Universität Bonn
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Fusion Algebras and Characters of Rational Conformal Field Theories

by
Wolfgang Eholzer

Abstract

We introduce the notion of (nondegenerate) strongly-modular fusion algebras. Here strongly-modular means that the fusion algebra is induced via Verlinde's formula by a representation of the modular group whose kernel contains a congruence subgroup. Furthermore, nondegenerate means that the conformal dimensions of possibly underlying rational conformal field theories do not differ by integers. Our first main result is the classification of all strongly-modular fusion algebras of dimension two, three and four and the classification of all nondegenerate strongly-modular fusion algebras of dimension less than 24.

Secondly, we show that the conformal characters of various rational models of $\mathcal{W}$-algebras can be determined from the mere knowledge of the central charge and the set of conformal dimensions. We also describe how to actually construct conformal characters by using theta series associated to certain lattices. On our way we develop several tools for studying representations of the modular group on spaces of modular functions. These methods, applied here only to certain rational conformal field theories, are in general useful for the analysis of rational models.

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Meinen Eltern

Das große Lalulā

Kroklokwafzi? Sem̄emem̄i!
Seiokrontro - prafriplo:
Bifzi, bafzi; hulalem̄i:
quasti basti bo …
Lalu lalu lalu lalu la!

Hontraruru miromente
zasku zes rü rü?
Entepente, leiolente
klekwapufzi lü
Lalu lalu lalu lalu la!

Simarar kos malzipempu
silzuzankunkrei (;)!
Majomar dos: Quempu Lempu
Siri Suri Sei []!
Lalu lalu lalu lalu la!

Christian Morgenstern

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Typeset by $\mathcal{AMS}$-TEX

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1. Introduction

One of the most successful insights in modern physics is that symmetry is fundamental. Perhaps, this is most apparent in gauge theories: The standard model -formulated as a gauge field theory- constitutes the most prominent example of a quantum field theory. It provides a description of all known fundamental forces besides gravity. Furthermore, some of its predictions have been checked experimentally and are by far the most accurate ones in the history of physics. However, there are still a lot of unsolved problems concerning the standard model itself and, even more, concerning the unification of the standard model with general relativity to a theory of everything. Many of the problems of the standard model are mathematical in nature and only a few can be treated in a mathematically rigorous way. For example, Minkowskian theories typically formulated in the language of path integrals seem not to make sense mathematically. Many different lines of research have been developed to overcome these problems. We only want to mention very shortly two of them and their relation to conformal field theories (CFTs), the topic of this thesis.

Firstly, algebraic quantum field theory (AQFT) (developed by Haag, Kastler and Borchers in the fifties and sixties (see e.g. [Ha] and references therein)) starts at a very fundamental level and encodes the basic features of quantum field theories in a very clear and mathematically rigorous way. So far, however, the treatment of 'realisitic' quantum field theories on the basis of algebraic quantum field theory is not possible. The best understood examples of quantum field theories that can -at least to some extend- be described within this framework are euclidean two dimensional conformal field theories.

Secondly, string theories, originally developed to describe strong interactions, are considered nowadays as one of the most promising candidates for theories of everything unifying the standard model and general relativity (for a review see e.g. [GSW]). Although a lot of progress has been made in string theory in the last two decades, the description of 'realistic' states of matter and something like a derivation of the standard model from string theory are far from being solved. Formulating field theory on the 'world sheet' of the strings gives rise to a two dimensional conformal field theory.

In this thesis we will be concerned with two dimensional chiral conformal field theories. In the last ten years two dimensional CFTs have played a profound role in theoretical physics as well as in mathematics. Starting with the work of A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov [BPZ] in 1984 it was shown that all correlation functions of chiral rational conformal field theories (RCFTs), i.e. conformal field field theories depending only on one of the two light cone coordinates and having only finitely many primary fields, are determined by the symmetry of the theory and can -at least in principle- be calculated. Using conformal field theory many new results connecting statistical mechanics and string theory with the theory of topological invariants of 3-manifolds or with number theory were found (see e.g [Wi,C]).

The classification of RCFTs became one of the important problems in mathematical physics. However, a complete classification seems to be an impossible task since, for example, all self dual double even lattices lead to RCFTs and there is at this stage no hope to classify all such lattices of rank greater than 24. Nevertheless, it might be possible to classify all RCFTs with 'small' effective central charge $\tilde{c}$. (The effective central charge is given by the difference of the central charge and 24 times the smallest conformal dimension of the rational model under consideration.) In particular, for $\tilde{c} \leq 1$ a classification of RCFTs can be obtained by using a theorem of Serre-Stark describing all modular forms of weight 1/2 on congruence subgroups if one assumes that the corresponding conformal characters are modular functions on a congruence subgroup.

With this thesis we want to contribute to the classification program of RCFTs with only a few primary fields and for low values of the effective central charge. Our investigations concern mainly two different directions:

Firstly, we investigate the structure of modular fusion algebras associated to RCFTs using the known classification of the irreducible representations of the finite groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$. For $\tilde{c} > 1$ only partial results have been obtained so far. One of the possibilities is to look at RCFTs where the corresponding fusion algebra has a 'small' dimension. In the special case of a trivial fusion algebra the RCFT has only one superselection sector and a classification of the corresponding modular invariant partition functions for unitary theories with $c \leq 24$ has been obtained [Sche]. As a next step in the classification one can try to classify the nontrivial fusion algebras of low dimension first and then investigate corresponding RCFTs. Indeed, the modular fusion algebras of dimension less than or equal to three satisfying the so-called Fuchs conditions have been classified (see e.g. [MMS,CPR]). In this thesis we develop several tools, following the ideas of references [E2, E3], which enable us to classify all strongly-modular fusion algebras of dimension less than or equal to four (for a definition of strongly-modular fusion algebras see §2.2). Our approach is based on the known classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ [NW].

Another possibility is to investigate theories where the corresponding fusion algebra has a certain structure but may have arbitrary or 'big' dimension. Here, a classification of all selfconjugate fusion algebras which are isomorphic to a polynomial ring in one variable, where the distinguished basis has a certain form and where the structure constants are less than or equal to one, has been obtained (see e.g. [CPR][1]). Furthermore, a classification of all fusion algebras which are isomorphic to a polynomial ring in one variable and where the quantum dimension of the elementary field is smaller or equal to 2 is known (this classification contains the fusion algebras occurring in the classification of ref. [CPR]; for a review see e.g. [F]). With the tools developed in this thesis we obtain another partial classification, namely of those strongly-modular fusion algebras of dimension less than 24 where

[1]More precisely, in [CPR] all selfconjugate modular fusion algebras with $N_{ij}^k \leq 1$, which are isomorphic to $\mathbb{Q}[x]/ < P(x) >$ and $\Phi_0 \cong 1, \Phi_1 \cong x, \Phi_j \cong p_j(x)$ $(j = 2, \ldots, n-1)$ for some polynomials P and p_j and where the degree of P is n and the degree of the p_j is j, have been classified (the assumption on the degree of p_j was used implicitly in loc. cit.).

the corresponding representation ρ of the modular group is such that $\rho(T)$ has nondegenerate eigenvalues. The nondegeneracy of the eigenvalues of $\rho(T)$ means that the difference of any two conformal dimensions of a possibly underlying RCFT is not an integer. The restriction on the dimension is of purely technical nature so that it should be possible to obtain a complete classification of all nondegenerate strongly-modular fusion algebras with the methods described in this thesis by systematical use of Galois theory.

Secondly, we discuss properties of conformal characters related to rational models which are an important tool in the study of rational models of $\mathcal{W}$-algebras. These conformal characters χ_h form a finite set of modular functions satisfying a transformation law

$$\chi_h(A\tau) = \sum_{h'} \rho(A)_{h,h'} \chi_{h'}(\tau).$$

Here A runs through the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ or through a certain subgroup $G(2)$ (if the underlying $\mathcal{W}$-algebra is fermionic), and ρ is a matrix representation of Γ or $G(2)$, which depends on the rational model under consideration.

It already has been noticed in the literature that conformal characters are very distinguished modular functions: First of all, similar to the j-function, their Fourier coefficients are nonnegative integers and they have no poles in the upper half plane. They sometimes admit interesting sum formulas: These formulas, which allow an interpretation as generating functions of the spectrum of certain quasi-particles, can be used to deduce dilogarithm-identities (see e.g. [NRT,KRV]). In some cases the conformal characters have simple product expansions. If one has both, sum and product expansions, the resulting identities are what is known in combinatorics as Rogers-Ramanujan type identities.

In this thesis we add one more piece to this theme. We show that in a number of cases the conformal characters of some RCFT are uniquely determined by the corresponding central charge and set of conformal dimensions. More precisely, we shall state a few general and simple axioms which are satisfied by the conformal characters of all known rational models of $\mathcal{W}$-algebras. These axioms state essentially not more than the $\mathrm{SL}(2,\mathbb{Z})$-invariance of the space of functions spanned by the conformal characters, the rationality of their Fourier coefficients and an upper bound for the order of their poles. The only data of the underlying rational model occurring in these axioms are the central charge and the conformal dimensions, which give the upper bound for the pole orders and a certain restriction on the $\mathrm{SL}(2,\mathbb{Z})$-invariance. We then prove that, for various sets of central charges and conformal dimensions, there is at most one set of modular functions which satisfies these axioms (cf. the main theorem 3 in §4.1).

Finally, we describe a mean which can be used to construct conformal characters using theta series associated to certain lattices. In particular, we shall apply our method to the case of five special rational models. The reason for the choice of these models is that the $\mathrm{SL}(2,\mathbb{Z})$-representations on their conformal characters can be treated in some generality, and that the conformal characters of one of these models (of type $\mathcal{W}(2,8)$ with central charge $c = -\frac{3164}{23}$) could not be computed explicitly by the so far known methods.

This thesis is organized as follows: In section 2 we give a short introduction into the theory of vertex operator algebras and present basic (working) definitions of $\mathcal{W}$-algebras and RCFTs or rational models. Furthermore, this section contains the abstract definition of fusion algebras and some of their basic properties. Section 3 contains two of our main results: The classification of the strongly-modular fusion algebras of dimension less than or equal to four and the classification of the nondegenerate strongly-modular fusion algebras of dimension less than 24. In the other subsections we prove our results and comment on the realization of the fusion algebras occurring in our classifications contained in §3.1. In §4 we present and prove another main result of this thesis, namely theorem 3 on uniqueness of conformal characters which states that, for several rational models, the central charge and the set of conformal dimensions together with a set of axioms fulfilled by all known RCFTs uniquely determine the conformal characters of the rational model under consideration. In order to prove the main theorem 3 we develop in §4.2 and §4.3 some mathematical tools which may be of independent interest. In the next section, we describe how one can actually construct conformal characters transforming under a certain congruence representation of the modular group. After presenting a general construction procedure we discuss concrete examples by constructing explicitly the conformal characters of certain rational models. Finally, we draw some conclusions and discuss open questions in §6.

Parts of this thesis have already been published:

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Notation. We use $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$, $\mathfrak{H}$ for the complex upper half plane, τ as a variable in $\mathfrak{H}$, $q = e^{2\pi i\tau}$, $q^\delta = e^{2\pi i\delta\tau}$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, Γ for the group $\mathrm{SL}(2,\mathbb{Z})$, and

$$\Gamma(n) = \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \mathbb{1} \pmod{n}\}$$

for the principal congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ of level n. Recall that a congruence subgroup of Γ is a subgroup containing $\Gamma(n)$ for some n. We use η for the Dedekind eta function

$$\eta(\tau) = \mathrm{e}^{\pi i\tau/12} \prod_{n\geq 1} (1-q^n).$$

The group Γ acts on $\mathfrak{H}$ by

$$A\tau = \frac{a\tau + b}{c\tau + d} \qquad (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

For a complex vector valued function $F(\tau)$ on $\mathfrak{H}$, and for an integer k we write $F|_k A$ for the function defined by

$$(F|_k A)(\tau) = (c\tau + d)^{-k} F(A\tau).$$

Finally, for a matrix representation $\rho\colon \Gamma \to \mathrm{GL}(n,\mathbb{C})$ and an integer k we use $M_k(\rho)$ for the vector space of all holomorphic maps $F\colon \mathfrak{H} \to \mathbb{C}^n$ (= column vectors) which satisfy $F|_k A = \rho(A)F$ for all $A \in \Gamma$, and which are bounded in any region $\mathrm{Im}(\tau) \geq r > 0$. Thus, if ρ is the trivial representation, then $M_k(\rho)$ is the space of ordinary modular forms on Γ and of weight k.

2. Rational conformal field theories and fusion algebras

In order to proceed towards a classification of RCFTs one needs precise definitions of the objects under consideration. One attempt to formulate the axioms of RCFTs mathematically rigorous starts with the definition of vertex operator algebras (see e.g. [FHL]). We summarize some basic facts about vertex operator algebras, their representations, intertwining operators and fusion algebras in this section. In particular, we concentrate on those aspects which are closely related to conformal field theory. We do not give all the mathematical details but rather try to describe the basic structures one needs for dealing with conformal field theory problems in the language of vertex operator algebras. The results in section §2.1 serve as (mathematically) motivating introduction and are not really needed in the following.

This section is organized as follows: the three parts of section 2.1 contain the definition of vertex operator algebras (VOAs), their representations and intertwining operators. Furthermore, we give working definitions of $\mathcal{W}$-algebras and rational models and review some basic theorems. In §2.2 we define various types of fusion algebras and comment on the relation of abstract fusion algebras to fusion algebras associated to RCFTs. Finally, in §2.3 we state and prove some basic lemmas on modular fusion algebras which we need in §3.

2.1 Vertex operator algebras, $\mathcal{W}$-algebras and rational models.

$\mathcal{W}$-algebras are a special kind of vertex operator algebras. For the reader's convenience we repeat the definition of vertex operator algebras and their representations (see e.g. [FHL,FZ]) and comment on their relation to conformal field theory.

Vertex algebras and vertex operator algebras.

Let us first comment on some basic properties of conformal field theories motivating the definition of vertex algebras below. Conformal field theories in two space time dimensions consist of fields $\phi(z,\bar{z})$ which are parameterized by coordinates z and $\bar{z}$. These theories live on a cylinder with time coordinate t and space coordinate x which is periodic with period 2π. The coordinates $z, \bar{z}$ are given by $z = e^{t+ix}$ and $\bar{z} = e^{t-ix}$, respectively. The fact that conformal field theories describe massless phenomena and that they live in two space time dimensions allows to consider right and left movers (i.e. holomorphic and antiholomorphic fields) separately (the corresponding fields are called chiral). We will concentrate in the following only on holomorphic fields. Holomorphy on the cylinder implies that a field $\phi(z)$ (corresponding to the formal power series $Y(\phi, z)$ below) has a Laurent series expansion

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\phi_n$$

where the 'modes' ϕ_n are given by $\phi_n = \mathrm{Res}_z\left(z^n\phi(z)\right)$. In addition to the holomorphic chiral fields there exists the vacuum state (denoted by $\mathbb{1}$) such that the map $\phi := \phi(0)\mathbb{1} \to \phi(z)$ is injective. Translational covariance is implemented by the generator L_{-1} which acts via

$$\left(L_{-1}\phi\right)(z) = \frac{d}{dz}\phi(z)$$

on the fields. Furthermore, locality implies that

$$\phi(z)\psi = \sum_{m\in\mathbb{Z}} z^{-m-1}\chi^m$$

where χ^m is given by $\chi^m = \mathrm{Res}_z\left(z^m\phi(z)\psi\right)$. Using translation invariance of the vacuum we find that

$$R(\phi(z)\psi(w))\mathbb{1} = \sum_{m\in\mathbb{Z}} (z-w)^{-m-1}\chi^m(w)\mathbb{1}$$

where the left hand side as to be understood as the radial ordered product of the two fields

$$R(\phi(z)\psi(w)) := \begin{cases} \phi(z)\psi(w) & |z|>|w| \\ \psi(w)\phi(z) & |z|<|w| \end{cases}$$

(for details see e.g. [Gi]). The fields $\chi^m(w)$ are given by

$$\chi^m(w) = \mathrm{Res}_{z-w}\left((z-w)^m R(\phi(z)\psi(w))\right).$$

Using Cauchy integration one can easily calculate the n-th mode of $\chi^k(w)$ and thus obtains the so-called Jacobi identity, i.e. the formula in axiom (3) in the definition of a vertex algebra below. For $k\leq -1$ the field $\chi^k(w)$ is (up to normalization) the 'normal ordered product of $\left(\frac{d}{dz}\right)^{-m-1}\phi(z)$ and $\psi(z)$' and usually denoted by $\frac{1}{(-m-1)!}N(\partial^{-m-1}\phi,\psi)(z)$ in the physical literature. Finally, time translations are implemented by the energy generator L_0 which gives rise to a grading with respect to the energy. Covariance with respect to this grading and the full conformal covariance, i.e. a representation of the Virasoro algebra on the space of fields, completes the properties of conformal field theories which motivate the mathematically rigorous definition of vertex algebras.

Definition (Vertex algebra). A vertex algebra is a complex $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_n$$

(an element $\phi\in V_n$ is said to be of dimension n), together with a linear map

$$V \to (\mathrm{End}\,V)[[z,z^{-1}]], \qquad \phi\mapsto Y(\phi,z) = \sum_{n\in\mathbb{Z}}\phi_n\, z^{-n-1},$$

(the elements of the image are called vertex operators), and two distinguished elements $1\in V_0$ (called the vacuum) and $\omega\in V_2$ (called the Virasoro element) satisfying the following axioms:

(1) The map $\phi\mapsto Y(\phi,z)$ is injective.
(2) For all $\phi,\psi\in V$ there exists an n_0 such that $\phi_n\psi = 0$ for all $n\geq n_0$.
(3) For all $\phi,\psi\in V$ and $m,n\in\mathbb{Z}$ one has

$$(\phi_m\psi)_n = \sum_{i\geq 0}(-1)^i\binom{m}{i}\left(\phi_{m-i}\psi_{n+i} - (-1)^m\psi_{m+n-i}\phi_i\right).$$

(For $m < 0$ the sum on the right hand side is infinite; in this case this identity has to be read argumentwise, i.e. it has to be understood in the sense that the left hand side applied to an arbitrary element of V equals the right hand side applied to the same element: Note that this makes sense since by (2) in the sum on the right hand side all but a finite number of terms become 0 when evaluated at an element of V.)

(4) $Y(1, z) = \mathbb{1}_V$.

(5) Writing $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ one has

$$L_0|_{V_n} = n\,\mathbb{1}_{V_n},$$

$$Y(L_{-1}\phi, z) = \frac{d}{dz}Y(\phi, z),$$

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\,(m^3 - m)\frac{c}{12}\mathbb{1}_V,$$

for all $n, m \in \mathbb{Z}$, $\phi \in V$, where c is a complex constant (called the central charge or rank).

Remarks.

(1) For $m \geq 0$ property (3) is equivalent to

$$[\psi_m, \phi_n] = \sum_{i\geq 0} \binom{m}{i} (\psi_i\phi)_{m+n-i}.$$

where the left hand side denotes the ordinary commutator of endomorphisms.

(2) This commutator identity implies in particular $[L_0, \phi_n] = (L_{-1}\phi)_{n+1} + (L_0\phi)_n$, hence $[L_0, \phi_n] = (d-1-n)\phi_n$ for $\phi \in V_d$ (here we used $(L_{-1}\phi)_{n+1} = (-n-1)\phi_n$ from axiom (5)). From this one obtains

$$\phi_n V_m \subseteq V_{m+d-n-1}.$$

(3) Although elements of negative dimension do not turn up directly in physical applications, ghosts (fields of negative dimension) quite often serve as an important tool in free field constructions. The corresponding structures can e.g. be described in terms of vertex algebras.

Symmetry algebras of conformal field theories have additional properties motivating the

Definition (Vertex operator algebra). A vertex algebra is called a vertex operator algebra (VOA) if

(1) the spectrum of L_0 is bounded from below by 0, and

(2) the graded components V_n of V are finite dimensional.

Of particular interest are special elements of VOAs which are lowest weights with respect to the $sl(2)$ or Virasoro algebra inside the VOA.

DEFINITION ((QUASI-)PRIMARY ELEMENTS OF A VOA). An element $\psi \in V_d$ of a VOA V is called quasi-primary of dimension d if $L_1\psi = 0$, i.e. $\psi \in \ker(L_1)$, and primary of dimension d if $L_n\psi = 0$ for all $n > 0$.

REMARK. Note that for a quasi-primary element ψ of dimension d one has $\psi_n \cdot 1 = 0$ for $n \geq 0$ and $[L_m, \psi_n] = ((d-1)(m+1) - n)\psi_{n+m}$ $(m = 0, \pm 1)$. For primary elements this formula holds true without any restrictions on m.

All known symmetry algebras arising in conformal field theory are generated by quasi-primary elements.

