha^+_{-k_1}\cdots\alpha^+_{-k_l}\overline{\alpha}^+_{-k_{l+1}}\cdots\overline{\alpha}^+_{-k_{2j}}|\frac{1}{8},\frac{1}{8}\rangle_f \\
\mathcal{H}^-_t &= \alpha^+_{-k_1}\cdots\alpha^+_{-k_l}\overline{\alpha}^+_{-k_{l+1}}\cdots\overline{\alpha}^+_{-k_{2j+1}}|\frac{1}{8},\frac{1}{8}\rangle_f.
\end{aligned}
$$

where k_i takes half positive integer values. By (2.32), the twisted sector partition function is

$$
\begin{aligned}
Z_t &= (q\overline{q})^{-\frac{1}{12}} tr_{\mathcal{H}_t}\frac{1}{2}(1+\gamma)q^{L_0}\overline{q}^{\overline{L}_0} \\
&= 4\cdot\frac{1}{2}(q\overline{q})^{-\frac{1}{12}}\left(\left|q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^{n-1/2})^{-2}\right|^2 + \left|q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1+q^{n-1/2})^{-2}\right|^2\right) \\
&= 4\cdot\frac{1}{2}\left(\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + \left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2\right) = \frac{1}{2}\left(2Z(0,1,0,2) - Z(0,1,0,1)\right).
\end{aligned}
\tag{2.42}
$$

The complete $\mathbb{Z}_2$ orbifold partition function is

$$
\begin{aligned}
Z_{\mathbb{Z}_2-orb} &= \frac{1}{2}\left(Z(\tau,\rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2\right) \\
&= \frac{1}{2}\left(Z(\tau,\rho) + 2Z(0,1,0,2) - Z(0,1,0,1)\right)
\end{aligned}
\tag{2.43}
$$

Lattice 3: The $\mathbb{Z}_3$ orbifold

Lattice 3 has $\tau = e^{2\pi i/3}$ and by (2.40) we may pick arbitrary $\rho \in \mathbb{H}/PSL(2,\mathbb{Z})$. The action of $\gamma \in \mathbb{Z}_3$ on the bosonic Hilbert space sectors is given by

$$\gamma|m_1, m_2, n_1, n_2\rangle = |m_2, -m_1 - m_2, n_2 - n_1, -n_1\rangle. \tag{2.44}$$

By (2.30), the untwisted sector partition function is

$$Z_u = (q\overline{q})^{-\frac{1}{12}} tr_{\mathcal{H}_u} \frac{1}{3}(1+\gamma+\gamma^2) q^{L_0}\overline{q}^{\overline{L}_0}.$$

By taking into account the equations (2.36), and (2.44), the first term in the trace is equal to the original patition function (2.16), the second and third term receives only contribution from the Hilbert space of states obtained by acting creation operators α^+_{-k} or $\overline{\alpha}^+_{-k}$ on the vacuum $|0,0,0,0\rangle$. Therefore the untwisted sector partition function is

$$\begin{aligned} Z_u &= \frac{1}{3}\left(Z(\tau = e^{2\pi i/3}, \rho) \; + \; \frac{(q\overline{q})^{-\frac{1}{12}}}{\sum_{l=1}^{2}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{3}} q^n\right)\left(1 - e^{-\frac{2\pi i l}{3}} q^n\right)\right|^2} \right) \\ &= \frac{1}{3}\left(Z(\tau = e^{2\pi i/3}, \rho) + 6\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 \right). \end{aligned}$$

$\mathbb{Z}_3$ does not act freely on the hexagonal torus. Thus one must consider new sectors, the twisted ones. In the $T^2/\mathbb{Z}_3$ ($\tau = e^{2\pi i/3}$) manifold, there are three fixed points, and one can obtain three Hilbert space sectors corresponding to the expansion of the field about each of these fixed points. However these three sectors give the same physics. Thus by (2.38), there are three twisted ground states of dimensions $(h, \overline{h}) = (1/9, 1/9)$ in each of the twisted sectors $\mathcal{H}_\gamma$ and $\mathcal{H}_{\gamma^2}$. By (2.32), the twisted sector partition function is

$$\begin{aligned} Z_t &= \frac{1}{3}\sum_{l=1}^{3}\left(A(\tfrac{2\pi l}{3}) \underset{A(2\pi/3)}{\square} \; + \; A(\tfrac{2\pi l}{3}) \underset{A(4\pi/3)}{\square} \right) \\ &= \frac{1}{3}\left(\frac{(q\overline{q})^{\frac{1}{36}}}{\sum_{l=1}^{3}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{3}} q^{n-\frac{1}{3}}\right)\left(1 - e^{-\frac{2\pi i l}{3}} q^{n-\frac{2}{3}}\right)\right|^2} \right. \\ &\quad \left. + \; \frac{(q\overline{q})^{\frac{1}{36}}}{\sum_{l=1}^{3}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{3}} q^{n-\frac{2}{3}}\right)\left(1 - e^{-\frac{2\pi i l}{3}} q^{n-\frac{1}{3}}\right)\right|^2} \right) \\ &= 2\cdot 3\cdot \frac{1}{3}(q\overline{q})^{-\frac{1}{18}} \sum_{l=1}^{3}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{3}+\frac{l}{3},\sigma)}\right|^2. \end{aligned} \tag{2.45}$$

The complete $\mathbb{Z}_3$ orbifold partition function is

$$Z_{\mathbb{Z}_3-orb} = \frac{1}{3}\left(Z(\tau = e^{\frac{2\pi i}{3}}, \rho) + 6\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 + 6(q\overline{q})^{-\frac{1}{18}}\sum_{l=1}^{3}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{3}+\frac{l}{3},\sigma)}\right|^2 \right). \tag{2.46}$$

Lattice 4: The $\mathbb{Z}_4$ orbifold

Lattice 4 has $\tau = i$, and by (2.40) we may pick arbitrary $\rho \in \mathbb{H}/PSL(2,\mathbb{Z})$. The action of $\gamma \in \mathbb{Z}_4$ on the bosonic ground state sectors is given as

$$\gamma|m_1, m_2, n_1, n_2\rangle = |m_2, -m_1, n_2, -n_1\rangle. \tag{2.47}$$

By (2.30), and taking into account of the equations (2.36), and (2.47), the untwisted sector partition function of the $\mathbb{Z}_4$ orbifold can be written as

$$\begin{aligned} Z_u &= \frac{1}{4}\left(Z(\tau = i, \rho) \; + \; \frac{(q\overline{q})^{-\frac{1}{12}}}{\sum_{l=1}^{3}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{4}}q^n\right)\left(1 - e^{-\frac{2\pi i l}{4}}q^n\right)\right|^2}\right) \\ &= \frac{1}{4}\left(Z(\tau = i, \rho) \; + \; 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{4}, \sigma)}\right|^2 \; + \; 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right). \end{aligned}$$

Tori with $\tau = i$ have three fixed points under the rotation group $\mathbb{Z}_4$, one of which corresponds to a $\mathbb{Z}_2$ twist and two to $\mathbb{Z}_4$ twists. Hence the total $\mathbb{Z}_4$ orbifold partition function is the sum of untwisted, $\mathbb{Z}_2$, and $\mathbb{Z}_4$ twisted sector partition functions

$$Z_{\mathbb{Z}_4-orb}(\tau = i, \rho) = Z_u \; + \; Z_{2t} \; + \; Z_{4t}.$$

The $\mathbb{Z}_2$ twisted sector partition function Z_{2t} can be read off from (2.42) by omitting the factor of four. By (2.38), the two ground states in each of the twisted sectors $\mathcal{H}_\gamma$, $\mathcal{H}_{\gamma^2}$ have dimensions $(h, \overline{h}) = (3/32, 3/32)$. By (2.32), and considering (2.37), (2.36), as well as (2.47), the $\mathbb{Z}_4$ twisted sector partition function therefore is

$$\begin{aligned} Z_{4t} &= \frac{1}{4}\sum_{l=1}^{4}\left(A(\tfrac{2\pi l}{4}) \underbrace{\square}_{A(\pi/2)} \; + \; A(\tfrac{2\pi l}{4}) \underbrace{\square}_{A(\pi)} \; + \; A(\tfrac{2\pi l}{4}) \underbrace{\square}_{A(3\pi/2)}\right) \\ &= \frac{1}{4}\left(\frac{(q\overline{q})^{\frac{1}{96}}}{\sum_{l=1}^{4}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{4}}q^{n-\frac{1}{4}}\right)\left(1 - e^{-\frac{2\pi i l}{4}}q^{n-\frac{3}{4}}\right)\right|^2}\right. \\ &\qquad + \frac{(q\overline{q})^{\frac{1}{24}}}{\sum_{l=1}^{4}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{4}}q^{n-\frac{1}{2}}\right)\left(1 - e^{-\frac{2\pi i l}{4}}q^{n-\frac{1}{2}}\right)\right|^2} \\ &\qquad \left. + \frac{(q\overline{q})^{\frac{1}{96}}}{\sum_{l=1}^{4}\prod_{n=1}^{\infty}\left|\left(1 - e^{\frac{2\pi i l}{4}}q^{n-\frac{3}{4}}\right)\left(1 - e^{-\frac{2\pi i l}{4}}q^{n-\frac{1}{4}}\right)\right|^2}\right) \\ &= \frac{1}{4}\left(4\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{4}, \sigma)}\right|^2 \; + \; 2\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2 \; + \; 2\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2\right. \\ &\qquad \left. + \; 4(q\overline{q})^{-\frac{1}{32}}\sum_{l=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{4} + \frac{l}{4}, \sigma)}\right|^2\right). \end{aligned}$$

Altogether, we find

$$Z_{\mathbb{Z}_4-orb} = \frac{1}{4}\left(Z(\tau = i, \rho) + 4\sum_{i=2}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{4}, \sigma)}\right|^2\right.$$

$$+4\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{4},\sigma)}\right|^2+4(q\overline{q})^{-\frac{1}{32}}\sum_{l=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{\sigma}{4}+\frac{l}{4},\sigma)}\right|^2\Bigg) \tag{2.48}$$

Lattice 5: The $\mathbb{Z}_6$ orbifold

Lattice 5 has $\tau = e^{2\pi i/3}$, and by (2.40) we may pick arbitrary $\rho \in \mathbb{H}/(PSL(2,\mathbb{Z})$. The bosonic ground state sectors transform as follows under the action of $\gamma \in \mathbb{Z}_6$

$$\gamma|m_1,m_2,n_1,n_2\rangle = |m_1+m_2,-m_1,n_2,-n_1+n_2\rangle. \tag{2.49}$$

By (2.30), (2.36), and (2.49) the untwisted sector partition function is

$$\begin{aligned} Z_u &= \frac{1}{6}\left(Z(\tau=e^{\frac{2\pi i}{3}},\rho)\ +\ \frac{(q\overline{q})^{-\frac{1}{12}}}{\sum_{l=1}^{5}\prod_{n=1}^{\infty}\left|\left(1-e^{\frac{2\pi il}{6}}q^n\right)\left(1-e^{-\frac{2\pi il}{6}}q^n\right)\right|^2}\right) \\ &= \frac{1}{6}\left(Z(\tau=e^{\frac{2\pi i}{3}},\rho)+2\left|\frac{\eta(\sigma)}{\vartheta_2(\frac{1}{3},\sigma)}\right|^2+6\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2+4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2\right). \end{aligned}$$

Tori with $\tau = e^{2\pi i/3}$ have three fixed points under the $\mathbb{Z}_6$ rotation symmetry, one corresponding to $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ twists each. The $\mathbb{Z}_6$ orbifold partition function therefore is the sum of untwisted, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ twisted sector partition functions

$$Z_{\mathbb{Z}_6-orb}(\tau=e^{\frac{2\pi i}{3}},\rho) = Z_u + Z_{2t} + Z_{3t} + Z_{6t}.$$

The $\mathbb{Z}_2$ twisted sector partition function Z_{2t} is obtained from (2.42) by omitting the factor of four. The $\mathbb{Z}_3$ twisted sector partition function Z_{3t} can be read off from (4.22) by omitting the factor of three. By (2.38) the ground states in each of the twisted sectors $\mathcal{H}_\gamma$, $\mathcal{H}_{\gamma^5}$ have dimensions $(h,\overline{h}) = (5/72,5/72)$. By (2.32), the $\mathbb{Z}_6$ twisted sector partition function is

$$\begin{aligned} Z_{6t} &= \frac{1}{6}\sum_{l=1}^{6}\Bigg(A(\tfrac{2\pi l}{6})\underbrace{\square}_{A(\pi/3)}+A(\tfrac{2\pi l}{6})\underbrace{\square}_{A(2\pi/3)}+A(\tfrac{2\pi l}{6})\underbrace{\square}_{A(\pi)} \\ &\qquad +A(\tfrac{2\pi l}{6})\underbrace{\square}_{A(4\pi/3)}+A(\tfrac{2\pi l}{6})\underbrace{\square}_{A(5\pi/3)}\Bigg) \\ &= \frac{1}{6}\Bigg(\frac{2(q\overline{q})^{-\frac{1}{72}}}{\sum_{l=1}^{6}\prod_{n=1}^{\infty}\left|\left(1-e^{\frac{2\pi il}{6}}q^{n-\frac{1}{6}}\right)\left(1-e^{-\frac{2\pi il}{6}}q^{n-\frac{5}{6}}\right)\right|^2} \\ &\qquad +\frac{2(q\overline{q})^{\frac{1}{36}}}{\sum_{l=1}^{6}\prod_{n=1}^{\infty}\left|\left(1-e^{\frac{2\pi il}{6}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{6}}q^{n-\frac{2}{3}}\right)\right|^2} \\ &\qquad +\frac{(q\overline{q})^{\frac{1}{24}}}{\sum_{l=1}^{6}\prod_{n=1}^{\infty}\left|\left(1-e^{\frac{2\pi il}{6}}q^{n-\frac{1}{2}}\right)\left(1-e^{-\frac{2\pi il}{6}}q^{n-\frac{1}{2}}\right)\right|^2}\Bigg) \end{aligned}$$

$$= \frac{1}{6}\left(2\left|\frac{\eta(\sigma)}{\vartheta_3(\frac{1}{3},\sigma)}\right|^2 + 2\left|\frac{\eta(\sigma)}{\vartheta_4(\frac{1}{3},\sigma)}\right|^2 + \left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2 + \left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + 2(q\overline{q})^{-\frac{1}{18}}\sum_{l=-1}^{1}\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right).$$

Altogether, we obtain

$$\begin{aligned} Z_{\mathbb{Z}_6-orb} &= \frac{1}{6}\Big(Z(\tau=e^{\frac{2\pi i}{3}},\rho) + \\ &\quad 4\sum_{j=2}^{4}\left|\frac{\eta(\sigma)}{\vartheta_j(\sigma)}\right|^2 + 2\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{1}{3},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 \\ &\quad + (q\overline{q})^{-\frac{1}{18}}\sum_{l=-1}^{1}\left(2\sum_{i=1}^{4}\left|\frac{\eta(\sigma)}{\vartheta_i(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right)\Big). \end{aligned} \tag{2.50}$$

Lattices 6 to 17: Modding out by R or $T_R^{(\prime)}$ reflection symmetries

The reflection symmetry R is a symmetry for every lattice with $\tau_1 \in \{0, \frac{1}{2}\}$. By inspection of the action on the respective fundamental cell one easily checks that an exchange of R_1 and R_2 is equivalent to a transformation of τ_2; we define

$$S, T \in PSL(2,\mathbb{Z}): \qquad S:\zeta\mapsto -\frac{1}{\zeta}, \quad T:\zeta\mapsto\zeta+1;$$
$$\Theta := \begin{cases} S & \text{if } \zeta_1 = 0, \\ TST^2S & \text{if } \zeta_1 = \frac{1}{2}. \end{cases}$$

Then

$$R_1 \leftrightarrow R_2 \text{ is equivalent to } \tau\mapsto\Theta\tau, \text{ where } \begin{cases} \Theta(i\tau_2) &= \frac{i}{\tau_2}, \\ \Theta(\frac{1}{2}+i\tau_2) &= \frac{1}{2}+\frac{i}{4\tau_2}. \end{cases} \tag{2.51}$$

We can therefore restrict ourselves to the discussion of the symmetry R_1 in the following. To extend R_1 to the charge lattice (2.10), the B-field B must obey (2.35) which is true iff $\rho_1 \in \frac{1}{2}\mathbb{Z}$. Then by using (2.14) for the R_1 action $|m_1,m_2,n_1,n_2\rangle \mapsto |m_1',m_2',n_1',n_2'\rangle$ we obtain

$$\begin{aligned} m_1' &= -m_1\,, & n_1' &= -n_1 + 2\tau_1 n_2 + 2\rho_1 m_2 + 4\tau_1\rho_1 m_1, \\ m_2' &= m_2 + 2\tau_1 m_1\,, & n_2' &= n_2 + 2\rho_1 m_1, \end{aligned} \tag{2.52}$$

and the invariant vectors $(p,\overline{p})$ of the charge lattice correspond to $|0,m_2,n_1,n_2\rangle$,

$$\overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}}\begin{pmatrix} n_2\tau_2 \pm m_2\rho_2 \\ 0 \end{pmatrix}, \quad n_2,m_2\in\mathbb{Z} \text{ such that } n_1 = n_2\tau_1 + m_2\rho_1 \in \mathbb{Z}. \tag{2.53}$$

The Hilbert space ground states $|m_1,m_2,n_1,n_2\rangle$ are pairwise orthogonal, so the only states that give a contribution to ${}_{R_1}\square_{1}$ are the ones that are built by an action of creation operators on ground states corresponding to vertex operators with R_1–invariant charge vectors (2.53).

Because (2.53) only depends on $\rho_1 \bmod \mathbb{Z}$ the same is true for the resulting orbifold theory and we can pick $\rho_1 \in \{0, \frac{1}{2}\}$. Note that in the case $\rho_1 = \frac{1}{2}$ the B-field of our theory is effectively shifted by an integer form if we apply R_1. This will be of some importance below.

To understand the action of the symmetry $T_R^{(\prime)} = RT_{\delta^{(\prime)}}$ on the Hilbert space of a toroidal conformal field theory observe that $T_{\delta^{(\prime)}}$ only acts on the ground state sectors and leaves the oscillator modes invariant. On a state $|m_1, m_2, n_1, n_2\rangle$ corresponding to the charge vector $(p, \overline{p})(\lambda, \mu)$ the action of $T_{R_1}^{(\prime)}$ is given by the action (2.52) of R_1 combined with multiplication by $exp[2\pi i(p, \overline{p})(\lambda, \mu) \cdot \frac{1}{2}(p, \overline{p})(2\delta^{(\prime)}, 0)] = (-1)^{\langle \mu, 2\delta^{(\prime)} \rangle}$, where we used (2.15). It is therefore a priori clear that as for the action of R we need to restrict the possible B–field values to $\rho_1 \in \{0, \frac{1}{2}\}$ for consistency of the action of $T_R^{(\prime)}$. By (2.28), $T_R^{(\prime)}$ actions are only needed in the case $\tau_1 = 0$. Using (2.14) one now checks that only for $\rho_1 = 0$ the order of $T_R^{(\prime)}$ is two, whereas for $\rho_1 = 1/2$ we find that $T_R^{(\prime)}$ generates a $\mathbb{Z}_4$ type group. The action of $g := (T_R^{(\prime)})^2$ is given by multiplication with ± 1 on the different Hilbert space sectors. To mod out a toroidal theory A by this $\mathbb{Z}_4$ then is equivalent to performing a $\mathbb{Z}_2$ orbifold procedure on $A/\{1, g\}$. But $A/\{1, g\}$ is another toroidal theory, because both generic torus currents are invariant under g and give conserved currents in $A/\{1, g\}$ as well. The $T_R^{(\prime)}$ action with $\rho_1 = 1/2$ hence need not be considered separately. For $\rho_1 = 0$ by (2.51) we now have

$$T_{R_1}^{(\prime)} \leftrightarrow T_{R_2}^{(\prime)} \text{ is equivalent to } \tau\,(= i\tau_2) \mapsto \Theta\tau \left(= \frac{i}{\tau_2}\right). \tag{2.54}$$

Since by (2.27) $\delta^{(\prime)} = \sqrt{\frac{\rho_2}{\tau_2}} \begin{pmatrix} 1/2 \\ * \end{pmatrix}$, if $|m_1, m_2, n_1, n_2\rangle$ is R_1-invariant, then by (2.53) $m_1 = 0, n_1 = n_2\tau_1 + m_2\rho_1$, and $T_R^{(\prime)}$ acts by

$$T_R^{(\prime)}: \quad |m_1, m_2, n_1, n_2\rangle \mapsto (-1)^{n_2} |m_1, m_2, n_1, n_2\rangle. \tag{2.55}$$

Below we will construct the families of conformal field theories obtained by the orbifold procedure with a group G which corresponds to one of the lattices 6 to 17. This will yield irreducible components $\mathcal{C}_{G-orb}^{(\tau_1, \rho_1)}$ of the moduli space $\mathcal{C}^2$ with $\tau_1, \rho_1 \in \{0, \frac{1}{2}\}$. In some cases discrete torsion gives additional degrees of freedom, increasing the number of irreducible components to $\mathcal{C}_{G^{\pm}-orb}^{(\tau_1, \rho_1)}$ or even $\mathcal{C}_{G^{\pm\pm}-orb}^{(\tau_1, \rho_1)}$. The Teichmüller space (For the case of the torus, Teichmüller space is the upper half plane) of each such irreducible component is $(\mathbb{R}^+)^k$, where $k = 1$ if τ_2 must be fixed for the particular lattice, too, and $k = 2$ otherwise. To find the correct parameter spaces, we must determine the subgroup $\mathcal{P}$ of $PSL(2, \mathbb{Z})^2 \times \mathbb{Z}_2^2$ in (2.19) which maps the respective Teichmüller space $(\mathbb{R}^+)^k$ onto itself. Then we must discuss which elements of $\mathcal{P}$ map equivalent orbifold theories onto each other.

Restrict $\mathcal{P}$ to one of the factors $\mathbb{R}^+ \subset \mathbb{H}$ of the Teichmüller space $(\mathbb{R}^+)^k$, specified by $\zeta_1 = 0$ or $\zeta_1 = \frac{1}{2}$. We claim that

$$\mathcal{P} \cap PSL(2, \mathbb{Z}) = \{1, \Theta\}. \tag{2.56}$$

As stated in (2.51), Θ acts on $I^0 := \{\zeta \in \mathbb{H} \mid \zeta_1 = 0\}$ by $\zeta_2 \mapsto \frac{1}{\zeta_2}$ and on $I^+ := \{\zeta \in \mathbb{H} \mid \zeta_1 = \frac{1}{2}\}$ by $\zeta_2 \mapsto \frac{1}{4\zeta_2}$. Now $I^0 = J^0 \cup \Theta J^0$, where $J^0 := \{\zeta \in I^0 \mid \zeta_2 \geq 1\}$. Because J^0 does not contain any two points identified by Möbius transformations, the assertion follows for the case $\zeta_1 = 0$. For $\zeta_1 = \frac{1}{2}$ observe that $I^+ = (J^+ \cup TSTJ^1) \cup \Theta(J^+ \cup TSTJ^1)$, where $J^+ := \{\zeta \in I^+ \mid \zeta_2 \geq \frac{\sqrt{3}}{2}\}$ and $J^1 := \{\zeta \in \mathbb{H} \mid \|\zeta\| = 1, \zeta_1 \in [-\frac{1}{2}, 0]\}$. Because no two

points in $J^+ \cup J^1$ are related by Möbius transformations, the assertion follows. For the respective factor of the Teichmüller space under discussion, Θ will be called T–duality. By our convention to fix $\tau_1, \rho_1 \in \{0, \frac{1}{2}\}$ it is clear that target space orientation $V : (\tau, \rho) \mapsto (-\overline{\tau}, -\overline{\rho})$ in (2.18) can only be contained in $\mathcal{P}$ if $\tau_1 = \rho_1 = 0$, in which case it acts trivially. Mirror symmetry $U : (\tau, \rho) \mapsto (\rho, \tau)$ is contained iff $\tau_1 = \rho_1$ and τ_2 is not fixed. Inspection of the charge lattice (2.10) and the action (2.52) of R_1 shows that mirror symmetry commutes with R_1, R_2 on toroidal conformal field theories. But a priori it is not clear whether it indeed commutes with the action of each of the symmetry groups corresponding to lattices 6 to 17. Therefore, a case by case study is necessary to decide which of Θ, U map a G orbifold onto an equivalent one and thus determine all the parameter spaces $\mathcal{C}^{(\bullet)}_{G^\bullet-orb}$. We will also see that not all of the lattices yield different components of the moduli space $\mathcal{C}^2$.

Lattices 6 and 7: The $\mathbb{Z}_2(R)$ reflection orbifold

Lattices 6 ($\tau_1 = 0$) and 7 ($\tau_1 = \frac{1}{2}$) have reflection symmetry $\mathbb{Z}_2(R)$. For $\tau_1 = \rho_1 = 0$ the torus theory is the product of two $c = 1$ theories corresponding to compactification on a circle each (see (2.17)). The symmetry R_1 by (2.26) leaves the first factor invariant and acts on the second as ordinary $\mathbb{Z}_2$ orbifold. Therefore the resulting partition function is the product of circle and circle orbifold partition function; namely, setting $r := \sqrt{\rho_2/\tau_2}$, $r' := \sqrt{\tau_2\rho_2}$ as in (2.17),

$$Z_{R_1-orb}(0, \tau_2, 0, \rho_2) = Z^{c=1}(r) Z^{c=1}_{orb}(r'), \tag{2.57}$$

where $Z^{c=1}$ and $Z^{c=1}_{orb}$ are given in (1.11) and (1.12), respectively. If we mod out by R_2 instead of R_1, by (2.51) we use $\tau_2 \mapsto \frac{1}{\tau_2}$, i.e. the radii r and r' are interchanged in (2.57). Application of T–duality to both τ and ρ simultaneously, which will be denoted by

$$\mathcal{S} : \ (\tau, \rho) \mapsto \left(-\frac{1}{\tau}, -\frac{1}{\rho}\right)$$

and called simultaneous T–duality in the following, amounts to $r \mapsto \frac{1}{r}$, $r' \mapsto \frac{1}{r'}$ in both cases, leaving (2.57) invariant. Mirror symmetry $\tau_2 \leftrightarrow \rho_2$ acts by $r \mapsto \frac{1}{r}$, $r' \mapsto r'$, which (2.57) is invariant under, too.
By (2.33) the general reflection orbifold partition function can be written as

$$Z_{R_1-orb} = \frac{1}{2}\left(1\,\underset{1}{\square} + R_1\,\underset{1}{\square} + 1\,\underset{R_1}{\square} + R_1\,\underset{R_1}{\square} \right). \tag{2.58}$$

As explained in our general discussion for lattices 6 to 17, the second term in (2.58) gets contributions only from the states built by an action of creation operators on such Hilbert space ground states that are invariant under the action of R_1. The corresponding charge vectors are given in (2.53), namely for lattice 6 with $\tau_1 = 0, \rho_1 = \frac{1}{2}$ we obtain

$$\overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} n\tau_2 \pm 2m\rho_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{n}{r} \pm mr \\ 0 \end{pmatrix}, \ m, n \in \mathbb{Z}, r := \sqrt{2\rho_2/\tau_2}. \tag{2.59}$$

Therefore for the untwisted sector partition function

$$\begin{aligned} Z_u(0, \tau_2, \frac{1}{2}, \rho_2) &= \frac{1}{2}\left(Z + \frac{1}{|\eta|^2} \cdot 2 \left|\frac{\eta}{\vartheta_2}\right| q^{\frac{1}{2}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{2}\left(\frac{n}{r}-mr\right)^2} \right) \\ &= \frac{1}{2}\left(Z + \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{2}\left(\frac{n}{r}-mr\right)^2} \right), \end{aligned}$$

where we have used the relation $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$. The twisted sector partition function can be calculated by modular transformations, and the complete reflection orbifold partition function is

$$\begin{aligned} Z_{R_1-orb}(0,\tau_2,\frac{1}{2},\rho_2) = \frac{1}{2}\Bigg(Z \;&+\; \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n} q^{\frac{1}{2}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{2}\left(\frac{n}{r}-mr\right)^2} \\ &+\; \left|\frac{\vartheta_3\vartheta_2}{2\eta^4}\right| \sum_{m,n} q^{\frac{1}{8}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{8}\left(\frac{n}{r}-mr\right)^2} \\ &+\; \left|\frac{\vartheta_4\vartheta_2}{2\eta^4}\right| \sum_{m,n} (-1)^{mn} q^{\frac{1}{8}\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\frac{1}{8}\left(\frac{n}{r}-mr\right)^2} \Bigg), \end{aligned} \tag{2.60}$$

with $r = \sqrt{2\rho_2/\tau_2}$, and for R_2 instead of R_1 with $r = \sqrt{2\tau_2\rho_2}$ by (2.51). Simultaneous T–duality $\mathcal{S}$ amounts to $r \mapsto \frac{1}{r}$ in both cases. This obviously leaves (2.60) invariant.
Mirror symmetry $U : (\tau,\rho) \mapsto (\rho,\tau)$ commutes with the R_1, R_2 actions on a toroidal theory, so

$$Z_{R-orb}(\frac{1}{2},\tau_2,0,\rho_2) = Z_{R-orb}(0,\rho_2,\frac{1}{2},\tau_2). \tag{2.61}$$

Hence the partition function $Z_{R-orb}(\frac{1}{2},\tau_2,0,\rho_2)$ for lattice 7 with $\rho_1 = 0$ is given by (2.60), but now with $r = \sqrt{\rho_2/2\tau_2}$ for R_1, $r = \sqrt{2\tau_2\rho_2}$ for R_2. Again, $\mathcal{S}$ acts by $r \mapsto \frac{1}{r}$ in both cases and leaves the partition function invariant.
The case $\tau_1 = \rho_1 = 1/2$ (lattice 7) is more subtle. The charge lattice (2.14) of the toroidal theory is generated by the four vectors

$$v_{\delta,\epsilon} := \frac{1}{2\sqrt{2\tau_2\rho_2}} \left(\begin{pmatrix} \tau_2 + \delta\rho_2 \\ \epsilon\,(1/2 - 2\delta\tau_2\rho_2) \end{pmatrix}, \begin{pmatrix} \tau_2 - \delta\rho_2 \\ \epsilon\,(1/2 + 2\delta\tau_2\rho_2) \end{pmatrix} \right),\ \delta,\epsilon \in \{\pm 1\}$$

which are pairwise interchanged by R_1 ($v_{\delta,1} \leftrightarrow v_{\delta,-1}$). Denote the corresponding vertex operators by $V(\pm v_{\delta,\epsilon})$. The R_1 invariant part of the charge lattice by (2.53) is given by

$$\begin{aligned} \overset{(-)}{p} = \frac{1}{\sqrt{2\tau_2\rho_2}} \begin{pmatrix} n_2\tau_2 \pm m_2\rho_2 \\ 0 \end{pmatrix} &= \sqrt{2} \begin{pmatrix} \frac{n}{r} \pm mr \\ 0 \end{pmatrix}, \\ n_2 &= 2n, m_2 = 2m, n_1 = n+m \in \mathbb{Z},\ r = \sqrt{\rho_2/\tau_2}. \end{aligned} \tag{2.62}$$

Because $\langle v_{\delta,\epsilon}, v_{\delta,-\epsilon}\rangle = 1$, the vertex operators corresponding to generators of the invariant part of the charge lattice are obtained from operator product expansions

$$\left(V(v_{\delta,1}) + V(-v_{\delta,1})\right) \times \left(V(v_{\delta,-1}) - V(-v_{\delta,-1})\right).$$

Since this is a product between an R_1 even and an R_1 odd operator, the resulting vertex operators are R_1 odd. It follows that R_1 acts on ground states corresponding to invariant charge vectors (2.62) by $|m_1,m_2,n_1,n_2\rangle \mapsto (-1)^{n_2}|m_1,m_2,n_1,n_2\rangle$. Thus for the untwisted sector partition function we find

$$Z_u(\frac{1}{2},\tau_2,\frac{1}{2},\rho_2) = \frac{1}{2}\left(Z \;+\; \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \left(\sum_{m,n\in\mathbb{Z}} - \sum_{m,n\in\mathbb{Z}+1/2} \right) q^{\left(\frac{n}{r}+mr\right)^2} \overline{q}^{\left(\frac{n}{r}-mr\right)^2} \right)$$

with $r = \sqrt{\rho_2/\tau_2}$. We remark that although not stated explicitly above, one may check that in none of the other cases of R_1 actions such additional signs on Hilbert space ground

states occur. Here, they are due to the fact that the action of R_1 effectively shifts the B–field by an integer form, as was already mentioned above. In the discussion of the bicritical point (C14) we will point out a very natural confirmation of the above result. By applying modular transformations to ${}_{R_1}\square_{1}$, we find

$$\begin{aligned} Z_{R_1-orb}(\frac{1}{2},\tau_2,\frac{1}{2},\rho_2) &= \frac{1}{2}\Bigg(Z + \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \left(\sum_{m,n} - \sum_{m,n\in\mathbb{Z}+\frac{1}{2}}\right) q^{\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\left(\frac{n}{r}-mr\right)^2} \\ &\quad + \left|\frac{\vartheta_3\vartheta_2}{4\eta^4}\right| \sum_{m,n\in\mathbb{Z}} \beta\; q^{\frac{1}{16}\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\frac{1}{16}\left(\frac{n}{r}-mr\right)^2} \\ &\quad + \left|\frac{\vartheta_4\vartheta_2}{4\eta^4}\right| \sum_{m,n\in\mathbb{Z}} \beta\; (i)^{nm}\; q^{\frac{1}{16}\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\frac{1}{16}\left(\frac{n}{r}-mr\right)^2}\Bigg), \end{aligned} \tag{2.63}$$

where $\beta = (1-(-1)^{n+m})$; $r = \sqrt{\rho_2/\tau_2}$ for the reflection R_1, and therefore $r = 2\sqrt{\tau_2\rho_2}$ for the reflection R_2 by (2.51). To apply simultaneous T–duality $\mathcal{S}$ amounts to $r \mapsto \frac{1}{r}$, yielding (2.63) invariant. Invariance under mirror symmetry $\tau_2 \leftrightarrow \rho_2$ is also obvious.
By our general discussion for lattices 6 to 17, to find the correct parameter space for the irreducible components of $\mathcal{C}^2$ obtained by $\mathbb{Z}_2(R)$ orbifolding, the Teichmüller spaces are constructed by considering $R = R_1$ only. T–duality applied to τ alone, which by (2.51) is equivalent to $R_1 \leftrightarrow R_2$, does not generically map onto an isomorphic theory. Our calculations above show that simultaneous T–duality $\mathcal{S}$ actually identifies isomorphic theories (see (2.57), (2.60) and (2.63)) as well as mirror symmetry. In particular, lattice 6 ($\tau_1 = 0$) with $\rho_1 = \frac{1}{2}$ and lattice 7 ($\tau_1 = \frac{1}{2}$) with $\rho_1 = 0$ correspond to families of isomorphic orbifold conformal field theories.
Summarizing, we have constructed the following three irreducible components of the moduli space:

$$\begin{aligned} \mathcal{C}^{(0,0)}_{\mathbb{Z}_2(R)-orb} &\cong \left(\mathbb{R}^+\right)^2/\{U,\mathcal{S}\}, \qquad \mathcal{C}^{(\frac{1}{2},\frac{1}{2})}_{\mathbb{Z}_2(R)-orb} \cong \left(\mathbb{R}^+\right)^2/\{U,\mathcal{S}\} \\ \mathcal{C}^{(0,\frac{1}{2})}_{\mathbb{Z}_2(R)-orb} &= \mathcal{C}^{(\frac{1}{2},0)}_{\mathbb{Z}_2(R)-orb} \cong \left(\mathbb{R}^+\right)^2/\mathcal{S}. \end{aligned}$$

Lattices 8 and 9: The D_2 orbifold

Lattices 8 ($\tau_1 = 0$) and 9 ($\tau_1 = 1/2$) have a $D_2 = \{1, A(\pi), R_1, R_2\}$ symmetry. By (2.33) for both $\tau_1, \rho_1 \in \{0, \frac{1}{2}\}$ the D_2 orbifold partition function is

$$\begin{aligned} Z_{D_2-orb}(\tau_1,\tau_2,\rho_1,\rho_2) &= \frac{1}{4}\sum_{g,h\in D_2} {}_g\square_h \\ &= \frac{1}{4}\left(2Z_{\mathbb{Z}_2-orb} + 2Z_{R_1-orb} + 2Z_{R_2-orb} - 2Z\right) \\ &\quad + {}_{R_1}\square_{A(\pi)} + {}_{A(\pi)}\square_{R_1} + {}_{R_2}\square_{R_1} \\ &\quad + {}_{R_2}\square_{A(\pi)} + {}_{A(\pi)}\square_{R_2} + {}_{R_2}\square_{R_1}\Bigg), \end{aligned} \tag{2.64}$$

where we have subtracted $Z = {}_1\square_1$ from the second and third term in the second line to avoid overcounting the contribution of the identity element which appears in

each reflection group. Observe that by (2.51) separate T–duality (2.56) on τ, or by mirror symmetry equivalently on ρ, interchanges $\mathbb{Z}_2(R_1)$ and $\mathbb{Z}_2(R_2)$. Therefore it maps isomorphic D_2 orbifold conformal field theories onto each other.
The terms in the third and fourth line of (2.64) form a modular orbit. To determine them we compute ${}_{R_1}\boxed{}_{A(\pi)}$. Denote by $\mathcal{H}_{A(\pi)}$ the twisted sector Hilbert space of the ordinary $\mathbb{Z}_2$ orbifold which by (2.37) corresponds to fields φ with half integer modes and $\varphi(z=0) = q_j$, $j \in \{1,2,3,4\}$, a $\mathbb{Z}_2$ fixed point on T^2. Assume that k of the four corresponding $\mathbb{Z}_2$ twisted ground states are eigenstates of R_1. There eigenvalues must agree and be ± 1 in order for the $\mathbb{Z}_2$ action on the twisted sector to be well defined. Since by (2.38) the twisted ground states have dimensions $(h,\overline{h}) = (1/8, 1/8)$, we find

$$\begin{aligned}
{}_{R_1}\boxed{}_{A(\pi)} &= tr_{\mathcal{H}_{A(\pi)}} R_1 q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \\
&= \pm k \cdot (q\overline{q})^{-\frac{1}{12}} \frac{(q\overline{q})^{\frac{1}{8}}}{\prod_{n=1}^{\infty}(1-q^{n-1/2})(1-\overline{q}^{n-1/2})(1+q^{n-1/2})(1+\overline{q}^{n-1/2})} \\
&= \pm k \left| \frac{\eta^2}{\vartheta_3 \vartheta_4} \right| = \pm \frac{k}{2} \left| \frac{\vartheta_2}{\eta} \right| . \qquad (2.65)
\end{aligned}$$

All in all by modular transformations the third line in (2.64) is equal to $\pm 2k Z_{Ising}$, where the Ising partition function was given in (1.22), and $k \in \{0,2,4\}$, because R_1 must map twisted ground states onto twisted ground states. To determine the correct factor k we first note that in case $\tau_1 = \rho_1 = 0$ the original toroidal theory decomposes into a tensor product of two $c=1$ theories. The action of D_2 respects the product structure, hence

$$Z_{D_2^+ - orb}(0,\tau_2,0,\rho_2) = Z_{orb}^{c=1}(\sqrt{\tau_2\rho_2}) Z_{orb}^{c=1}(\sqrt{\rho_2/\tau_2}), \qquad (2.66)$$

where $Z_{orb}^{c=1}$ was given in (1.12). One now checks that in this case $k=4$, in agreement with the geometric observation that all the four $\mathbb{Z}_2$ fixed points on T^2 are invariant under the R actions. For $\tau_1 = 1/2, \rho_1 = 0$ one can argue that only two of the four fixed points are invariant, thus $k=2$. If $\rho_1 = 1/2$, this geometric argument breaks down since, as noted in our general discussion for lattices 6 to 17, in this case the symmetries R_1, R_2 effectively shift the B–field by an integer form. The correct factor for $\tau_1 = 0, \rho_1 = 1/2$ is $k=2$, as well. This follows from the construction of the D_4 orbifold conformal field theory (lattice 15), where we will see that the D_2 orbifold at $\tau_1 = 0, \rho_1 = 1/2$ must always contain an even number of fields with dimensions $h = \overline{h} = 1/16$. For $\tau_1 = \rho_1 = 1/2$ we find $k=0$. This follows from the fact that ${}_1\boxed{}_{R_1}$ by (2.63) generically does not get any contributions from fields with dimensions $h = \overline{h} = 1/16$. Hence ${}_{A(\pi)}\boxed{}_{R_1} = \pm\frac{k}{2}\left|\frac{\vartheta_3}{\eta}\right|$ cannot give such contributions either. In summary,

$$\begin{aligned}
Z_{D_2^\pm - orb}(\tau_1,\tau_2,\rho_1,\rho_2) &= \frac{1}{2}(Z_{\mathbb{Z}_2 - orb} + Z_{R_1 - orb} + Z_{R_2 - orb} \pm k Z_{Ising} - Z), \\
k &= 4(1-\tau_1-\rho_1), \qquad (2.67)
\end{aligned}$$

where $Z_{\mathbb{Z}_2 - orb}$ is given in (2.43), and Z_{R-orb} is given in (2.57), (2.60) or (2.63), respectively. In particular, for this orbifold construction discrete torsion has a nontrivial effect, and we can produce two non–equivalent theories corresponding to lattice 8 and each possible value of ρ_1. We stress that we have been discussing a perhaps counterintuitive effect of "turning on the B–field": The action of R_1, R_2 on twisted ground states depends

severely on the value of ρ_1. In particular, they must not be interpreted from a purely geometric point of view.
Because $\mathbb{Z}_2(R)$ orbifold conformal field theories as well as the formula for k in (2.67) are invariant under mirror symmetry, the same is true for D_2 orbifolds. Hence we have constructed five irreducible components of $\mathcal{C}^2$,

$$\mathcal{C}^{(0,0)}_{D_2^+-orb} \cong \left(\mathbb{R}^+/\Theta\right)^2/U, \quad \mathcal{C}^{(0,\frac{1}{2})}_{D_2^+-orb} \cong \mathcal{C}^{(\frac{1}{2},0)}_{D_2^+-orb} \cong \left(\mathbb{R}^+/\Theta\right)^2,$$

$$\mathcal{C}^{(\frac{1}{2},\frac{1}{2})}_{D_2-orb} \cong \left(\mathbb{R}^+/\Theta\right)^2/U.$$

Lattice 10: The $\mathbb{Z}_2(T_R)$ reflection plus shift orbifold

Lattice 10 ($\tau_1 = 0$) has reflection plus shift symmetry $\mathbb{Z}_2(T_{R_1}) = \{1, T_{R_1} = R_1 e^{2\pi i p\cdot\frac{\delta_1}{\sqrt{2}}}\}$, where $\delta_1 = \sqrt{\rho_2/\tau_2}(1/2, 0)$. From our general discussion on the $T_R^{(\prime)}$ action for lattices 6 to 17 we know that we only have to consider the case $\rho_1 = 0$. By (2.33) the general reflection plus shift orbifold partition function is

$$Z_{T_{R_1}-orb} = \frac{1}{2}\left({}_1\square_1 + T_{R_1}\square_1 + {}_1\square_{T_{R_1}} + T_{R_1}\square_{T_{R_1}} \right). \tag{2.68}$$

The torus theory is a tensor product of two $c = 1$ circle theories (2.17): ${}_1\square_1 = {}_1\square_1(\varphi_1){}_1\square_1(\varphi_2)$. The $\mathbb{Z}_2(T_{R_1})$ action respects the product structure, thus we have ${}_{T_{R_1}}\square_1 = {}_{t_\epsilon}\square_1(\varphi_1){}_{(-1)}\square_1(\varphi_2)$ with $t_\epsilon = e^{2\pi i p\cdot\frac{\epsilon}{\sqrt{2}}}, \epsilon = \frac{1}{2}\sqrt{\rho_2/\tau_2}$. From the circle orbifold theory (1.12) one has ${}_{(-1)}\square_1(\varphi_2) = 2\left|\frac{\eta}{\vartheta_2}\right|$. As explained in our general discussion for lattices 6 to 17, the translation symmetry t_ϵ does not affect oscillator modes. By (2.53) it acts on the Hilbert space ground states $|0, m, 0, n\rangle$ of the circle theory ${}_1\square_1(\varphi_1)$ via multiplication with $(-1)^n$. So using (1.11) and (2.53) with $r := \sqrt{\rho_2/\tau_2}$ we find

$${}_{t_\epsilon}\square_1(\varphi_1) = \frac{1}{\eta\overline{\eta}}\sum_{m,n}(-1)^n q^{\frac{1}{4}(\frac{n}{r}+mr)^2}\overline{q}^{\frac{1}{4}(\frac{n}{r}-mr)^2}.$$

The remaining boxes in (2.68) are obtained by modular transformations. Thus the complete partition function is

$$\begin{aligned} Z_{T_{R_1}-orb}(0,\tau_2,0,\rho_2) = \frac{1}{2}\Bigg(Z &+ \left|\frac{\vartheta_3\vartheta_4}{\eta^4}\right| \sum_{m,n}(-1)^n q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2} \\ &+ \left|\frac{\vartheta_3\vartheta_2}{\eta^4}\right| \sum_{n\in\mathbb{Z}, m\in\mathbb{Z}+1/2} q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2} \\ &+ \left|\frac{\vartheta_4\vartheta_2}{\eta^4}\right| \sum_{\substack{n\in\mathbb{Z},\\ m\in\mathbb{Z}+1/2}} (-1)^n q^{\frac{1}{4}\left(\frac{n}{r}+mr\right)^2}\overline{q}^{\frac{1}{4}\left(\frac{n}{r}-mr\right)^2} \Bigg), \end{aligned} \tag{2.69}$$

where $r := \sqrt{\rho_2/\tau_2}$. If we mod out by $\mathbb{Z}_2(T_{R_2})$ instead of $\mathbb{Z}_2(T_{R_1})$ by (2.54) we have to set $r := \sqrt{\tau_2\rho_2}$. Simultaneous T–duality $\mathcal{S}$ amounts to $r \mapsto \frac{1}{r}$ which does not leave (2.69)

invariant. But for $\mathbb{Z}_2(T_{R_1})$ orbifolds, $\mathcal{S}$ combined with mirror symmetry maps onto an isomorphic theory, whereas for $\mathbb{Z}_2(T_{R_2})$ orbifolds mirror symmetry maps onto isomorphic theories. Therefore the $\mathbb{Z}_2(T_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{\mathbb{Z}_2(T_R)-orb} \cong \left(\mathbb{R}^+\right)^2 / U.$$

Lattice 11: The $D_2(T_R)$ orbifold

Lattice 11 ($\tau_1 = 0$) has a $D_2(T_R) = \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}$ symmetry as defined in (2.27), and we can generally set $\rho_1 = 0$. By (2.33) the partition function has the form

$$\begin{aligned} &Z_{D_2(T_R)-orb}(0,\tau_2,0,\rho_2) \\ &\quad = \frac{1}{4}\Big(2Z_{\mathbb{Z}_2-orb} + 2Z_{T_{R_1}-orb} + 2Z_{\widehat{T}_{R_2}-orb} - 2Z \\ &\qquad + T_{R_1}\underset{A(\pi)}{\square} + \widehat{T}_{R_2}\underset{A(\pi)}{\square} + A(\pi)\underset{T_{R_1}}{\square} + A(\pi)\underset{\widehat{T}_{R_2}}{\square} + T_{R_1}\underset{\widehat{T}_{R_2}}{\square} + \widehat{T}_{R_2}\underset{T_{R_1}}{\square}\Big) . \end{aligned} \tag{2.70}$$

The terms in the second line can be computed by a similar argument as those in the third line of (2.64). Only here none of the four ordinary $\mathbb{Z}_2$ fixed points is invariant under T_{R_1} or $\widehat{T}_{R_2}$, so the first two boxes vanish. The others are obtained by modular transformations from these and therefore vanish as well. In particular, in this case discrete torsion has no effect on the partition function.