DEFINITION (QUASI-PRIMARY GENERATED VOA). A vertex operator algebra V is called quasi-primary generated if

$$V = \oplus_{n=0}^{\infty} (L_{-1})^n \ker(L_1).$$

REMARK. All homogeneous elements of a quasi-primary generated VOA are linear combinations of terms of the form $\psi_n \cdot 1$ for some quasi-primary ψ.

In order to make closer contact with physics we need the notion of an 'invariant' bilinear form on VOAs.

DEFINITION (INVARIANT BILINEAR FORM OF A VOA). A bilinear form $(\cdot,\cdot)$ on a VOA V is said to be invariant if it satisfies the condition:

$$(\psi_n u, v) = (-1)^d \sum_{m \geq 0} \frac{1}{m!} \left(u, (L_1^m \psi)_{-n-m-2(d-1)} v\right)$$

for all $u, v \in V$ and $\psi \in V_d$.

REMARK. Note that for $\psi \in V_d$ quasi-primary the invariance condition reads

$$(\psi_n u, v) = (u, \psi_{-n-2(d-1)} v).$$

Therefore, one defines for a quasi-primary element $\psi \in V_d$

$$\psi_n^\dagger := \psi_{-n-2(d-1)},$$

in particular $L_n^\dagger = L_{-n}$.

For quasi-primary generated VOAs there always exists a 'natural' invariant bilinear form.

THEOREM (EXISTENCE OF AN INVARIANT BILINEAR FORM OF A VOA [L]). *Let V be a quasi-primary generated simple VOA. Then one has*

(1) $\dim(V_0) = 1$, *and*
(2) *the invariant bilinear form $(\cdot,\cdot)$ on V defined by:*

$$(V_n, V_m) = 0, \qquad n \neq m$$

and

$$(\psi_m \cdot 1, u) \cdot 1 = \psi_m^\dagger u, \qquad u, \psi_m \cdot 1 \in V_n \text{ and quasi-primary } \psi$$

is nondegenerate.

Of course we also need the notion of

Modules of vertex operator algebras and intertwining operators.

Definition (Representation of a VOA). A representation of a VOA V is a linear map

$$\rho: V \to (\operatorname{End} M)[[z, z^{-1}]], \qquad \phi \mapsto Y_M(\phi, z) = \sum_{n\in\mathbb{Z}} \rho(\phi)_n z^{-n-1},$$

where M is an $\mathbb{N}$-graded complex vector space

$$M = \bigoplus_{n\in\mathbb{N}} M_n,$$

such that the following axioms are satisfied:

(1) For all $\phi \in V_d$ and m, n one has $\rho(\phi)_n M_m \subset M_{m-n-1+d}$.
(2) For all $\phi \in V$ and $v \in M$ there exists an n_0 such that $\rho(\phi)_n v = 0$ for all $n \geq n_0$.
(3) For all $\phi, \psi \in V$ and all $m, n \in \mathbb{Z}$ one has

$$\rho(\phi_m\psi)_n = \sum_{i\geq 0} (-1)^i \binom{m}{i} \left(\rho(\phi)_{m-i}\rho(\psi)_{n+i} - (-1)^m \rho(\psi)_{m+n-i}\rho(\phi)_i\right),$$

where again this identity has to be read argumentwise.
(4) $Y_M(1, z) = \mathbb{1}_M$.
(5) Using $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} \rho(L)_n z^{-n-2}$, i.e. $\rho(L)_n = \rho(\omega)_{n+1}$ (note that this equality is not an identity involving some special $L \in V$, but introduces only a suggestive abbreviation for the right hand side), one has

$$Y_M(L_{-1}\phi, z) = \frac{d}{dz} Y_M(\phi, z),$$

$$[\rho(L)_m, \rho(L)_n] = (m-n)\rho(L)_{m+n} + \delta_{m+n,0}\,(m^3 - m)\frac{c}{12}\mathbb{1}_M,$$

for all $n, m \in \mathbb{Z}$, $\phi \in V$, where c is the central charge of V.

The representation ρ is called irreducible if there is no nontrivial subspace of M which is invariant under all $\rho(\phi)_n$.

In the following we shall occasionally use the term V-module M instead of representation $\rho: V \to \operatorname{End}(M)[[z, z^{-1}]]$.

Remarks.

(1) Note that a vertex operator algebra V is a V-module itself via $\phi \mapsto Y(\phi, z)$ (use remark (2) after the definition of vertex operator algebra for verifying axiom (1) of a representation).
(2) A VOA is called simple if it is irreducible as a module of itself.
(3) For a given module M of a VOA V there exists a dual module M' of V given by:

$$M' = \oplus_{n\in\mathbb{N}} M'_n := \oplus_{n\in\mathbb{N}} M^*_n$$

and

$$< Y_{M'}(\phi, z)w', w > = < w', Y_M(e^{zL_1}(-z^{-2})^{L_0}\phi, z^{-1})w >$$

where $< \cdot, \cdot >$ is the natural pairing between M' and M. Furthermore, (Y_M, M) is irreducible if and only if $(Y_{M'}, M')$ is irreducible and $(Y_{M'}, M')$ is isomorphic to (Y_M, M) [FHL].

As a simple consequence of the definition we have the

LEMMA. *Let* $\rho: V \to \mathrm{End}(M)[[z, z^{-1}]]$ *be an irreducible representation of VOA with* $\dim(M_n) < \infty$ $(n \in \mathbb{N})$. *Then there exists a complex constant* h_m *such that*

$$\rho(L)_0|_{M_n} = (h_M + n)\, 1\!\!I_{M_n}$$

for all $n \in \mathbb{N}$.

PROOF. By axiom (1) of a vertex operator algebra representation we have that $\rho(L)_0 M_0 \subseteq M_0$. Hence, since M_0 is finite dimensional, there exists an eigenvector v of $\rho(L)_0$ in M_0. Let h_M be the corresponding eigenvalue. Since ρ is irreducible the vector space M is generated by the vectors $\rho(\phi)_n v$ $(\phi \subset V_d,\ d \in \mathbb{N},\ n \in \mathbb{Z})$; for proving this, note that the subspace spanned by the latter vectors is invariant under all $\rho(\phi)_n$ as can be deduced from axiom (3)). For $m \in \mathbb{N}$ let M'_m be the subspace generated by all $\rho(\phi)_n v$ with $\phi \in M_d$ and $d - n - 1 = m$. By axiom (1) we have $M'_m \subseteq M_m$, and since M is the sum of all the M'_m we conclude $M'_m = M_m$.

On the other hand, one has $[\rho(L)_0, \rho(\phi)_n] = (d-n-1)\phi_n$ for all n and all $\phi \in V_d$ (similarly as in remark (2) after the definition of vertex operator algebras). From this we obtain $\rho(L)|_{M'_n} = (h_M + n)\, 1\!\!I_{M'_n}$. This proves the lemma. □

The lemma suggests the following

DEFINITION (CHARACTER OF A VOA MODULE). Let M be an irreducible module of the vertex operator algebra V (with respect to the representation ρ) and assume that $\dim(M_n) < \infty$ $(n \in \mathbb{N})$. Then the character χ_M of M is the formal power series defined by

$$\chi_M(q) := \mathrm{tr}_M(q^{\rho(L)_0 - c/24}) := q^{h_M - c/24} \sum_{n \in \mathbb{N}} \dim(M_n) q^n$$

where c is the central charge of V and h_M the conformal dimension of M.

The most important class of VOAs is given by 'rational' vertex operator algebras:

DEFINITION (RATIONALITY OF A VOA). A vertex operator algebra V is called rational if the following axioms are satisfied:

(1) V has only finitely many inequivalent irreducible representations M.
(2) For all inequivalent irreducible representations M one has $\dim(M_n) < \infty$ $(n \in \mathbb{N})$.
(3) Every finitely generated representation of V is equivalent to a direct sum of finitely many irreducible representations.

Here the notions equivalence and direct sum are to be understood in the obvious sense. Furthermore, finitely generated means that there exists a finite dimensional subspace V' of V such that the smallest vectorspace containing V' which is invariant under all $\rho(\psi)_n$ $(n \in \mathbb{Z}, \psi \in V')$ equals V (this should not be confused with the (different) notion of finitely generated $\mathcal{W}$-algebras cf. below).

The importance of the rational algebras becomes clear by the following theorem:

THEOREM (ZHU [Zh]). *Let M_i $(i = 0, \ldots, n-1)$ be a complete set of inequivalent irreducible modules of the rational vertex operator algebra V. Assume, furthermore, that Zhu's finiteness condition is satisfied, i.e.*

$$\dim(V/(V)_{-2}V) < \infty$$

where $(V)_{-2}V \subset V$ is defined by $(V)_{-2}V := \{\phi_{-2}\psi | \phi, \psi \in V\}$. Then the conformal characters χ_{M_i} become holomorphic functions on the upper complex half plane $\mathfrak{H}$ by setting $q = e^{2\pi i \tau}$ with $\tau \in \mathfrak{H}$. Furthermore, the space spanned by the conformal characters χ_{M_i} $(i = 0, \ldots, n-1)$ is invariant under the natural action $(\chi(\tau), A) \mapsto \chi(A\tau)$ of the modular group $\mathrm{SL}(2, \mathbb{Z})$.

Naively one would like to talk about the multiplicity of a certain representation of a VOA in the tensor product of two VOA representations. However, the tensor product of two representations does in general not carry the structure of a VOA representation. Instead, we use the notion of intertwining operators and fusion rules.

DEFINITION (INTERTWINING OPERATOR). An intertwining operator $\mathcal{I}$ of three irreducible modules $(\rho^i, M^i), (\rho^j, M^j), (\rho^k, M^k)$ of a VOA V satisfying $\dim(M^\alpha_n) < \infty$ $(n \in \mathbb{N}; \alpha = i, j, k)$ is a linear map

$$\begin{aligned} \mathcal{I}\colon M^i &\to z^{-h_i - h_j + h_k}\ \mathrm{Hom}(M^j, M^k)[[z, z^{-1}]], \\ v &\mapsto I(v, z) = z^{-h_i - h_j + h_k} \sum_{n \in \mathbb{Z}} I(v)_n z^{-n-1}, \end{aligned}$$

such that the following axioms are satisfied:

(1) For all $v \in M^i_d$ and m, n one has $I(v)_n M^j_m \subset M^k_{m-n-1+d}$.

(2) For all $\phi \in V$, $v \in M^i$ and all $m, n \in \mathbb{Z}$ one has

$$I(\rho^i(\phi)_m v)_n = \sum_{l \geq 0} (-1)^l \binom{m}{l} \left(\rho^k(\phi)_{m-l} I(v)_{n+l} - (-1)^m I(v)_{m+n-l} \rho^j(\phi)_l \right),$$

where again this identity has to read argumentwise.

(3) For all $v \in M^i$ one has $I(L_{-1}v, z) = \frac{d}{dz} I(v, z)$.

We call (M^i, M^j, M^k) the type of the intertwining operator $\mathcal{I}$.

REMARK. Note that an irreducible representation ρ of a simple VOA is an intertwining operator of type (V, M, M).

We are now able to define the fusion rule coefficients which will be the starting point of our results on the classification of fusion algebras in §3.

DEFINITION (FUSION RULE COEFFICIENT). The fusion rule coefficient $N_{i,j}^k$ of three irreducible modules $(\rho^i, M^i), (\rho^j, M^j), (\rho^k, M^k)$ of a VOA V which satisfy $\dim(M_n^\alpha) < \infty$ $(n \in \mathbb{N}; \alpha = i, j, k)$ is the dimension of the space of the corresponding intertwining operators.

This definition can be viewed as a natural generalization of the situation for simple Lie algebras. In the case of simple Lie algebras the dimension of the space of intertwing operators between three irreducible representations gives exactly the multiplicity of the third representation in the tensor product of the first two representations. This also provides us with a motivation for calling property (3) in the definition of vertex algebras and in the definition of representation of VOAs and property (2) in the definition of intertwining operators 'Jacobi identity'.

REMARK. Let M^i $(i = 0, \ldots, n-1)$ be a complete set of inequivalent irreducible modules of a simple rational VOA and assume that all fusion coefficients are finite, i.e. $N_{i,j}^k < \infty$. It is then proven -under certain further assumptions- that (for details see [HL,Hu]):

(1) The representation ρ^0 of the VOA acting on itself is isomorphic to its dual representation $\rho^{0'}$.
(2) The following equalities for the fusion coefficients hold true

$$N_{0,i}^j = \delta_{i,j}, \qquad N_{i,j}^0 = \delta_{i,j'}, \qquad N_{i,j}^k = N_{j,i}^k, \qquad N_{i,j}^k = N_{i',j'}^{k'},$$
$$\sum_{k=0}^n N_{i,j}^k N_{k,l}^m = \sum_{k=0}^n N_{i,k}^m N_{j,l}^k,$$

where i, j, l, m run from 0 to $n-1$.

In this case one can interpret the fusion coefficients as structure constants of a unital associative commutative algebra, the fusion algebra (see §2.2 for a abstract definition of fusion algebras).

$\mathcal{W}$-algebras and rational models.

One of our aims in this section is to make the notion of $\mathcal{W}$-algebras and rational models mathematically precise. Note, however, that the definitions below just collect the basic properties of the objects called '$\mathcal{W}$-algebras' and 'rational models' in the physical literature. We would like to stress therefore that our definitions can only serve as working definitions and it might be necessary to change them in the future. Nevertheless, we think that the definitions below will among others help to clarify notions.

DEFINITION ($\mathcal{W}$-ALGEBRA). A simple vertex operator algebra V is called $\mathcal{W}$-algebra if it is quasi-primary generated.

All known $\mathcal{W}$-algebras which define rational models are 'finitely generated'. We make the notion of being 'finitely generated' precise as follows. For any subspace $W \subset V$ of a $\mathcal{W}$-algebra V denote by $U(W)$ the smallest subspace of V which is invariant under ψ_m $(\psi \in W; m \leq -1)$ and contains 1. A subspace V' of a $\mathcal{W}$-algebra V is called a generating subspace if $V' \subset \ker(L_1)$ and $V = U(V')$.

Definition (Finitely generated $\mathcal{W}$-algebra). A $\mathcal{W}$-algebra V is called finitely generated if there exists a finite dimensional subspace $V' \subset \ker(L_1)$ generating V, i.e. V is the smallest vectorspace which contains 1 and is invariant under all ψ_m $(\psi \in V'; m \leq -1)$.

Remark. A generating subspace V is called minimally generating if

$$V_n \cap U(\oplus_{i<n} V_i) = \{0\} \qquad (n \in \mathbb{N}).$$

For any generating subspace V there obviously exists a minimal generating subspace $\hat{V}$ contained in V. For a generating subspace V with $V = \oplus_{i=1}^{n} V_{d_i}$ and $\dim(V_{d_i}) = k_i$ define the type of V by $(d_1^{k_1}, \ldots, d_n^{k_n})$. Furthermore, define an ordering on the type of generating subspaces by:

$(d_1^{k_1}, \ldots, d_n^{k_n}) < ({d'}_1^{{k'}_1}, \ldots, {d'}_{n'}^{{k'}_{n'}})$ if

(1) $d_i^{k_i} = {d'}_i^{{k'}_i}$ for $i < i_0 \leq \min(n, n')$ and either $d_{i_0} < d'_{i_0}$ or $d_{i_0} = {d'}_{i_0}$ and $k_{i_0} < {k'}_{i_0}$ or

(2) $n < n'$ and $d_i^{k_i} = {d'}_i^{{k'}_i}$ for $i = 1, \ldots, n$.

In the physical literature the type of $\mathcal{W}$-algebras is used frequently:

Definition (Type of a $\mathcal{W}$-algebra). A finitely generated $\mathcal{W}$-algebra V is said to be of type $\mathcal{W}(d_1^{k_1}, \ldots, d_n^{k_n})$ if the minimum (with respect to the order above) of the type of all generating subspaces of V is given by $(d_1^{k_1}, \ldots, d_n^{k_n})$.

Remarks.

(1) Examples of $\mathcal{W}$-algebras can be constructed directly from the Virasoro and Kac-Moody algebras. They are of type $\mathcal{W}(1^n)$, respectively $\mathcal{W}(2)$ for the Virasoro algebra [FZ].

(2) Starting from a Kac-Moody algebra associated to a simple Lie algebra $\mathcal{K}$ one can construct a 1-parameter family $\mathcal{WK}$ of $\mathcal{W}$-algebras, the parameter being the central charge (see e.g. [BS]) (Note that this construction is different from the one mentioned in (1)). For all but a finite number of central charges these $\mathcal{W}$-algebras are of type $\mathcal{W}(d_1, \ldots, d_n)$ where n is the rank of $\mathcal{K}$ and the d_i $(i = 1, \ldots, n)$ are the orders of the Casimir operators of $\mathcal{K}$. The remaining ones, called truncated, are of type $\mathcal{W}(d_{i_1}, \ldots, d_{i_r})$ where the d_{i_k} form a proper subfamily of the d_i above. Note that the $\mathcal{W}$-algebras constructed from the Virasoro algebra mentioned in (1) are exactly the Casimir $\mathcal{W}$-algebras associated to $\mathcal{A}_1$. The rational models of Casimir $\mathcal{W}$-algebras (sometimes called minimal models) have been determined, assuming certain conjectures, in [FKW] (some corresponding data can be also be found in Appendix 7.4).

(3) In the physical literature one refers to a finitely generated $\mathcal{W}$-algebra by giving their type (although this does not specify the $\mathcal{W}$-algebra uniquely in many cases).

One can try to construct finitely generated $\mathcal{W}$-algebras directly from their axioms (see e.g. [KW,BFKNRV]). In this direct construction of $\mathcal{W}$-algebras one starts with a finite number of 'simple' elements which are defined by

DEFINITION (SIMPLE ELEMENTS OF A $\mathcal{W}$-ALGEBRA). A quasi-primary element ϕ of a simple $\mathcal{W}$-algebra is called simple if

$$(\phi, \psi_m \chi_{-1} \cdot 1) = 0 \qquad (m \leq -1; \psi, \chi \in \ker(L_1))$$

where $(\cdot, \cdot)$ is the invariant bilinear form whose existence is guaranteed by one of the theorems given above.

One of the main ingredients in the direct construction of $\mathcal{W}$-algebras is the following commutator formula (see e.g. [Na,FRT]):

THEOREM (COMMUTATOR FORMULA FOR $\mathcal{W}$-ALGEBRAS). *Let ϕ and ψ be two quasi-primary elements of a $\mathcal{W}$-algebra V of dimension $d, d' \geq 1$, respectively. Then there exist quasi-primary elements $\chi^{d''} \in V_{d''}$ $(0 \leq d'' < d + d')$ such that*

$$[\phi_n, \psi_m] = \sum_{d''=0}^{d+d'-1} p(d, d', d'', m, n)\, \chi^{d''}_{d''+1-d-d'+m+n}$$

where the $p(d, d', d'', m, n)$ are universal polynomials given by

$$p(d, d', d'', m, n) = \sum_{r+s=d+d'-1-d''} (-1)^r\ r!\ s! \binom{D-r}{s} \binom{D'-s}{r} \binom{D-m}{r} \binom{D'-n}{s}$$

and $D = 2(d-1)$ and $D' = 2(d'-1)$.

PROOF. For the proof we need the simple

LEMMA. *For a quasi-primary element $\psi \in V_d$ and integers $0 \leq a \leq b$ one has*

$$\left(L_{-1}^a \psi\right)_b = (-1)^{a+1} a! \binom{b}{a} \psi_{b-a}$$

and

$$[L_1^a, L_{-1}^b]\psi = (a!)^2 \binom{b}{a} \binom{2d-1+b}{a} L_{-1}^{b-a} \psi.$$

PROOF. We leave the easy calculation to the reader.