The original toroidal theory decomposes into the tensor product of two $c = 1$ theories (2.17). By (2.27) $\widehat{T}_{R_2}$ leaves the second factor invariant. Since on the Hilbert space ground states $|0,m,0,n\rangle$ of the first factor $\widehat{T}_{R_2}|0,m,0,n\rangle = \pm|0,-m,0,-n\rangle = \pm R_2|0,m,0,n\rangle$, $\widehat{T}_{R_2}\underset{1}{\square} = R_2\underset{1}{\square}$ and thus we have $Z_{\widehat{T}_{R_2}-orb} = Z_{R_2-orb}$. All in all

$$Z_{D_2(T_R)-orb}(0,\tau_2,0,\rho_2) = \frac{1}{2}(Z_{\mathbb{Z}_2-orb} + Z_{R_2-orb} + Z_{T_{R_1}-orb} - Z), \tag{2.71}$$

where $Z_{\mathbb{Z}_2-orb}, Z_{R_2-orb}, Z_{T_{R_1}-orb}$ are given in (2.43), (2.57), and (2.69), respectively.

By the discussion of $\mathbb{Z}_2(T_R)$ orbifold conformal field theories (lattice 10), only combined $\mathcal{S}$ with mirror symmetry leaves $Z_{T_{R_1}-orb}$ invariant. This also maps isomorphic R orbifolds onto each other (lattice 6), so the $D_2(T_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{D_2(T_R)-orb} \cong \left(\mathbb{R}^+\right)^2 / U\mathcal{S}.$$

Lattice 12: The $D_2(T'_R)$ orbifold

Lattice 12 ($\tau_1 = 0$) has a $D_2(T'_R) = \{1, A(\pi), T'_{R_1}, T'_{R_2}\}$ symmetry as defined in (2.27), and we may set $\rho_1 = 0$. The calculation of the partition functions is analogous to that for lattice 11, where in (2.70) we now replace T_{R_1} by T'_{R_1} and $\widehat{T}_{R_2}$ by T'_{R_2}. Again, none of the ordinary $\mathbb{Z}_2$ fixed points is invariant under a symmetry T'_R. So the second line in $(2.70)'$ vanishes, too, and discrete torsion has no effect.

For $\tau_1 = \rho_1 = 0$ analogously to $Z_{\widehat{T}_{R_2}-orb} = Z_{R_2-orb}$ in the partition function for lattice 11 we now find $Z_{T'_R-orb} = Z_{T_R-orb}$. So we have

$$Z_{D_2(T'_R)-orb}(0,\tau_2,0,\rho_2) = \frac{1}{2}(Z_{\mathbb{Z}_2-orb} + Z_{T_{R_1}-orb} + Z_{T_{R_2}-orb} - Z), \tag{2.72}$$

where $Z_{\mathbb{Z}_2-orb}$ and Z_{T_R-orb} are given in (2.43) and (2.69), respectively. Since T–duality (2.54) applied to τ interchanges $\mathbb{Z}_2(T_{R_1})$ and $\mathbb{Z}_2(T_{R_2})$, but neither simultaneous T–duality $\mathcal{S}$ nor mirror symmetry leaves invariant both of them, $D_2(T'_R)$ orbifold conformal field theories form a family

$$\mathcal{C}_{D_2(T'_R)-orb} \cong \left(\mathbb{R}^+/\Theta\right) \times \mathbb{R}^+ .$$

Lattice 13: The $D_3(R_1)$ orbifold

Lattice 13 ($\tau = e^{2\pi i/3}$) has a $D_3(R_1) = \mathbb{Z}_3 \cup \{R_1, A(2\pi/3)R_1, A(4\pi/3)R_1\}$ symmetry. By our general discussion for lattices 6 to 17 and since for lattice 13 the value of τ_2 must be fixed to $\tau_2 = \frac{\sqrt{3}}{2}$, the components of the moduli space $\mathcal{C}^2$ obtained by $D_3(R_1)$ orbifolding are

$$\mathcal{C}^{(\rho_1)}_{D_3(R_1)-orb} \cong \mathbb{R}^+, \qquad \rho_1 \in \{0, \tfrac{1}{2}\}.$$

The maximal abelian subgroups of $D_3(R_1)$ are $\mathbb{Z}_3$, and three order two groups $\{1, R_1\}$, $\{1, A(2\pi/3)R_1\}$, $\{1, A(4\pi/3)R_1\}$. These groups give identical contributions to the partition function since they are conjugate within $D_3(R_1)$. Thus we find

$$\begin{aligned} Z_{D_3(R_1)-orb}(1/2, \sqrt{3}/2, \rho_1, \rho_2) &= \frac{1}{6}\left(3Z_{\mathbb{Z}_3-orb} + 3(2Z_{R_1-orb} - Z)\right) \\ &= \frac{1}{2}\left(Z_{\mathbb{Z}_3-orb} + 2Z_{R_1-orb} - Z\right), \end{aligned} \tag{2.73}$$

where $Z_{\mathbb{Z}_3-orb}$ is given in (2.46), and Z_{R_1-orb} is given in (2.61) or in (2.63) for $\rho_1 = 0$ or $\rho_1 = 1/2$, respectively.

Lattice 14: The $D_3(R_2)$ orbifold

Lattice 14 ($\tau = e^{2\pi i/3}$) has a $D_3(R_2) = \mathbb{Z}_3 \cup \{R_2, A(2\pi/3)R_2, A(4\pi/3)R_2\}$ symmetry. Analogously to lattice 13 we find

$$Z_{D_3(R_2)-orb}(1/2, \sqrt{3}/2, \rho_1, \rho_2) = \frac{1}{2}\left(Z_{\mathbb{Z}_3-orb} + 2Z_{R_2-orb} - Z\right), \tag{2.74}$$

where $Z_{\mathbb{Z}_3-orb}$ is given in (2.46), and Z_{R_2-orb} is given in (2.61) and (2.63) for $\rho_1 = 0$ or $\rho_1 = 1/2$, respectively.
From our discussion of lattices 6 and 7 we know that Z_{R_2-orb} is obtained from Z_{R_1-orb} by application of T–duality (2.56) on τ. Using mirror symmetry we see that we can equally apply T–duality to ρ and find

$$\begin{aligned} Z_{D_3(R_2)-orb}(1/2, \sqrt{3}/2, 0, \rho_2) &= Z_{D_3(R_1)-orb}(1/2, \sqrt{3}/2, 0, 1/\rho_2), \\ Z_{D_3(R_2)-orb}(1/2, \sqrt{3}/2, 1/2, \rho_2) &= Z_{D_3(R_1)-orb}(1/2, \sqrt{3}/2, 1/2, 1/4\rho_2). \end{aligned}$$

The above actually is the equation for T–duality on $\mathcal{C}^{(\rho_1)}_{D_3(R_1)-orb}$. In particular, the $D_3(R_2)$ orbifold procedure does not yield a new component of the moduli space $\mathcal{C}^2$ but only reproduces $\mathcal{C}^{(\rho_1)}_{D_3(R_1)-orb}, \rho_1 \in \{0, \frac{1}{2}\}$.

Lattice 15: The D_4 orbifold

Lattice 15 ($\tau = i$) has a $D_4 = \mathbb{Z}_4 \cup \{R_1, A(\pi/2)R_1, R_2, A(\pi/2)R_2\}$ symmetry. The maximal abelian subgroups of D_4 are $\mathbb{Z}_4$, $D_2 = \{1, A(\pi), R_1, R_2\}$, and $D'_2 = \{1, A(\pi), A(\pi/2)R_1, A(\pi/2)R_2\}$. The two order four groups D_2 and D'_2 give different contributions to the

partition function, since these groups are not conjugate in D_4. The fundamental cells of lattice 15 we have to pick in order to interprete them as reflections along the edges of the cell have different shape. For D_2 it is a unit square giving a contribution $Z_{D_2-orb}(\tau = i, \rho)$, whereas for D_2' it is a rhombus giving a contribution $Z_{D_2-orb}(\tau = \frac{1}{2} + \frac{i}{2}, \rho)$. Note that by (2.67) for $\rho_1 = 0$ we have an independent choice of sign for the discrete torsion parts of D_2, D_2', and for $\rho_1 = 1/2$ discrete torsion enters for D_2 only. The partition function for this theory is

$$\begin{aligned} Z_{D_4^{\delta\epsilon}-orb}(0,1,0,\rho_2) &= \frac{1}{2}\Big(Z_{\mathbb{Z}_4-orb}(0,1,0,\rho_2) + Z_{D_2^{\delta}-orb}(0,1,0,\rho_2) \\ &\quad + Z_{D_2^{\epsilon}-orb}(\frac{1}{2},\frac{1}{2},0,\rho_2) \;-\; Z_{\mathbb{Z}_2-orb}(0,1,0,\rho_2)\Big), \\ Z_{D_4^{\pm}-orb}(0,1,\frac{1}{2},\rho_2) &= \frac{1}{2}\Big(Z_{\mathbb{Z}_4-orb}(0,1,\frac{1}{2},\rho_2) \;+\; Z_{D_2^{\pm}-orb}(0,1,\frac{1}{2},\rho_2) \\ &\quad + Z_{D_2-orb}(\frac{1}{2},\frac{1}{2},\frac{1}{2},\rho_2) \;-\; Z_{\mathbb{Z}_2-orb}(0,1,\frac{1}{2},\rho_2)\Big), \end{aligned} \tag{2.75}$$

where $\delta, \epsilon \in \{\pm\}$; $Z_{\mathbb{Z}_2-orb}, Z_{\mathbb{Z}_4-orb}$ and $Z_{D_2^{\pm}-orb}$ are given in (2.43), (2.48), and (2.67). We remark that in case $\rho_1 = 1/2$ the $Z_{\mathbb{Z}_4}, Z_{D_2}, Z_{\mathbb{Z}_2}$ parts of (2.75) always contribute even numbers of fields with dimensions $h = \overline{h} = 1/16$. This shows that for the D_2 orbifold with $\tau_1 = 0, \rho_1 = 1/2$ we must indeed have $k = 2$ in (2.67).
Since for the D_2 orbifold by our discussion of lattices 8 and 9 separate T–duality may be performed on τ, ρ without changing the theory, the D_4 orbifold conformal field theories form six families

$$\mathcal{C}^{(0)}_{D_4^{\delta\epsilon}-orb} \cong \mathbb{R}^+/\Theta,\ \delta,\epsilon \in \{\pm\}, \quad \mathcal{C}^{(1/2)}_{D_4^{\pm}-orb} \cong \mathbb{R}^+/\Theta.$$

Lattice 16: The $D_4(T_R')$ orbifold

Lattice 16 ($\tau = i$) has a $D_4(T_R') = \mathbb{Z}_4 \cup \{T_{R_1}', A(\pi/2)T_{R_1}', T_{R_2}', A(\pi/2)T_{R_2}'\}$ symmetry as defined in (2.27), and we may set $\rho_1 = 0$. The maximal abelian subgroups of $D_4(T_R')$ are $\mathbb{Z}_4$, $D_2(T_R') = \{1, A(\pi), T_{R_1}', T_{R_2}'\}$, and $D_2 = \{1, A(\pi), A(\pi/2)T_{R_1}', A(\pi/2)T_{R_2}'\}$. Anologously to lattice 15 we find

$$\begin{aligned} Z_{D_4(T_R')^{\pm}-orb}(0,1,0,\rho_2) &= \frac{1}{2}\big(Z_{\mathbb{Z}_4-orb}(0,1,0,\rho_2) + Z_{D_2(T_R')-orb}(0,1,0,\rho_2) \\ &\quad + Z_{D_2^{\pm}-orb}(\frac{1}{2},\frac{1}{2},0,\rho_2) - Z_{\mathbb{Z}_2-orb}(0,1,0,\rho_2)\Big), \end{aligned} \tag{2.76}$$

where $Z_{\mathbb{Z}_2-orb}$, $Z_{\mathbb{Z}_4-orb}$, $Z_{D_2(T_R')-orb}$ and $Z_{D_2^{\pm}-orb}$ are given in (2.43), (2.48), (2.72), and (2.67). The discussion of the $D_2(T_R')$ orbifold (lattice 12) shows that T–duality does not map equivalent $D_4(T_R')$ orbifold theories onto each other, thus

$$\mathcal{C}_{D_4(T_R')^{\pm}-orb} \cong \mathbb{R}^+.$$

Lattice 17: The D_6 orbifold

Lattice 17 ($\tau = e^{2\pi i/3}$) has a

$$D_6 = \mathbb{Z}_6 \cup \{R_1, A(\pi/3)R_1, A(2\pi/3)R_1, R_2, A(\pi/3)R_2, A(2\pi/3)R_2\}$$

symmetry. The maximal abelian subgroups of D_6 are $\mathbb{Z}_6$, and three groups of type D_2, namely

$$\{1, A(\pi), R_1, R_2\}, \quad \{1, A(\pi), A(\pi/3)R_1, A(\pi/3)R_2\},$$
$$\{1, A(\pi), A(2\pi/3)R_1, A(2\pi/3)R_2\}.$$

These order four groups give identical contributions to the partition function since they are conjugate in D_6. This also means that in order for the action of D_6 on the twisted sector to be well defined, discrete torsion must be the same for all the three of them. Thus we find

$$\begin{aligned} Z_{D_6^{(\pm)}-orb}(1/2, \sqrt{3}/2, \rho_1, \rho_2) &= \frac{1}{12}(6Z_{\mathbb{Z}_6-orb} + 3(4Z_{D_2^{(\pm)}-orb} - 2Z_{\mathbb{Z}_2-orb})) \\ &= \frac{1}{2}(Z_{\mathbb{Z}_6-orb} + 2Z_{D_2^{(\pm)}-orb} - Z_{\mathbb{Z}_2-orb}), \end{aligned} \tag{2.77}$$

where $Z_{\mathbb{Z}_2-orb}$, $Z_{\mathbb{Z}_6-orb}$ and $Z_{D_2^{(\pm)}-orb}$ are given in (2.43), (2.50), and (2.67). Analogously to lattice 15, the three families of D_6 orbifold conformal field theories are

$$\mathcal{C}^{(0)}_{D_6^{\pm}-orb} \cong \mathbb{R}^+/\Theta, \quad \mathcal{C}^{(1/2)}_{D_6-orb} \cong \mathbb{R}^+/\Theta.$$

Chapter 3

Multicritical lines and points

In this chapter we determine intersections of the 28 nonexceptional components $\mathcal{C}^{(\bullet)}_{G^{(\bullet)}-orb}$ of the moduli space that we constructed in section 2.5. We find that all but four of them can be connected directly or indirectly to the moduli space $\mathcal{T}^2$ of toroidal theories. No further intersections are expected.
I remark that all the identifications below have been confirmed by myself on the level of partition functions numerically. On the level of operator algebra, a few identifications were known in [21, 40, 48], the others have been obtained by K. Wendland.
The procedure closely follows the proof for the isomorphy of the $c = 1$ circle theory at radius $r = 2$ to the orbifold theory at radius $r = 1$ which we studied briefly in section 1.2 (see, e.g., [21, 40]). The main idea is to exploit the enhanced $SU(2)$ symmetry of the circle theory at radius $r = 1$. Namely, $SU(2)$ relates two generically different $\mathbb{Z}_2$ actions in this theory by conjugation. Thus the resulting orbifold theories are isomorphic. One of them is the circle theory at doubled radius $r = 2$, the other is the ordinary $\mathbb{Z}_2$ orbifold theory at radius $r = 1$.
As anticipated in [14] each of the toroidal multicritical points generates a series of further multicritical points or lines, since we can mod out further symmetries. But even better, this procedure will lead to the determination of intersection points: By the discussion in sections 2.1 and 2.3, all the 28 nonexceptional components of $\mathcal{C}^2$ are obtained by modding out solvable groups from toroidal theories. This means that we can always regain the original toroidal theory by performing another orbifold procedure. In particular, any intersection point between nonexceptional nonisolated components of $\mathcal{C}^2$ corresponds to a multicritical point on $\mathcal{T}^2$.
One can simplify things by stepwise modding out [19]: If a symmetry group G contains a normal subgroup H, then the G orbifold conformal field theory A/G of a theory A is isomorphic to the G/H orbifold conformal field theory of A/H. Moreover, the G/H action on A/H translates to an action on any other theory A' which was identified with A/H. For $H' \subset G/H, G' \cong H \times H'$ this leads to possibly new identifications $A/G' \cong A'/H'$ which need not correspond to conjugate actions on the original A. In $A/H \cong A'$ we may have gotten rid of all states which the G' action has no consistent conjugate on.
This chapter is organised as follows, in section 3.1 we discuss the multicritical points and lines obtainable by modding out conjugate $\mathbb{Z}_2$ symmetries of tori with enhanced $SU(2)$ symmetry. In sections 3.2–3.6 we determine multicritical points and lines obtainable from those identifications we found in 3.1 by modding out further symmetries. In section 3.7 we follow the same procedure for the $SU(3)$ torus theory at $\tau = \rho = e^{2\pi i/3}$. The slightly technical discussion results in a list of multicritical points and lines in nonexceptional nonisolated components of $\mathcal{C}^2$. In the following we will denote the $G^{(\bullet)}$ orbifold theory

of the toroidal theory $A_T(\tau_1,\tau_2,\rho_1,\rho_2)$ with parameters (τ,ρ) by $A_{G^{(\bullet)}-orb}(\tau_1,\tau_2,\rho_1,\rho_2)$.