With remark (1) after the definition of vertex algebras we have

$$[\phi_m, \psi_n] = \sum_{i \geq 0} \binom{m}{i} (\phi_i \psi)_{m+n-i}.$$

Since V is quasi-primary generated we know that there exist quasi-primary elements $\chi^{d''} \in V_{d''}$ such that

$$\phi_0 \psi = \sum_{d''=0}^{d+d'-1} L_{-1}^{d+d'-1-d''}\, \chi^{d''}.$$

Assume w.l.o.g. that $d \geq d'$. Using that ϕ is quasi-primary and applying the Lemma above we find

$$\begin{aligned}
\phi_i\psi &= \frac{1}{\prod_{n=0}^{i}(2d-2-n)}\left(\mathrm{ad}(L_1)^i\phi_0\right)\psi = \left(i!\binom{2d-2}{i}\right)^{-1} L_1^i\phi_0\psi \\
&= \left(i!\binom{2d-2}{i}\right)^{-1}\sum_{d''}[L_1^i, L_{-1}^{d+d'-1-d''}]\chi^{d''} \\
&= i!\binom{2d-2}{i}^{-1}\sum_{d''}\binom{d+d'-1-d''}{i}\binom{d+d'+d''-2}{i}L_{-1}^{d+d'-1-d''-i}\chi^{d''}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(L_{-1}^{d+d'-1-d''-i}\chi^{d''}\right)_{m+n-i} = \\
&(-1)^{d+d'-1-d''-i+1}(d+d'-1-d''-i)!\binom{m+n-i}{d+d'-1-d''-i}\chi^{d''}_{m+n+d'+1-d-d''}
\end{aligned}$$

we find

$$[\phi_n, \psi_m] = \sum_{d''=0}^{d+d'-1} p(d, d', d'', m, n)\,\chi^{d''}_{m+n+d'+1-d-d''}$$

where

$$\begin{aligned}
&p(d, d', d'', m, n) = (-1)^{d+d'-d''}(d+d'-1-d'')!\,\cdot \\
&\sum_{i=0}^{d+d'-1-d''}\binom{m}{i}\binom{2d-2}{i}^{-1}\binom{d+d'+d''-2}{i}\binom{-m-n+d+d'-d''-2}{d+d'-1-d''-i}
\end{aligned}$$

(here we have used $\binom{x}{r} = (-1)^r\binom{r-x-1}{r}$). Finally, note that this polynomial equals (up to a constant factor depending on d, d' and d'')

$$\sum_{r+s=d+d'-1-d''}(-1)^r\, r!\, s!\binom{D-r}{s}\binom{D'-s}{r}\binom{D-m}{r}\binom{D'-n}{s}$$

where $D = 2(d-1)$ and $D' = 2(d'-1)$ as one can e.g. see by comparing the zeros of the two polynomials. $\square$

REMARK. Note that polynomials similar to the polynomials above occur in the theory of modular forms (cf. [BEH^3] and [Za2]).

We make the notion of 'rational models' precise.

DEFINITION (RATIONAL MODEL). A rational model (or rational model of a $\mathcal{W}$-algebra) is a rational $\mathcal{W}$-algebra V which satisfies Zhu's finiteness condition. The *effective central charge* of a rational model is defined by

$$\tilde{c} = c - 24\min\{h_{M_i}\}$$

where M_i runs through a complete set of inequivalent irreducible representations of V.

REMARKS.

(1) In the literature rational models are frequently called rational conformal field theories (RCFTs) and we will also do so.
(2) Examples of rational models are given by certain vertex operator algebras constructed from Kac-Moody algebras [FZ] or the Virasoro algebra [Wa] (for more details see also below).
(3) One can show that the effective central charge of a rational model with a minimal generating subspace of dimension n lies in the range [EFH²NV]

$$0 \leq \tilde{c} < n.$$

(4) Historically the term 'rational models' was used in the physical literature [BPZ] for field theories in which the operator product expansion of any two local quantum fields decomposes into finitely many conformal families from a finte set.

The following theorem justifies the terminology 'rational models':

THEOREM ([AM]). *Assume that the representation of the modular group acting on the space spanned by the conformal characters of a rational model is unitary. Then the central charge and the conformal dimensions of the rational model are rational numbers.*

2.2 Definition of fusion algebras.

Consider a rational model consisting of a $\mathcal{W}$-algebra V and its (finitely many) inequivalent irreducible modules M_i ($i = 0, \ldots, n-1$). Here M_0 denotes the vacuum representation, i.e. the representation of V acting on itself. Recall, that for a module M of V there is the notion of the dual (or adjoint or conjugate) module M' and that one has $(M')' \cong M$. Since V is rational the conjugation defines a permutation π of order two of the irreducible modules $M_i' \cong M_{\pi(i)}$.

The structure constants $N_{i,j}^k$ of the 'fusion algebra' associated to V are given by the dimension of the corresponding space of intertwining operators of three modules (cf. §2.1). From now on we will assume that the fusion coefficients related to the rational models under consideration are always finite.

One of the important properties of the $N_{i,j}^k$ which is well known in the physical literature is the fact that the numbers $N_{i,j}^k$ can be viewed as the structure constants of an associative commutative algebra, the fusion algebra. In the terminology of vertex operator algebras a corresponding statement is proven under certain assumptions in a recent series of papers [Hu] (see also the remark below the definition of fusion coefficients in §2.1). In the abstract definition of fusion algebras the properties of all known examples associated to RCFTs are collected.

DEFINITION (FUSION ALGEBRA). A **fusion algebra** $\mathcal{F}$ is a finite dimensional algebra over $\mathbb{Q}$ with a distinguished basis $\Phi_0 = \mathbb{I}, \ldots, \Phi_{n-1}$ ($n = \dim(\mathcal{F})$) satisfying the following axioms:

(1) $\mathcal{F}$ is associative and commutative.

(2) The structure constants $N_{i,j}^k$ $(i,j,k = 0,\dots,n-1)$ with respect to the distinguished basis Φ_i are nonnegative integers.
(3) There exists a permutation $\pi \in S_n$ of order two such that for the structure constants in (2) one has

$$N_{i,j}^0 = \delta_{i,\pi(j)}, \qquad N_{\pi(i),\pi(j)}^{\pi(k)} = N_{i,j}^k, \qquad i,j,k = 0,\dots,n-1.$$

REMARKS.

(1) An isomorphism ϕ of two fusion algebras $\mathcal{F}, \mathcal{F}'$ is an isomorphism of unital algebras which maps the distinguished basis to the distinguished basis, i.e. there exists a permutation $\sigma \in S_n$ such that $\phi(\Phi_i) = \Phi'_{\sigma(i)}$ $(i = 0,\dots,n-1)$.
(2) The tensor product of two fusion algebras $\mathcal{F}$ and $\mathcal{F}'$ is again a fusion algebra, its distinguished basis is given by $\Phi_{i_1} \otimes \Phi'_{i_2}$ $(i_1 = 0,\dots,\dim(F)-1,\ i_2 = 0,\dots,\dim(F')-1)$.
(3) The permutation π of order two is called charge conjugation. Fusion algebras with trivial charge conjugation are called selfconjugate.
(4) Note that it is an open question whether two nonisomorphic fusion algebras can be isomorphic as unital algebras.

It is known that fusion algebras arising from RCFTs have additional properties believed to be generic. One of these additional properties is their relation to conformal characters. Recall, that one can show for rational vertex operator algebras satisfying Zhu's finiteness condition [Zh] that the conformal characters defined in §2.1 become holomorphic functions in the upper complex half plane by setting $q = e^{2\pi i \tau}$. Furthermore, for these RVOAs the space spanned by the finitely many conformal characters is invariant under the action of the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ (note that it is conjectured that Zhu's finiteness condition is not a necessary assumption for rational VOAs). It was conjectured in 1988 by E. Verlinde [Ve] that for any rational model there exists a representation $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ of Γ such that

$$\begin{aligned}
\chi_i(A\tau) = (\chi_i|A)(\tau) &= \sum_{m=0}^{n-1} \rho(A)_{j,i}\chi_j(\tau) \qquad A \in \Gamma \\
N_{i,j}^0 &= \rho(S^2)_{i,j} \\
N_{i,j}^k &= \sum_{m=0}^{n-1} \frac{\rho(S)_{i,m}\rho(S)_{j,m}\rho(S^{-1})_{m,k}}{\rho(S)_{0,m}}.
\end{aligned}$$

We will refer to this formula as 'Verlinde's formula' in the following. The above conjecture motivates the definition of modular fusion algebras.

DEFINITION (MODULAR FUSION ALGEBRA). A **modular fusion algebra** $(\mathcal{F},\rho)$ is a fusion algebra $\mathcal{F}$ together with a unitary representation $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ satisfying the following additional axioms:

(1) $\rho(S)$ is a symmetric and $\rho(T)$ is a diagonal matrix.
(2) $N_{i,j}^0 = \rho(S^2)_{i,j}$,
(3) $N_{i,j}^k = \sum_{m=0}^{n-1} \frac{\rho(S)_{i,m}\rho(S)_{j,m}\rho(S^{-1})_{m,k}}{\rho(S)_{0,m}}$

where $N_{i,j}^k$ $(i,j,k = 0,\dots,n-1)$ are the structure constants of $\mathcal{F}$ with respect to the distinguished basis.

REMARKS.

(1) Note that property (3) already implies that $\mathcal{F}$ is associative and commutative.
(2) Two modular fusion algebras $(\mathcal{F},\rho)$ and $(\mathcal{F}',\rho')$ are called isomorphic if: 1) $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as fusion algebras, 2) ρ and ρ' are equivalent, 3) $\rho(T)_{i,j} = \rho'(T)_{\sigma(i)\sigma(j)}$ where $\sigma \in S_n$ is the permutation defined by the isomorphism of the fusion algebras.
(3) The tensor product of two modular fusion algebras $(\mathcal{F},\rho),(\mathcal{F}',\rho')$ is defined by $(\mathcal{F}\otimes\mathcal{F}',\rho\otimes\rho')$ and is again a modular fusion algebra.
(4) A (modular) fusion algebra is called composite if it is isomorphic to a tensor product of two nontrivial (modular) fusion algebras. Here a (modular) fusion algebra is called trivial if it is one dimensional. A noncomposite (modular) fusion algebra is also called simple.
(5) Note that for a modular fusion algebra with trivial charge conjugation $(\rho(S^2) = \mathbb{1})$ the matrix $\rho(S)$ is real.
(6) For modular fusion algebras associated to rational models the eigenvalues of $\rho(T)$ are given by the conformal dimensions h_i $(i = 0,\dots,n-1)$ of the irreducible modules M_i (h_i is the smallest L_0 eigenvalue in the module M_i) and the central charge c of the theory:

$$\rho(T) = \mathrm{diag}(e^{2\pi i(h_0 - c/24)},\dots,e^{2\pi i(h_{n-1}-c/24)}).$$

(7) Quite often nonisomorphic modular fusion algebras are isomorphic as fusion algebras.

In the later sections we will investigate which representations of Γ are related to modular fusion algebras.

DEFINITION (ADMISSIBLE REPRESENTATION OF $\mathrm{SL}(2,\mathbb{Z})$). A representation of the modular group $\rho : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ is called **conformally admissible** or simply **admissible** if there exists a fusion algebra $\mathcal{F}$ such that $(\mathcal{F},\rho)$ is a modular fusion algebra.

It is known that modular fusion algebras associated to rational models have many additional properties. In particular, the central charge and the conformal dimensions are rational [Va,AM]. Furthermore, there exist certain compatibility conditions between the central charge c, the conformal dimensions h_i and the fusion coefficients N_{ij}^k (the so-called Fuchs conditions) (see e.g. [MMS],[CPR][2]):

[2]Note that the formula connecting the central charge with the conformal dimension in [CPR] contains a misprint.

$$\frac{n(n-1)}{12} - \sum_{m=0}^{n-1} \left(h_i - \frac{c}{24}\right) \in \frac{1}{6}(\mathbb{N}\backslash\{1\}),$$

$$\sum_{m=0}^{n-1} \left((h_i + h_j + h_k + h_l)N_{i,j}^m N_{k,m}^l - h_m(N_{i,j}^m N_{k,m}^l + N_{i,k}^m N_{j,m}^l + N_{i,l}^m N_{k,j}^m)\right)$$
$$- \frac{1}{2}\left(\sum_{m=0}^{n-1} N_{i,j}^m N_{k,m}^l\right)\left(1 - \sum_{m=0}^{n-1} N_{i,j}^m N_{k,m}^l\right) \in \mathbb{N}$$

In many contexts the so-called quantum dimensions of the irreducible representations of the symmetry algebra are of particular interest.

Definition (Quantum dimension). Let V be a rational model. Then the real number

$$\delta_{M_i} := \lim_{\tau \to i\infty} \frac{\chi_i(\tau)}{\chi_0(\tau)}$$

associated to an irreducible representation M_i of V is called the quantum dimension of M_i.

Of course, the quantum dimensions are nonnegative. If there exists a unique irreducible representation M_λ with minimal conformal dimension h_λ then the quantum dimensions are given by

$$\delta_{M_i} = \frac{\rho(S)_{i,\lambda}}{\rho(S)_{0,\lambda}}.$$

In the rest of this thesis we will extensively rely on the observation that in all known examples of RCFTs the conformal characters are modular functions on some congruence subgroup of Γ. Therefore, the corresponding representation ρ factors through a representation of $\Gamma(N)$. Here we have used $\Gamma(N)$ for the principal congruence subgroup of Γ of level N

$$\Gamma(N) = \{\ A \in \Gamma \mid A \equiv \mathbb{1} \bmod N\ \}.$$

Definition (Strongly-modular fusion algebra). A modular fusion algebra $(\mathcal{F}, \rho)$ is called **strongly-modular** if the kernel of the representation ρ contains a congruence subgroup of Γ.

In this case ρ defines a representation of $\mathrm{SL}(2, \mathbb{Z}_N)$ and is called a level N representation of Γ (here and in the following we use $\mathbb{Z}_N$ for $\mathbb{Z}/N\mathbb{Z}$). A level N representation ρ will be called even or odd if $\rho(S^2) = \mathbb{1}$ or $\rho(S^2) = -\mathbb{1}$, respectively. Furthermore, one can show that for strongly-modular fusion algebras associated to rational models the representation ρ is defined over the field K of N-th roots of unity, i.e. $\rho : \Gamma \to \mathrm{GL}(n, K)$ if the corresponding conformal characters are modular functions on some congruence subgroup [ES1]. Indeed, we expect that this is true for all RCFTs thus motivating the following definition and conjecture.

Definition (K-rational representation of $\mathrm{SL}(2,\mathbb{Z})$). A level N representation $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{C})$ is called K-rational if it is defined over the field K of the N-th roots of unity, i.e. $\rho : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{GL}(n, K)$.

CONJECTURE. All modular fusion algebras associated to rational models are strongly-modular fusion algebras and the corresponding representations of the modular group are K-rational.

2.3 Some simple properties of modular fusion algebras.
In this section we prove some simple lemmas about modular fusion algebras which will be needed in the proofs of the main theorems in §3.

LEMMA 1. *Let $(\mathcal{F}, \rho)$ be a modular fusion algebra. Assume that $\rho(T)$ has nondegenerate eigenvalues. Then ρ is irreducible.*

PROOF. Assume that ρ is reducible and $\rho(T)$ has nondegenerate eigenvalues. Then $\rho(S)$ has block diagonal form and therefore $\rho(S)_{0,m} = 0$ for some m. This is a contradiction to property (3) in the definition of modular fusion algebras.

DEFINITION ((NON-)DEGENERATE MODULAR FUSION ALGEBRA). A modular fusion algebra $(\mathcal{F}, \rho)$ is called **degenerate** or **nondegenerate** if $\rho(T)$ has degenerate or nondegenerate eigenvalues, respectively.

LEMMA 2. *Let $\rho, \rho' : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be equivalent, irreducible, unitary representations of the modular group. Assume that $\rho(T) = \rho'(T)$ is a diagonal matrix with nondegenerate eigenvalues. Then there exists a unitary diagonal matrix D such that $\rho = D^{-1}\rho' D$.*

PROOF. Since ρ and ρ' are equivalent there exists a matrix D' such that $\rho = {D'}^{-1}\rho' D'$. Since $\rho(T) = \rho'(T)$ is a diagonal matrix with nondegenerate eigenvalues D' is diagonal. Finally, the irreducibility of ρ implies by Schur's lemma that ${D'}^{+}D' = \alpha\mathbb{1}$ for some positive real number α so that $D = \frac{1}{\sqrt{\alpha}}D'$ satisfies the desired properties.

LEMMA 3. *Let $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ be two nondegenerate modular fusion algebras. Assume that ρ is equivalent to ρ' and $\rho(T) = \rho'(T)$. Then $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic as fusion algebras.*

PROOF. The lemma follows directly from the definition of (modular) fusion algebras and Lemma 2.

LEMMA 4. *Let $(\mathcal{F}, \rho)$ be a modular fusion algebra. Then ρ is not isomorphic to a direct sum of one dimensional representations.*

PROOF. If ρ is the direct sum of one dimensional representations $\rho(S)$ is also a diagonal matrix. This implies that one cannot apply Verlinde's formula giving a contradiction since we have assumed that $(\mathcal{F}, \rho)$ is a modular fusion algebra.

Since there are exactly 12 one dimensional representations of Γ one has the following trivial lemma.

LEMMA 5.

(1) *Let ρ be a one dimensional representation of Γ. Then ρ is equivalent to one of the following representations*

$$\rho(S) = e^{2\pi i \frac{3n}{4}}, \qquad \rho(T) = e^{2\pi i \frac{n}{12}}, \qquad n = 0, \ldots, 11.$$

(2) *Let $(\mathcal{F}, \rho)$ be a one dimensional modular fusion algebra. Then $(\mathcal{F}, \rho)$ is strongly-modular, $\mathcal{F}$ is trivial and ρ is given by*

$$\rho(S) = (-1)^n, \qquad \rho(T) = e^{2\pi i \frac{n}{6}}, \qquad n = 0, \ldots, 5.$$

LEMMA 6. *Let $(\mathcal{F}, \rho)$ be a strongly-modular fusion algebra associated to a rational model. Then ρ is K-rational.*

PROOF. For a rational vertex operator algebra satisfying Zhu's finiteness condition the characters are holomorphic functions on the upper complex half plane. Since we have assumed that $(\mathcal{F}, \rho)$ is strongly-modular, ρ is a level N representation for some N. This implies that the characters are modular functions on $\Gamma(N)$. Moreover, their Fourier coefficients are positive integers so that one can apply the theorem on K-rationality of ref. [ES1] implying that ρ is K-rational.

Although Lemma 6 will not be used in the following it provides us with a good motivation for looking at K-rationality of level N representations.

3. On the classification of modular fusion algebras

In this section we develop several tools, following references [E2,E3], which enable us to classify all strongly-modular fusion algebras of dimension less than or equal to four (for a definition of strongly-modular fusion algebras see §2.2). Our approach is based on the known classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ [NW].

With the tools developed in this section we obtain another partial classification, namely of those strongly-modular fusion algebras of dimension less than 24 where the corresponding representation ρ of the modular group is such that $\rho(T)$ has nondegenerate eigenvalues. The nondegeneracy of the eigenvalues of $\rho(T)$ means that the difference of any two conformal dimensions of a possibly underlying RCFT is not an integer. The restriction on the dimension is of purely technical nature so that it should be possible to obtain a complete classification of all nondegenerate strongly-modular fusion algebras with the methods described in this thesis by using systematically Galois theory.

This section is organized as follows: In §3.1 state our main results on the classification of strongly-modular fusion algebras. Section 3.2 contains some remarks about the realization of strongly-modular fusion algebras in rational models. In the next four subsections we give a short review of the classification of the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ which will be the main tool in the proof of the main theorems. The proofs of our main theorems 1 and 2 are contained in the last three subsections. Finally, the three Appendices 7.1-7.3 at the end of the thesis contain the explicit form of the modular fusion algebras occurring in our classifications as well as the explicit form of the irreducible level p^λ representations of dimension less than or equal to four.

3.1 Results on the classification of strongly-modular fusion algebras.

We summarize our results on the classification of low dimensional strongly-modular fusion algebras in the following two main theorems (note that the terminology used for the fusion algebras will be explained in detail in §3.2):

Main theorem 1 (Classification of strongly-modular fusion algebras of dimension ≤ 4). *Let $(\mathcal{F}, \rho)$ be a two, three or four dimensional simple strongly-modular fusion algebra. Then $\mathcal{F}$ is isomorphic to one of the following fusion algebras:*

$$\mathbb{Z}_2,\ "(2,5)",\ \mathbb{Z}_3,\ "(2,7)",\ "(3,4)",\ \mathbb{Z}_4,\ \mathbb{Z}_2 \otimes \mathbb{Z}_2,\ "(2,9)".$$

Furthermore, $(\mathcal{F}, \rho)$ is isomorphic to the tensor product of a one dimensional modular fusion algebra with one of the modular fusion algebras in Table 7.2a or 7.2b (see Appendix).

In the nondegenerate case we have the

Main theorem 2 (Classification of nondegenerate strongly-modular fusion algebras of dimension < 24). *Let $(\mathcal{F}, \rho)$ be a simple nondegenerate strongly-modular fusion algebra of dimension less than 24. Then $\mathcal{F}$ is isomorphic to one of the following types of fusion algebras*

$$\mathbb{Z}_2,\ "(3,4)",\ "(2,9)",\ "(2,q)",\ B_9,\ B_{11},\ G_9,\ G_{17},\ E_{23}$$

where $q < 47$ is an odd prime. Moreover, $\mathcal{F}$ is isomorphic to $\mathbb{Q}[x]/ < P(x) >$ with distinguished basis $p_j(x)$ ($j = 0, \ldots, n-1$). Here P and p_j are the unique polynomials satisfying

$$P(x) = \det(\mathcal{N}_1 - x)$$

$$p_0(x) = 1, \qquad p_1(x) = x, \qquad p_j(x) = \sum_{k=0}^{n-1} (\mathcal{N}_1)_{j,k}\, p_k(x).$$

where the $(\mathcal{N}_1)_{j,k} := N_{1,j}^k$ are the fusion matrices given in Appendix 7.3. Furthermore, ρ is isomorphic to the tensor product of an even one dimensional representation of Γ with one of the representations in Table 7.3 (see Appendix).