3.1 Multicritical lines on the torus moduli space $\mathcal{T}^2$: Conjugate $\mathbb{Z}_2$ actions

To compare all $\mathbb{Z}_2$ symmetries of the $SU(2)^2$ torus theory at $\tau=\rho=i$ we discuss their action on the $(1,0)$ fields. In section 2.2 the conserved currents of the generic toroidal theory are denoted j_μ; the additional vertex operators of dimensions $(1,0)$ are denoted $j_\mu^\pm, \mu\in\{1,2\}$, such that each triple $j_\mu, j_\mu^\pm$ generates an $SU(2)_1$ Kac–Moody algebra. Each of these $SU(2)_1$ Kac–Moody algebras belongs to one of the $c=1$ factors of the torus theory. Let us list all $\mathbb{Z}_2$ symmetries with two positive and four negative eigenvalues on the set of $(1,0)$ fields. By $\widetilde{\mathbb{Z}_2(R)}$ we denote the $\mathbb{Z}_2(R)$ symmetry applied to the torus theory with fundamental cell such that $\tau=\rho=1/2+i/2$ (remember the phases on Hilbert space ground states that were discussed for lattice 9):

$\mathbb{Z}_2$ rotational group :	$j_\mu\mapsto -j_\mu,$	$j_\mu^\pm\leftrightarrow j_\mu^\mp,$
shift orbifold by $\delta'=\frac{1}{2}\binom{1}{1}$:	$j_\mu\mapsto j_\mu,$	$j_\mu^\pm\mapsto -j_\mu^\pm,$
$\widetilde{\mathbb{Z}_2(R_1)}$:	$j_1\mapsto j_1, j_2\mapsto -j_2,$	$j_1^\pm\mapsto -j_1^\pm, j_2^+\leftrightarrow j_2^-,$
$\mathbb{Z}_2(T_{R_1})$:	$j_1\mapsto j_1, j_2\mapsto -j_2,$	$j_1^\pm\mapsto -j_1^\pm, j_2^+\leftrightarrow j_2^-.$

None of the above symmetries mixes currents from different $c=1$ factors of the torus theory or j_μ with $j_\mu^\pm$ currents. Moreover, their eigenvalue spectrum is identical on each $c=1$ factor, so we may use the corresponding $c=1$ result to show that the four $\mathbb{Z}_2$ orbifolds by the above listed symmetries give isomorphic theories when applied to the $SU(2)^2$ theory. This generates a quadrucritical point. The shift orbifold by the half lattice vector δ', as usual, results in a torus theory with additional generator δ' of the lattice and half volume and B–field ($A_T(0,1,0,2)=A_T(0,1,0,1/2)$ by T–duality):

$$\begin{aligned} A_T(0,1,0,2) &= A_{T_R-orb}(0,1,0,1) \\ &= A_{\mathbb{Z}_2-orb}(0,1,0,1)=A_{R-orb}(1/2,1/2,1/2,1/2). \end{aligned} \tag{Q1}$$

The equality $A_T(0,1,0,2)=A_{\mathbb{Z}_2-orb}(0,1,0,1)$ has already been proven in [48], both on the level of partition function and operator algebra.
The above quadrucritical point turns out to actually be the intersection of four bicritical lines. First consider the family of torus theories at parameters $\tau=\rho=it, t\in\mathbb{R}^+$ which decompose into tensor products of two $c=1$ circle theories at radii $r=1$ and $r'=t$, respectively. For all values of t the first factor possesses an $SU(2)$ symmetry. Since the actions of T_{R_2} and the shift by $\delta'=\frac{1}{2}\binom{1}{t}$ only differ on this first factor, where they are generally conjugate by the $SU(2)$ symmetry, we find ($A_T(1/2,t/2,0,t/2)=A_T(0,t/2,1/2,t/2)$ by mirror symmetry)

$$\forall\, t\in\mathbb{R}^+ :\quad A_T(0,t/2,1/2,t/2)=A_{T_{R_2}-orb}(0,t,0,t), \tag{L1}$$

and analogously

$$\forall\, t\in\mathbb{R}^+ :\quad A_{T_{R_1}-orb}(0,t,0,t)=A_{\mathbb{Z}_2-orb}(0,t,0,t). \tag{L2}$$

Next consider the family of toroidal theories at parameters $\tau=\rho=1/2+it, t\in\mathbb{R}^+$. We also have a generic $SU(2)\times U(1)$ symmetry for this family. Inspection of the charge

lattice shows that as before we have conjugate $\mathbb{Z}_2$ symmetries now giving bicritical lines

$$\begin{aligned} \forall\, t \in \mathbb{R}^+ : \quad A_{\mathbb{Z}_2-orb}(1/2,t,1/2,t) &= A_{R_1-orb}(1/2,t,1/2,t), && \text{(L3)} \\ A_{R_2-orb}(1/2,t,1/2,t) &= A_T(0,2t,1/4,t/2). && \text{(L4)} \end{aligned}$$

There are two more $\mathbb{Z}_2$ symmetries which are conjugate on the entire family of toroidal theories with parameters $\tau = \rho = it, t \in \mathbb{R}^+$ by $SU(2)$ symmetry on the first factor. They have four positive and two negative eigenvalues on $(1,0)$ fields:

$$\begin{array}{lllll} \mathbb{Z}_2(R_2): & j_1 \mapsto -j_1, j_2 \mapsto j_2, & j_1^+ \leftrightarrow j_1^-, & j_2^\pm \mapsto j_2^\pm, \\ \text{shift orbifold by } \delta_1 = \frac{1}{2}\binom{1}{0}: & j_\mu \mapsto j_\mu, & j_1^\pm \mapsto -j_1^\pm, & j_2^\pm \mapsto j_2^\pm. \end{array}$$

In particular,

$$\forall\, t \in \mathbb{R}^+ : \quad A_{R_2-orb}(0,t,0,t) = A_T(0,t/2,0,2t). \qquad \text{(L5)}$$

We remark that $\mathbb{Z}_2(R)$ applied to the theory with fundamental cell such that $\tau = 1/2 + i/2, \rho = i$ has three positive and three negative eigenvalues on the set of $(1,0)$ fields. Hence it is not conjugate to any other crystallographic symmetry of $A_T(0,1,0,1)$.

3.2 Series of multicritical lines and points obtainable from (L1) and (L5)

We are now going to mod out further symmetries on both sides of the equalities obtained above. The main problem is to find the correct translation for the action of a symmetry from one model to the other. The simplest case is (L5) from which we mod out R_1 on both sides. Because all the symmetries used so far respect the factorization of $A_T(0,t,0,t)$ into a tensor product of two circle theories and commute, we directly get

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^+-orb}(0,t,0,t) = A_{R_1-orb}(0,t/2,0,2t). \qquad \text{(L6)}$$

Note that by mirror symmetry and T–duality (2.51) we have

$$A_{R_1-orb}(0,2,0,1/2) = A_{R_2-orb}(0,2,0,2),$$

hence the above multicritical line and the one found in (L5) intersect in a tricritical point:

$$A_{D_2^+-orb}(0,1,0,1) = A_{R_1-orb}(0,1/2,0,2) = A_T(0,1,0,4). \qquad \text{(T1)}$$

We now systematically mod out all possible symmetries of the torus theory $A_T(0,t/2,0,2t)$ in (L5). The procedure is similar in all cases, namely, the charge lattices of the underlying toroidal theories on both sides of an identification must be determined, as well as twisted ground states, if present. After having performed a state by state identification, symmetries can be translated from one side to the other. This way the details which we partly omit in the proofs below can easily be filled.

As to (L5), by (2.28) the actions we can generically mod out on the torus theory $A_T(0,t/2,0,2t)$ are $\mathbb{Z}_2, \mathbb{Z}_2(R), \mathbb{Z}_2(T_R), D_2^\pm, D_2(T_R)$ and $D_2(T_R')$. At $t=2$ one has additional $\widetilde{Z_2(R)}$ and $\mathbb{Z}_4$ actions which give no new identifications, though.

Modding out by $\mathbb{Z}_2(R_1)$ gives the bicritical line (L6) as discussed above. The reflection R_2 on the torus side acts as a shift by $\delta_1 = \frac{1}{2}\binom{1}{0}$ on the underlying torus theory of $A_{R_2}(0,t,0,t)$ leading to a trivial identity. The symmetry T_{R_1} applied to the torus side differs in its action from R_1 by additional signs on those vertex operators (of lowest

dimension) in $A_T(0,t/2,0,2t)$ which correspond to twisted ground states in $A_{R_2}(0,t,0,t)$. Therefore, comparison with (L6) shows

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^- -orb}(0,t,0,t) = A_{T_{R_1}-orb}(0,t/2,0,2t). \tag{L7}$$

Modding out by T_{R_2} instead of T_{R_1} again gives a trivial identity, since T_{R_2} acts on the underlying torus of $A_{R_2-orb}(0,t,0,t)$ by the shift T_{δ_1}. Note that a comparison of (L7) with (L6) gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion. Since $A_{T_{R_1}-orb}(0,1/2,0,2) = A_{T_{R_2}-orb}(0,2,0,2)$ by T–duality (see the discussion of lattice 10), the multicritical lines (L7) and (L1) intersect in a tricritical point:

$$A_{D_2^- -orb}(0,1,0,1) = A_{T_{R_2}-orb}(0,2,0,2) = A_T(0,1,1/2,1). \tag{T2}$$

Next, we mod out the ordinary $\mathbb{Z}_2$ action on (L5). The multicritical line (L5) can also be written as $A_{\widehat{T}_{R_2}-orb}(0,t,0,t) = A_T(0,t/2,0,2t)$. Recall from (2.17) that $A_T(0,t,0,t)$ as well as $A_T(0,2t,0,t/2)$ are tensor products of circle theories at radii $r=1, r'=t$ and $r=2, r'=t$, respectively. Now consider the residual action of $D_2(T_R)$ of the original torus theory $A_T(0,t,0,t)$ on the orbifoldized theory $A_{\widehat{T}_{R_2}-orb}(0,t,0,t)$ and note that it acts as ordinary $\mathbb{Z}_2$ on the invariant sector. The twisted ground states of the first circle factor are interchanged, so all in all we get an ordinary $\mathbb{Z}_2$ action on the torus theory $A_T(0,t/2,0,2t)$. This yields

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2(T_R)-orb}(0,t,0,t) = A_{\mathbb{Z}_2-orb}(0,t/2,0,2t). \tag{L8}$$

By analogous arguments one finds that modding out (L1) by $\mathbb{Z}_2$ on the torus side yields

$$\forall\, t \in \mathbb{R}^+ : \quad A_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2) = A_{D_2(T'_R)-orb}(0,t,0,t). \tag{L9}$$

As mentioned above, R_2 applied to the torus theory $A_T(0,t/2,1/2,t/2)$ acts as shift T_{δ_1} on the underlying torus theory of $A_{R_2-orb}(0,t,0,t)$. Applying this to the bicritical line (L8), if R_2 acts with positive sign on the $\mathbb{Z}_2$ twisted ground states of the right hand side we obtain a trivial identity. On the other hand, if we use negative discrete torsion on the right hand side we find

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2(T_R)-orb}(0,t/2,0,2t) = A_{D_2^- -orb}(0,t/2,0,2t). \tag{L10}$$

Note that the bicritical lines (L7) and (L10) intersect in a tricritical point which can be interpreted as the result of modding out (T1) by T_{R_1}:

$$A_{T_{R_1}-orb}(0,1,0,4) = A_{D_2^- -orb}(0,2,0,2) = A_{D_2(T_R)-orb}(0,1/2,0,2). \tag{3.1}$$

To mod out (L5) by $D_2(T'_R)$ on the torus side amounts to modding out (L7) by T'_{R_2} which acts as shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{t}$ on the underlying torus of $A_{D_2^- -orb}(0,t,0,t)$. Thus

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^- -orb}(1/2,t/2,0,t/2) = A_{D_2(T'_R)-orb}(0,t/2,0,2t), \tag{L11}$$

Note that because of T–duality $A_{D_2(T'_R)-orb}(0,2,0,2) = A_{D_2(T'_R)-orb}(0,1/2,0,2)$ as discussed for lattice 12, so (L11) intersects (L9) in a tricritical point which can be understood as the result of modding out (T2) by $\mathbb{Z}_2$:

$$A_{D_2^- -orb}(1/2,1/2,0,1/2) = A_{D_2(T'_R)-orb}(0,1/2,0,2) = A_{\mathbb{Z}_2}(0,1,1/2,1). \tag{T3}$$

We now turn to a systematic discussion of intersection lines and points obtained from (L1). From $A_T(0,t/2,1/2,t/2)$ we can generically mod out $\mathbb{Z}_2, \mathbb{Z}_2(R)$ and $D_2^{\pm}$. The additional symmetries for $t=1$ and $t=2$ produce nothing new.
Modding out by the ordinary $\mathbb{Z}_2$ action on the torus side gives the bicritical line (L9), as was mentioned above. We claim that the result of modding out a $\mathbb{Z}_2(R_1)$ action leads to the bicritical line

$$\forall\, t \in \mathbb{R}^+ : \quad A_{R_1-orb}(0,t/2,1/2,t/2) = A_{D_2(T_R)-orb}(0,1/t,0,t). \tag{L12}$$

Actually, the slightly surprising parameters on the right hand side are due to an apparent asymetry in the definition of $D_2(T_R) = \{1, A(\pi), T_{R_1}, \widehat{T}_{R_2}\}$. If we use $\widehat{D_2(T_R)} = \{1, A(\pi), T_{R_2}, \widehat{T}_{R_1}\}$ instead, then by T–duality (see the discussion of lattice 11) the parameters on the right hand side of (L12) are $(0,t,0,t)$. Our claim thus amounts to the fact that R_1 as applied to $A_T(0,t/2,1/2,t/2)$ induces an ordinary $\mathbb{Z}_2$ action (or equivalently $\widehat{T}_{R_1}$) on $A_{T_{R_2}-orb}(0,t,0,t)$. For the $(1,0)$ fields this is easy to check: R_1 leaves one of the abelian currents of the torus theory invariant and multiplies the other by -1. So do $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $A_{T_{R_2}-orb}(0,t,0,t)$, where the T_{R_2} invariant generic abelian current of the underlying torus theory is multiplied by -1, and the T_{R_2} invariant combination of vertex operators remains invariant. To give a full proof for (L12), note that the charge lattice of $A_T(1/2,t/2,0,t/2)$ by (2.14) is generated by vectors

$$(p,\overline{p}) \in \left\{ \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\-\frac{1}{t}\end{pmatrix},\begin{pmatrix}1\\-\frac{1}{t}\end{pmatrix}\right), \quad \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\right), \right.$$
$$\left. \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\\frac{2}{t}\end{pmatrix},\begin{pmatrix}0\\\frac{2}{t}\end{pmatrix}\right), \quad \tfrac{1}{\sqrt{2}}\left(\begin{pmatrix}\frac{1}{2}\\\frac{t}{2}\end{pmatrix},\begin{pmatrix}-\frac{1}{2}\\-\frac{t}{2}\end{pmatrix}\right) \right\}.$$

The four vertex operators of dimension $\frac{1}{4}(1+\frac{1}{t^2})$ given by

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi^1+\delta\varphi^2/t)} e^{\frac{i\epsilon}{\sqrt{2}}(\overline{\varphi}^1+\delta\overline{\varphi}^2/t)}, \ \epsilon,\delta \in \{\pm 1\}$$

correspond to the following T_{R_2} invariant vertex operators of $A_T(0,t,0,t)$:

$$e^{\frac{i\epsilon}{\sqrt{2}}(\varphi_I^1+\varphi_I^2/t)} e^{\frac{i\epsilon}{\sqrt{2}}(\delta\overline{\varphi}_I^1+\overline{\varphi}_I^2/t)} - e^{\frac{i\epsilon}{\sqrt{2}}(-\varphi_I^1+\varphi_I^2/t)} e^{\frac{i\epsilon}{\sqrt{2}}(-\delta\overline{\varphi}_I^1+\overline{\varphi}_I^2/t)}, \quad \epsilon,\delta \in \{\pm 1\}$$

(see (2.14) to determine the charge lattice of the torus theory $A_T(0,t,0,t)$; φ_I^μ denote the bosonic fields in this torus theory to distinguish them from φ^μ on $A_T(0,t/2,1/2,t/2)$). Both R_1 on $A_T(1/2,t/2,0,t/2)$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on $A_{T_{R_2}-orb}(0,t,0,t)$ pairwise interchange these vertex operators. The four vertex operators of dimension $\frac{1}{16}(1+t^2)$ given by

$$e^{\frac{i}{2\sqrt{2}}(\epsilon\varphi_1+\delta t\varphi_2)} e^{-\frac{i}{2\sqrt{2}}(\epsilon\overline{\varphi}_1+\delta t\overline{\varphi}_2)}, \ \epsilon,\delta \in \{\pm 1\}$$

correspond to the twisted ground states on $A_{T_{R_2}-orb}(0,t,0,t)$, both being pairwise interchanged by R_1 on $A_T(\frac{1}{2},\frac{t}{2},0,\frac{t}{2})$ and $\mathbb{Z}_2$ and $\widehat{T}_{R_1}$ on reflection plus shift orbifold $A_{T_{R_2}-orb}(0,t,0,t)$ as well. This proves (L12). Modding out R_2 instead of R_1 gives the same result, up to T–duality. Note that the point (3.1) actually lies on (L12), hence we have found another quadrucritical point:

$$\begin{aligned} A_{T_R-orb}(0,1,0,4) &= A_{D_2^--orb}(0,2,0,2) \\ &= A_{D_2(T_R)-orb}(0,1/2,0,2) = A_{R-orb}(0,1,1/2,1). \end{aligned} \tag{Q2}$$

Moreover, (L12) intersects the bicritical lines (L2) and (L8), so there is another quadrucritical point:

$$\begin{aligned} A_{R-orb}(1/2,1/2,0,2) &= A_{D_2(T_R)-orb}(0,1,0,1) \\ &= A_{\mathbb{Z}_2-orb}(0,2,0,2) = A_{T_{R_1}-orb}(0,2,0,2). \end{aligned} \tag{Q3}$$

We proceed with the above reasoning to see that the $\mathbb{Z}_2$ action on $A_T(0,\frac{t}{2},\frac{1}{2},\frac{t}{2})$ translates to a T'_{R_1} action on $A_{T_{R_2}-orb}(0,t,0,t) = A_{T'_{R_2}-orb}(0,t,0,t)$ (this is the proof of (L9)). Therefore, to determine the action induced by D_2^+ on $A_T(0,t/2,1/2,t/2)$, we note that on $A_{T_{R_2}-orb}(0,t,0,t)$ the additional symmetry to mod out compared to (L12) on the underlying torus theory $A_T(0,t,0,t)$ is the combination $T'_{R_1}\widehat{T}_{R_1}$, i.e. a shift by $\delta_1 = \frac{1}{2}\binom{1}{0}$. Moreover, the $\mathbb{Z}_2$ twisted ground states in $A_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2)$ are given by vertex operators which are $\widehat{T}_{R_1}$ invariant, and therefore

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2^+-orb}(0,t/2,1/2,t/2) = A_{D_2(T_R)-orb}(0,2/t,0,2t). \tag{L13}$$

This can also be seen by applying R_1 to $A_{\mathbb{Z}_2-orb}(0,t/2,1/2,t/2)$ in (L9). Modding out the D_2^- action on the torus side analogously gives (L11), again. Note that the bicritical line (L13) intersects (L8) and (L10), so we have found two more tricritical points:

$$A_{D_2^+-orb}(0,1/2,1/2,1/2) = A_{D_2(T_R)-orb}(0,2,0,2) = A_{\mathbb{Z}_2-orb}(0,1,0,4), \tag{T4}$$

$$A_{D_2^+-orb}(0,1,1/2,1) = A_{D_2(T_R)-orb}(0,1,0,4) = A_{D_2^--orb}(0,1,0,4), \tag{T5}$$

3.3 Series of multicritical lines and points obtainable from (L2)-(L4)

To gain further identifications from (L2) we can only mod out further symmetries of the underlying torus theory $A_T(0,t,0,t)$. If we add generators of order four we only get trivial identities. An action of $\mathbb{Z}_2(R)$ type basically acts as a shift on the $A_{T_{R_1}}(0,t,0,t)$ theory, so we arrive at the bicritical lines (L6) and (L7) again. All other symmetries give trivial identities.

Next we consider (L3). The symmetries we can generically mod out are $\mathbb{Z}_2, \mathbb{Z}_2(R)$ and $\mathbb{Z}_2(T_R)$, all giving trivial identities. For $t = \sqrt{3}/2$ we can mod out additional symmetries containing a $\mathbb{Z}_3$ action, but this does not produce anything new. For the special value $t = 1/2$, where we have $A_{\mathbb{Z}_2-orb}(0,1,0,1) = A_{R_1-orb}(1/2,1/2,1/2,1/2)$ all but the modding out of T'_{R_1} give trivial identities as well. The symmetry T'_{R_1} multiplies both $\mathbb{Z}_2$ invariant $(1,0)$ fields in $A_{\mathbb{Z}_2-orb}(0,1,0,1)$ by -1, and the generators of the invariant part of the $A_T(0,1,0,1)$ charge lattice are pairwise interchanged. The same is true for the $\mathbb{Z}_2$ twisted ground states. We claim that this translates to an R_2 action on $A_{R_1-orb}(1/2,1/2,1/2,1/2)$. Namely, as a result of the discussion for lattice 7 we found that on $A_T(1/2,1/2,1/2,1/2)$ the action of D_2 leaves invariant none of the combinations of vertex operators of dimensions $(1,0)$. The respective $(1/8,1/8)$ and $(1/2,1/2)$ fields in $A_{R_1-orb}(1/2,1/2,1/2,1/2)$ are also pairwise interchanged, thus

$$A_{D_2(T'_R)-orb}(0,1,0,1) = A_{D_2-orb}(1/2,1/2,1/2,1/2).$$

By (L9) and $A_{\mathbb{Z}_2-orb}(0,1/2,1/2,1/2) = A_{\mathbb{Z}_2-orb}(0,1,0,2)$ we see that we have actually found a tricritical point on a bicritical line:

$$A_{D_2-orb}(1/2,1/2,1/2,1/2) = A_{\mathbb{Z}_2-orb}(0,1,0,2) = A_{D_2(T'_R)-orb}(0,1,0,1). \tag{T6}$$

We remark that the above can be seen more directly by showing that in the notation of section 3.1 the groups $\widetilde{\mathbb{Z}_2(R_1)} \times \widetilde{\mathbb{Z}_2(R_2)}, \mathbb{Z}_2 \times \mathbb{Z}_2(T_{\delta'})$ and $D_2(T'_R)$ are conjugate symmetry groups of type D_2 of the $SU(2)^2$ torus theory.
In the discussion of lattice 15 we found that $D_4^{\pm}$ acting on $A_T(0,1,\frac{1}{2},\frac{1}{2})$ has a subgroup $D'_2 \subset D_4^{\pm}$ which effectively acts on $A_T(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. By the above this is conjugate to the $D_2(T'_R)$ action on $A_T(0,1,0,1)$, where $D_2(T'_R) \subset D_4^{\pm}(T'_R)$ generically exactly gives the distinction between $D_4^{\pm}(T'_R)$ and $D_4^{\pm}$. This means

$$A_{D_4^+-orb}(0,1,1/2,1/2) = A_{D_4(T'_R)^+-orb}(0,1,0,1), \tag{3.2}$$

$$A_{D_4^--orb}(0,1,\frac{1}{2},\frac{1}{2}) = A_{D_4(T'_R)^--orb}(0,1,0,1). \tag{C1}$$

Let us now turn to the discussion of (L4). Generically, we can only mod out a $\mathbb{Z}_2$ action on $A_T(0,2t,1/4,t/2)$. This leads to another bicritical line:

$$\forall\, t \in \mathbb{R}^+ : \quad A_{D_2-orb}(1/2,t,1/2,t) = A_{\mathbb{Z}_2-orb}(0,2t,1/4,t/2), \tag{L14}$$

as follows directly from (L3) and (L4). Note that (L14) intersects the bicritical line (L9) in (T6).
We can mod out additional symmetries of (L4) at special values of t, namely if $\rho = 1/4 + it/2$ is equivalent to ρ' with $\rho'_1 \in \{0, 1/2\}$ by Möbius transformations. This is true for $t \in \{1/2, \sqrt{3}/2, \sqrt{7}/2, \sqrt{5/12}, \sqrt{3/20}, \sqrt{1/28}\}$, but only for $t = \sqrt{3}/2$ we produce a new identification by our methods. Here, (L4) gives $A_{R_2-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) = A_T(0,\sqrt{3},0,\sqrt{3})$, and the torus theory decomposes into a tensor product of two $c=1$ circle theories at radii $r=1$ and $r'=\sqrt{3}$, respectively. The latter only contains one $(1,0)$ field which is identified with the vertex operator $e^{i\sqrt{2/3}\,\varphi_1} e^{i\sqrt{2/3}\,\overline{\varphi}_1} + e^{-i\sqrt{2/3}\,\varphi_1} e^{-i\sqrt{2/3}\,\overline{\varphi}_1}$ in the $A_{R_2-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$ model. The $SU(2)$ generators of the first circle factor are identified with the two other R_2 invariant vertex operators and the abelian current j_2 of $A_T(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$. The only symmetry we can mod out to find a new identification is T_{R_1}. Then by definition, of the $(1,0)$ fields on the torus side only one is invariant, namely the abelian current of the first factor theory. The same is true for the R_1 action on $A_{R_2-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2)$, where only one combination of vertex operators is invariant. Actually, the actions match entirely, showing

$$A_{D_2-orb}(1/2,\sqrt{3}/2,1/2,\sqrt{3}/2) = A_{T_{R_1}-orb}(0,\sqrt{3},0,\sqrt{3}). \tag{C2}$$