In the next subsection we comment on the question which of the strongly-modular fusion algebras described by the the main theorems 1 and 2 occur in known RCFTs.

3.2 Realization of strongly-modular fusion algebras in RCFTs and data of certain rational models.

Let us first comment on the fusion algebras related to the theorems in §3.1.

The fusion algebras of type "$(2, q)$" occur in the Virasoro minimal models with central charge $c = c(2, q)$. Here the rational models of the Virasoro vertex operator algebra for $c = c(p, q) = 1 - 6\frac{(p-q)^2}{pq}$ ($p, q > 1$, $(p, q) = 1$) [BPZ,Wa] are called Virasoro minimal models and the corresponding fusion algebras are denoted by "(p, q)". A list of conformal dimensions for these models can be found at the end of this subsection.

The fusion algebra of type $\mathbb{Z}_n$ occurs in the so-called $\mathbb{Z}_n$-models (see e.g. [De]). This fusion algebra are isomorphic to the group algebra of $\mathbb{Z}_n$ with the distinguished basis given by the group elements. We will call the fusion algebra given by the group algebra of $\mathbb{Z}_n$ in the following $\mathbb{Z}_n$ fusion algebra.

For all fusion algebras in the main theorem 2 apart from B_9 there indeed exist RCFTs where the associated fusion algebras are isomorphic to the ones in Table 7.3: The fusion algebra in the first row occurs in the so-called $\mathbb{Z}_2$-model, the ones in row 2, 3 and 4 in the corresponding Virasoro minimal models (see above) and, finally, the ones in row 6, 7, 8 and 9 occur as fusion algebras of certain rational models, so-called minimal models of Casimir $\mathcal{W}$-algebras (cf. §2.1 and Appendix 7.4), namely for $\mathcal{W}B_2$ and $c = -\frac{444}{11}$, $\mathcal{W}G_2$ and $c = -\frac{590}{9}$, $\mathcal{W}G_2$ and $c = -\frac{1420}{17}$ and $\mathcal{W}E_7$ and $c = -\frac{3164}{23}$ [E2] (central charges, conformal dimensions and characters of Casimir $\mathcal{W}$-algebras are described in Appendix 7.4; the data for five of the particular rational models mentioned here is also collected at the end of the subsection in Table 3.2c). The fusion algebras of type B_9 seems to be related to $\mathcal{W}B_2$ and $c = -24$. However, in this case the model is not rational.

The fact that we do not know examples of RCFTs for all of the **modular** fusion algebras in our classification can be understood as follows. The classification of the strongly-modular fusion algebras implies restrictions on the central charge and the conformal dimensions of possibly underlying RCFTs. In Table 3.2a we have collected the possible values of c and the h_i for the simple strongly-modular fusion algebras of dimension less than or equal to four. Note, however, that these restrictions are not as strong as the ones in [Ki] for the two dimensional case or in [CPR] for the two and three dimensional case. A natural way to obtain stronger restrictions than the ones presented in Table 3.2a is to look whether there exist vector valued modular functions transforming under the corresponding representation of the modular group which have the correct pole order at $i\infty$. This can be done using the methods which will be developed in §4 and indeed leads to much stronger restrictions on c and the h_i as we shall discuss elsewhere. Of course, we expect that for any RCFT the corresponding characters are modular functions so that these stronger restrictions have to be valid explaining that our classification contains modular fusion algebras for which we do not know any realization in RCFTs.

Table 3.2a: Central charges and conformal dimensions related to simple strongly-modular fusion algebras of dimension ≤ 4

$\mathcal{F}$	$c \pmod 4$	$h_i \pmod{\mathbb{Z}}$
$\mathbb{Z}_2$	1	$0, \frac{1}{4}$
	3	$0, \frac{3}{4}$
$\mathbb{Z}_3$	2	$0, \frac{1}{3}, \frac{1}{3}$ or $0, \frac{2}{3}, \frac{2}{3}$
$\mathbb{Z}_4$	1	$0, \frac{1}{8}, \frac{1}{2}, \frac{1}{8}$ or $0, \frac{5}{8}, \frac{1}{2}, \frac{5}{8}$
	3	$0, \frac{3}{8}, \frac{1}{2}, \frac{3}{8}$ or $0, \frac{7}{8}, \frac{1}{2}, \frac{7}{8}$
$\mathbb{Z}_2 \otimes \mathbb{Z}_2$	0	$0, 0, 0, \frac{1}{2}$ or $0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$
"$(2,5)$"	$\frac{6}{5}$	$0, \frac{3}{5}$
	$\frac{14}{5}$	$0, \frac{2}{5}$
	$\frac{2}{5}$	$0, \frac{1}{5}$
	$\frac{18}{5}$	$0, \frac{4}{5}$
"$(2,7)$"	$\frac{16}{7}$	$0, \frac{4}{7}, \frac{5}{7}$
	$\frac{12}{7}$	$0, \frac{3}{7}, \frac{2}{7}$
	$\frac{4}{7}$	$0, \frac{3}{7}, \frac{1}{7}$
	$\frac{24}{7}$	$0, \frac{4}{7}, \frac{6}{7}$
	$\frac{8}{7}$	$0, \frac{6}{7}, \frac{2}{7}$
	$\frac{20}{7}$	$0, \frac{1}{7}, \frac{5}{7}$
"$(2,9)$"	$\frac{10}{3}$	$0, \frac{1}{3}, \frac{2}{3}, \frac{2n}{9}$
	$\frac{2}{3}$	$0, \frac{1}{3}, \frac{2}{3}, \frac{n}{9}$
		$n = 1, 4, 7$
"$(3,4)$"	$\frac{3n}{2}$	$0, \frac{1}{2}, \frac{3n}{16}$
	$n = 0, \ldots, 15$	

We finally give concrete lists of the central charges and conformal dimensions of certain rational models which will appear again in the main theorem 3 on uniqueness of conformal characters in §4 and by the main theorem 5 on theta formulas for conformal characters in §5.

Central charges and conformal dimensions of certain rational models .

Note that some of the results summarized in this section are not yet proved on a mathematically rigorous level. However, taken as an input into the formalism developed in §4 the central charges and sets of conformal dimensions given below will lead to consistent representations of the modular group on spaces of modular functions. This section serves rather as a motivation than as a background for the considerations in the subsequent sections.

Firstly, we review some known rational models with effective central charge less than 1. The simplest $\mathcal{W}$-algebras are those which can be constructed from the Virasoro algebra (as already mentioned in§2.1). The rational models among these are called the Virasoro minimal models (see e.g. [BPZ,RC,Wa]). They can be parameterized by a set of two coprime integers $p, q \geq 2$. The rational model corresponding to such a set p, q has central charge

$$c = c(p,q) = 1 - 6\frac{(p-q)^2}{pq}$$

and its conformal dimensions are given by:

$$h(p,q,r,s) = \frac{(rp-sq)^2-(p-q)^2}{4pq} \quad (1 \leq r < q,\ (2,r)=1,\ 1 \leq s < p),$$

where we assume q to be odd.

The Virasoro minimal models are special examples of the larger class of rational models with $\tilde{c} < 1$ which emerges from the ADE-classification of modular invariant partition functions [CIZ,EFH^{2}NV]. Their central charges and conformal dimensions are given in Table 3.2b: The first column describes the type of modular invariant partition function, the central charge is always $c = c(p,q)$ where p and q are the parameters of the respective row under consideration. Moreover, $c(p,q)$ and $h(p,q,\cdot,\cdot)$ are as defined above. Note that the listed models exist also for p, q, m not necessarily prime. The primality restrictions have been added for technical reasons only which will become clear in the next section.

Table 3.2b: Data of certain rational $\mathcal{W}$-algebras related to the ADE-classification

type	type of $\mathcal{W}$-algebra	$H_{c(p,q)} \qquad (I_n := \{1,\ldots,n\})$
(A_{q-1}, A_{p-1})	$\mathcal{W}(2)$ $p > q$ odd primes	$\{h(p,q,r,s) \mid r \in I_{q-1},\ s \in I_{p-1},\ (2,r)=1\}$
(A_{q-1}, D_{m+1})	$\mathcal{W}(2, \frac{(m-1)(q-2)}{2})$ $p = 2m$ q and m odd primes	$\{h(p,q,r,s) \mid r \in I_{(q-1)/2},\ s \in I_m,\ (2,s)=1\}$
(A_{q-1}, E_6)	$\mathcal{W}(2, q-3)$ $p = 12,\ q \geq 5$ q prime	$\{\min(h(p,q,r,1), h(p,q,r,\ 7)) \mid r \in I_{(q-1)/2}\} \cup$ $\{\min(h(p,q,r,5), h(p,q,r,11)) \mid r \in I_{(q-1)/2}\} \cup$ $\{h(p,q,r,4) \mid r \in I_{(q-1)/2}\}$
(A_{q-1}, E_8)	$\mathcal{W}(2, q-5)$ $p = 30,\ q \geq 7$ q prime	$\{\min(h(p,q,r,1), h(p,q,r,11)) \mid r \in I_{(q-1)/2}\} \cup$ $\{\min(h(p,q,r,7), h(p,q,r,13)) \mid r \in I_{(q-1)/2}\}$

The second list of rational models which we shall consider are special cases of the so-called Casimir $\mathcal{W}$-algebras (cf. §2.1).

In Table 3.2c we list the central charges c, effective central charge $\tilde{c}$ and the sets of conformal dimensions H_c of 5 rational models with $\tilde{c} > 1$.

The last three are minimal models of Casimir $\mathcal{W}$-algebras associated to $\mathcal{B}_2, \mathcal{G}_2$ and $\mathcal{E}_7$.

The first two $\mathcal{W}$-algebras are 'tensor products' of the rational $\mathcal{W}$-algebra with $c = -22/5$ constructed from the Virasoro algebra and the rational $\mathcal{W}$-algebras with $c = 14/5$ or $c = 26/5$ constructed from the Kac-Moody algebras associated to $\mathcal{G}_2$ or $\mathcal{F}_4$, respectively. We denote them by $\mathcal{W}_{\mathcal{G}_2}(2, 1^{14})$ and $\mathcal{W}_{\mathcal{F}_4}(2, 1^{26})$, respectively. Here the construction of the $\mathcal{W}$-algebras in question is the one mentioned in §2.1 in the remark after the definition of the type of $\mathcal{W}$-algebras in (2).

Table 3.2c: Data of the five rational models

$\mathcal{W}$-algebra	c	$\tilde{c}$	H_c
$\mathcal{W}_{G_2}(2, 1^{14})$	$-\frac{8}{5}$	$\frac{16}{5}$	$\frac{1}{5}\{0, -1, 1, 2\}$
$\mathcal{W}_{F_4}(2, 1^{26})$	$\frac{4}{5}$	$\frac{28}{5}$	$\frac{1}{5}\{0, -1, 2, 3\}$
$\mathcal{W}(2, 4)$	$-\frac{444}{11}$	$\frac{12}{11}$	$-\frac{1}{11}\{0, 9, 10, 12, 14, 15, 16, 17, 18, 19\}$
$\mathcal{W}(2, 6)$	$-\frac{1420}{17}$	$\frac{20}{17}$	$-\frac{1}{17}\{0, 27, 30, 37, 39, 46, 48, 49, 50, 52, 53, 55, 57, 58, 59, 60\}$
$\mathcal{W}(2, 8)$	$-\frac{3164}{23}$	$\frac{28}{23}$	$-\frac{1}{23}\{0, 54, 67, 81, 91, 94, 98, 103, 111, 112, 116, 118, 119, 120, 122, 124, 125, 129, 130, 131, 132, 133\}$

We give some comments on these 5 rational models. Using [RC] and [Ka] the central charges, conformal characters and dimensions of the two composite rational models can be computed. For the rational models of type $\mathcal{W}(2, d)$ lists of the associated conformal dimension can be found in [EFH^{2}NV].

As it will be seen in the next section the first five rational models in Table 3.2c do have a common feature: The representations of Γ afforded by their conformal characters belong, up to multiplication by certain 1-dimensional Γ-representations, to one and the same series ρ_l (cf. §4.4 and §5.2 for details). So one could ask whether there exist more rational models with this property. A more detailed investigation of the fusion algebras associated to such potentially existing models showed that this is not the case [E2] (cf. also the speculation in [EfH^{2}NV]).

3.3 Some theorems on level N representations of $\mathrm{SL}(2, \mathbb{Z})$.

In this subsection we will consider level N representations of $\mathrm{SL}(2, \mathbb{Z})$. Firstly, we review the fact that all irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_N)$ can be obtained by those of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ where p is a prime and λ is a positive integer. Secondly, we discuss the construction of level p^λ representations using Weil representations (in this part we follow ref. [NW]).

Lemma 7. *Let ρ be a finite dimensional representation of* $\mathrm{SL}(2,\mathbb{Z}_N)$ *where N is a positive integer. Then the representation ρ is completely reducible. Furthermore, each irreducible component ω of ρ has a unique product decomposition*

$$\omega \cong \otimes_{j=1}^{n} \pi(p_j^{\lambda_j})$$

where $N = \prod_{j=1}^{n} p_j^{\lambda_j}$ is the prime factor decomposition of N and the $\pi(p_j^{\lambda_j})$ are irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p_j^{\lambda_j}})$.

Proof. Since $\mathrm{SL}(2,\mathbb{Z}_N)$ is a finite group, ρ is completely reducible.
For a proof of the second statement note that

$$\mathrm{SL}(2,\mathbb{Z}_N) = \mathrm{SL}(2,\mathbb{Z}_{p_1^{\lambda_1}}) \times \cdots \times \mathrm{SL}(2,\mathbb{Z}_{p_n^{\lambda_n}})$$

where $N = \prod_{j=1}^{n} p_j^{\lambda_j}$ (see e.g. [G]). Obviously, the tensor product of irreducible representations $\pi(p_j^{\lambda_j})$ of $\mathrm{SL}(2,\mathbb{Z}_{p_j^{\lambda_j}})$ is an irreducible representation of $\mathrm{SL}(2,\mathbb{Z}_N)$. Using now Burnside's lemma we obtain the second statement.

In order to deal with the representations of the groups $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ we describe their structure by the following theorem.

Theorem (Structure of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ [NW, Satz 1, p. 466]). *The finite group* $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ *is generated by the elements*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and the relations

$$T^{p^\lambda} = 1\!\!1, \qquad S^2 = H(-1)$$
$$H(a)H(a') = H(aa'), \qquad H(a)T = T^{a^2}H(a), \qquad SH(a) = H(a^{-1})S$$

where $H(a) := T^{-a}ST^{-a^{-1}}S^{-1}T^{-a}S^{-1}$ and $a, a' \in \mathbb{Z}_{p^\lambda}^*$.

Remark. As elements of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ the $H(a)$ $(a \in \mathbb{Z}_{p^\lambda}^*)$ are given by

$$H(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

We will now describe the construction of representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ by means of Weil representations.

Definition (Quadratic form). Let M be a finite $\mathbb{Z}_{p^\lambda}$ module. A quadratic form Q of M is a map $Q : M \to p^{-\lambda}\mathbb{Z}/\mathbb{Z}$ such that

(1) $Q(-x) = Q(x)$ for all $x \in M$.
(2) $B(x,y) := Q(x+y) - Q(x) - Q(y)$ defines a $\mathbb{Z}_{p^\lambda}$-bilinear map from $M \times M$ to $p^{-\lambda}\mathbb{Z}/\mathbb{Z}$.

Definition (Quadratic module). A finite $\mathbb{Z}_{p^\lambda}$ module M together with a quadratic form Q is called a quadratic module of $\mathbb{Z}_{p^\lambda}$.

Definition (Weil representation). Let (M,Q) be a quadratic module. Define a right action of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ on the space of $\mathbb{C}$ valued functions on M by

$$\begin{aligned}(f|T)(x) &= e^{2\pi i Q(x)}\ f(x)\\ (f|H(a))(x) &= \alpha_Q(a)\alpha_Q(-1)\ f(x) \qquad \forall a\in\mathbb{Z}_{p^\lambda}^*\\ (f|S^{-1})(x) &= \frac{\alpha_Q(-1)}{|M|^{1/2}}\sum_{y\in M} e^{2\pi i B(x,y)}\ f(y)\end{aligned}$$

where $|M|$ denotes the order of M,

$$\alpha_Q(a) = \frac{1}{|M|}\sum_{x\in M} e^{2\pi i a Q(x)}$$

and f is any $\mathbb{C}$ valued function on M.

If this right action of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ defines a representation of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ it is called the (proper) Weil representation associated to the quadratic module (M,Q) and denoted by $W(M,Q)$.

Note that the above right action always defines a projective representation of Γ. A necessary and sufficient condition for it to define a proper representation is given by the following theorem.

Theorem (Proper Weil representation [NW, Satz 2, p. 467]). *The above right action of* $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ *defines a representation of* $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ *if and only if*

$$\alpha_Q(a)\alpha_Q(a') = \alpha_Q(1)\alpha_Q(aa') \qquad a,a'\in\mathbb{Z}_{p^\lambda}^*.$$

In the following we will only deal with proper Weil representations and, therefore, call them simply Weil representations.

3.4 Weil representations associated to binary quadratic forms.

Although the classification of the irreducible representations of the finite groups $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ is contained in [NW] we will give a short review here. Our main motivation for this is the fact that we will strongly rely on this classification in the proofs of the main theorems 1 and 2 in §3.7 and §3.9. (Furthermore, ref. [NW] is not written in English but in German.)

In the this subsection we describe how to obtain irreducible level p^λ representations as subrepresentations of Weil representations. In subsections §3.5 and §3.6 we give complete lists of the irreducible representations for the cases $p\neq 2$ and $p=2$, respectively.

In addition to the review we investigate in some cases whether the irreducible representations are K-rational or not.

Most of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of Weil representations $W(M,Q)$ associated to a module M of rank one or two. The following two theorems describe the Weil representations needed in the later sections.

Theorem (Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ $(p \neq 2)$ [NW, Lemma 1, Satz 3, p. 474]). *Let $p \neq 2$ be a prime. Then the following quadratic modules of $\mathbb{Z}_{p^\lambda}$ define Weil representations:*

(1)	$M = \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda} r x^2$	$(\lambda \geq 1)$	$(R_\lambda(r))$
(2)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda} x_1 x_2$	$(\lambda \geq 1)$	(D_λ)
(3)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^\lambda}$,	$Q(x) = p^{-\lambda}(x_1^2 - u x_2^2)$	$(\lambda \geq 1)$	(N_λ)
(4)	$M = \mathbb{Z}_{p^\lambda} \oplus \mathbb{Z}_{p^{\lambda-\sigma}}$,	$Q(x) = p^{-\lambda} r(x_1^2 - p^\sigma t x_2^2)$	$(\lambda \geq 2)$	$(R_\lambda^\sigma(r,t))$

where r, t run through $\{1, u\}$ with $(\frac{u}{p}) = -1$ ($(\frac{\cdot}{\cdot})$ denotes the Legendre symbol), where $\sigma = 1, \ldots, \lambda - 1$ and where the last column contains the name of the corresponding Weil representation.

Theorem (Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ [NW, Satz 4, p. 474]). *Let $p = 2$. Then the following quadratic modules of $\mathbb{Z}_{2^\lambda}$ define Weil representations:*

(1)	$M = \mathbb{Z}_{2^\lambda} \oplus \mathbb{Z}_{2^\lambda}$,	$Q(x) = 2^{-\lambda} x_1 x_2$	$(\lambda \geq 1)$	(D_λ)
(2)	$M = \mathbb{Z}_{2^\lambda} \oplus \mathbb{Z}_{2^\lambda}$,	$Q(x) = 2^{-\lambda}(x_1^2 + x_1 x_2 + x_2^2)$	$(\lambda \geq 1)$	(N_λ)
(3)	$M = \mathbb{Z}_{2^{\lambda-1}} \oplus \mathbb{Z}_{2^{\lambda-\sigma-1}}$,	$Q(x) = 2^{-\lambda} r(x_1^2 + 2^\sigma t x_2^2)$	$(\lambda \geq 2)$	$(R_\lambda^\sigma(r,t))$

where $\sigma = 0, \ldots, \lambda - 2$, where (r,t) run through a system of representatives of the classes of pairs defined by $(r_1, t_1) \cong (r_2, t_2)$

if $t_1 \equiv t_2 \bmod \min(8, 2^{\lambda-\sigma})$ and

$$\begin{cases} r_2 \equiv r_1 \bmod 4 \quad \textit{or} \quad r_2 \equiv r_1 t_1 \bmod 4 & \textit{for} \quad \sigma = 0 \\ r_2 \equiv r_1 \bmod 8 \quad \textit{or} \quad r_2 \equiv r_1 + 2 r_1 t_1 \bmod 8 & \textit{for} \quad \sigma = 1 \\ r_2 \equiv r_1 \bmod 4 & \textit{for} \quad \sigma = 2 \\ r_2 \equiv r_1 \bmod 8 & \textit{for} \quad \sigma \geq 3 \end{cases}$$

and where the last column contains the name of the corresponding Weil representation.

All irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of Weil representations $W(M,Q)$. One possibility to extract subrepresentations of such representations is to use characters of the automorphism group of the quadratic form Q:

Theorem (Subrepresentation of a Weil representation [NW, p. 480]). *Let $W(M,Q)$ be a Weil representation described by one of the theorems on Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ above, $\mathcal{U}$ an abelian subgroup of $Aut(M,Q)$ and χ a character of $\mathcal{U}$. Then the subspace*

$$V(\chi) := \{\ f : M \to \mathbb{C} \mid f(\epsilon x) = \chi(\epsilon) f(x), \quad x \in M,\ \epsilon \in \mathcal{U}\ \}$$

of $\mathbb{C}^M$ is invariant under $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$. The corresponding subrepresentation is denoted by $W(M,Q,\chi)$.

REMARKS.

(1) The space $V(\chi)$ is spanned by $V(\chi) =< f_x(\chi) >_{x\in M}$ where
$$f_x(\chi)(y) = \sum_{\epsilon\in\mathcal{U}} \chi(\epsilon)\delta_{\epsilon x,y}, \qquad \delta_{x,y} = \begin{cases} 1 & \text{for x=y} \\ 0 & \text{otherwise.} \end{cases}$$

(2) The automorphism group of the quadratic forms in Theorem 4 contain a conjugation κ: $\kappa(x_1,x_2) = (x_2,x_1)$ in case (1) and $\kappa(x_1,x_2) = (x_1,-x_2)$ in the cases (2) and (3). In these cases the space
$$V(\chi)_\pm := \{\ f \in V(\chi) \mid f(\kappa x) = \pm f(x), \quad x \in M\ \}$$
is invariant under $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$. The corresponding subrepresentation is denoted by $W(M,Q,\chi)_\pm$.

From now on we will denote the trivial character $\chi \equiv 1$ by χ_1. Indeed, almost all irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ can be obtained as subrepresentations of the Weil representations described by the main theorems on Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ using 'primitive' characters:

DEFINITION (PRIMITIVE CHARACTER OF A WEIL REPRESENTATION)[3].
Let $W(M,Q)$ be a Weil representation described by one of the two theorems on Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ above and let $\mathcal{U} = \mathrm{Aut}(M,Q)$. A character χ of $\mathcal{U}$ is called primitive iff there exists an element $\epsilon \in \mathcal{U}$ with $\chi(\epsilon) \neq 1$ such that each element of pM is a fixed point of ϵ. The set of primitive characters of $\mathcal{U}$ will be denoted by $\mathbb{P}$.

THEOREM (ISOMORPHY OF WEIL REPRESENTATIONS [NW, Hauptsatz 1, p. 492]). *Let $W(M,Q)$ and $W(M',Q')$ be Weil representation described by one of the two theorems on Weil representations above and χ,χ' primitive characters. Then one has*

(1) *$W(M,Q,\chi)$ is an irreducible level p^λ representation.*
(2) *$W(M,Q,\chi)$ and $W(M',Q',\chi')$ are isomorphic if and only if the quadratic modules (M,Q) and (M',Q') are isomorphic and $\chi = \chi'$ or $\chi = \bar{\chi}'$.*

The second main theorem of ref. [NW] describes the classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$.

THEOREM (CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ [NW, Hauptsatz 2, p. 493]). *The Weil representations described by the two theorems on Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ above contain all irreducible representations of the groups $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ (in general they are of the form $W(M,Q,\chi)$ for a primitive character χ) apart from 18 exceptional representations for $p = 2$. These exceptional representations can be obtained as tensor products of two representations contained in some $W(M,Q)$ (described by the theorem on Weil representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ above).*

§3.5 and §3.6 contain lists of all irreducible level p^λ representations.

[3] In the case of $M = \mathbb{Z}_{2^{\lambda-1}}\oplus\mathbb{Z}_2$ ($\lambda \geq 5$) the definition of primitive characters is slightly different [NW, p. 491]: Here $\mathcal{U} \cong< -1 >< \alpha >$ with $\alpha = \begin{cases} 1+4t+\sqrt{-8t} & \lambda = 5 \\ 1-2^{\lambda-3}+\sqrt{-2^{\lambda-2}t} & \lambda > 5 \end{cases}$ and χ is primitive if $\chi(\alpha) = -1$.

3.5 The irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ for $p \neq 2$.

In the classification of the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$ for $p \neq 2$ one has to distinguish the cases $\lambda = 1$ and $\lambda > 1$. Therefore, we treat them separately.

Following [NW] we denote the trivial representation by C_1.

Theorem (Classification of irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_p)$ ($p \neq 2$) [NW]). *A complete set of irreducible representations of* $\mathrm{SL}(2, \mathbb{Z}_p)$ *for a prime p with $p \neq 2$ is given by the representations collected in Table 3.5a. In Table 3.5a the χ run through the set of characters of $\mathcal{U}$ and χ_{-1} is the unique nontrivial character of $\mathcal{U}$ taking values in ± 1. Furthermore, we denote by # (here and in the following) the number of inequivalent representations.*

Table 3.5a: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_p)$ for $p \neq 2$

type of rep.		dimension	#
$D_1(\chi)$	$\chi \in \mathbb{P}$	$p+1$	$\frac{1}{2}(p-3)$
$N_1(\chi)$	$\chi \in \mathbb{P}$	$p-1$	$\frac{1}{2}(p-1)$
$R_1(r, \chi_1)$	$\left(\frac{r}{p}\right) = \pm 1$	$\frac{1}{2}(p+1)$	2
$R_1(r, \chi_{-1})$	$\left(\frac{r}{p}\right) = \pm 1$	$\frac{1}{2}(p-1)$	2
$N_1(\chi_1)$		p	1

We will denote the 3 one dimensional level 3 representations C_1, $R_1(1, \chi_{-1})$ and $R_1(2, \chi_{-1})$ by B_1, B_2 and B_3, respectively.

The explicit form of these representations is well known (see e.g. [E2]) and one can address the question which of these representations are K-rational (see also §4). Note that, in view of the results in §2, this question is natural in the context of admissible representations.

Lemma 8. *Let $p \neq 2$ be a prime.*

(1) *For $p \equiv 1 \pmod 3$ there is exactly one and for $p \not\equiv 1 \pmod 3$ there is no K-rational representation of type $D_1(\chi)$.*
(2) *For $p \equiv 2 \pmod 3$ there is exactly one and for $p \not\equiv 2 \pmod 3$ there is no K-rational representation of type $N_1(\chi)$ ($\chi \in \mathbb{P}$).*
(3) *The representations of type $R_1(r, \chi_{\pm 1})$ and $N_1(\chi_1)$ are K-rational.*

Proof. Using a character table for the above representations (see e.g. [Do]) one easily finds that the characters of representations of type $D_1(\chi)$ or $N_1(\chi)$ take values in the field of p-th roots of unity only if $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$ and if χ is a character of order 3. Therefore, there is at most one K-rational representation of type $D_1(\chi)$ or $N_1(\chi)$ for the corresponding values of p. Using the explicit form of these representations (see e.g. [E2]) one finds that these two representations are indeed K-rational. For the other two types of representations the K-rationality follows directly from the fact that $\chi_{\pm 1}$ takes values in ± 1.

THEOREM (CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ ($p\neq 2; \lambda>1$) [NW]). *A complete set of irreducible representations of* $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ *for $p\neq 2$ prime and $\lambda>1$ is given by the representations in Table 3.5b. Where χ_{-1} is the unique nontrivial character with values in ± 1 and $R_\lambda(r,\chi_{\pm 1})_1$ is the unique level p^λ subrepresentation of $R_\lambda(r,\chi_{\pm 1})$ which has dimension $\frac{1}{2}(p^2-1)p^{\lambda-2}$.*

Table 3.5b: Irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ for $p\neq 2$ and $\lambda>1$

type of rep.		dimension	#
$D_\lambda(\chi)$	$\chi\in\mathbb{P}$	$(p+1)p^{\lambda-1}$	$\frac{1}{2}(p-1)^2p^{\lambda-2}$
$N_\lambda(\chi)$	$\chi\in\mathbb{P}$	$(p-1)p^{\lambda-1}$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$
$R_\lambda^\sigma(r,t,\chi)$	$\left(\frac{r}{p}\right)=\pm 1,\ \left(\frac{t}{p}\right)=\pm 1$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$	$4\sum_{\sigma=1}^{\lambda-1}(p-1)p^{\lambda-\sigma-1}$
$R_\lambda(r,\chi_{\pm 1})_1$	$\left(\frac{r}{p}\right)=\pm 1$	$\frac{1}{2}(p^2-1)p^{\lambda-2}$	4

LEMMA 9. *Let $p\neq 2$ be a prime and $\lambda>1$ an integer.*

(1) *The representations of type $R_\lambda^\sigma(r,t,\chi)$ are K-rational for $p\neq 2$ and $\lambda>1$.*
(2) *The representations of type $R_\lambda(r,\chi_{\pm 1})_1$ are K-rational for $p\neq 2$ and $\lambda>1$. Furthermore, the image of T under these representations has nondegenerate eigenvalues only if $p=3$ and $\lambda=2$.*

PROOF. Since the automorphism group of the quadratic form of $R_\lambda^\sigma(r,t,\chi)$ is given by [NW, p. 495] $\mathcal{U}\cong\mathbb{Z}_2\times\mathbb{Z}_{p^{\lambda-\sigma}}$ we obtain (1). In the second case one obviously has $\mathcal{U}\cong\mathbb{Z}_2$ so that the K-rationality follows directly. The statement concerning the eigenvalues of the image of T for the representations of type $R_\lambda(r,\chi_{\pm 1})_1$ is proved in Satz 4 of [NW].

3.6 The irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$**.**

The classification of the irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ is complicated since there are a lot of exceptional representations for $\lambda<6$ [NW]. Since these representations have small dimensions and we will be interested in such representations in §3.7 and §3.9 we describe them in the rest of this subsection. The Tables 3.6a-3.6f list complete sets of inequivalent irreducible representations of the groups $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ for the corresponding values of λ.

For $\lambda=1$ there are only two irreducible representations (see Table 3.6a). The representation C_2 is given by $C_2(S)=C_2(T)=-1$ and both level 2 representations are K-rational.

For $\lambda=2$ there are seven irreducible representations (see Table 3.6b). The representation C_3 is given by $C_3(S)=C_3(T)=-i$, C_4 by $C_4(S)=C_4(T)=i$ and $R_2^0(1,3)_1$ is defined by $R_2^0(1,3)\cong R_2^0(1,3)_1\oplus C_1$. All level 4 representations are K-rational.

For $\lambda=3$ there are 20 irreducible representations (see Table 3.6c). Here $\hat{\chi}$ is one of the two characters of $\mathcal{U}$ of order 4 and the representation $R_3^0(1,3,\chi_1)_1$ is defined by $R_3^0(1,3,\chi_1)\cong R_3^0(1,3,\chi_1)_1\oplus N_1(\chi_1)\oplus C_2\oplus C_2$.

For $\lambda=4$ there are 46 irreducible representations (see Table 3.6d). Here the representation $R_4^2(r,3,\chi_1)_1$ is given by the equality $R_4^2(r,3,\chi_1)\cong R_4^2(r,3,\chi_1)_1\oplus R_2^0(r,t)$.

Table 3.6a: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_2)$

type of rep.		dim	#
$C_2 = N_1(\chi)$	$\chi \in \mathbb{P}$	1	1
$N_1(\chi_1)$		2	1

Table 3.6b: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^2})$

type of rep.		dim	#
$D_2(\chi)_+$	$\chi \not\equiv 1$	3	1
$D_2(\chi)_-$	$\chi \not\equiv 1$	3	1
$R_2^0(1,3)_1$		3	1
$C_2 \otimes R_2^0(1,3)_1$		3	1
$N_2(\chi)$	$\chi \in \mathbb{P};\ \chi \not\equiv 1$	2	1
$C_3 = R_2^0(3,1,\chi)$	$\chi \not\equiv 1$	1	1
$C_4 = R_2^0(1,1,\chi)$	$\chi \not\equiv 1$	1	1

Table 3.6c: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^3})$

type of rep.		dim	#
$D_3(\chi)_\pm$	$\chi \in \mathbb{P}$	6	4
$R_3^0(1,3,\chi_1)_1$		6	1
$C_3 \otimes R_3^0(1,3,\chi_1)_1$		6	1
$N_3(\chi)$	$\chi \in \mathbb{P};\ \chi^2 \not\equiv 1$	4	2
$N_3(\chi)_\pm$	$\chi \in \mathbb{P};\ \chi^2 \equiv 1$	2	4
$R_3^0(r,t,\hat{\chi})$	$r = 1,3;\ t = 1,5$	3	4
$R_3^0(1,t,\chi)_\pm$	$\chi \not\equiv 1;\ t = 3,7$	3	4

Table 3.6d: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^4})$

type of rep.		dim	#
$D_4(\chi)$	$\chi \in \mathbb{P}$	24	2
$N_4(\chi)$	$\chi \in \mathbb{P}$	8	6
$R_4^0(r,t,\chi)$	$\chi \in \mathbb{P};\ \chi \not\equiv 1;\ r = 1,3;\ t = 1,5$	6	4
$R_4^0(r,t,\chi)_\pm$	$\chi \in \mathbb{P};\ \chi^2 \equiv 1;\ r = 1,3;\ t = 1,5$	3	16
$R_4^0(1,t,\chi)_\pm$	$\chi \in \mathbb{P};\ t = 3,7$	6	8
$R_4^2(r,t,\chi)$	$\chi \not\equiv 1;\ r,t \in \{1,3\}$	6	4
$C_2 \otimes R_4^2(r,3,\chi)$	$\chi \not\equiv 1;\ r = 1,3$	6	2
$R_4^2(r,3,\chi_1)_1$	$r = 1,3$	6	2
$N_3(\chi)_+ \otimes R_4^0(1,7,\psi)_+$	$\chi \in \mathbb{P};\ \chi^2 \equiv 1;\ \psi \not\equiv 1;$ $\psi^2 \equiv 1;\ \psi(-1) = 1$	12	2

For $\lambda = 5$ there are 92 irreducible representations (see Table 3.6e). Here for fixed $r = 1, 3$ the 2 irreducible representations of type $R_5^2(\cdot, 1, \chi)_1$ ($\chi \notin \mathbb{P}$) are given by the 2 two dimensional irreducible level 5 subrepresentations of $R_5^2(r, 1)$.

Table 3.6e: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^5})$

type of rep.		dim	#
$D_5(\chi)$	$\chi \in \mathbb{P}$	48	4
$N_5(\chi)$	$\chi \in \mathbb{P}$	16	12
$R_5^0(r, t, \chi)$	$\chi \in \mathbb{P};\ r = 1, 3;\ t = 1, 5$	12	16
$R_5^0(1, t, \chi)_\pm$	$\chi \in \mathbb{P};\ t = 3, 7$	24	4
$R_5^1(r, t, \chi)_\pm$	$\chi \in \mathbb{P};\ r, t \in \{1, 5\}$ or $r = 1, 3$ and $t = 3, 7$	12	16
$R_5^2(r, t, \chi)_\pm$	$\chi \in \mathbb{P};\ r = 1, 3;\ t = 1, 3, 5, 7$	6	32
$R_5^2(r, 1, \chi)_1$	$\chi \notin \mathbb{P};\ r = 1, 3$	12	4
$C_3 \otimes R_5^2(r, 1, \chi)_1$	$\chi \notin \mathbb{P};\ r = 1, 3$	12	4

For $\lambda > 5$ there are the following irreducible representations (see Table 3.6f). Here χ are always primitive characters and $R_\lambda^{\lambda-3}(r, t, \chi_{\pm 1})_1$ is the unique irreducible level 2^λ subrepresentation of $R_\lambda^{\lambda-3}(r, t, \chi_{\pm 1})$ which has dimension $3 \cdot 2^{\lambda-4}$.

Table 3.6f: Irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^\lambda})$ for $\lambda > 5$

type of rep.[4]		dim	#
$D_\lambda(\chi)$		$3 \cdot 2^{\lambda-1}$	$2^{\lambda-3}$
$N_\lambda(\chi)$		$2^{\lambda-1}$	$3 \cdot 2^{\lambda-3}$
$R_\lambda^0(1, 7, \chi)$	$t = 3, 7$	$3 \cdot 2^{\lambda-2}$	$2^{\lambda-3}$
$R_\lambda^\sigma(r, t, \chi)$	$\begin{cases} r = 1, 3;\ t = 1, 5 & \text{for } \sigma = 0 \\ r, t \in \{1, 5\} \text{ or} & \\ r = 1, 3 \text{ and } t = 3, 7 & \text{for } \sigma = 1 \\ r = 1, 3;\ t = 1, 3, 5, 7 & \text{for } \sigma = 2 \end{cases}$	$3 \cdot 2^{\lambda-3}$	$5 \cdot 2^{\lambda-2}$
$R_\lambda^\sigma(r, t, \chi)$	$\sigma = 3, \ldots, \lambda - 3;\ r, t \in \{1, 3, 5, 7\}$	$3 \cdot 2^{\lambda-4}$	$4 \cdot \sum_{\sigma=3}^{\lambda-3} 2^{\lambda-\sigma}$
$R_\lambda^{\lambda-2}(r, t, \chi)$	$r = 1, 3, 5, 7;\ t = 1, 3$	$3 \cdot 2^{\lambda-4}$	16
$R_\lambda^{\lambda-3}(r, t, \chi_{\pm 1})_1$	$r = 1, 3, 5, 7;\ t = 1, 3$	$3 \cdot 2^{\lambda-4}$	16

[4]For $\lambda = 6$ one has to use representation of type $R_6^4(r, t, \chi_1)_1$ and $C_2 \otimes R_6^4(r, t, \chi_1)_1$ ($r = 1, 3$) instead of those of type $R_\lambda^{\lambda-3}(r, t, \chi_{\pm 1})_1$. The representations $R_6^4(r, t, \chi_1)_1$ are the unique level 6 subrepresentations of $R_6^4(r, t, \chi_1)$ with dimension 12.

3.7 Proof of the classification of the strongly-modular fusion algebras of dimension less than or equal to four.

Proof of the main theorem 1 for $\dim(\mathcal{F}) = 2$.

Let $(\mathcal{F}, \rho)$ be a two dimensional strongly-modular fusion algebra. Lemma 4 implies that ρ is irreducible. Therefore, we have to consider all irreducible two dimensional representations of Γ which factor through a congruence subgroup. By Lemma 7 we know that these representations can be obtained by taking the tensor products of all irreducible two dimensional level p^λ representations with all one dimensional representations of Γ.

There are exactly 11 inequivalent irreducible two dimensional level p^λ representations. Their explicit form is given in Appendix 7.1. We are interested in the classification of the two dimensional strongly-modular fusion algebras up to tensor products with one dimensional fusion algebras. Therefore, we can restrict our investigation to one of the two dimensional representations of level 2, 2^3, 3 and the two representations of level 5 (see Appendix 7.1). For the remaining 5 two dimensional representations the eigenvalues of the image of T are nondegenerate. Hence, Lemma 2 implies that the corresponding matrix representations are unique up to conjugation with unitary diagonal matrices and permutation of the basis elements. One can easily apply Verlinde's formula and check whether the resulting coefficients $N_{i,j}^k$ have integer absolute values for the two possible choices of the basis element Φ_0 corresponding to the vacuum (conjugation with a unitary diagonal matrix does not change the absolute value of $N_{i,j}^k$). In particular for the level 2 representation $N_1(\chi_1)$ and the level 3 representation $N_1(\chi)$ we obtain for both possible choices of the distinguished basis elements Φ_0 and Φ_1

$$|N_{1,1}^1| = \begin{cases} \frac{2}{\sqrt{3}}, & \text{for } N_1(\chi_1),\ p = 2 \\ \frac{1}{\sqrt{2}}, & \text{for } \ N_1(\chi),\ p = 3. \end{cases}$$

Since $|N_{1,1}^1|$ is not an integer we can exclude these two representations. For the level 2^3 and 5 representations one obtains integer values for the $N_{i,j}^k$. Moreover, in all three cases both possible choices of the distinguished basis elements Φ_0 and Φ_1 lead to isomorphic fusion algebras. We conclude that the representation of the modular group given by a two dimensional strongly-modular fusion algebra is isomorphic to the tensor product of a one dimensional representation and $N_3(\chi)_+$ ($p^\lambda = 2^3$) or $R_1(r, \chi_{-1})$ ($r = 1, 2; p^\lambda = 5$). Using that $\rho(S^2)$ should be a matrix consisting of nonnegative integers one can determine the one dimensional representation of Γ up to an even one dimensional representation. Therefore, $(\mathcal{F}, \rho)$ is determined up to tensor products with one dimensional modular fusion algebras. The resulting representations and fusion algebras are collected in Table 7.2a. $\square$

Proof of the main theorem 1 for $\dim(\mathcal{F}) = 3$.