3.4 Series of multicritical points obtainable from (Q1)

The identifications in section 3.1 we have not yet used by our discussions of the bicritical lines (L1)-(L5) are $A_T(0,1,0,2) = A_{\mathbb{Z}_2-orb}(0,1,0,1)$ and $A_{T_{R_1}-orb}(0,1,0,1) = A_{R-orb}(1/2,1/2,1/2,1/2)$, taken from (Q1). In the latter case we can mod out additional symmetries on the underlying tori, but this produces no new identifications. Namely, the ordinary $\mathbb{Z}_2$ action applied to the left hand side gives the identification $A_{D_2(T_R)-orb}(0,1,0,1) = A_{R_1-orb}(0,1/2,1/2,1/2)$ on (L12), and $\mathbb{Z}_2$ applied to the right hand side gives $A_{D_2(T'_R)-orb}(0,1,0,1) = A_{D_2-orb}(1/2,1/2,1/2,1/2)$, see (T6). In fact, by the discussion at the beginning of the section we know that it suffices to mod out further symmetries of identities that contain toroidal theories.
We are now going to mod out further symmetries on both sides of the equality $A_{\mathbb{Z}_2-orb}(0,1,0,1) = A_T(0,1,0,2)$. We mostly use the description in terms of the toroidal theory

$A_T(0,1,0,2)$, which by (2.14) has charge vectors

$$\overset{(-)}{p} = \frac{1}{2}\left\{ \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \pm 2 \begin{pmatrix} m_2 \\ m_1 \end{pmatrix} \right\}, \quad m_i, n_i \in \mathbb{Z}. \tag{3.3}$$

On the $A_{\mathbb{Z}_2-orb}(0,1,0,1)$ side, the torus currents J_1, J_2 of $A_T(0,1,0,2)$ are $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h,\overline{h}) = (1,0)$ in the two $c=1$ factors of $A_T(0,1,0,1)$. The states $|0,0,\pm 1,0\rangle$, $|0,0,0,\pm 1\rangle$ in $A_T(0,1,0,2)$ by (3.3) correspond to the $(1/8,1/8)$ fields of the theory and therefore are identified with the four twisted ground states of the $\mathbb{Z}_2$ orbifold $A_{\mathbb{Z}_2-orb}(0,1,0,1)$. Further generators of the Hilbert space of $A_T(0,1,0,2)$ are vertex operators corresponding to $|\pm 1,0,0,0\rangle$, $|0,\pm 1,0,0\rangle$ which are identified with the $\mathbb{Z}_2$ invariant combinations of vertex operators with dimensions $(h,\overline{h}) = (1/2,1/2)$ of the $A_T(0,1,0,1)$ side. These do not live in one of the separate factor theories.

The $\mathbb{Z}_2$ action on $A_T(0,1,0,2)$ induces a $\widetilde{\mathbb{Z}_2(R)}$ action on the underlying torus of $A_{\mathbb{Z}_2-orb}$ $(0,1,0,1)$, and we arrive at $A_{D_2-orb}(1/2,1/2,1/2,1/2) = A_{\mathbb{Z}_2-orb}(0,1,0,2)$ reproducing part of (T6). The R_1 action on $A_T(0,1,0,2)$ translates to $A_{\mathbb{Z}_2-orb}(0,1,0,1)$ in the following way: Among the $(1,0)$ fields in $A_{\mathbb{Z}_2-orb}(0,1,0,1)$ only the combination in the first factor of $A_T(0,1,0,1)$ is invariant; two of the twisted ground states of the $\mathbb{Z}_2$ orbifold are exchanged, whereas two of them are fixed. Among the $(\frac{1}{2},\frac{1}{2})$ fields, again two are fixed and two are exchanged; this is just the R_1 action on $A_{\mathbb{Z}_2-orb}(\frac{1}{2},\frac{1}{2},0,1)$, hence

$$A_{D_2^+-orb}(1/2,1/2,0,1) = A_{R-orb}(0,1,0,2). \tag{C3}$$

If we combine the $\mathbb{Z}_2$ and $\mathbb{Z}_2(R)$ actions on $A_T(0,1,0,2)$, the $\mathbb{Z}_2$ now will act as a shift on the underlying torus of $A_{\mathbb{Z}_2-orb}(0,1,0,1)$. It is easier to understand the resulting identification by considering the $\mathbb{Z}_2$ orbifold theory $A_{\mathbb{Z}_2-orb}(0,1,0,2)$. T'_{R_1} acts on $A_{\mathbb{Z}_2-orb}(0,1,0,2)$ by pairwise interchanging the $\mathbb{Z}_2$ twisted ground states and multiplying the $\mathbb{Z}_2$ invariant vertex operators of dimensions $(1/8,1/8)$ in $A_T(0,1,0,2)$ by -1. On the other hand, R_1 with negative discrete torsion will multiply the two T'_{R_1} invariant twisted ground state combinations by -1 but leave invariant the two $\mathbb{Z}_2$ invariant $(1/8,1/8)$ fields of $A_T(0,1,0,2)$. These $\mathbb{Z}_2$ actions are conjugate, since the action on the invariant $\mathbb{Z}_2$ twisted ground state combinations of $A_{\mathbb{Z}_2-orb}(0,1,0,1) = A_T(0,1,0,2)$ is merely exchanged with that on two combinations of twisted ground states of $A_{\mathbb{Z}_2-orb}(0,1,0,2)$. This again is possible because of the $c=1$ identification between the circle theory at radius $r=1$ and the orbifold theory at radius $r=2$. In summary,

$$A_{D_2(T'_R)-orb}(0,1,0,2) = A_{D_2^- -orb}(0,1,0,2). \tag{C4}$$

The T_{R_1} action on $A_T(0,1,0,2)$ differs from the R_1 action by a sign in the action on the $(1/8,1/8)$ fields, i.e. the twisted ground states of the $\mathbb{Z}_2$ orbifold on the $A_{\mathbb{Z}_2-orb}(1/2,1/2,0,1)$ side. Therefore by comparison with (C3)

$$A_{D_2^- -orb}(1/2,1/2,0,1) = A_{T_R-orb}(0,1,0,2). \tag{C5}$$

Comparison of (C3) with (C5) also gives a fairly natural explanation for the additional degree of freedom we have due to discrete torsion.

If we mod out $\widetilde{\mathbb{Z}_2(R)}$ and the corresponding D_2 type symmetries on $A_T(0,1,0,2)$, i.e. consider $\mathbb{Z}_2(R)$ on $A_T(\frac{1}{2},\frac{1}{2},0,2)$ we only reproduce identities we have found already above: $A_{D_2(T_R)-orb}(0,1,0,1) = A_{R-orb}(\frac{1}{2},\frac{1}{2},0,2)$ on (L12), as well as $A_{D_2(T_R^{(\prime)})-orb}(0,2,0,2) = A_{D_2^{\pm}-orb}(\frac{1}{2},\frac{1}{2},0,2)$ on (L13) and (L11), respectively.

Next we discuss the action of T_{R_1} on $A_T(0,1,0,1/2)$ instead of $A_T(0,1,0,2)$. In (3.3) this exchanges the roles of m_i and n_i, such that compared to the action of R_1 on $A_T(0,1,0,2)$ we now have additional signs on $(1/2,1/2)$ fields. In particular, only one combination of $(1/2,1/2)$ fields is invariant, as well as three of the twisted ground state combinations in $A_{\mathbb{Z}_2-orb}(0,1,0,1)$. We claim that this is the residual action of an ordinary $\mathbb{Z}_4$ rotation on $A_T(0,1,0,1)$. It acts by interchanging the two circle factors of $A_T(0,1,0,1)$, but the generators of the Hilbert space of the second factor are multiplied with an additional sign. Indeed, this is exactly the T_{R_1} action on a torus whose lattice has an additional generator $(1/2,1/2)$ compared to $\mathbb{Z}^2$ for $A_T(0,1,0,1)$, i.e. on $A_T(0,1,0,1/2)$. Hence,

$$A_{\mathbb{Z}_4-orb}(0,1,0,1) = A_{T_R-orb}(0,1,0,1/2). \tag{C6}$$

Using (C6) we can further mod out T_{R_2} on the underlying torus theory of the above $A_{T_{R_1}-orb}(0,1,0,1/2)$. This translates to a $\widetilde{\mathbb{Z}_2(R_2)}$ action on the underlying torus theory of $A_{\mathbb{Z}_4-orb}(0,1,0,1)$, so

$$A_{D_4^+-orb}(0,1,1/2,1/2) = A_{D_2(T_R)-orb}(0,1,0,1/2).$$

By (3.2) we see that we have actually found a tricritical point:

$$A_{D_4^+-orb}(0,1,1/2,1/2) = A_{D_2(T_R)-orb}(0,1,0,1/2) = A_{D_4(T_R')^+-orb}(0,1,0,1). \tag{T7}$$

We now rewrite (C6) as $A_{\mathbb{Z}_4-orb}(0,1,0,1) = A_{T'_{R_1}-orb}(0,1,0,1/2)$ and mod out by T'_{R_2} on the underlying torus of the right hand side. Analogously to the T_{R_2} action on $A_{T_{R_1}-orb}(0,t/2,0,2t)$ in (L7), which induced a shift on the underlying torus theory of $A_{D_2^--orb}(0,t,0,t)$, in (C6) we get a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $A_{\mathbb{Z}_4-orb}(0,1,0,1)$. Then we obtain

$$A_{\mathbb{Z}_4-orb}(0,1,0,2) = A_{D_2(T_R')-orb}(0,1,0,1/2). \tag{C7}$$

Back to the identification $A_{\mathbb{Z}_2-orb}(0,1,0,1) = A_T(0,1,0,1/2)$ in (Q1) we now mod out groups containing $\mathbb{Z}_4$ on the torus side. With the ordinary $\mathbb{Z}_4$ action we reproduce the above bicritical point (C7), but in combination with $D_2(T'_R)$, the $\mathbb{Z}_4$ generator acts as a shift on the underlying torus theory of $A_{\mathbb{Z}_4-orb}(0,1,0,2)$ in (C7):

$$A_{D_4(T_R')^+-orb}(0,1,0,1/2) = A_{\mathbb{Z}_4-orb}(0,1,0,4) \tag{C8}$$
$$A_{D_4(T_R')^--orb}(0,1,0,1/2) = A_{\mathbb{Z}_4-orb}(0,1,1/2,1). \tag{C9}$$

The latter identification is more easily understood when we mod out symmetries on the tricritical point (T3), as we will do in section 3.6.
The effect of D_4 type actions is most easily understood from the fact that by (C4) the action of $D_2(T'_R) \subset D_4(T'_R)^\pm$ on $A_T(0,1,0,2)$ is conjugate to that of $D_2^- \subset D_4^{-\pm}$. Therefore,

$$A_{D_4(T_R')^+-orb}(0,1,0,2) = A_{D_4^{-+}-orb}(0,1,0,2) \tag{3.4}$$
$$A_{D_4(T_R')^--orb}(0,1,0,2) = A_{D_4^{--}-orb}(0,1,0,2). \tag{3.5}$$

3.5 Series of multicritical points obtainable from (T1)

From the multicritical points and lines determined so far we can find further multicritical points by modding out further symmetries. By the systematic procedure we followed

above, this can only give something new, if we use an identification obtained as intersection of bicritical lines. Moreover, because by the discussion at the beginning of the section it suffices to use identifications containing a toroidal theory, only (T1) and (T2) are left to be discussed in this and the following section.
For the point (T1) only the identification $A_T(0,1,0,4) = A_{D_2^+}(0,1,0,1)$ has not been used yet. By modding out $\mathbb{Z}_2$ we yield (T4) from (T1), in particular $A_{\mathbb{Z}_2-orb}(0,1,0,4) = A_{D_2^+-orb}(0,1/2,1/2,1/2)$. Modding out a $\mathbb{Z}_2(R)$ action yields $A_{R-orb}(0,1,0,4) = A_{D_2^+-orb}$ $(0,2,0,2)$ on (L6). Note that this shows that $\mathbb{Z}_2$ and R on $A_T(0,1,0,4)$ both induce shifts on the underlying torus theory of $A_{D_2^+}(0,1,0,1)$, namely $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$, and $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, respectively. The combined action gives a trivial identity for D_2^+, and $A_{D_2^--orb}(0,1,0,4) = A_{D_2^+-orb}(0,1,1/2,1)$ in (T5). Modding out $\mathbb{Z}_2(T_R), D_2(T_R)$ and $D_2(T_R')$ reproduces the points at $t=2$ in (L7), (L10), and (L11), respectively. Modding out $\mathbb{Z}_4$ reproduces (C8). To determine the result of modding out D_4 actions, note that by the above the action of R induces a shift T_{δ_1} on the underlying torus theory of $A_{D_2^+}(0,1,0,1)$, so from (C8) we obtain

$$A_{D_4^{--}-orb}(0,1,0,4) = A_{D_4(T_R')^+-orb}(0,1,0,4). \tag{3.6}$$

All the other choices of discrete torsion give trivial identities. Modding out by $D_4(T_R')^{\pm}$ gives the same or a trivial identity again.
Next we mod out $\widetilde{\mathbb{Z}_2(R)}$, i.e. $\mathbb{Z}_2(R)$ on $A_T(1/2,1/2,0,4)$. This interchanges the two circle factors of the original $A_T(0,1,0,1)$ in $A_{D_2^+-orb}(0,1,0,1)$ above and thus is equivalent to adding a $\mathbb{Z}_4$ generator to D_2. Therefore,

$$A_{R-orb}(1/2,1/2,0,4) = A_{D_4^{++}-orb}(0,1,0,1). \tag{C10}$$

To mod out the corresponding D_2 actions we again use the above observation that $\mathbb{Z}_2$ on $A_T(1/2,1/2,0,4)$ acts as T_{δ_1} on the underlying torus theory of $A_{D_2^+-orb}(0,1,0,1)$ to find

$$A_{D_2^+-orb}(1/2,1/2,0,4) = A_{D_4^{++}-orb}(0,1,0,2), \tag{C11}$$

and

$$A_{D_2^--orb}(1/2,1/2,0,4) = A_{D_4(T_R')^+-orb}(0,1,0,2),$$

where the latter together with (3.4) gives a tricritical point

$$A_{D_2^--orb}(1/2,1/2,0,4) = A_{D_4(T_R')^+-orb}(0,1,0,2) = A_{D_4^{-+}-orb}(0,1,0,2). \tag{T8}$$

3.6 Series of multicritical points obtainable from (T2)

We now discuss additional identifications that can be obtained from (T2). The only identity not used up to now is $A_T(0,1,1/2,1) = A_{D_2^--orb}(0,1,0,1)$. If we mod out a $\mathbb{Z}_2$ action from the torus theory, (T2) is transformed into (T3), in particular we yield $A_{\mathbb{Z}_2-orb}(0,1,1/2,1) = A_{D_2^--orb}(1/2,1/2,0,1/2)$. The $\mathbb{Z}_2$ action thus induces a shift $T_{\delta'}, \delta' = \frac{1}{2}\binom{1}{1}$ on the underlying torus theory of $A_{D_2^--orb}(0,1,0,1)$. The R action on $A_T(0,1,1/2,1)$ induces a shift as well, now by $T_{\delta_1}, \delta_1 = \frac{1}{2}\binom{1}{0}$, yielding $A_{R-orb}(0,1,1/2,1) = A_{D_2^--orb}(0,2,0,2)$ in (Q2). The combined R and $\mathbb{Z}_2$ actions thus yield a trivial identity for D_2^- and $A_{D_2^+-orb}(0,1,1/2,1) = A_{D_2(T_R)-orb}(0,1,0,4)$ on (L13). Modding out $\mathbb{Z}_4$ is equivalent to modding out another $\mathbb{Z}_2$ action on $A_{\mathbb{Z}_2-orb}(0,1,1/2,1) = A_{D_2^--orb}(1/2,1/2,0,1/2)$ which interchanges the circle factors of the underlying geometric torus (i.e. $\mathbb{Z}_2$ invariant

vertex operators with $h = \overline{h}$). The action matches a $\widetilde{D_4}$ action on $A_{D_2^- - orb}(1/2, 1/2, 0, 1/2)$, where the additional D_2^- invariant vertex operators as compared to $A_{D_2^- - orb}(1/2, 1/2, 0, 1)$ correspond to the $\mathbb{Z}_2$ twisted ground states of $A_{\mathbb{Z}_2 - orb}(0, 1, 1/2, 1)$. We thus obtain $A_{\mathbb{Z}_4 - orb}(0, 1, 1/2, 1) = A_{D_4(T_R')^- - orb}(0, 1, 0, 1/2)$ reproducing (C9). Since by the above we know that R_1 on $A_T(0, 1, 1/2, 1)$ induces a T_{δ_1} shift on the underlying torus theory of $A_{D_2^- - orb}(0, 1, 0, 1)$, it also follows that

$$A_{D_4^- - orb}(0, 1, 1/2, 1) = A_{D_4(T_R')^- - orb}(0, 1, 0, 4). \tag{C12}$$

Flipping the sign of discrete torsion on both sides of the above equivalence we find

$$A_{D_4^+ - orb}(0, 1, 1/2, 1) = A_{D_4(T_R')^+ - orb}(0, 1, 0, 4),$$

which together with (3.6) yields a tricritical point:

$$A_{D_4^+ - orb}(0, 1, 1/2, 1) = A_{D_4(T_R')^+ - orb}(0, 1, 0, 4) = A_{D_4^- - - orb}(0, 1, 0, 4). \tag{T9}$$

We now mod out $\widetilde{\mathbb{Z}_2(R)}$ on $A_T(0, 1, 1/2, 1)$, i.e. $\mathbb{Z}_2(R)$ on $A_T(1/2, 1/2, 1/2, 1)$. Similarly to (C10) we find

$$A_{R - orb}(1/2, 1/2, 1/2, 1) = A_{D_4^- - - orb}(0, 1, 0, 1). \tag{C13}$$

Because by the above, $\mathbb{Z}_2$ on $A_T(1/2, 1/2, 1/2, 1)$ induces a shift $T_{\delta'}$ on the underlying torus theory of $A_{D_4^- - - orb}(0, 1, 0, 1)$ in (C13), we find

$$A_{D_2 - orb}(1/2, 1/2, 1/2, 1) = A_{D_4(T_R')^- - orb}(0, 1, 0, 2).$$

Together with (3.5) this gives another tricritical point:

$$A_{D_2 - orb}(1/2, 1/2, 1/2, 1) = A_{D_4(T_R')^- - orb}(0, 1, 0, 2) = A_{D_4^- - - orb}(0, 1, 0, 2). \tag{T10}$$

3.7 Multicritical points obtained from conjugate $\mathbb{Z}_3$, D_3, $\mathbb{Z}_6$ and D_6 type actions

We start by comparing all $\mathbb{Z}_3$ type symmetries of the $SU(3)$ torus theory at parameters $\tau = \rho = \omega$, $\omega := e^{2\pi i/3}$. The generically conserved currents of the torus theory we call j_1, j_2, and k_1, k_2, k_3 together with $l_\mu = k_\mu^\dagger, \mu \in \{1, 2, 3\}$ denote the additional vertex operators with dimensions $(h, \overline{h}) = (1, 0)$. The fields j_μ, k_μ, l_μ generate an $SU(3)_1$ Kac–Moody algebra, and $\{k_\mu\}$, $\{l_\mu\}$ form closed orbits under the ordinary $\mathbb{Z}_3$ action. In passing we remark that among all possible $\mathbb{Z}_2$ symmetries of $A_T(\omega, \omega)$, those conjugate only reproduce (L3).

Among the $\mathbb{Z}_3$ actions on one hand we have the ordinary rotational $\mathbb{Z}_3$ which leaves two fields $k_1 + k_2 + k_3$ and $l_1 + l_2 + l_3$ invariant, three fields $j^+ = j_1 + ij_2, k_1 + \omega k_2 + \omega^2 k_3, l_1 + \omega l_2 + \omega^2 l_3$ have eigenvalue ω. On the other hand, the shift orbifold by $\delta = \frac{1}{2}(\lambda_1 - \lambda_2)$ exhibits the same spectrum, where the λ_i as usual denote a basis of the lattice associated to the parameters $\tau = \rho = \omega$. Here, j_1, j_2 are invariant, and k_1, k_2, k_3 have eigenvalue ω. We particularly see that the two $\mathbb{Z}_3$ actions are conjugate, thus modding out $A_T(\omega, \omega)$ by these two symmetries gives isomorphic theories. The shift orbifold again produces a torus theory with same parameter $\tau = \omega$, but ρ reduced by a factor of three; in the following we use $\alpha := 1/2 + i3\sqrt{3}/2$ which is related to $\omega/3$ by the Möbius transformation T^2S and state

$$A_{\mathbb{Z}_3 - orb}(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}) = A_T(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}). \tag{C14}$$

We will now mod out additional symmetries on both sides of the above equality. Only those of order two give new identifications. Note that both R_2 and the ordinary $\mathbb{Z}_2$ on $A_T(\omega,\omega)$ interchange the two $\mathbb{Z}_3$–invariant $(1,0)$ fields $k_1+k_2+k_3$ and $l_1+l_2+l_3$. Thus $R_2, \mathbb{Z}_2$ must act as R_1, R_2 on the torus theory $A_T(\omega,\alpha)$. Study the action on the charge lattice to check that the order above is indeed correct. This means that the R_1 action on $A_T(\omega,\omega)$ must induce the ordinary $\mathbb{Z}_2$ action on $A_T(\omega,\alpha)$. In particular, the fields $k_1+k_2+k_3$ and $l_1+l_2+l_3$ are multiplied by -1 under R_1. Here we can confirm our result of the discussion of lattice 7: The signs obtained there occur in a completely natural way in the present example.
All in all for the $\mathbb{Z}_2$ actions on $A_T(\omega,\omega)$ compared to $A_T(\omega,\alpha)$ we have found $(R_1, R_2, \mathbb{Z}_2) \mapsto (R_2, \mathbb{Z}_2, R_1)$ and therefore directly obtain the following bicritical points:

$$A_{D_3(R_1)-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}) = A_{\mathbb{Z}_2-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}), \tag{C15}$$

$$A_{D_3(R_2)-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}) = A_{R_1-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}), \tag{C16}$$

$$A_{\mathbb{Z}_6-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}) = A_{R_2-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}), \tag{C17}$$

$$A_{D_6-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{\sqrt{3}}{2}) = A_{D_2-orb}(\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{2},\frac{3\sqrt{3}}{2}). \tag{C18}$$

Chapter 4

The Orbifolds of $N = 2$ Unitary Superconformal Field Theories with Central Charge $c = 3$

The complete understanding of the moduli space of $N = 2$ superconformal field theories with central charge $c = 3$ needs a description of all its orbifold theories. In a non-linear σ-model description, this concerns two dimensional tori and their orbifolds. For $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds, C. Vafa and N. Warner [67] made predictions for (chiral, chiral) and (antichiral, antichiral) fields based on Landau-Ginzburg descriptions. Apparently, they never had been checked explicitely. The moduli spaces of those orbifold theories were obtained in [51]. Landau-Ginzburg descriptions for the three orbifolds, we use the superpotentials $\Phi_1^3 + \Phi_2^3 + \Phi_3^3 + 6a\Phi_1\Phi_2\Phi_3$, $\Phi_1^4 + \Phi_2^4 + a\Phi_1^2\Phi_2^2$, and $\Phi_1^3 + \Phi_2^6 + a\Phi_1^2\Phi_2^2$, respectively. Note that we are interested in one dimensional moduli spaces, such that one needs superpotentials with one free parameter a or, in other words, singularities of modality one. Correlation functions for these potentials have been studied in [31][52]. Here we calculate the $\mathbb{Z}_M$ orbifold partition functions and verify the predictions of C. Vafa and N. Warner. For $c = 6$ similar calculations have been formulated by T. Eguchi et al [28]. There, charges behave in a simpler way than for $c = 3$. When fermions are omitted from the $c = 3$ superconformal theories, one obtains $c = 2$ bosonic theories. In this case the partition function for the $\mathbb{Z}_2$ orbifold was given in [48].

The $N = 2$ superconformal field theories with $c = 3$ are described by a free chiral scalar superfield containing two real bosons or a single complex left (right) boson (each of $c = 1$) and two Majorana-Weyl (MW) fermions or a free complex left(right) fermion $\psi^{\pm}(z) = \psi^1(z) \pm i\psi^2(z)$ ($\overline{\psi}^{\pm}(\overline{z}) = \overline{\psi}^1(\overline{z}) \pm i\overline{\psi}^2(\overline{z})$) (each of $c = \frac{1}{2}$). The action for this system may be written as

$$S = \frac{1}{2\pi} \int d^2z (G_{\mu\nu}\partial\varphi^\mu\overline{\partial}\varphi^\nu + B_{\mu\nu}\partial\varphi^\mu\overline{\partial}\varphi^\nu + \psi^-\overline{\partial}\psi^+ + \psi^+\overline{\partial}\psi^-). \tag{4.1}$$

This action has $N = 2$ superconformal symmetry. Directly from the action, we can determine the generators of the $N = 2$ superconformal algebra, the stress-energy tensor $T(z)$, its super partners $G^\alpha(z) = G^1(z) \pm iG^2(z)$,$\alpha = 1, 2$, and the $U(1)$ current $J(z)$. They have the form

$$\begin{aligned} T(z) &= -\frac{1}{2}\partial\varphi^-(z)\partial\varphi^+(z) - \frac{1}{4}\psi^-\partial\psi^+(z) - \frac{1}{4}\psi^+(z)\partial\psi^-(z), \\ G^{\pm}(z) &= \psi^{\mp}(z)\partial\varphi^{\pm}(z), \quad J(z) = \frac{1}{2}\psi^-(z)\psi^+(z) = \frac{i}{2}\varepsilon^{\mu\nu}\psi^\mu(z)\psi^\nu(z). \end{aligned} \tag{4.2}$$

Similarly relation holds for the antiholomorphic generators.
A field satisfying $h = \pm\frac{Q}{2}$ is a chiral or antichiral primary field. Note that fields $\{\psi^{\pm}(z), \overline{\psi}^{\pm}(\overline{z})\}$ all satisfy this condition since they have charge ± 1 and conformal dimension $\frac{1}{2}$ for both the left and right movers. The left primary chiral fields are $\{1, \psi^{+}(z)\}$; the right chiral primary fields are $\{1, \overline{\psi}^{+}(\overline{z})\}$. The left and right antichiral primary fields are obtained from these by complex conjugation. The basic linearly independent elements of the (c,c) ring of the $N = 2$ superconformal field theories with $c = 3$ is given by

$$\mathcal{R}_{(c,c)} = \{1,\ \psi^{+}(z),\ \overline{\psi}^{+}(\overline{z}),\ \psi^{+}(z)\overline{\psi}^{+}(\overline{z})\}. \tag{4.3}$$

Similarly, for the (a,c) ring one has

$$\mathcal{R}_{(a,c)} = \{1,\ \psi^{-}(z),\ \overline{\psi}^{+}(\overline{z}),\ \psi^{-}(z)\overline{\psi}^{+}(\overline{z})\}. \tag{4.4}$$

The elements of the two other rings $\mathcal{R}_{(a,a)}$ and $\mathcal{R}_{(c,a)}$ are obtained from $\mathcal{R}_{(c,c)}$ and $\mathcal{R}_{(a,c)}$ by complex conjugation. By (1.52), the Poincaré polynomial is

$$P(t,\overline{t}) = 1 + t + \overline{t} + t\overline{t}. \tag{4.5}$$

and satisfies a duality relation (1.53). By (1.50), the Witten index is $Tr(-1)^F = 0$. Note that the conformal dimension of the unique highest left–right chiral and antichiral primary field is $(\frac{1}{2}, \frac{1}{2})$. One notes that the Poincaré polynomial and the ring structure for the (c,c) and (a,c) primary fields isomorphic. However, this is not true in general. The conformal dimensions and $U(1)$ charges of the ground states of Ramond sector are $(h, \overline{h}) = (1/8, 1/8)$ and $(Q, \overline{Q}) = (\pm 1/2, \pm 1/2)$, respectively. Under the spectral flow (1.49) with flow parameter $\theta = \frac{1}{2}$, they flow to the chiral primary fields with conformal dimension $(h, \overline{h}) = (1/2, 1/2)$ and charge $(Q, \overline{Q}) = (+1, +1)$ or $(h, \overline{h}) = (Q, \overline{Q}) = (0, 0)$. The partition function for the $N = 2$ superconformal theories with $c = 3$ is constructed by tensoring the theory of a complex free boson defined on a 2-dimentional torus $T^2 = S^1 \times S^1$ with the theory of a single complex Dirac fermion. The corresponding partition function is

$$Z(\tau, \rho, z) := Z(\tau, \rho) Z_{Dirac}(\sigma, z), \tag{4.6}$$

where the torus partition function are given in (2.16) and the Dirac partition function will be constructed in the next section.

4.1 Dirac fermions on the torus

We now consider the Dirac fermions on the torus. This theory has central charge $c = \overline{c} = 1$ and modular invariant. The action for a complex left and right Dirac fermions is

$$S = \frac{1}{2\pi} \int d^2 z (\psi_\mu(z) \overline{\partial} \psi_\mu(z) + \overline{\psi}_\mu(\overline{z}) \partial \overline{\psi}_\mu(\overline{z})).$$

The operator product expansions between $\psi^\mu(z)$ and itself is

$$\psi^\mu(z)\psi^\nu(w) = -\frac{\delta^{\mu\nu}}{z - w}.$$

The holomorphic (left) energy momentum tensor is

$$T(z) = -\frac{1}{4} : \psi^{+}(z) \partial \psi^{-}(z) : \ .$$

The Dirac fermions have mode expansion

$$\psi^{\pm}(z) = \sum_n d_n^{\pm} z^{-n} \begin{cases} n \in \mathbb{Z} & (R) \\ n \in \mathbb{Z} + \frac{1}{2} & (NS) \end{cases} . \tag{4.7}$$

Since the fermionic theory split into Neveu-Schwarz and Ramond sector the fermionic Virasoro generators takes the following form

$$L_m = \frac{1}{2}\sum_n (n + \frac{m}{2}) : d^{\mu}_{-n} d^{\mu}_{m+n} \begin{cases} n \in \mathbb{Z} & (R) \\ n \in \mathbb{Z} + \frac{1}{2} & (NS) \end{cases} . \tag{4.8}$$

The fermionic Virasoro zero mode operator is then

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_n : d^{\mu}_{-n} d^{\mu}_n : \\ &= \sum_{n>0} n d^{\mu}_{-n} d^{\mu}_n + \begin{cases} \frac{1}{12} & n \in \mathbb{Z} \quad (R) \\ -\frac{1}{24} & n \in \mathbb{Z} + \frac{1}{2} \quad (NS) \end{cases} . \end{aligned} \tag{4.9}$$

One can see that the result for the Neveu-Schwarz ($n \in \mathbb{Z} + \frac{1}{2}$) sector agrees with $(L_0)_{cyl} = L_0 - \frac{c}{24}$. For the Ramond sector the vacuum energy is shifted by $\frac{1}{8}$. The modes d^{μ}_n and $\overline{d}^{\mu}_n$ of the holomorphic and antiholomorphic fermions satisfy anticommutation relation

$$\{d^{\mu}_n, d^{\nu}_m\} = \delta_{n+m,0}\ \delta^{\mu\nu}, \quad \{\overline{d}^{\mu}_n, \overline{d}^{\nu}_m\} = \delta_{n+m,0}\ \delta^{\mu\nu}, \quad \{d^{\mu}_n, \overline{d}^{\nu}_m\} = 0 \quad \forall n, m. \tag{4.10}$$

In the Neveu-Schwarz sector, d^{μ}_{-n} and d^{μ}_n are creation and annihilation operators for $n > 0$, and satisfy $d^{\mu}_n|0\rangle = 0$ for $n > 0$. The NS vacuum state $|0\rangle$ is the unique $SL(2,\mathbb{C})$ invariant state, i.e. it is annihilated by the generators L_0, $L_{\pm 1}$ of $SL(2,\mathbb{C})$ transformation. The NS Hilbert space states are built up by applying creation operators d^{μ}_{-n} to $|0\rangle$, namely

$$\mathcal{H}_{NS} = d^{\mu}_{-n_1} \cdots d^{\mu}_{-n_l} \overline{d}^{\nu}_{-m_1} \cdots \overline{d}^{\nu}_{-m_k} |0\rangle_{NS}.$$

The Ramond Hilbert space is constructed by the action of creation operators d^{μ}_{-n} to the ground states $|\frac{1}{8}, \frac{1}{8}\rangle_{\pm}$, namely

$$\mathcal{H}_R = d^{\mu}_{-n_1} \cdots d^{\mu}_{-n_l} \overline{d}^{\nu}_{-m_1} \cdots \overline{d}^{\nu}_{-m_k} |\frac{1}{8}, \frac{1}{8}>_{\pm} , \quad n, m \in \mathbb{Z}.$$

In general partition function for the Dirac fermion on the torus is written as

$$Z_{Dirac}(\sigma, z) = trq^{L_0 - \frac{1}{24}} \overline{q}^{\overline{L}_0 - \frac{1}{24}} y^{J_0} \overline{y}^{\overline{J}_0}. \tag{4.11}$$

A torus is identified by two periods which by rescaling coordinates we take as 1 and σ, where σ is the modular parameter of the torus. The choice of periodic (R) and antiperiodic (NS) boundary conditions on the two cycles of the torus known as spin structure for the fermion. So fermions obey the following boundary conditions in the space and in the time direction of the torus

$$\begin{aligned} \psi^{\mu}(z+1) &= \pm\psi^{\mu}(z) \\ \psi^{\mu}(z+\sigma) &= \pm\psi^{\mu}(z). \end{aligned} \tag{4.12}$$

The same boundary conditions are true for the antiholomorphic components $\overline{\psi}^{\mu}(\overline{z})$. We have chosen four possible boundary condition (R,R), (R,NS), (NS,R), and (NS,NS)

for the fermions. The spin structure with antiperiodic boundary conditions in the time and space direction of the torus simbolically denoted by $A^2\underset{A^2}{\square}$, according to (4.11), it is equivalent to the trace in the antiperiodic sector

$$\begin{aligned} A^2\underset{A^2}{\square} &= q^{-\frac{1}{24}} tr_{NS} q^{L_0} y^{J_0} = q^{-\frac{1}{24}} tr_{NS} q^{\sum_{n+\frac{1}{2}>0} n d^\mu_{-n} d^\mu_n} y^{J_0} \\ &= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}}) = \frac{\vartheta_3(z,\sigma)}{\eta(\sigma)}, \end{aligned}$$

where $y = e^{2\pi i z}$. The spin structure with periodic boundary condition in the time and antiperiodic boundary condition in the space direction denoted by $P^2\underset{A^2}{\square}$. It corresponds to taking the trace of the insertion of a fermion number operator $(-1)^F$ in the antiperiodic sector

$$\begin{aligned} P^2\underset{A^2}{\square} &= q^{-\frac{1}{24}} tr_{NS} (-1)^F q^{L_0} y^{J_0} = q^{-\frac{1}{24}} tr_{NS} (-1)^F q^{\sum_{n+\frac{1}{2}>0} n d^\mu_{-n} d^\mu_n} y^{J_0} \\ &= q^{-\frac{1}{24}} \prod_{n=1}^{\infty} (1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}}) = \frac{\vartheta_4(z,\sigma)}{\eta(\sigma)}. \end{aligned}$$

Similarly for the periodic sector one may write the following relation

$$\begin{aligned} A^2\underset{P^2}{\square} &= q^{\frac{1}{12}} tr_R q^{L_0} y^{J_0} = q^{\frac{1}{12}} tr_R q^{\sum_{n>0} n d^\mu_{-n} d^\mu_n} y^{J_0} \\ &= q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1+yq^{n})(1+y^{-1}q^{n}) = \frac{\vartheta_2(z,\sigma)}{\eta(\sigma)} \end{aligned}$$

and

$$\begin{aligned} P\underset{P^2}{\square} &= q^{\frac{1}{12}} tr_R q^{L_0} y^{J_0} = q^{\frac{1}{12}} tr_R q^{\sum_{n>0} n d^\mu_{-n} d^\mu_n} y^{J_0} \\ &= q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1-yq^{n})(1-y^{-1}q^{n}) = \frac{\vartheta_1(z,\sigma)}{\eta(\sigma)}. \end{aligned}$$

Therefore partition function for the Dirac fermions can be obtained by taking equal spin structures for the left and right fermion

$$\begin{aligned} Z_{Dirac}(\sigma,z) &= \frac{1}{2}\left(P^2\overline{P}^2\underset{P^2\overline{P}^2}{\square} + A^2\overline{A}^2\underset{P^2\overline{P}^2}{\square} + P^2\overline{P}^2\underset{A^2\overline{A}^2}{\square} + A^2\overline{A}^2\underset{A^2\overline{A}^2}{\square} \right) \\ &= \frac{1}{2}\left(\left|\frac{\vartheta_1(z,\sigma)}{\eta(\tau)}\right|^2 + \left|\frac{\vartheta_2(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_3(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_4(z,\sigma)}{\eta(\sigma)}\right|^2 \right). \end{aligned} \tag{4.13}$$

One notes that the Dirac partition function is given by the modular invariant combination of the theta function. The partition function of the $N = 2$ superconformal field theories with $c = 3$ is thus given by

$$Z(\tau,\rho,z) : = Z(\tau,\rho) Z_{Dirac}(\sigma,z)$$

$$
\begin{aligned}
&= \frac{1}{|\eta^2(\sigma)|^2} q^{\frac{1}{4\tau_2\rho_2}|n_1-\tau n_2-\rho(m_2+\tau m_1)|^2} \overline{q}^{\frac{1}{4\tau_2\rho_2}|n_1-\tau n_2-\overline{\rho}(m_2+\tau m_1)|^2} \cdot \\
&\frac{1}{2}\left(\left|\frac{\vartheta_1(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_2(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_3(z,\sigma)}{\eta(\sigma)}\right|^2 + \left|\frac{\vartheta_4(z,\sigma)}{\eta(\sigma)}\right|^2\right).
\end{aligned}
\tag{4.14}
$$

In the following section we will construct $N = 2$ superconformal $\mathbb{Z}_M$, $M = 2, 3, 4$ and 6 orbifolds of (4.14) by following the orbifold prescription in section 2.4. Since we want to construct superconformal orbifold theories, we should specify how world fermions are affected by the $\mathbb{Z}_M$ rotation symmetry. Under the $\mathbb{Z}_M$ rotations the fermionic fields and its modes transform as

$$
(\gamma^k\psi)^{\pm}(z) = e^{\pm\frac{2\pi ik}{M}}\psi^{\pm}(z), \qquad \gamma^k d_n^{\pm}\gamma^{-k} = e^{\pm\frac{2\pi ik}{M}} d_n^{\pm}, \quad k \in \{1, 2, \cdots M\}. \tag{4.15}
$$

The $\mathbb{Z}_M$ rotations are the symmetries both the action (4.1) and $N = 2$ world sheet supersymmetry generators (4.2). The two dimensional $N = 2$ superconformal orbifold models $T^2/\mathbb{Z}_M$ may be constructed by identifying points of the two-dimensional torus T^2 under the symmetry group $\mathbb{Z}_M$ of the theory. In the twisted sector, Ramond or Neveu-Schwarz fermions obey

$$
\begin{aligned}
\psi^+(1) &= \pm h\psi^+(0) \\
\psi^+(\sigma) &= \pm\gamma\psi^+(0),
\end{aligned}
\tag{4.16}
$$

where h and γ are twists on the fields in the space and time direction of the torus. The mode expansion of the fermionic field which satisfies the above boundary conditions is

$$
\psi^+(z) = \sum_{n\in\mathbb{Z}+k/M+1/2-t/2} d_n^+ z^{-n}, \quad k \in \{1, 2, \cdots M-1\}, \tag{4.17}
$$

where integer t is equal to zero in the Neveu-Schwarz sector, and to one in the Ramond sector. The conformal dimension of the bosonic twisted ground state is given in (2.38). For the fermion one has

$$
h_s = \overline{h}_s = \frac{1}{2}\left(\frac{k}{M}\right)^2. \tag{4.18}
$$

Complete modular invariant $N = 2$ superconformal $\mathbb{Z}_M$ orbifold partition function has the form

$$
\begin{aligned}
Z_{\mathbb{Z}_M-orb} &= \frac{1}{|\mathbb{Z}_M|}\sum_{\gamma\in\mathbb{Z}_M} \gamma\boxed{}_{1} + \frac{1}{|\mathbb{Z}_M|}\sum_{\gamma,h\in\mathbb{Z}_M, h\neq 1} \gamma\boxed{}_{h} \\
&= \frac{1}{|\mathbb{Z}_M|}\sum_{\substack{\gamma,h\in\mathbb{Z}_M,\\ [\gamma,h]=0}} \gamma\boxed{}_{h} = \sum_{h\in\mathbb{Z}_M} tr_{\mathcal{H}_h} P_h q^{L_0-\frac{1}{8}}\overline{q}^{\overline{L}_0-\frac{1}{8}} y^{J_0}\overline{y}^{\overline{J}_0}.
\end{aligned}
\tag{4.19}
$$

4.2 The $\mathbb{Z}_2$ Orbifold

We have constructed the bosonic $G = \mathbb{Z}_2$ orbifold partition function in section 2.5. Since the sum over the spin structures for the Dirac fermion is invariant under $\psi^{\pm} \to -\psi^{\pm}$, we see that the $N = 2$ superconformal $\mathbb{Z}_2$ orbifold partition function is equal to that for an

$c = 2$ bosonic orbifold partition function (2.43) times the partition function (4.13) for the Dirac fermion

$$Z_{\mathbb{Z}_2-orb} = \frac{1}{2}\left(Z(\tau,\rho) + 4\left|\frac{\eta(\sigma)}{\vartheta_2(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_4(\sigma)}\right|^2 + 4\left|\frac{\eta(\sigma)}{\vartheta_3(\sigma)}\right|^2 \right) \cdot Z_{Dirac}(\sigma, z). \qquad (4.20)$$

The (chiral, chiral), (antichiral, chiral) and their complex conjugates, Ramond ground states as well as Witten index for the $\mathbb{Z}_2$ orbifold superconformal field theories are the same as those for the $N = 2$ theories with $c = 3$.