Let $(\mathcal{F}, \rho)$ be a three dimensional strongly-modular fusion algebra. By Lemma 7, ρ is either irreducible or isomorphic to a sum of a two dimensional and a one dimensional irreducible representation. We will now consider these two cases separately.

Firstly, assume that ρ is irreducible. By Lemma 7, ρ is isomorphic to the tensor product of a one dimensional representation and one of the three dimensional irreducible level p^λ representations. There are exactly 33 inequivalent irreducible 3 dimensional level p^λ representations. Their explicit form is given in Appendix 7.1. We are interested in the classification up to tensor products with one dimensional modular fusion algebras. Therefore, we can restrict our investigation to a set of irreducible representations which are not related via tensor products with one dimensional representations. This means that we have to consider one representation of level 3 and 2^2, two representations of level 5 and 7 and, finally, four representations of level 2^4 (see Appendix 7.1).

For these representations the eigenvalues of the image of T are nondegenerate so that we can proceed now as in the proof of the main theorem 1.

Using Verlinde's formula for the representation $N_1(1,\chi_1)$ ($p=3$) we obtain $|N_{1,1}^1| = \frac{1}{2}$ for all possible choices of the distinguished basis. In the same way one finds for $R_1(r,\chi_1)$ ($r=1,2; p=5$) that

$$\begin{cases} |N_{1,1}^2| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(1, e^{2\pi i \frac{r}{5}}, e^{2\pi i \frac{4r}{5}}) \\ & \quad\text{or } \rho(T) = \mathrm{diag}(1, e^{2\pi i \frac{4r}{5}}, e^{2\pi i \frac{r}{5}}) \\ |N_{1,1}^1| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(e^{2\pi i \frac{r}{5}}, 1, e^{2\pi i \frac{4r}{5}}) \\ |N_{1,1}^1| = \frac{1}{\sqrt{2}} & \text{for } \rho(T) = \mathrm{diag}(e^{2\pi i \frac{4r}{5}}, 1, e^{2\pi i \frac{r}{5}}). \end{cases}$$

Here the different cases correspond to the different possible choices of the distinguished basis. We conclude that ρ cannot be isomorphic to a tensor product of a one dimensional representation and $N_1(1,\chi_1)$ ($p=3$) or $R_1(r,\chi_1)$ ($r=1,2;p=5$).

An analogous calculation shows that for the representations of type $R_1(r,\chi_{-1})$ one has $|N_{i,j}^k| \in \mathbb{N}$ for all 3 possible choices of the distinguished basis. For the remaining representations one also has $|N_{i,j}^k| \in \mathbb{N}$ for the two possible choices of the distinguished basis (here the matrix $\rho(S)$ contains a zero so that there are only two possible choices of the distinguished basis).

Hence, ρ is isomorphic to a tensor product of a one dimensional representation with one of these 7 representations. Using that for a modular fusion algebra $\rho(S^2)_{i,j}$ equals $N_{i,j}^0$ one can determine the possible one dimensional representations. The corresponding strongly-modular fusion algebras are contained in Table 10 in the second and third row.

Secondly, assume that ρ decomposes into a direct sum of two irreducible representations $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_j) = j$. Then ρ_2 is isomorphic to the tensor product of a one dimensional representation with one of the two dimensional irreducible level p^λ representations contained in Table 7.1a.

Using Lemma 1 we conclude that $\rho(T)$ has degenerate eigenvalues so that $\rho_2(T)$ must have an eigenvalue of the form $e^{2\pi i \frac{n}{12}}$. Hence, ρ_2 cannot be isomorphic to the tensor product of a one dimensional representation and one of the two dimensional irreducible level 5 and 2^3 representations in Table 7.1a. Using once more that $\rho(T)$ has degenerate eigenvalues we obtain that ρ is isomorphic to the tensor product of a one dimensional representation with either $N_1(\chi_1) \oplus C_j$ ($j=1,2;p=2$) or $N_1(\chi) \oplus B_j$ ($j=2,3;p=3$). In order find out whether these four representations are admissible we have to look for distinguished bases.

Let us first consider the case $\rho \cong C \otimes (N_1(\chi) \oplus B_j)$ $(j = 2, 3; p = 3)$ where C is a one dimensional representation. Here $\rho(S^2)$ has two different eigenvalues since $N_1(\chi)$ is odd and the representations B_j are even. Since the vacuum is selfconjugate, i.e. $\rho(S^2)_{00} = 1$ the representation C has to be odd. Without loss of generality we choose $C = C_4$ for $j = 2$ and $C = C_3$ for $j = 3$. Furthermore, the fact that $\rho(S^2)$ has two different eigenvalues implies that we must have

$$\rho(S^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using these two conditions it follows that in a basis in which $\rho(S^2)$ has this form and $\rho(T)$ is diagonal we must have

$$\rho(S) = \frac{1}{\sqrt{3}} \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & e^{2\pi i \frac{1}{3}} & e^{2\pi i \frac{2}{3}} \\ \epsilon & e^{2\pi i \frac{2}{3}} & e^{2\pi i \frac{1}{3}} \end{pmatrix}, \qquad \epsilon^2 = 1$$

and

$$\rho(T) = \begin{cases} \operatorname{diag}(e^{2\pi i \frac{5}{12}}, e^{2\pi i \frac{1}{12}}, e^{2\pi i \frac{1}{12}}) & \text{or} \\ \operatorname{diag}(e^{2\pi i \frac{7}{12}}, e^{2\pi i \frac{11}{12}}, e^{2\pi i \frac{11}{12}}) \end{cases}$$

up to conjugation with a unitary diagonal matrix (the two possibilities for $\rho(T)$ correspond to the two possible choices of the distinguished basis).

Applying now Verlinde's formula leads to a modular fusion algebra iff $\epsilon = 1$ for both choices of the distinguished basis. The corresponding fusion algebra, $\rho(S)$ and $\rho(T)$ are listed in the third row of Table 7.2a.

Finally, consider the case $\rho \cong C \otimes (N_1(\chi_1) \oplus C_j)$ $(j = 1, 2)$. Since $N_1(\chi_1)$ $(p = 2)$ and C_j $(j = 1, 2)$ are even ρ has to be even, too. Therefore, C is even and w.l.o.g. we choose $C = C_1$ for $j = 1$ and $C = C_2$ for $j = 2$. Since ρ is even one must have $\rho(S^2) = \mathbb{1}$ and, therefore, $\rho(S)$ is real (c.f. the second remark in §2.2). Plugging this in we find (up to permutation of the basis elements) that

$$\rho(S) = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3}a & \sqrt{3}b \\ -\sqrt{3}a & 2 - 3a^2 & 3ab \\ \sqrt{3}b & 3ab & 3a^2 - 1 \end{pmatrix}, \qquad \rho(T) = (-1)^j \operatorname{diag}(1, -1, -1)$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$. Using Verlinde's formula we obtain as conditions for ρ to be admissible

$$\begin{cases} \frac{(1-3a^2)(3a^2-2)}{\sqrt{3}a} \in \mathbb{N} & \text{for } \rho(T) = (-1)^j \operatorname{diag}(1, -1, -1) \\ \frac{1}{\sqrt{3}a(3a^2-2)}, \frac{3a^2-1}{\sqrt{3}a(3a^2-2)} \in \mathbb{N} & \text{for } \rho(T) = (-1)^j \operatorname{diag}(-1, -1, 1). \end{cases}$$

The first case implies that $a^2 = \frac{1}{3}$ or $a^2 = \frac{2}{3}$ and the second one $a^2 = \frac{1}{3}$, respectively. Inserting these values of a in the explicit form of $\rho(S)$ above we indeed obtain modular fusion algebras if we choose the signs of a and b correctly. The resulting modular fusion algebras are contained in the fifth row of Table 7.1a. As fusion algebras they are of type "$(3,4)$", also called Ising fusion algebra.

This completes the proof in the three dimensional case. □

PROOF OF THE MAIN THEOREM 1 FOR $\dim(\mathcal{F}) = 4$.

Let $(\mathcal{F}, \rho)$ be a strongly-modular fusion algebra. Then, by Lemma 4, we have the following possibilities for ρ:

(1) ρ is irreducible,
(2) $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = 3$, $\dim(\rho_2) = 1$,
(3) $\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = \dim(\rho_2) = 2$,
(4) $\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$ with $\dim(\rho_1) = 2$, $\dim(\rho_2) = \dim(\rho_3) = 1$

where ρ_i $(i = 1, 2, 3)$ are irreducible representations.

(1) ρ is irreducible

Assume that ρ is irreducible. Then ρ is either isomorphic to the tensor product of 2 two dimensional representations of coprime levels or it is isomorphic to the tensor product of a one dimensional representation with a four dimensional irreducible level p^λ representation. In the first case we obviously have that ρ is only admissible iff both two dimensional representations are admissible (look at Table 7.1a). In this case the corresponding modular fusion algebra is a tensor product of two modular fusion algebras contained in Table 7.2a. Let us now consider the other case, namely that $\rho \cong C \otimes \rho_1$ where C is a one dimensional representation and ρ_1 is a four dimensional irreducible level p^λ representation. In this case ρ_1 is given by one of the 9 representations in Table 7.1c. Note that for all of these representations the eigenvalues of the image of T are nondegenerate so that we can use the argumentation used in the proof of the main theorem 1 for $\dim(\mathcal{F}) = 2$.

For the representation $N_1(\chi)$ $(\chi^3 \not\equiv 1; p = 5)$ we find by Verlinde's formula

$$|N_{1,1}^1| = \sqrt{3}, \qquad \text{for} \quad \rho(T) = \operatorname{diag}(e^{2\pi i \frac{n}{5}}, e^{2\pi i \frac{3n}{5}}, e^{2\pi i \frac{2n}{5}}, e^{2\pi i \frac{4n}{5}}) \qquad (n = 1, \ldots, 4)$$

where again the different possibilities for $\rho(T)$ correspond to the different possible distinguished basis. This shows that ρ_1 cannot be isomorphic to this representation.

Since the representation $N_1(\chi)$ $(\chi^3 \equiv 1; p = 5)$ is isomorphic to the tensor product of the two different level 5 representations in Table 7.1a it is clear that this representation is admissible. Since the image of T under this representation has nondegenerate eigenvalues the corresponding modular fusion algebras are isomorphic to the tensor product of 2 two dimensional modular fusion algebras (as fusion algebras they are of type "$(2,5)$").

Consider now the representations $R_1(r, \chi_1)$ $(r = 1, 2; p = 7)$. Here Verlinde's formula implies that

$$|N_{1,1}^1| = \frac{1}{\sqrt{2}} \quad \text{for} \quad \rho(T) = \operatorname{diag}(e^{2\pi i \frac{n}{7}}, 1, \cdot, \cdot) \quad (n = 1, \ldots 6)$$

and

$$|N_{1,1}^2| = \frac{1}{\sqrt{2}} \quad \text{for} \quad \rho(T) = \begin{cases} \operatorname{diag}(1, e^{2\pi i \frac{2}{7}}, e^{2\pi i \frac{4}{7}}, e^{2\pi i \frac{1}{7}}) & \text{or} \\ \operatorname{diag}(1, e^{2\pi i \frac{5}{7}}, e^{2\pi i \frac{3}{7}}, e^{2\pi i \frac{6}{7}}). \end{cases}$$

As above this removes these representations from the list of candidates leading to modular fusion algebras.

For the representation $N_3(\chi)$ ($\chi^3 \not\equiv 1; p = 2^3$) one has

$$|N_{1,1}^1| = \sqrt{\frac{4}{3}} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{2n+1}{8}}, e^{2\pi i \frac{2n+5}{8}}, \cdot, \cdot) \quad (n = 1, \ldots, 4)$$

so that this representation is also excluded.

Consider now the representations $R_2^1(r, 1, \chi)$ ($r = 1, 2; \chi^3 \not\equiv 1; p = 3^2$). Here one has

$$|N_{1,1}^1| = \frac{1}{\sqrt{3}} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{rn^2}{9}}, e^{2\pi i \frac{r}{3}}, \cdot, \cdot) \quad (n = 1, 2, 3).$$

The basis element in the representation space corresponding to the $\rho(T)$ eigenvalue of order three cannot correspond to Φ_0 since in the corresponding row of $\rho(S)$ contains a zero.

Finally, the only remaining four dimensional irreducible level p^λ representations that might lead to modular fusion algebras are those of type $R_2^1(r, 1, \chi)$ ($r = 1, 2; \chi^3 \equiv 1; p^\lambda = 3^2$). Indeed, these representations lead to modular fusion algebras. To be more precise one has to consider the tensor product of an odd one dimensional representation with them because the $R_2^1(r, 1, \chi)$ ($\chi^3 \equiv 1$) are odd themselves. The corresponding fusion algebras are of type "$(2, 9)$" and the explicit form is given in Table 7.2b. The different modular fusion algebras result from the two different representations and the fact that the distinguished basis can be chosen in different ways.

$\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = 3$, $\dim(\rho_2) = 1$

Assume that ρ is isomorphic to the direct sum of a one dimensional and an irreducible three dimensional representation. Then one has $\rho \cong C \otimes (\rho_1 \oplus D)$ where C and D are one dimensional representations and ρ_1 is one of the three dimensional irreducible level p^λ representations in Table 7.1b. By Lemma 1 we know that $\rho(T)$ has degenerate eigenvalues. Therefore, ρ_1 is of type $N_1(\chi_1)$ ($p = 3$), $R_1(r, \chi_1)$ ($r = 1, 2; p = 5$), $D_2(\chi)_+$ ($p^\lambda = 2^2$) or $R_3^0(1, 3)_\pm$ ($p^\lambda = 2^3$).

Consider first the representation $N_1(\chi_1)$ ($p = 3$). In this case we can have $D = B_j$ ($j = 1, 2, 3$). Since B_j and $N_1(\chi_1)$ are even we can choose without loss of generality $C = C_1$. Using Verlinde's formula we find that

$$|N_{1,1}^1| = \frac{1}{2} \quad \text{for} \quad \rho(T) = \mathrm{diag}(e^{2\pi i \frac{j+1}{3}}, e^{2\pi i \frac{j+2}{3}}, e^{2\pi i \frac{j}{3}}, e^{2\pi i \frac{j}{3}})$$

giving a contradiction for these choices of the distinguished basis. For $\rho(T) = \mathrm{diag}(e^{2\pi i \frac{j}{3}}, e^{2\pi i \frac{j}{3}}, e^{2\pi i \frac{j+1}{3}}, e^{2\pi i \frac{j+2}{3}})$ the line of reasoning is a little bit more involved. Here $N_{i,j}^0 = \rho(S^2)_{i,j} = \delta_{i,j}$ implies that $\rho(S)$ is given by

$$\rho(S) = \frac{1}{3} \begin{pmatrix} 4b^2 - 1 & 4ab & 2a & 2a \\ 4ab & 3 - 4b^2 & -2b & -2b \\ 2a & -2b & -1 & 2 \\ 2a & -2b & 2 & -1 \end{pmatrix}$$

up to conjugation with an orthogonal diagonal matrix, with $a, b \in \mathbb{R}$ and $a^2+b^2 = 1$. With the explicit form of $\rho(S)$ we find as conditions for ρ to be admissible

$$N_{1,1}^1 = \frac{1}{2a(3-4a^2)} \in \mathbb{Z}, \qquad N_{1,1}^2 = \frac{2a^2-1}{2a(3-4a^2)} \in \mathbb{Z}.$$

However, the only solutions that satisfy these two conditions are those a which equal $\frac{1}{2m}$ for an integer m and satisfy $m^3 \equiv 0 \bmod 3m^2-1$. It follows that $m \equiv 0 \bmod 3m^2-1$ which gives a contradiction. Therefore, the representations $N_1(\chi_1) \oplus B_j$ $(p = 3)$ do not lead to modular fusion algebras.

Next we consider the representations $R_1(r, \chi_1)$ $(r = 1, 2; p = 5)$. In this case the one dimensional representation D has to be the trivial one. Since these two representations are even we can choose without loss of generality $C = C_1$, too. Using that $N_{i,j}^0 = \delta_{i,j}$ we find that the matrix which describes the basis in the two dimensional eigenspace corresponding to the eigenvalue 1 of $\rho(T)$ is orthogonal. Furthermore, by looking at suitable $N_{i,j}^k$ we find that there are only two possibilities for this matrix. In the corresponding basis we indeed find modular fusion algebra given by the tensor product of two modular fusion algebras of type "(2, 5)". That ρ is admissible can also be interfered from the equality $R_1(r, \chi_1) \oplus C \cong R_1(r, \chi_{-1}) \otimes R_1(r, \chi_{-1})$ $(r = 1, 2; p = 5)$.

Finally, we have to consider $D_2(\chi)_+$ $(p^\lambda = 2^2)$ and $R_3^0(1, 3, \chi)_\pm$ $(p^\lambda = 2^3)$. The corresponding possibilities for ρ are $C_3 \otimes D_2(\chi)_+ \oplus C_j$ $(j = 1, 3, 4)$, $C_4 \otimes R_3^0(1, 3, \chi)_+ \oplus C_3$ or $C_3 \otimes R_3^0(1, 3, \chi)_- \oplus C_4$. For the case $\rho \cong C_3 \otimes D_2(\chi)_+ \oplus C_1$ we obtain a modular fusion algebra given by the tensor product of two $\mathbb{Z}_2$ fusion algebras. This can also be seen by looking at the identity

$$C_3 \otimes D_2(\chi)_+ \oplus C_1 \cong D_2(\chi)_+ \otimes D_2(\chi)_+.$$

For $C_4 \otimes R_3^0(1, 3, \chi)_+ \oplus C_3$ or $C_3 \otimes R_3^0(1, 3, \chi)_- \oplus C_4$ we obtain $\mathbb{Z}_4$ type fusion algebras (see Table 7.2b). The other two representations $(C_3 \otimes D_2(\chi)_+ \oplus C_j$ $(j = 3, 4))$ are not admissible as one can easily check by applying Verlinde's formula.

$\rho \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = \dim(\rho_2) = 2$

Assume that ρ decomposes into a direct sum of 2 two dimensional irreducible representations. In this case we have $\rho = C \otimes (\rho_1 \oplus D \otimes \rho_2)$ where C and D are one dimensional representations and ρ_1, ρ_2 are some level p^λ representations contained in Table 7.1a. Since ρ is reducible we know that $\rho(T)$ has degenerate eigenvalues. This together with the parity of the representations in Table 7.1a implies that ρ equals (up to a tensor product with an even one dimensional representation) one of the following representations:

$$\begin{aligned}
&N_1(\chi_1) \oplus N_1(\chi_1) \\
&C_3 \otimes (N_1(\chi) \oplus B_i \otimes N_1(\chi)) \quad (i = 1, 2) \\
&C_4 \otimes (R_1(r, \chi_{-1}) \oplus R_1(r, \chi_{-1})) \quad (r = 1, 2) \\
&C_4 \otimes (N_3(\chi)_+ \oplus N_3(\chi)_+).
\end{aligned}$$

In all cases we have that $\rho(S)$ is conjugate to a matrix of block diagonal form. More precisely, this matrix consists of two identical two by two matrices. A simple calculation shows now that conjugation of $\rho(S)$ with a matrix which leaves $\rho(T)$ diagonal leads to a matrix which has at least one zero element in every row. This is a contradiction since we have assumed that ρ is admissible and one can apply Verlinde's formula.

$\rho \cong \rho_1 \oplus \rho_2 \oplus \rho_3$ with $\dim(\rho_1) = 2$, $\dim(\rho_2) = \dim(\rho_3) = 1$

Assume that ρ decomposes into a direct sum of an irreducible two dimensional and 2 one dimensional representations. Then, again by Lemma 1, $\rho(T)$ has degenerate eigenvalues and a simple parity argument shows that the only possibilities for ρ are (up to a tensor product with an even one dimensional representation):

$$N_1(\chi_1) \oplus C_1 \oplus C_1 \qquad \text{or} \qquad N_1(\chi_1) \oplus C_1 \oplus C_2$$

where $N_1(\chi_1)$ is the level 2 representation in Table A1. We have to consider these two cases separately.

Firstly, let ρ be conjugate to $N_1(\chi_1) \oplus C_1 \oplus C_1$. Then the requirements that $\rho(S)$ has to be symmetric and real and that $\rho(T)$ has to be diagonal imply that (up to permutation of the basis elements and conjugation with an orthogonal diagonal matrix):

$$\rho(S) = -\frac{1}{2}\begin{pmatrix} -1 & \sqrt{3}a & \sqrt{3}b & \sqrt{3}c \\ \sqrt{3}a & 3a^2-2 & 3ab & 3ac \\ \sqrt{3}b & 3ab & 3b^2-2 & 3bc \\ \sqrt{3}c & 3ac & 3bc & 3c^2-2 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 = 1$ and $\rho(T) = \mathrm{diag}(-1, 1, 1, 1)$.

Fixing the distinguished basis such that Φ_0 corresponds to the eigenvector of $\rho(T)$ with eigenvalue -1 we obtain

$$N_{11}^1 = \frac{(2-3a^2)(1-3a^2)}{\sqrt{3}a}, \quad N_{22}^2 = \frac{(2-3b^2)(1-3b^2)}{\sqrt{3}b}, \quad N_{33}^3 = \frac{(2-3c^2)(1-3c^2)}{\sqrt{3}c}$$
$$N_{11}^2 = \sqrt{3}(3a^2-1)b, \qquad N_{11}^3 = \sqrt{3}(3a^2-1)c$$
$$N_{22}^1 = \sqrt{3}(3b^2-1)a, \qquad N_{22}^3 = \sqrt{3}(3b^2-1)c.$$

This implies that $a^2 = b^2 = c^2 = \frac{1}{3}$. The resulting structure constants indeed define a fusion algebra, namely the tensor product of two fusion algebras of type $\mathbb{Z}_2$. As a modular fusion algebra this fusion algebra is **simple**, i.e. it is not a tensor product of two nontrivial modular fusion algebras. The resulting modular fusion algebra is contained in Table 7.2b.

For the other choice of the distinguished basis where Φ_0 corresponds to an eigenvector $\rho(T)$ with eigenvalue 1 we find

$$N_{33}^1 = \frac{(3a^2-1)b}{a(3a^2-2)}, \qquad N_{33}^2 = \frac{(3a^2-1)c}{a(3a^2-2)},$$
$$N_{33}^3 = \frac{3a^2-1}{\sqrt{3}a(3a^2-2)}, \qquad N_{22}^3 = \frac{1-3b^2}{\sqrt{3}a(3a^2-2)}$$

where the basis was chosen such that $\rho(T) = \operatorname{diag}(1,1,1,-1)$. Let now $n := (N_{33}^1)^2 + (N_{33}^2)^2$ and $m := (N_{33}^3)^2$. It is now easy to verify that n and m satisfy the equation

$$m^3 + (1-5n)m^2 + (4n^2+7n)m + 4n^2 - 3n^3 = 0.$$

By Lemma 10 in §3.8 below the only nonnegative integer solution of this equation with m being a square is given by $n = m = 0$. Therefore, we find as the only possible solution $a^2 = b^2 = c^2 = \frac{1}{3}$. The resulting structure constants define a fusion algebra isomorphic to the tensor product of two $\mathbb{Z}_2$ fusion algebras. However, analogous to the case of the other distinguished basis discussed above this modular fusion algebra is **simple** and contained in Table 7.2b.

Secondly, assume that ρ is conjugate to $N_1(\chi_1) \oplus C_1 \oplus C_2$. Requiring that $\rho(S)$ is a symmetric real matrix and that $\rho(T)$ is diagonal implies (up to a permutation of the basis elements and conjugation with an orthogonal diagonal matrix)

$$\rho(S) = \frac{1}{2}\begin{pmatrix} 3b^2-1 & -3ab & -\sqrt{3}ac & \sqrt{3}ad \\ -3ab & 3a^2-1 & -\sqrt{3}bc & \sqrt{3}bd \\ -\sqrt{3}ac & -\sqrt{3}bc & 3c^2-2 & -3cd \\ \sqrt{3}ad & \sqrt{3}bd & -3cd & 3d^2-2 \end{pmatrix}$$

where $a,b,c,d \in \mathbb{R}$ and $a^2+b^2 = 1$, $c^2+d^2 = 1$ and $\rho(T) = \operatorname{diag}(1,1,-1,-1)$. Using Verlinde's formula we obtain for the choice of the distinguished basis in which Φ_0 corresponds to the eigenvector of $\rho(T)$ with eigenvalue 1

$$(N_{11}^1)^2 = \frac{(3a-1)^2(6a-5)^2}{9a^2(1-a^2)(3a-2)^2}, \quad (N_{11}^2)^2 = \frac{c^2}{3a^2(3a^2-2)}, \quad (N_{11}^3)^2 = \frac{d^2}{3a^2(3a^2-2)}.$$

For the other choice of the distinguished basis (Φ_0 corresponding to eigenvalue -1) one finds the same expressions with a and c exchanged.

Let $n := (N_{11}^2)^2 + (N_{11}^3)^2$ and let $m := (N_{11}^1)^2$. It is easy to verify that the following equation for n and m holds true

$$\begin{aligned}(1-3n)m^3 &+ (12-37n+31n^2)m^2 + (48-152n+155n^2-53n^3)m \\ &+ 64 - 208n + 249n^2 - 130n^3 + 25n^4 = 0.\end{aligned}$$

By Lemma 10 in §3.8 below the only nonnegative integer solution of this equation with m being a square is given by $m = 0, n = 1$. This is a contradiction to the explicit form of n and m in terms of a above. Hence the representation $N_1(\chi_1) \oplus C_1 \oplus C_2$ is not admissible.

This proves the main theorem 1. $\square$

3.8 Proof of a Lemma on diophantic equations.

LEMMA 10[5]. *Let n be a nonnegative integer, m a square of an integer and n, m solutions of*

(1) $m^3 + (1-5n)m^2 + (4n+7n^2)m + 4n^2 - 3n^3 = 0$ *or*
(2) $(1-3n)m^3 + (12-37n+31n^2)m^2 + (48-152n+155n^2-53n^3)m + 64 - 208n + 249n^2 - 130n^3 + 25n^4 = 0$

Then either $n = m = 0$ for (1) or $m = 0, n = 1$ for (2).

PROOF. Firstly, consider the equation (1). It can be written in the form

$$(3n-m)(m-n)^2 = (m+2n)^2.$$

If $n = m$ then $m = n = 0$. Otherwise, set $t = \frac{m+2n}{m-n}$ implying

$$m = \frac{(t+2)t^2}{2t-5}, \qquad n = \frac{(t-1)t^2}{2t-5}.$$

If m and n are integral then also t has to be integral (any prime factor of the denominator of t would divide the denominator of m and n). Then $N = 2t-5$ divides $(t-1)t^2 = \frac{1}{8}(N+5)^2(N+3)$ so that N divides $3 \cdot 5^2$. None of the resulting 12 possibilities leads to a nonnegative integer solution of n, m where $m \neq n$ and m is a square.

Secondly, consider the equation (2). Set $k = m - n + 4$, then (2) is equivalent to

$$k^3 + 2k^2n - 3k^3n + 125n^2 - 92kn^2 + 22k^2n^2 - 11n^3 = 0.$$

If $k = 0$ then $n = 0$ and $m = -4$ is not a square. Otherwise, (2) is equivalent to

$$(-3t+22t^2)k^2 + (1+2t-92t^2-11t^3)k + 125t^2 = 0, \quad k \neq 0$$

where $t = \frac{n}{k}$. This equation has discriminant $(1+18t+t^2)(1-7t+11t^2)^2$ and this must be a square. Setting $\frac{p}{q} := (1-t-(1+18t+t^2)^{1/2})/(10t) \in \mathbb{Q}$ (with coprime p, q and $q > 0$) we get

$$t = \frac{q(p+q)}{p(5p+q)}.$$

Hence, using the quadratic equation in k we finally have

$$m = \frac{(2p+q)^2(p-q)^2}{p^2(2q-p)^2(p+q)}, \qquad n = \frac{q^3}{p^2(2q-p)}.$$

The parameterization of n implies that $p = \pm 1$ and, furthermore, that $q^3 \equiv 0 \bmod (2q-p)$. Therefore, we have $p^3 \equiv 0 \bmod (2q-p)$ so that $2q - p = \pm 1$. From the resulting four possibilities only $p = q = 1$ satisfies the desired properties and leads to $m = 0, n = 1$. □

REMARK. Note that the proof of Lemma 10 relies essentially on the fact that the curves defined by the two above equations are rational.

[5]I would like to thank D. Zagier for discussion on this lemma [Za]

3.9 Proof of the classification of the nondegenerate strongly-modular fusion algebras of dimension less than 24.

Proof of the main theorem 2.

Let $(\mathcal{F},\rho)$ be a simple nondegenerate strongly-modular fusion algebra of dimension less than 24. Lemma 1 implies that ρ is irreducible. Furthermore, since $(\mathcal{F},\rho)$ is strongly-modular we have to consider all irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_N)$ of dimension less than 24. Since $(\mathcal{F},\rho)$ is simple and nondegenerate simple Lemma 7 shows that we can restrict our investigation to irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$. Once again, since $(\mathcal{F},\rho)$ is nondegenerate we can follow the line of reasoning in the proof of the main theorem 1 for the two dimensional case.

Therefore, we can directly apply Verlinde's formula to any such matrix representation $\hat{\rho}$ and look whether the resulting coefficients $N_{i,j}^k$ have integer absolute values for the different choices of the basis element corresponding to Φ_0. If the resulting numbers $N_{i,j}^k$ do not have integer absolute values we can conclude that there exists no nondegenerate strongly-modular fusion algebra $(\mathcal{F},\rho)$ where ρ is conjugate to the tensor product of a one dimensional representation of Γ and $\hat{\rho}$. We have investigated this for all irreducible representations of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ of dimension less than 24 by constructing them explicitly[6].

The proof of the theorem will consist of three separate cases: We consider representations of $\mathrm{SL}(2,\mathbb{Z}_p)$ and $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ and $\mathrm{SL}(2,\mathbb{Z}_{2^\lambda})$ separately.

Firstly, let ρ be isomorphic to a tensor product of a one dimensional representation and an irreducible representation $\hat{\rho}$ of $\mathrm{SL}(2,\mathbb{Z}_p)$ $(p \neq 2)$. Note that this case was already discussed in [E2].

For the representations of type $D_1(\chi)$ the matrix $\rho(T)$ has degenerate eigenvalues so that we can leave out this type of representation.

For the representations of type $N_1(\chi)$ we find modular fusion algebras only for $p = 5, 11, 17$ and 23 and $\chi^3 \equiv 1$. For $p = 5$ the modular fusion algebra is not simple but equals the tensor product of two modular fusion algebras where the corresponding fusion algebras are of type "$(2,5)$" (cf. also the proof of the main theorem 3). The modular fusion algebras corresponding to $p = 11, 17, 23$ are contained in the last three rows of Table 7.3. As was already mentioned in [E2] these four representations are probably the only admissible ones of type $N_1(\chi)$. However, we do not have a proof of this statement but numerical checks show that there is no other admissible representation of this type for $p < 167$ [E2].

The representations of type $R_1(r,\chi_1)$ and $N_1(\chi_1)$ do not lead to modular fusion algebras [E2].

For all $\hat{\rho}$ of type $R_1(r,\chi_{-1})$ we obtain modular fusion algebras. Here $\rho \cong (C_4)^{\frac{p+1}{2}} \otimes R_1(r,\chi_{-1})$ is admissible for all odd primes p. The corresponding modular fusion algebras are of type "$(2,p)$". They are contained in the third row of Table 7.3.

Secondly, let ρ be isomorphic to a tensor product of a one dimensional representation and a irreducible representation $\hat{\rho}$ of $\mathrm{SL}(2,\mathbb{Z}_{p^\lambda})$ $(p \neq 2,\ \lambda > 1)$.

[6] Here we have used the computer algebra system PARI-GP [GP].

For the representations of type $D_\lambda(\chi)$ the matrix $\rho(T)$ has degenerate eigenvalues excluding these representations from our investigation.

The only representations of type $N_\lambda(\chi)$ which have dimension less than 24 are those corresponding to $(p = 3; \lambda = 2, 3)$ and $(p = 5; \lambda = 2)$. A calculation shows that exactly one of these representations leads to a modular fusion algebra. This is the representation with $(p = 3; \lambda = 2)$ and $\chi^3 \equiv 1$. The corresponding strongly-modular fusion algebra is contained in Table 7.3.

Only those representations of type $R^\sigma_\lambda(r, t, \chi)$ and $R_\lambda(r, \chi_{\pm 1})_1$ with $(p = 3; \lambda = 2, 3)$ or $(p = 5; \lambda = 2)$ have dimension less than 24. The representations $R^1_2(r, 1, \chi)$ $(p^\lambda = 3^2; \chi^3 \equiv 1)$ lead to nondegenerate modular fusion algebras (cf. the proof of the main theorem 3). From the other representations only those with $p^\lambda = 3^3; r = 1, 2; \chi^3 \equiv 1$ lead to modular fusion algebras (see Table 7.3).

Thirdly, consider the irreducible representations of $\mathrm{SL}(2, \mathbb{Z}_{2^\lambda})$. All irreducible representations of dimension less than or equal to 4 have been considered in the main theorems 1 to 3. The corresponding admissible representations with nondegenerate eigenvalues of $\rho(T)$ are contained in Table 7.3.

For $\lambda = 1, 2$ all irreducible representations have dimension less than or equal to 3.

For $\lambda = 3$ we have to consider the representations of type $R^0_3(1, 3, \chi_1)_1$ and $D_3(\chi)_\pm$. The former representation does not lead to a modular fusion algebra but the representations $D_3(\chi)_\pm$ lead to modular fusion algebras of type $\mathbb{Z}_2 \otimes$ "(3, 4)". The corresponding modular fusion algebras are composite and therefore not contained in Table 7.3.

For $\lambda = 4$ only the irreducible representations of type $R^0_4(r, t, \chi)_\pm$, $R^2_4(r, 3, \chi_1)_1$ and $R^2_4(r, t, \chi)$ lead to modular fusion algebras. The first one leads to a fusion algebra of type "(3, 4)" (see main theorem 2). The other two representations lead to composite modular fusion algebras. These fusion algebras are of type $\mathbb{Z}_2 \otimes$"(3, 4)" and are not contained in Table 7.3.

For $\lambda = 5, 6$ there are no irreducible representations of dimension less than 24 leading to modular fusion algebras (some of them correspond to 'fermionic fusion algebras' of $N = 1$-Super-Virasoro minimal models which we do not discuss here). $\square$

4. Uniqueness of conformal characters

In this section we show that given the central charge and the finite set of conformal dimensions of certain rational models the conformal characters are already uniquely determined. More precisely, we shall state a few general and simple axioms which are satisfied by the conformal characters of all known rational models of $\mathcal{W}$-algebras. These axioms state essentially not more than the $\mathrm{SL}(2,\mathbb{Z})$-invariance of the space of functions spanned by the conformal characters, the rationality of their Fourier coefficients and an upper bound for the order of their poles. The only data of the underlying rational model occurring in these axioms are the central charge and the conformal dimensions, which give the upper bound for the pole orders and a certain restriction on the $\mathrm{SL}(2,\mathbb{Z})$-invariance. We then prove that, for various sets of central charges and conformal dimensions, there is at most one set of modular functions which satisfies these axioms (cf. the main theorem 3 in §4.1).

In this section we restrict our attention to rational models of $\mathcal{W}$-algebras where the associated representation ρ turns out to be irreducible. This restriction is mainly of technical nature: It simplifies the identification of ρ. However, we believe that our main theorem on uniqueness of conformal characters can be generalized, i.e. that it can be extended to rational models with composite ρ, possibly with a slightly more restrictive set of axioms.

We have organized §4 as follows: In §4.1 we state and comment on our main result: The theorem on uniqueness of conformal characters. The sections §4.2 and §4.3, where we develop the necessary tools needed for the proof of the main theorem 3, may be of independent interest for those studying representations ρ arising from conformal characters. Finally, in §4.4 we prove the main theorem 3.

4.1 Results on the uniqueness of conformal characters of certain rational models.

MAIN THEOREM 3 (UNIQUENESS OF CONFORMAL CHARACTERS). *Let c be any of the central charges of Table 3.2b or 3.2c, let H_c denote the set of corresponding conformal dimensions, and let H be a subset of H_c containing 0. Assume that there exist nonzero functions $\xi_{c,h}$ $(h \in H)$, holomorphic on the upper half plane, which satisfy the following conditions:*

1. *The functions $\xi_{c,h}$ are modular functions for some congruence subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$.*
2. *The space of functions spanned by the $\xi_{c,h}$ $(h \in H)$ is invariant under Γ with respect to the action $(A,\xi) \mapsto \xi(A\tau)$.*
3. *For each $h \in H$ one has $\xi_{c,h} = \mathcal{O}(q^{-\tilde{c}/24})$ as $\mathrm{Im}(\tau)$ tends to infinity, where $\tilde{c} = c - 24 \min H$.*
4. *For each $h \in H$ the function $q^{-(h-\frac{c}{24})}\xi_{c,h}$ is periodic with period 1.*
5. *The Fourier coefficients of the $\xi_{c,h}$ are rational numbers.*

Then $H = H_c$, and, for each $h \in H$, the function $\xi_{c,h}$ is unique up to multiplication by a scalar.

REMARKS.

(1) The theorem only ensures the uniqueness of the functions $\xi_{c,h}$ not their existence. However, they do exist. For Table 3.2b the existence of the corresponding functions is a well-known fact [CIZ,EFH^{2}NV]: explicit formulas for them can be given in terms of the Riemann-Jacobi theta series

$$\sum_{\substack{x\in\mathbb{Z}\\ x\equiv\lambda \bmod 2k}} \exp(2\pi i\tau x^2/4k).$$

The existence of the functions $\xi_{c,h}$ related to Table 3.2c will be proved in §5.

(2) The conformal characters χ_M of a rational model with H as set of conformal dimensions satisfy the properties listed under (2) – (5) by the very definition of rational models and Zhu's theorem if we set $\xi_{c,h} = \chi_M$ (h = conformal dimension of M). Property (1) is not part of this definition, and it is not clear whether it is implied by the axioms for rational models. However, there are indications that it always holds true (cf. the discussion below).

(3) If we assume for a rational model corresponding to a row in Table 3.2b or Table 3.2c that its conformal characters satisfy (1) we can conclude from our theorem that the corresponding set H_c is exactly the set of its conformal dimensions and that the properly normalized functions $\xi_{c,h}$ ($h \in H_c$) are its conformal characters.

(4) For the proof of the theorem for the five models of Table 3.2c the assumption $0 \in H$ is not needed, and it can possibly be dropped in all cases. However, we did not pursue this any further: From the physical point of view the assumption $0 \in H$ is natural since $h = 0$ corresponds to the vacuum representation of the underlying $\mathcal{W}$-algebra, i.e. the representation given by the action of the algebra on itself.

For the first two cases of Table 3.2c the requirement that the $\xi_{c,h}$ are modular functions on some congruence subgroup is not necessary. Here we have the

SUPPLEMENT TO THE MAIN THEOREM. *For $c = -\frac{8}{5}$ and $c = \frac{4}{5}$ and with H_c as in Table 3.2c the equality $H = H_c$ and the uniqueness of the $\xi_{c,h}$ ($h \in H$) are already implied by properties (2) to (5).*

For the other cases we do not know whether the statement about the uniqueness of H and the $\xi_{c,h}$ remains true if one also takes into account non-modular functions or non-congruence subgroups.

However, as already mentioned, it seems to be reasonable to expect that the conformal characters associated to rational models satisfy (1). Support for this is given by the following:

There is no example of a conformal character of any rational model which is not a modular function on a congruence subgroup.

As mentioned above the functions $\xi_{c,h}$, whose uniqueness is ensured by the main theorem, exist. As it turns out they can be normalized so that their Fourier coefficients are always nonnegative integers (for the case of Table 3.2c cf. §5). This gives further evidence that they are identical with the conformal characters of the corresponding $\mathcal{W}$-algebra models whence the latter therefore satisfy (1).

According to the main theorem 3, for each H_c of Table 3.2b and 3.2c the Γ-module spanned by the $\xi_{c,h}$ is uniquely determined. In particular the S-matrix (i.e. the matrix representing the action of S with respect to the basis given by the $\xi_{c,h}$ with the normalization indicated in the preceding remark) is unique. Closed formulas for the S-matrices corresponding to the first four rows of Table 3.2c will be given in §5.2 (cf. [ES2]). They can be compared with the S-matrix of the corresponding $\mathcal{W}(2,4)$ rational model with $c = -\frac{444}{11}$ as numerically computed in [E1] using so-called direct calculations in the $\mathcal{W}$-algebra. Both S-matrices coincide within the numerical precision.

The last three rational models listed in Table 3.2c are minimal models of Casimir $\mathcal{W}$-algebras for which formulas for the corresponding conformal characters have been obtained in [FKW] under the assumption of a certain conjecture. Once more, the conformal characters obtained in this way are modular functions on congruence subgroups (cf. Appendix 7.4 and the discussion in §5.3).

In the other subsections of §4 we prove our main theorem. To this end we will develop some general tools dealing with modular representations, i.e. with representations of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ on spaces of modular functions or forms. These methods are introduced in the next two subsections. In §4.4 we conclude with the proof of the main theorem 3.