4.3 The $\mathbb{Z}_3$ Orbifold

By dividing the $\mathbb{Z}_3$ symmetry from (4.14) for $\tau = e^{2\pi i/3}$ and arbitrary $\rho \in \mathbb{H}/PSL(2,\mathbb{Z})$, we construct $\mathbb{Z}_3$ superconformal orbifold partition function. By taking into account the equations (2.36), (4.15), (4.1), and (2.44), the untwisted sector partition function is

$$\begin{aligned}
Z_u &= \frac{1}{3}\sum_{k=0}^{2} {}^{\frac{2\pi k}{3}}\Big[\ \Big]_0 = (q\overline{q})^{-\frac{1}{8}} tr_{\mathcal{H}_u} \frac{1}{3}(1+\gamma+\gamma^2) q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0}\overline{y}^{\overline{J}_0} \\
&= \frac{1}{3}\Bigg(Z(\tau = e^{\frac{2\pi i}{3}}, \rho, z) \ + \ \frac{1}{2}\sum_{l=1}^{2}\prod_{n=1}^{\infty}\Bigg(\left|\left(e^{\frac{2\pi i l}{3}}y\right)^{\frac{1}{2}} - \left(e^{\frac{2\pi i l}{3}}y\right)^{-\frac{1}{2}}\right|^2 \cdot \\
&\qquad \left|\frac{\left(1-e^{\frac{2\pi i l}{3}}q^n y\right)\left(1-e^{-\frac{2\pi i l}{3}}q^n y^{-1}\right)}{\left(1-e^{\frac{2\pi i l}{3}}q^n\right)\left(1-e^{-\frac{2\pi i l}{3}}q^n\right)}\right|^2 \\
&\quad + \ \left|\left(e^{\frac{2\pi i l}{3}}y\right)^{\frac{1}{2}} + \left(e^{\frac{2\pi i l}{3}}y\right)^{-\frac{1}{2}}\right|^2 \left|\frac{\left(1+e^{\frac{2\pi i l}{3}}q^n y\right)\left(1+e^{-\frac{2\pi i l}{3}}q^n y^{-1}\right)}{\left(1-e^{\frac{2\pi i l}{3}}q^n\right)\left(1-e^{-\frac{2\pi i l}{3}}q^n\right)}\right|^2 \\
&\qquad + \ (q\overline{q})^{\frac{1}{8}}\left|\frac{\left(1+e^{\frac{2\pi i l}{3}}q^{n-\frac{1}{2}}y\right)\left(1+e^{-\frac{2\pi i l}{3}}q^{n-\frac{1}{2}}y^{-1}\right)}{\left(1-e^{\frac{2\pi i l}{3}}q^n\right)\left(1-e^{-\frac{2\pi i l}{3}}q^n\right)}\right|^2 \\
&\qquad + \ (q\overline{q})^{\frac{1}{8}}\left|\frac{\left(1-e^{\frac{2\pi i l}{3}}q^{n-\frac{1}{2}}y\right)\left(1-e^{-\frac{2\pi i l}{3}}q^{n-\frac{1}{2}}y^{-1}\right)}{\left(1-e^{\frac{2\pi i l}{3}}q^n\right)\left(1-e^{-\frac{2\pi i l}{3}}q^n\right)}\right|^2\Bigg)\Bigg) \\
&= \frac{1}{3}\Bigg(Z(\tau = e^{\frac{2\pi i}{3}}, \rho, z) \\
&\qquad + \ \frac{1}{2}\sum_{i=1}^{4}\sum_{l=1}^{2}\left|(e^{\frac{2\pi i l}{3}})^{1/2} - (e^{\frac{2\pi i l}{3}})^{-1/2}\right|^2 \left|\frac{\theta_i(z+\frac{l}{3},\sigma)}{\theta_1(\frac{l}{3},\sigma)}\right|^2\Bigg) \\
&= \frac{1}{3}\left(Z(\tau = e^{\frac{2\pi i}{3}}, \rho, z) + \frac{3}{2}\sum_{i=1}^{4}\left(\left|\frac{\theta_i(z+\frac{1}{3},\sigma)}{\theta_1(\frac{1}{3},\sigma)}\right|^2 + \left|\frac{\theta_i(z-\frac{1}{3},\sigma)}{\theta_1(\frac{1}{3},\sigma)}\right|^2\right)\right).
\end{aligned} \qquad (4.21)$$

As we knew from the bosonic $\mathbb{Z}_3$ orbifold construction, the conformal weight of the bosonic ground states is $(\frac{1}{9}, \frac{1}{9})$. By (4.18), the conformal weight of the fermionic twisted ground states is $(\frac{1}{18}, \frac{1}{18})$. Thus the total conformal weight of the twisted sector is then

$(\frac{1}{6},\frac{1}{6})$. By considering the equations (2.36), (4.15), (2.37), (4.17) and (2.44), the twisted sector partition function may be written as

$$
\begin{aligned}
Z_{3t} &= \frac{1}{3}\sum_{l=0}^{2}\left(A(\tfrac{2\pi l}{3})\underset{A(\frac{2\pi}{3})}{\square} + A(\tfrac{2\pi l}{3})\underset{A(\frac{4\pi}{3})}{\square}\right) \\
&= 3\times\frac{1}{3}\times\frac{1}{2}\sum_{l=0}^{2}\prod_{n=1}^{\infty}\left(\left|y^{\frac{1}{6}}\frac{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}y\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2\right. \\
&\quad + \left|y^{\frac{1}{6}}\frac{\left(1+e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}y\right)\left(1+e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left|q^{\frac{1}{24}}y^{-\frac{1}{3}}\frac{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{5}{6}}y\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{1}{6}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left|q^{\frac{1}{24}}y^{-\frac{1}{3}}\frac{\left(1+e^{\frac{2\pi il}{3}}q^{n-\frac{5}{6}}y\right)\left(1+e^{-\frac{2\pi il}{3}}q^{n-\frac{1}{6}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left|y^{-\frac{1}{6}}\frac{\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}y\right)\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left|y^{\frac{1}{6}}\frac{\left(1+e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}y\right)\left(1+e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left|q^{\frac{1}{24}}y^{\frac{1}{3}}\frac{\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{1}{6}}y\right)\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{5}{6}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2 \\
&\quad + \left.\left|q^{\frac{1}{24}}y^{\frac{1}{3}}\frac{\left(1+e^{-\frac{2\pi il}{3}}q^{n-\frac{1}{6}}y\right)\left(1+e^{\frac{2\pi il}{3}}q^{n-\frac{5}{6}}y^{-1}\right)}{\left(1-e^{\frac{2\pi il}{3}}q^{n-\frac{1}{3}}\right)\left(1-e^{-\frac{2\pi il}{3}}q^{n-\frac{2}{3}}\right)}\right|^2\right) \\
&= \frac{1}{2}\sum_{i=1}^{4}\sum_{l=-1}^{1}\left(\left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2 + \left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right).
\end{aligned}
\tag{4.22}
$$

Then the complete superconformal $\mathbb{Z}_3$ orbifold partition function has the form

$$
\begin{aligned}
Z_{\mathbb{Z}_3-orb} &= \frac{1}{3}\left(Z(\tau=e^{\frac{2\pi i}{3}},\rho,z) + \frac{3}{2}\sum_{i=1}^{4}\left(\left|\frac{\vartheta_i(z+\frac{1}{3},\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2 + \left|\frac{\vartheta_i(z-\frac{1}{3},\sigma)}{\vartheta_1(\frac{1}{3},\sigma)}\right|^2\right.\right. \\
&\quad \left.\left. + \sum_{l=-1}^{1}\left(\left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2 + \left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3}\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\right)\right)\right)
\end{aligned}
\tag{4.23}
$$

.

One can easily check the modular invariance of this partition function by using theta function modular transformation properties which are given in appendix 5.3.

We find eight Ramond ground states with conformal dimension $(h,\overline{h}) = (1/8, 1/8)$ and with charges $(\pm 1/2, \pm 1/2)$, $3 \times (\pm 1/6, \pm 1/6)$, eight NS chiral primary states with conformal dimensions $(0,0)$, $(1/2,1/2)$, $3\times(1/6,1/6)$, $3\times(1/3,1/3)$ and with charges $(0,0)$, $(1,1)$, $3\times(1/3,1/3)$, $3\times(2/3,2/3)$, as well as eight NS antichiral primary states having the same conformal dimensions but the opposite charges as the NS chiral fields. By (1.49) with $\eta = 1/2$, the ground states of the Ramond sector flow to the (chiral, chiral) primary states of the NS sector, namely

$$\begin{array}{rcl} \textit{Ramond ground states} & \longleftrightarrow & \textit{NS chiral states} \\ q^{1/8}\overline{q}^{1/8}y^{-1/2}\overline{y}^{-1/2} & \longleftrightarrow & 1 \\ q^{1/8}\overline{q}^{1/8}y^{1/2}\overline{y}^{1/2} & \longleftrightarrow & q^{1/2}\overline{q}^{1/2}y\overline{y} \\ 3\times q^{1/8}\overline{q}^{1/8}y^{-1/6}\overline{y}^{-1/6} & \longleftrightarrow & 3\times q^{1/6}\overline{q}^{1/6}y^{1/3}\overline{y}^{1/3} \\ 3\times q^{1/8}\overline{q}^{1/8}y^{1/6}\overline{y}^{1/6} & \longleftrightarrow & 3\times q^{1/3}\overline{q}^{1/3}y^{2/3}\overline{y}^{2/3}. \end{array} \tag{4.24}$$

If we revers the direction of the spectral flow, we get an isomorphism between the (antichiral, antichiral) primary states and the ground states of the Ramond sector. By (1.50), (1.52) and (4.24) the Witten index and the Poincaré polynomial for the (c,c) states are

$$Tr(-1)^F = 8, \quad P(t,\overline{t})_{(c,c)} = Tr_{\mathcal{R}}t^{J_0}\overline{t}^{\overline{J}_0} = 1 + t\overline{t} + 3t^{\frac{1}{3}}\overline{t}^{\frac{1}{3}} + 3t^{\frac{2}{3}}\overline{t}^{\frac{2}{3}}. \tag{4.25}$$

Now we consider the spectral flow from the NS sector to the NS sector with flow parameter $\theta = 1$. Under this flow (antichiral, antichiral) primary states flow to (chiral, chiral) primary states

$$\begin{array}{rcl} \text{(antichiral, antichiral)} & \longleftrightarrow & \text{(chiral, chiral) states} \\ \\ q^{1/2}\overline{q}^{1/2}y^{-1}\overline{y}^{-1} & \longleftrightarrow & 1 \\ 3\times q^{1/6}\overline{q}^{1/6}y^{-1/3}\overline{y}^{-1/3} & \longleftrightarrow & 3\times q^{1/3}\overline{q}^{1/3}y^{2/3}\overline{y}^{2/3} \\ 3\times q^{1/3}\overline{q}^{1/3}y^{-2/3}\overline{y}^{-2/3} & \longleftrightarrow & 3\times q^{1/6}\overline{q}^{1/6}y^{1/3}\overline{y}^{1/3}\ . \end{array}$$

The Poincaré polynomial for the (a,a) ring is

$$P(t,\overline{t})_{a,a} = Tr_{\mathcal{R}}t^{J_0}\overline{t}^{\overline{J}_0} = 1 + t^{-1}\overline{t}^{-1} + 3t^{-\frac{1}{3}}\overline{t}^{-\frac{1}{3}} + 3t^{-\frac{2}{3}}\overline{t}^{-\frac{2}{3}}\ . \tag{4.26}$$

In the spectrum, there are no nontrivial (antichiral, chiral) or its conjugate (chiral, antichiral) states.

4.4 The $\mathbb{Z}_4$ Orbifold

By dividing the $\mathbb{Z}_4$ symmetry from (4.14) for $\{\tau = i, \rho \in \mathbb{H}/PSL(2,\mathbb{Z})\}$, we formulate a superconformal $\mathbb{Z}_4$ orbifold model. As we knew from bosonic $\mathbb{Z}_4$ orbifold construction, the conformal weight of the bosonic $\mathbb{Z}_4$ twisted ground state is $(\frac{3}{32}, \frac{3}{32})$. By (4.18), the conformal weight of the fermionic $\mathbb{Z}_4$ twisted ground state is $(\frac{1}{32}, \frac{1}{32})$. Thus the total conformal weight of the $\mathbb{Z}_4$ twisted sector is then $(\frac{1}{8}, \frac{1}{8})$. Again the total superconformal $\mathbb{Z}_4$ orbifold partition function can be obtained by summing over untwisted, $\mathbb{Z}_2$, and $\mathbb{Z}_4$ twisted sectors partition functions

$$Z_{\mathbb{Z}_4-orb}(\tau = i, \rho, z) = Z_u + Z_{2t} + Z_{4t}.$$

By (2.36), (4.15), (4.1), and (2.47), we obtain the following untwisted sector partition function

$$
\begin{aligned}
Z_u &= \frac{1}{4}\left(A(0)\underset{A(0)}{\boxed{}} + A(\tfrac{\pi}{2})\underset{A(0)}{\boxed{}} + A(\pi)\underset{A(0)}{\boxed{}} + A(\tfrac{3\pi}{2})\underset{A(0)}{\boxed{}} \right) \\
&= \frac{1}{4}\left(Z(\tau=i,\rho,z) + \frac{1}{2}\sum_{l=1}^{3}\left|(e^{\frac{2\pi i l}{4}})^{\frac{1}{2}} - (e^{\frac{2\pi i l}{4}})^{-\frac{1}{2}}\right|^2 \left|\frac{\theta_i(z+\frac{l}{4},\sigma)}{\theta_1(\frac{l}{4},\sigma)}\right|^2 \right) \\
&= \frac{1}{4}\left(Z(\tau=i,\rho,z) + \sum_{i=1}^{4}\sum_{l=1}^{3}\left|\frac{\vartheta_i(z+\frac{l}{4},\sigma)}{\vartheta_1(\frac{l}{4},\sigma)}\right|^2 + \sum_{i=1}^{4}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_2(\sigma)}\right|^2 \right).
\end{aligned}
$$

By (2.36), (4.15), (2.37), (4.17) and (2.47), $\mathbb{Z}_4$ twisted sector partition function may has the form

$$
\begin{aligned}
Z_{4t} &= \frac{1}{4}\sum_{i,l=1}^{4}\left(\left| y^{-\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}-\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}-\frac{\sigma}{4},\sigma)}\right|^2 + \left| y^{\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}+\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}+\frac{\sigma}{4},\sigma)}\right|^2 \right. \\
&\qquad \left. + \left|\frac{\vartheta_i(z+\frac{l}{4},\sigma)}{\vartheta_4(\frac{l}{4},\sigma)}\right|^2 \right).
\end{aligned}
$$

The $\mathbb{Z}_2$ twisted sector partition function can be read off from (4.20) by ommiting the factor of four, namely

$$
Z_{2t} = \frac{1}{4}\left(\left|\frac{\theta_i(z,\sigma)}{\theta_3(\sigma)}\right|^2 + \left|\frac{\theta_i(z,\sigma)}{\theta_4(\sigma)}\right|^2 \right).
$$

Thus, we may write the modular invariant $\mathbb{Z}_4$ orbifold partition function in the following form

$$
\begin{aligned}
Z_{\mathbb{Z}_4-orb} &= \frac{1}{4}\sum_{i,l=1}^{4}\left(Z(\tau=i,\rho,z) + \sum_{j=2}^{4}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_j(\sigma)}\right|^2 + \sum_{s=1}^{3}\left|\frac{\vartheta_i(z+\frac{s}{4},\sigma)}{\vartheta_1(\frac{s}{4},\sigma)}\right|^2 \right. \\
&\quad + \left|\frac{\vartheta_i(z+\frac{l}{4},\sigma)}{\vartheta_4(\frac{l}{4},\sigma)}\right|^2 + \left| y^{-\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}-\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}-\frac{\sigma}{4},\sigma)}\right|^2 \\
&\quad \left. + \left| y^{\frac{1}{4}}\frac{\vartheta_i(z+\frac{l}{4}+\frac{\sigma}{4},\sigma)}{\vartheta_1(\frac{l}{4}+\frac{\sigma}{4},\sigma)}\right|^2 \right). \qquad (4.27)
\end{aligned}
$$

In the spectrum there are nine Ramond ground states which flow to the NS chiral states under the spectral flow operation (1.49)with flow parameter $\eta = 1/2$

$$
\begin{aligned}
\textit{Ramond ground states} &\longleftrightarrow \textit{NS chiral states} \\
q^{1/8}\overline{q}^{1/8}y^{-1/2}\overline{y}^{-1/2} &\longleftrightarrow 1 \\
q^{1/8}\overline{q}^{1/8}y^{1/2}\overline{y}^{1/2} &\longleftrightarrow q^{1/2}\overline{q}^{1/2}y\overline{y} \\
2\times q^{1/8}\overline{q}^{1/8}y^{-1/4}\overline{y}^{-1/4} &\longleftrightarrow 2\times q^{1/8}\overline{q}^{1/8}y^{1/4}\overline{y}^{1/4} \\
2\times q^{1/8}\overline{q}^{1/8}y^{1/4}\overline{y}^{1/4} &\longleftrightarrow 2\times q^{3/8}\overline{q}^{3/8}y^{3/4}\overline{y}^{3/4} \\
3\times q^{1/8}\overline{q}^{1/8} &\longleftrightarrow 3\times q^{1/4}\overline{q}^{1/4}y^{1/2}\overline{y}^{1/2}. \qquad (4.28)
\end{aligned}
$$

There are nine $(antichiral, antichiral)$ states which are given by the complex conjugation of $(chiral, chiral)$ states. As in the $\mathbb{Z}_3$ orbifold case, one can get isomorphism between the $(antichiral, antichiral)$ primary states and the ground states of Ramond sector by reversing the direction of the spectrral flow. By (1.50), (1.52) and (4.28), the Witten index and the Poincaré polynomial for the (c,c) states are

$$\begin{aligned} Tr(-1)^F &= 9 \\ P(t,\overline{t})_{(c,c)} &= 1 + t\overline{t} + 3t^{\frac{1}{2}}\overline{t}^{\frac{1}{2}} + 2t^{\frac{1}{4}}\overline{t}^{\frac{1}{4}} + 2t^{\frac{3}{4}}\overline{t}^{\frac{3}{4}}. \\ P(t,\overline{t})_{(a,a)} &= 1 + t^{-1}\overline{t}^{-1} + 3t^{-\frac{1}{2}}\overline{t}^{-\frac{1}{2}} + 2t^{-\frac{1}{4}}\overline{t}^{-\frac{1}{4}} + 2t^{-\frac{3}{4}}\overline{t}^{-\frac{3}{4}} \quad . \end{aligned} \tag{4.29}$$

With the spectral flow parameter $\theta = 1$, the NS sector comes back to the NS sector. One notes that the $\mathbb{Z}_4$ orbifold model contains only (c, c) and their conjugate (a,a) states. For this model, the (a,c) and (c,a) states are trivial and consist only of the vacumm state.

4.5 The $\mathbb{Z}_6$ Orbifold

We now construct a $\mathbb{Z}_6$ orbifold model with $\{\tau = e^{\frac{2\pi i}{3}}, \rho \in \mathbb{H}/PSL(2,\mathbb{Z})\}$ by dividing $\mathbb{Z}_6$ symmetry from (4.14). By (4.18), we find that the conformal dimension of the fermionic $\mathbb{Z}_6$ twisted ground state is $(\frac{1}{72}, \frac{1}{72})$. For bosons we have $(\frac{5}{72}, \frac{5}{72})$. Thus the total conformal weight of the $\mathbb{Z}_6$ twisted ground state is then $(\frac{1}{12}, \frac{1}{12})$. The $\mathbb{Z}_6$ orbifold partition function is the sum of partition functions of the untwisted, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_6$ twisted sectors

$$Z_{\mathbb{Z}_6-orb}(\tau = e^{\frac{2\pi i}{3}}, \rho, z) = Z_u + Z_{2t} + Z_{3t} + Z_{6t}.$$

By similar procedure, we obtain following untwisted $\mathbb{Z}_6$ orbifold partition function

$$\begin{aligned} Z_u &= \frac{1}{6}\sum_{l=0}^{5} A(\tfrac{2\pi}{3})\underbrace{\boxed{}}_{A(0)} = (q\overline{q})^{-\frac{1}{8}} tr_{\mathcal{H}_u} \frac{1}{6}(1+\gamma+\cdots+\gamma^5) q^{L_0}\overline{q}^{\overline{L}_0} y^{J_0}\overline{y}^{\overline{J}_0} \\ &= \frac{1}{6}\left(Z(\tau = e^{\frac{2\pi i}{3}}, \rho, z) + \frac{1}{2}\sum_{i=1}^{4}\sum_{l=1}^{5} \left|(e^{\frac{2\pi i l}{6}})^{\frac{1}{2}} - (e^{\frac{2\pi i l}{6}})^{-\frac{1}{2}}\right|^2 \left|\frac{\vartheta_i(z+\frac{l}{6},\sigma)}{\vartheta_1(\frac{l}{6},\sigma)}\right|^2 \right) \\ &= \frac{1}{6}\sum_{i=1}^{4}\left(Z(\tau = e^{\frac{2\pi i}{3}}, \rho, z) \;+\; \frac{3}{2}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_2(\sigma)}\right|^2 \right. \\ &\qquad \left. + \frac{3}{2}\sum_{s=1}^{2}\left|\frac{\vartheta_i(z+\frac{s}{3},\sigma)}{\vartheta_1(\frac{s}{3},\sigma)}\right|^2 \;+\; \frac{1}{2}\sum_{l=-1}^{1}\left|\frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_2(\frac{l}{3},\sigma)}\right|^2 \right). \end{aligned}$$

The $\mathbb{Z}_2$ and $\mathbb{Z}_3$ twisted sector partition functions can be read off from (4.20) and (4.22) by ommiting the factor of four and three, respectively. The $\mathbb{Z}_6$ twisted sector partition function may have the form

$$\begin{aligned} Z_{6t} &= \frac{1}{12}\sum_{i,k=1}^{4}\sum_{l=-1}^{1}\left(\left|\frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_3(\frac{l}{3},\sigma)}\right|^2 + \left|\frac{\vartheta_i(z+\frac{l}{3},\sigma)}{\vartheta_4(\frac{l}{3},\sigma)}\right|^2 \right. \\ &\qquad \left. + \left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_k(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2 + \left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_k(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2 \right). \end{aligned}$$

All in all we obtain the following modular invariant partition function

$$
\begin{aligned}
Z_{\mathbb{Z}_6-orb} &= \frac{1}{6}\sum_{i=1}^{4}\sum_{j=2}^{4}\sum_{l=-1}^{1}\Bigg(Z(\tau=e^{\frac{2\pi i}{3}},\rho,z)\;+\;\frac{3}{2}\left|\frac{\vartheta_i(z,\sigma)}{\vartheta_j(\sigma)}\right|^2 \\
&\quad+\frac{3}{2}\sum_{s=1}^{2}\left|\frac{\vartheta_i(z+\frac{s}{3},\sigma)}{\vartheta_1(\frac{s}{3},\sigma)}\right|^2+\left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2 \\
&\quad+\left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{\vartheta_1(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2+\left|\frac{\vartheta_i(z+\frac{l}{3},\sigma)}{2\vartheta_j(\frac{l}{3},\sigma)}\right|^2 \\
&\quad+\left|y^{-\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}-\frac{\sigma}{3},\sigma)}{2\vartheta_j(\frac{l}{3}-\frac{\sigma}{3},\sigma)}\right|^2+\left|y^{\frac{1}{3}}\frac{\vartheta_i(z+\frac{l}{3}+\frac{\sigma}{3},\sigma)}{2\vartheta_j(\frac{l}{3}+\frac{\sigma}{3},\sigma)}\right|^2\Bigg).
\end{aligned}
\tag{4.30}
$$

In this model there are ten Ramond ground states. Again we connect the ground states of Ramond sector with NS chiral primary states using eq. (1.49) with $\eta = 1/2$

$$
\begin{aligned}
\textit{Ramond ground states} &\longleftrightarrow \textit{NS chiral states}\\
q^{1/8}\overline{q}^{1/8}y^{-1/2}\overline{y}^{-1/2} &\longleftrightarrow 1\\
q^{1/8}\overline{q}^{1/8}y^{1/2}\overline{y}^{1/2} &\longleftrightarrow q^{1/2}\overline{q}^{1/2}y\overline{y}\\
2\times q^{1/8}\overline{q}^{1/8} &\longleftrightarrow 2\times q^{1/4}\overline{q}^{1/4}y^{1/2}\overline{y}^{1/2}\\
q^{1/8}\overline{q}^{1/8}y^{1/3}\overline{y}^{1/3} &\longleftrightarrow q^{5/12}\overline{q}^{5/12}y^{5/6}\overline{y}^{5/6}\\
q^{1/8}\overline{q}^{1/8}y^{-1/3}\overline{y}^{-1/3} &\longleftrightarrow q^{1/12}\overline{q}^{1/12}y^{1/6}\overline{y}^{1/6}\\
2\times q^{1/8}\overline{q}^{1/8}y^{1/6}\overline{y}^{1/6} &\longleftrightarrow 2\times q^{1/3}\overline{q}^{1/3}y^{2/3}\overline{y}^{2/3}\\
2\times q^{1/8}\overline{q}^{1/8}y^{-1/6}\overline{y}^{-1/6} &\longleftrightarrow 2\times q^{1/6}\overline{q}^{1/6}y^{1/3}\overline{y}^{1/3}.
\end{aligned}
\tag{4.31}
$$

The (a,a) states are given by the complex conjugation of (c,c) states. By (1.50), (1.52) and (4.31), The Witten index and the Poincaré polynomials for the (c, c) and (a,a) states are

$$
\begin{aligned}
Tr(-1)^F &= 10\\
P(t,\overline{t})_{(c,c)} &= 1+t\overline{t}+2t^{\frac{1}{2}}\overline{t}^{\frac{1}{2}}+t^{\frac{5}{6}}\overline{t}^{\frac{5}{6}}+t^{\frac{1}{6}}\overline{t}^{\frac{1}{6}}+2t^{\frac{2}{3}}\overline{t}^{\frac{2}{3}}+2t^{\frac{1}{3}}\overline{t}^{\frac{1}{3}}.\\
P(t,\overline{t})_{(a,a)} &= 1+t^{-\frac{1}{2}}\overline{t}^{-\frac{1}{2}}+2t^{-\frac{1}{2}}\overline{t}^{-\frac{1}{2}}+t^{-\frac{5}{6}}\overline{t}^{-\frac{5}{6}}\\
&\qquad+t^{-\frac{1}{6}}\overline{t}^{-\frac{1}{6}}+2t^{-\frac{2}{3}}\overline{t}^{-\frac{2}{3}}+2t^{-\frac{1}{3}}\overline{t}^{-\frac{1}{3}}\quad.
\end{aligned}
\tag{4.32}
$$

We found no (a, c) and (c,a) states in this model.

4.6 $N = 2$ Landau-Ginzburg model

In this section, we first review some of the facts of the $N = 2$ superconformal Landau-Ginzburg theories by following the articles [50][67], then we check the spectrum of the (c,c) fields and the Witten index. The $N = 2$ superconformal Landau-Ginzburg action takes the following form

$$
S=\int d^2zd^4\theta K(\Phi_i,\overline{\Phi}_i)+\left(\int d^2zd^2\theta W(\Phi_i)+h.c\right). \tag{4.33}
$$

$\Phi_i = \Phi_i(z,\overline{z},\theta^{\pm},\overline{\theta}^{\pm})$ $(i = 1, 2, ...n)$ are the $N = 2$ n chiral scalar superfields which satisfy the following condition

$$
\overline{D}_{\pm}\Phi_i = D_{\pm}\overline{\Phi}_i = 0.
$$

with the superderivative defined as $D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\mp \frac{\partial}{\partial z}$. The first term (K) is called Kähler potential. It includes derivatives of the superfields. The conformal dimension of those fields is greater than $(1,1)$. Such fields are called irrelevant. The second term (W) is called superpotential which is a holomorphic function of the superfields. It contains only relevant fields, i.e. fields with conformal dimension to $(1,1)$ or less than $(1,1)$. The superpotential $W(\Phi_i)$ is a quasi-homogeneous function with isolated singularities at $\Phi_i = 0$. In other words the superpotential is called quasi-homogeneous if it satisfies

$$W(\lambda^{w_i}\Phi_i) = \lambda^d W(\Phi_i)\ ,\ for \quad \Phi_i \to \lambda^{w_i}\Phi_i. \tag{4.34}$$

It has isolated singularity at $\Phi_i = 0$ if it satisfies

$$W(\Phi_i)|_0 = 0\ , \quad \partial_i W(\Phi_j)|_0 = 0.$$

For every isolated quasi-homegeneous superpotential, there exists an $N = 2$ superconformal field theory. One can read off the $U(1)$ charge of the lowest component of the chiral superfields Φ_i from the action (4.33). The θ -integrals in the first term have $(left, right)$ charges $(-1,-1)$. Because of neutrality of the action it follows that $W(\Phi_i)$ has charge $(1,1)$. As the result, the chiral superfield Φ_i must have charge $Q_i = \frac{w_i}{d}$ for both its left and right moving components. Now one notes that for any state in the Landau-Ginzburg theory $Q_L - Q_R$ is always an integer. This is true for the chiral superfield Φ_i, as it has equal left-right charges. Moreover, it is also true for the most general fields because they are obtained by taking products of Φ_i and $\overline{\Phi}_i$, as well as products of their super derivatives. This implies that one can apply spectral flow to the Landau-Ginzburg models.

The local *ring* $\mathcal{R}$ of the superpotential $W(\Phi_i)$ of the Landau-Ginzburg model is obtained by taking into account all monomials of chiral superfields Φ_i and setting $\partial_i W(\Phi_j)|_0 = 0$. The number of elements of the ring is denoted by $\mu = dim\mathcal{R}$. It is called multiplicity of $W(\Phi_i)$. It is also equal to the Witten index $Tr(-1)^F$.

The *modality* (or moduli is the number of free parameters in the theory.) m of a quasi-homogeneous superpotentials with isolated singularities is given by the number of chiral primary states with charge greater than or equal to one.

The *Poincaré polynomial* [50] for the Landau-Ginzburg theories is

$$P(t) = Tr_{\mathcal{R}} t^{dJ_0} = \prod_{i=1}^{n} \frac{1 - t^{d-w_i}}{1 - t^{w_i}}, \quad \text{or} \quad P(t,\overline{t}) = Tr_{\mathcal{R}} t^{J_0}\overline{t}^{\overline{J}_0}. \tag{4.35}$$

This polynomial is only function of $t\overline{t}$. (Because Landau-Ginzburg primary chiral fields have equal left-right charges.) For convenience, $t\overline{t}$ is replaced by the variable t^d, where d is defined in (4.34). The Witten index [50] is

$$Tr(-1)^F = P(t=1) = \mu = \prod_{i=1}^{n} \frac{d - w_i}{w_i} = \prod_{i=1}^{n} \left(\frac{1}{Q_i} - 1\right) \quad . \tag{4.36}$$

The chiral primary state $|\chi\rangle$ of the highest charge corresponds to the leading term in the Poincaré polynomial at $t \to \infty$, which gives $t^{\sum_i d - 2w_i}$. Therefore the highest charge and conformal dimension of chiral primary state $|\chi\rangle$ [50] are given as

$$Q_\chi = \sum_{i=1}^{\infty} \frac{d - 2w_i}{d} = \sum_i (1 - 2Q_i), \qquad h_\chi = \frac{Q_\chi}{2} = \sum_{i=1}^{n} \left(\frac{1}{2} - Q_i\right).$$

By using $h_\chi = \frac{c}{6}$ the central charge of the Landau-Ginzburg theory is given as

$$c = 6\beta = 6\sum_{i=1}^{n}\left(\frac{1}{2} - Q_i\right). \tag{4.37}$$

It well known [67] that the *quasi-homogeneous superpotentials* with isolated singularities for modality $m = 1$ of the Landau-Ginzburg theories at $c = 3$ are equivalent to the $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifolds of the $N = 2$ theories at $c = 3$. The corresponding superpotentials are given as

$$\begin{aligned} W_3(\Phi_1, \Phi_2, \Phi_3) &= \Phi_1^3 + \Phi_2^3 + \Phi_3^3 + 6a\Phi_1\Phi_2\Phi_3, \quad a^3 + 27 \neq 0 && (4.38)\\ W_4(\Phi_1, \Phi_2) &= \Phi_1^4 + \Phi_2^4 + a\Phi_1^2\Phi_2^2, \quad a^2 \neq 4 && (4.39)\\ W_6(\Phi_1, \Phi_2) &= \Phi_1^3 + \Phi_2^6 + a\Phi_1^2\Phi_2^2, \quad 4a^3 + 27 \neq 0. && (4.40) \end{aligned}$$

With the knowledge in this section, we may write the basic linearly independent elements of the (c,c) ring of superpotential (4.38) in the following form

chiral fields	1	Φ_1	Φ_2	Φ_3	$\Phi_1\Phi_2$	$\Phi_1\Phi_3$	$\Phi_2\Phi_3$	$\Phi_1\Phi_2\Phi_3$
charges	0	1/3	1/3	1/3	2/3	2/3	2/3	1
dimensions	0	1/6	1/6	1/6	1/3	1/3	1/3	1/2.

(4.41)

By (4.36), (4.35) and (4.41), the Witten index and Poincaré polynomial are

$$Tr(-1)^F = 8, \quad P(t,\overline{t})_{(c,c)} = Tr_{\mathcal{R}}\, t^{J_0}\overline{t}^{\overline{J}_0} = 1 + t\overline{t} + 3t^{\frac{1}{3}}\overline{t}^{\frac{1}{3}} + 3t^{\frac{2}{3}}\overline{t}^{\frac{2}{3}}. \tag{4.42}$$

For the superpotential (4.39) we have

chiral fields	1	Φ_1	Φ_2	$\Phi_1\Phi_2$	Φ_1^2	Φ_2^2	$\Phi_1^2\Phi_2$	$\Phi_1\Phi_2^2$	$\Phi_1^2\Phi_2^2$
charges	0	1/4	1/4	1/2	1/2	1/2	3/4	3/4	1
dimensions	0	1/8	1/8	1/4	1/4	1/4	3/8	3/8	1/2 .

(4.43)

By (4.36), (4.35) and (4.43), the Witten index and Poincaré polynomial are

$$Tr(-1)^F = 9, \quad P(t,\overline{t})_{(c,c)} = 1 + t\overline{t} + 3t^{\frac{1}{2}}\overline{t}^{\frac{1}{2}} + 2t^{\frac{1}{4}}\overline{t}^{\frac{1}{4}} + 2t^{\frac{3}{4}}\overline{t}^{\frac{3}{4}} \tag{4.44}$$

Similarly, for the superpotential (4.40) we may get

chiral fields	1	Φ_1	Φ_2	$\Phi_1\Phi_2$	Φ_2^2	Φ_2^3	Φ_2^4	$\Phi_1\Phi_2^2$	$\Phi_1\Phi_2^3$	$\Phi_1\Phi_2^4$
charges	0	1/6	1/3	1/2	2/3	1/3	1/2	2/3	5/6	1
dimensions	0	1/12	1/6	1/4	1/3	1/6	1/4	1/3	5/12	1/2.

(4.45)

By (4.36), (4.35) and (4.45), the Witten index and Poincaré polynomial are

$$Tr(-1)^F = 10, \quad P(t,\overline{t})_{(c,c)} = 1 + t\overline{t} + 2t^{\frac{1}{2}}\overline{t}^{\frac{1}{2}} + t^{\frac{5}{6}}\overline{t}^{\frac{5}{6}} + t^{\frac{1}{6}}\overline{t}^{\frac{1}{6}} + 2t^{\frac{2}{3}}\overline{t}^{\frac{2}{3}} + 2t^{\frac{1}{3}}\overline{t}^{\frac{1}{3}}. \tag{4.46}$$

Chapter 5

Appendix

5.1 The modular group

A torus may be defined by specifying two linearly independent lattice vectors on the plane and identfying points that differ by an integer combination of these vectors. On the complex plane these lattice vectors may be represented by two complex numbers w_1, and w_2, which is called the periods of the lattice. Let $w'_{1,2}$ be two periods describing the same lattice as $w_{1,2}$. Since the points w'_1 and w'_2 belong to the lattice they must be expressible as integer combination of w_1 and w_2 :

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{5.1}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad-bc = 1$. One can also write $w_{1,2}$ in terms of $w'_{1,2}$, which implies that the above matrix should have an invers with integer components. Since the unite cell of the lattice should have the same area whatever the period we use, the determinant of that matrix should be unity. Thus we are considering the group $SL(2,\mathbb{Z})$ which has integer, invertible matrices with unite determinat. The properties of conformal field theories defined on a torus neither depend on the overall scale of the lattice, nor absolute orientation of the lattice vectors. The relevant parameter is the ratio $\sigma = w_2/w_1$, which is called modular parameter. Under the change of period (5.1), the modular parameter σ transform as

$$\sigma \to \frac{a\sigma + b}{c\sigma + d} \qquad ad - bc = 1, \tag{5.2}$$

that is known as the modular group. The transformation of σ will not change if we revers the sign of all of a, b, c,d in (5.2). Therefore, the modular group is actually $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$. Usually we use the following particular modular transformations

$$T\colon \sigma \to \sigma + 1, \qquad S\colon \sigma \to -\frac{1}{\sigma}. \tag{5.3}$$

By modular transformation one can always take σ to lie in the fundemental domain

$$\sigma_2 > 0, \quad -\frac{1}{2} \le \sigma_1 \le 0 \quad and \quad |\sigma| \ge 1$$

or

$$\sigma_2 > 0, \quad 0 < \sigma_1 < \frac{1}{2} \quad and \quad |\sigma| > 1.$$

5.2 The Dedekind eta function, Jacobi theta functions

Dedekind eta function is defined as

$$\eta(\sigma) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \tag{5.4}$$

where we have defined $q = e^{2\pi i\sigma}$. The explicit expression of the Jacobi theta functions in terms of sums and products are

$$\begin{aligned}
\theta_1(\sigma) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} (=0) \\
\theta_2(\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} = 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 \\
\theta_3(\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 \\
\theta_4(\sigma) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2.
\end{aligned} \tag{5.5}$$

The relation between Dedekind eta function and Jacobi theta function is

$$2\eta^3(\sigma) = \vartheta_2(\sigma)\vartheta_3(\sigma)\vartheta_4(\sigma). \tag{5.6}$$

The generalizations of (5.5) are written as

$$\begin{aligned}
\theta_1(z,\sigma) &= -i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} \\
&\quad = -iy^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1}) \\
\theta_2(z,\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = y^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1}) \\
\theta_3(z,\sigma) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n = \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}}) \\
\theta_4(z,\sigma) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}} y^n = \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}}).
\end{aligned} \tag{5.7}$$

We may easily check the transformation of theta functions under $z \to z+\frac{1}{2}$

$$\begin{aligned}
\vartheta_1(z+\frac{1}{2},\sigma) &= \vartheta_2(z,\sigma) \qquad \vartheta_2(z+\frac{1}{2},\sigma) = \vartheta_1(z,\sigma) \\
\vartheta_3(z+\frac{1}{2},\sigma) &= \vartheta_4(z,\sigma) \qquad \vartheta_4(z+\frac{1}{2},\sigma) = \vartheta_3(z,\sigma).
\end{aligned} \tag{5.8}$$

Theta functions obey the following relation under $z \to z+1$ or $z \to z+\sigma$

$$\begin{aligned}
\vartheta_1(z+1,\sigma) &= -\vartheta_1(z,\sigma) & \vartheta_1(z+\sigma,\sigma) &= -\frac{1}{yq}\vartheta_1(z,\sigma) \\
\vartheta_2(z+1,\sigma) &= -\vartheta_2(z,\sigma) & \vartheta_2(z+\sigma,\sigma) &= \frac{1}{yq}\vartheta_2(z,\sigma) \\
\vartheta_3(z+1,\sigma) &= \vartheta_3(z,\sigma) & \vartheta_3(z+\sigma,\sigma) &= \frac{1}{yq^{1/2}}\vartheta_3(z,\sigma) \\
\vartheta_4(z+1,\sigma) &= \vartheta_4(z,\sigma) & \vartheta_4(z+\sigma,\sigma) &= -\frac{1}{yq^{1/2}}\vartheta_4(z,\sigma).
\end{aligned} \tag{5.9}$$

We may also check the following relation for $z \to z+\frac{\sigma}{2}$

$$\begin{aligned}
\vartheta_1(z+\frac{\sigma}{2},\sigma) &= -iy^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_4(z,\sigma) & \vartheta_2(z+\frac{\sigma}{2},\sigma) &= y^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_3(z,\sigma) \\
\vartheta_3(z+\frac{\sigma}{2},\sigma) &= y^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_2(z,\sigma) & \vartheta_4(z+\frac{\sigma}{2},\sigma) &= -iy^{-\frac{1}{2}}q^{-\frac{1}{8}}\vartheta_1(z,\sigma).
\end{aligned} \tag{5.10}$$

5.3 Modular transformations of theta functions

The modular transformation properties of theta under the T : $\sigma \to \sigma+1$ are easily derived from Eq. (5.5). By the infinite product expression for ϑ_i one obtains $\vartheta_2(\sigma+1) = e^{\pi i/4}\vartheta_2(\sigma)$, $\vartheta_3(\sigma+1) = \vartheta_4(\sigma)$, and $\vartheta_4(\sigma+1) = \vartheta_3(\sigma)$. By (5.4) one writes the modular transformation of the Dedekind's η function as $\eta(\sigma+1) = e^{\pi i/12}\eta(\sigma)$.
To check the modular transformation properties of the $\vartheta_i(\sigma)$ and $\eta(\sigma)$ under $\sigma \to -\frac{1}{\sigma}$ one needs the Poisson resummation formula :

$$\sum_n e^{-\pi a n^2+bn} = \frac{1}{\sqrt{a}}\sum_k e^{-\frac{\pi}{a}\left(k+\frac{b}{2\pi i}\right)}. \tag{5.11}$$

By applying (5.11) to the infinite series expression for $\vartheta_3(\sigma)$ one gets

$$\vartheta_3(-\frac{1}{\sigma}) = \sum_n e^{-\frac{\pi i n^2}{\sigma}} = \sqrt{-i\sigma}\sum_k e^{\pi i\sigma k^2} = \sqrt{-i\sigma}\vartheta_3(\sigma). \tag{5.12}$$

Similarly one easily obtains $\vartheta_2(-\frac{1}{\sigma}) = \sqrt{-i\sigma}\vartheta_4(\sigma)$ and $\vartheta_4(-\frac{1}{\sigma}) = \sqrt{-i\sigma}\vartheta_2(\sigma)$. By the above transformation properties and the relation (5.6) we get $\eta(-\frac{1}{\sigma}) = \sqrt{-i\sigma}\eta(\sigma)$. Here, we list the modular transformation properties of the $\vartheta_i(\sigma)$ and $\eta(\sigma)$ functions as follows:

$$\begin{aligned}
\vartheta_2(\sigma+1) &= e^{\pi i/4}\vartheta_2(\sigma) & \vartheta_2(-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}\vartheta_4(\sigma) \\
\vartheta_3(\sigma+1) &= \vartheta_4(\sigma) & \vartheta_3(-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}\vartheta_3(\sigma) \\
\vartheta_4(\sigma+1) &= \vartheta_3(\sigma) & \vartheta_4(-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}\vartheta_2(\sigma) \\
\eta(\sigma+1) &= e^{\pi i/12}\eta(\sigma) & \eta(-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}\eta(\sigma).
\end{aligned} \tag{5.13}$$

The modular transformation properties of (5.7) under $\sigma \to -\frac{1}{\sigma}$ and $z \to \frac{z}{\sigma}$ can be obtained by applying (5.11), namely

$$\begin{aligned}
\vartheta_1(\frac{z}{\sigma},-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}e^{\frac{\pi i z^2}{\sigma}}\vartheta_1(\sigma) & \vartheta_2(\frac{z}{\sigma},-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}e^{\frac{\pi i z^2}{\sigma}}\vartheta_4(\sigma) \\
\vartheta_3(\frac{z}{\sigma},-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}e^{\frac{\pi i z^2}{\sigma}}\vartheta_3(\sigma) & \vartheta_4(\frac{z}{\sigma},-\frac{1}{\sigma}) &= (-i\sigma)^{1/2}e^{\frac{\pi i z^2}{\sigma}}\vartheta_2(\sigma).
\end{aligned} \tag{5.14}$$

It clear that the $\vartheta_i(z,\sigma)$ transform in the same way as $\vartheta_i(\sigma)$ under $\sigma \to \sigma + 1$.

5.4 Doubling identities

The Jacobi thetafunctions satisfy the following doubling identities

$$\begin{aligned} \vartheta_2(2\sigma) &= \sqrt{\frac{\vartheta_3(\sigma)^2 - \vartheta_4(\sigma)^2}{2}} \\ \vartheta_3(2\sigma) &= \sqrt{\frac{\vartheta_3(\sigma)^2 + \vartheta_4(\sigma)^2}{2}} \\ \vartheta_4(2\sigma) &= \sqrt{\vartheta_3(\sigma)\vartheta_4(\sigma)}. \end{aligned} \tag{5.15}$$

Chapter 6

Conclusion

In chapter 1 we have recalled the $N = 0, 1, 2$ superconformal algebra and superconformal field theories.
In chapter 2 We have explicitly constructed the parameter spaces and the one loop partition functions of the sixteen types of crystallographic orbifold conformal field theories of toroidal theories with central charge $c = 2$. Taking into acount all possible choices of the B–field and all values of discrete torsion, this yields 28 different components of the moduli space $\mathcal{C}^2$ of unitary conformal field theories with central charge $c = 2$. In the construction of the various theories some unexpected effects of the B–field have occured which might lead to a better understanding of its properties, also for higher dimensional cases.
In chapter 3 We have determined the multicritical points and lines of the 28 components of $\mathcal{C}^2$ constructed before. We have found eight bicritical lines and 26 multicritical points, among them two quadrucritical and seven tricritical points. The proof of the multicriticality on the level of the operator algebra for all these lines and points were obtained by K. Wendland. By analysis of multicritical points, all but four of the irreducible crystallographic components of the moduli space are directly or indirectly connected to the moduli space of toroidal theories. The remaining four components are

$$\mathcal{C}^{(0)}_{D_4^{+-}-orb}, \mathcal{C}^{(0)}_{D_6^{\pm}-orb}, \mathcal{C}^{(0)}_{D_3(R)-orb}.$$

In chapter 4, The partition functions for $\mathbb{Z}_M$ orbifolds have been calculated. The Witten indexes, the spectrum of (chiral, chiral) fields for the $\mathbb{Z}_M$, $M \in \{3, 4, 6\}$, orbifolds and for the Landau-Ginzburg superpotentials (4.38–4.40) are given in equations (4.25), (4.29), (4.32), (4.24), (4.28), (4.31) and (4.41), (4.43), (4.45), (4.42), (4.44), (4.46), respectively. The results are in agreement with the Landau-Ginzburg predictions of C. Vafa and N. Warner.
In chapter 5.1 we have presented modular group, the modular transformation properties of Dedekind eta function and Jacobi theta functions.

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