4.2 A dimension formula for vector valued modular forms.

In this section we state dimension formulas for spaces of vector valued modular forms on $\mathrm{SL}(2,\mathbb{Z})$. These formulas are one of the main tools in the proof of the main theorem. It is quite natural in the context of conformal characters, or more generally in the context of modular representations, to ask for such formulas: The vector χ whose entries are the conformal characters of a rational model, multiplied by a suitable power of η, is exactly what we shall call a vector valued modular form, and is as such an element of a finite dimensional space. (The latter holds true at least in the case where the characters are invariant under a subgroup of finite index in Γ; see the assumptions in the theorem below).

Multiplying χ by an odd power of η yields a vector valued modular form of half-integral weight. However, because of the ambiguity of the square root of $c\tau+d$ (c, d being the lower entries of a matrix in Γ) we now do not deal with a vector valued modular form on $\mathrm{SL}(2,\mathbb{Z})$ but rather on a certain double cover $D\Gamma = \mathrm{DSL}(2,\mathbb{Z})$ of this group.

We now make these notions precise.

The double cover $D\Gamma$ is defined as follows: the group elements are the pairs (A, w), where A is a matrix in Γ and w is a holomorphic function on $\mathfrak{H}$ satisfying $w^2(\tau) = c\tau + d$ with c, d the lower row of A. The multiplication of two such pairs is defined by

$$(A, w(\tau)) \cdot (A', w'(\tau)) = (AA', w(A'\tau) \cdot w'(\tau)).$$

For any $k \in \mathbb{Z}$ we have an action of $D\Gamma$ on functions f on $\mathfrak{H}$ given by

$$(f|_k(A, w))(\tau) = f(A\tau)\, w(\tau)^{-2k}.$$

Note that for integral k this action factors to an action of Γ, which is nothing else than the usual '$|_k$'-action of Γ given by $(f|_k A)(\tau) = f(A\tau)(c\tau + d)^{-k}$.

For a subgroup Δ of Γ we will denote by $D\Delta \subset D\Gamma$ the preimage of Δ with respect to the natural projection $D\Gamma \to \Gamma$ mapping elements to their first component.

Special subgroups of $D\Gamma$ which we have to consider below are the groups

$$\Gamma(4m)^\sharp = \{(A, j(A,\tau)) | A \in \Gamma(4m)\}.$$

Here, for $A \in \Gamma(4m)$, we use

$$j(A,\tau) = \vartheta(A\tau)/\vartheta(\tau)$$

where $\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$. It is well-known that indeed $j(A,\tau) = \epsilon(A)\sqrt{c\tau + d}$ where c, d are the lower row of A and $\epsilon(A) = \pm 1$. Explicit formulas for $\epsilon(A)$ can be found in the literature, e.g. [Sk].

We can now define the notion of a vector valued modular form on Γ or $D\Gamma$.

Definition. For any representation $\rho: D\Gamma \to \mathrm{GL}(n,\mathbb{C})$ and any number $k \in \frac{1}{2}\mathbb{Z}$ denote by $M_k(\rho)$ the space of all holomorphic maps $F: \mathfrak{H} \to \mathbb{C}^n$ which satisfy $F|_k\alpha = \rho(\alpha)F$ for all $\alpha \in D\Gamma$, and which are bounded in any region $\mathrm{Im}(\tau) \geq r > 0$. Denote by $S_k(\rho)$ the subspace of all forms $F(\tau)$ in $M_k(\rho)$ which tend to 0 as $\mathrm{Im}(\tau)$ tends to infinity.

If ρ is a representation of Γ and k is integral we use $M_k(\rho)$ for $M_k(\rho \circ \pi)$, where π is the projection of $D\Gamma$ onto the first component. Clearly, in this case the transformation law for the functions F of $M_k(\rho)$ is equivalent to $F|_kA = \rho(A)F$ for all $A \in \Gamma$. In general, if k is integral, the group $D\Gamma$ may be replaced by Γ in all of the following considerations.

Finally, for a subgroup Δ of $D\Gamma$ or Γ we use $M_k(\Delta)$ for the space of modular forms of weight k on Δ in the usual sense. In the case $\Delta \subset \Gamma$ the weight k has of course to be integral. The reader may not mix the two kinds of spaces $M_k(\rho)$ and $M_k(\Delta)$; it will always be clear from the context whether ρ and Δ refer to a representation or a group.

Clearly, if the image of ρ is finite, i.e. if the kernel of ρ is of finite index in $D\Gamma$ then the components of an F in $M_k(\rho)$ are modular forms of weight k on this kernel. In particular, the space $M_k(\rho)$ is then finite dimensional. Formulas for the dimension of these spaces can be obtained as follows: Let V be the complex vector space of row vectors of length $n = \dim\rho$, equipped with the $D\Gamma$-right action $(z,\alpha) \mapsto z\rho(\alpha)$, where $()^t$ means transposition. The space $M_k(\rho)$ can then be identified with the space $\mathrm{Hom}_{D\Gamma}(V, M_k(\Delta))$ of $D\Gamma$-homomorphisms from V to $M_k(\Delta)$, where $\Delta = \ker\rho$, via the correspondence

$$M_k(\rho) \ni F \mapsto \text{ the map which associates } z \in V \text{ to } z \cdot F \in M_k(\Delta).$$

By orthogonality of group characters the dimension of $\mathrm{Hom}_{D\Gamma}(V, M_k(\Delta))$ can be expressed in terms of the traces of the endomorphisms defined by the action of elements of $D\Gamma$ on $M_k(\Delta)$. These traces in turn can be explicitly computed using the Eichler-Selberg trace formula. This way one can derive the following theorem (cf. [Sk, pp. 100] for a complete proof):

THEOREM (DIMENSION FORMULA [Sk]). *Let $\rho : \mathrm{DSL}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ be a representation with finite image and such that $\rho((\epsilon^2 \mathbb{1}, \epsilon)) = \epsilon^{-2k} \mathbb{1}$ for all fourth roots of unity ϵ. and let $k \in \frac{1}{2}\mathbb{Z}$. Then the dimension of $M_k(\rho)$ is given by the following formula*

$$\begin{aligned}\dim M_k(\rho) - \dim S_{2-k}(\overline{\rho}) =& \frac{k-1}{12} \cdot n + \frac{1}{4} \operatorname{Re}\left(\mathrm{e}^{\pi i k/2} \operatorname{tr} \rho((S, \sqrt{\tau}))\right) \\ &+ \frac{2}{3\sqrt{3}} \operatorname{Re}\left(\mathrm{e}^{\pi i (2k+1)/6} \operatorname{tr} \rho((ST, \sqrt{\tau+1}))\right) \\ &+ \frac{1}{2} a(\rho) - \sum_{j=1}^{n} \mathbb{B}_1(\lambda_j).\end{aligned}$$

Here the λ_j $(1 \leq j \leq n)$ are complex numbers such that $\mathrm{e}^{2\pi i \lambda_j}$ runs through the eigenvalues of $\rho(T)$, we use $a(\rho)$ for the number of j such that $\mathrm{e}^{2\pi i \lambda_j} = 1$, and we use $\mathbb{B}_1(x) = x' - 1/2$ if $x \in x' + \mathbb{Z}$ with $0 < x' < 1$, and $\mathbb{B}_1(x) = 0$ for x integral. Moreover, for $\tau \in \mathfrak{H}$, we use $\sqrt{\tau}$ and $\sqrt{\tau+1}$ for those square roots which have positive real parts.

REMARK. For $k \geq 2$ the theorem gives an explicit formula for $\dim M_k(\rho)$ since in this case $\dim(S_{2-k}(\rho)) = 0$ (the components of a vector valued modular form are ordinary modular forms on $\ker \rho$, and there exist no nonzero modular forms of negative weight and no cusp forms of weight 0).

For $k = 1/2$, $3/2$ and $\ker(\rho) \supset \Gamma(4m)^\sharp$ it is still possible to give an explicit formula for $M_k(\rho)$ [Sk]. However, we do not need those dimension formulas in full generality but need only the following corollary:

SUPPLEMENT TO THE DIMENSION FORMULA [Sk]. *Let $\rho : \mathrm{DSL}(2,\mathbb{Z}) \to \mathrm{GL}(n,\mathbb{C})$ be an irreducible representation with $\Gamma(4m)^\sharp \subset \ker(\rho)$ for some integer m. Then one has $\dim(M_{1/2}(\rho)) = 0, 1$. Furthermore, if $\dim(M_{1/2}(\rho)) = 1$ then the eigenvalues of $\rho(T)$ are of the form $\mathrm{e}^{2\pi i \frac{l^2}{4m}}$ with integers l.*

REMARK. A complete list of all those representations ρ for which $\dim(M_k(\rho)) = 1$ can be found in [Sk].

A proof of this supplement can be found in [Sk]. It uses a theorem of Serre-Stark describing explicitly the modular forms of weight $1/2$ on congruence subgroups.

4.3 Three basic lemmas on representations of $\mathrm{SL}(2,\mathbb{Z})$**.**

In this section we will prove some lemmas which are useful for identifying a given representation ρ of Γ if one has certain information about ρ, which can e.g. -for representations related to rational models- be easily computed from the central charge and the conformal dimensions.

Assume that the conformal characters of a rational model are modular functions on some a priori unknown congruence subgroup. Then the first step for determining the representation ρ, given by the action of Γ on the conformal characters, consists in finding a positive integer N such that ρ factors through $\Gamma(N)$. The next theorem tells us that the optimal choice of N is given by the order of $\rho(T)$.

THEOREM (FACTORIZATION CRITERION). *Let $\rho\colon\Gamma\to\mathrm{GL}(n,\mathbb{C})$ be a representation, and let $N>0$ be an integer. Assume that $\rho(T^N)=1$, and, if $N>5$, that the kernel of ρ is a congruence subgroup. Then ρ factors through a representation of $\Gamma/\Gamma(N)$.*

PROOF. The kernel Γ' of ρ contains the normal hull in Γ of the subgroup generated by T^N. Call this normal hull $\Delta(N)$. By a result of [Wo] (but actually going back to Fricke-Klein) one has $\Delta(N)=\Gamma(N)$ for $N\le 5$. If $N>5$ then by assumption we have $\Gamma'\supset\Gamma(N')$ for some integer N'. Thus Γ' contains $\Delta(N)\Gamma(NN')$, which, once more by [Wo], equals $\Gamma(N)$.

By the last theorem the determination of the representation ρ associated to a rational model with modular functions as conformal characters is reduced to the investigation of the finite list of irreducible representations of $\Gamma/\Gamma(N)\approx\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ with some easily computable N. The following theorem, or rather its subsequent corollary, allows to reduce this list dramatically.

THEOREM (K-RATIONALITY OF MODULAR REPRESENTATIONS). *Let k and $N>0$ be integers, let $K=\mathbb{Q}(\mathrm{e}^{2\pi i/N})$. Then the K-vector space $M_k^K(\Gamma(N))$ of all modular forms on $\Gamma(N)$ of weight k whose Fourier developments with respect to $\mathrm{e}^{2\pi i\tau/N}$ have coefficients in K is invariant under the action $(f,A)\mapsto f|_kA$ of Γ.*

PROOF. Let $j(\tau)$ denote the usual j-function, which has Fourier coefficients in $\mathbb{Z}$ and satisfies $j(A\tau)=j(\tau)$ for all $A\in\Gamma$. Assume that k is even. Then the map $f\mapsto f/{j'}^{k/2}$ defines an injection of the K-vector space $M_k^K(\Gamma(N))$ into the field of all modular functions on $\Gamma(N)$ whose Fourier expansions have coefficients in K. It clearly suffices to show that the latter field is invariant under Γ. A proof for this can be found in [Sh, p. 140, Prop. 6.9 (1), equ. (6.1.3)]. The case k odd can be reduced to the case k even by considering the squares of the modular forms in $M_k^K(\Gamma(N))$.

COROLLARY. *Let $\rho:\Gamma\to\mathrm{GL}(n,\mathbb{C})$ be a representation whose kernel contains $\Gamma(N)$ for some positive integer N, and let $K=\mathbb{Q}(\mathrm{e}^{2\pi i/N})$. If, for some integer k, there exists a nonzero element in $M_k(\rho)$ whose Fourier development has Fourier coefficients in K^n, then $\rho(\Gamma)\subset\mathrm{GL}(n,K)$.*

PROOF. If $F\in M_k(\rho)$ has Fourier coefficients in K^n then $F|_kA$, by the preceding theorem, has Fourier coefficients in K^n too. Here A is any element in Γ. From $F|_kA=\rho(A)F$ we deduce that $\rho(A)$ has entries in K.

REMARK. If one assumes that a vector valued modular form is related to the conformal characters of a rational model which are modular functions of some congruence subgroup then obviously all the Fourier coefficients are rational so that the corollary applies.

4.4 Proof of the theorem on uniqueness of conformal characters.

We will now prove our main theorem stated in §4.1. Pick one of the central charges c in Table 3.2b or Table 3.2c. Assume that for some $H\subset H_c$ containing 0 there exist functions $\xi_{c,h}$ ($h\in H$) which satisfy the properties (1) to (5) of the main theorem. Let ξ denote the vector whose components are the functions $\xi_{c,h}$ ordered with increasing h. Note that the h-values are pairwise different modulo 1.

By (4) the $\xi_{c,h}$ are thus linearly independent. Hence, we have a well-defined $|H|$-dimensional representation ρ of the modular group if we set $\xi(A\tau) = \rho(A)\xi(\tau)$ for $A \in \Gamma$. Finally, recall that the Dedekind eta function η is a modular form of weight $1/2$ for $D\Gamma$, more precisely, that there exists a one-dimensional representation θ of $D\Gamma$ on the group of 24-th roots of unity such that $\eta \in M_{\frac{1}{2}}(\theta)$.

For any half integer $k \in \frac{1}{2}\mathbb{Z}$ such that

$$k \geq \tilde{c}/2$$

we have $F := \eta^{2k}\xi \in M_k(\rho \otimes \theta^{2k})$, as is as an immediate consequence of property (3) and the assumption that the $\xi_{c,h}$ are holomorphic in the upper half plane. Let k be the smallest possible half integer satisfying this inequality. The actual value is given in Table 4.4 below.

We shall show that by property (1) to (5) the representation ρ is uniquely determined (up to equivalence). Its precise description can be read off from the last column of Table 3, respectively (notations will be explained below). In particular, ρ has dimension equal to the cardinality of H_c, and hence we conclude $H = H_c$. The h-values are pairwise incongruent modulo 1, i.e. $\rho(T)$ has pairwise different eigenvalues. Since $\rho(T)$ is a diagonal matrix the representation ρ is thus unique up to conjugation by diagonal matrices.

Finally, the kernel of ρ is a congruence subgroup by property (1). In particular, $\rho \otimes \theta^{2k}$ has a finite image. Thus we can apply the dimension formulas stated in §4.2. (For verifying the second assumption for the dimension formula note that ρ is even and that $\theta((\epsilon^2\mathbb{1}, \epsilon)) = \eta|_{\frac{1}{2}}(\epsilon^2\mathbb{1}, \epsilon)(\tau)/\eta(\tau) = \epsilon^{-1}$ for all $\epsilon^4 = 1$.) It will turn out that $M_k(\rho \otimes \theta^{2k})$ is one-dimensional. Thus, if there actually exist functions $\xi_{c,h}$ satisfying (1) to (5) then $M_k(\rho \otimes \theta^{2k}) = \mathbb{C} \cdot \xi\eta^{2k}$. Since ρ is unique up to conjugation by diagonal matrices we conclude that ξ is unique up to multiplication by such matrices, and this proves the theorem. We now give the details.

Determination of the representation ρ. We first determine the equivalence class of the representation ρ.

For an integer k' let $l(k')$ be the lowest common denominator of the numbers $h - c/24 + k'/12$ $(h \in H_c)$, i.e. let

$$l(k') = 12d/\gcd(12d, \ldots, 12n_j + k'd, \ldots),$$

where the n_j/d denote the rational numbers $h - c/24$ $(h \in H_c)$ with integers n_j, d. Clearly, the order of $(\rho \otimes \theta^{2k'})(T)$ divides $l(k')$. Let k' the smallest nonnegative integer such that $l = l(k')$ is minimal, and set $\widetilde{\rho} = \rho \otimes \theta^{2k'}$. The values of k' and l are given in Table 4.4.

Note that k' integral implies that $\widetilde{\rho}$ can be regarded as a representation of Γ (rather than $\mathrm{DSL}(2, \mathbb{Z})$). By property (1) its kernel is a congruence subgroup. Thus we can apply the factorization criterion of §4.3 to conclude that this kernel contains $\Gamma(l)$. Note that here the assumption (1), namely that the $\xi_{c,h}$ are invariant under a congruence subgroup is crucial if $l > 5$. For $l \leq 5$, this assumption is not necessary, which explains the supplement to the main theorem.

We shall say that a representation of Γ has level N if its kernel contains $\Gamma(N)$ (here N is not assumed to be minimal). Since any representation of level N factors to a representation of

$$\Gamma/\Gamma(N) \approx \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}),$$

it has a unique decomposition as sum of irreducible level N representations. Furthermore, there are only finitely many irreducible level N representation, and each such representation π has a unique product decomposition

$$\pi = \prod_{p^\lambda \| l} \pi_{p^\lambda}$$

with irreducible level p^λ representations π_{p^λ}. Here the product is to be taken over all prime powers dividing N and such that $\gcd(p^\lambda, N/p^\lambda) = 1$. Finally, $\pi_{p^\lambda}(T)$ has order dividing p^λ, i.e. its eigenvalues are p^λ-th roots of unity. Since any N-th root of unity ζ has a unique decomposition as product of the p^λ-th roots of unity $\zeta^{\frac{N}{p^\lambda}x_p}$ with $\frac{N}{p^\lambda}x_p \equiv 1 \bmod p^\lambda$, we conclude:

LEMMA. *Let ζ_j $(1 \le j \le n = \dim \pi)$ be the eigenvalues of $\pi(T)$. Then, for each $p^\lambda \| N$, the eigenvalues $\neq 1$ of $\pi_{p^\lambda}(T)$ (counting multiplicities) are exactly those among the numbers $\zeta_j^{\frac{N}{p^\lambda}x_p}$ $(1 \le j \le n)$ which are not equal to 1.*

Table 4.4: Representations of Γ and weights related to certain rational models

$\mathcal{W}$-algebra	c	k	k'	l	$\widetilde{\rho} = \rho \otimes \theta^{2k'}$
$\mathcal{W}(2)$	$1 - 6\frac{(p-q)^2}{pq}$	$\frac{1}{2}$	2	$8pq$	$R_1^p(q, \chi_{-1}) \otimes R_1^q(p, \chi_{-1}) \otimes D_8^{pq}$
$\mathcal{W}(2, \frac{(m-1)(q-2)}{2})$	$1 - 3\frac{(2m-q)^2}{mq}$	$\frac{1}{2}$	$\frac{1-3mq}{2} \bmod 12$	mq	$R_1^q(2m, \chi_{-1}) \otimes R_1^m(2q, \chi_1)$
$\mathcal{W}(2, q-3)$	$1 - \frac{(12-q)^2}{2q}$	$\frac{1}{2}$	$-1 - q \bmod 3$	$16q$	$R_1^q(3, \chi_{-1}) \otimes D_{16}^q$
$\mathcal{W}(2, q-5)$	$1 - \frac{(30-q)^2}{5q}$	$\frac{1}{2}$	$\frac{1-5q}{2} \bmod 12$	$5q$	$R_1^q(30, \chi_{-1}) \otimes R_1^5(q, \chi_{-1})$
$\mathcal{W}_{G_2}(2, 1^{14})$	$-\frac{8}{5}$	2	4	5	ρ_5
$\mathcal{W}_{F_4}(2, 1^{26})$	$\frac{4}{5}$	3	10	5	ρ_5
$\mathcal{W}(2, 4)$	$-\frac{444}{11}$	1	6	11	ρ_{11}
$\mathcal{W}(2, 6)$	$-\frac{1420}{17}$	1	2	17	ρ_{17}
$\mathcal{W}(2, 8)$	$-\frac{3164}{23}$	1	10	23	ρ_{23}

Recall that in Table 4.4 the integers p, q and m are primes with $q \neq p, m$.

The representation ρ in line 1 to 4 of Table 4.4. First, we consider the rational models corresponding to the first 4 rows of Table 4.4. By assumption $h = 0$ is in H, i.e. $\mu = \exp(2\pi i(-c/24 + k'/12))$ is an eigenvalue of $\widetilde{\rho}(T)$. Let π be that irreducible level l representation in the sum decomposition of $\widetilde{\rho}$ such that $\pi(T)$ has the eigenvalue μ. Since π is irreducible it has a decomposition as product of irreducible representations π_{p^λ} as above. Since μ is a primitive l-th root of unity the lemma implies that the π_{p^λ} are nontrivial.

The minimal dimension of a nontrivial irreducible level p^λ representation is 2, 3 or $(p-1)/2$ accordingly if p^λ equals 8, 16 or is an odd prime p (cf. §3 or [NW, p. 521ff]). Hence we have the inequalities

$$\dim \pi \geq \begin{cases} (p-1)(q-1)/2 & \text{for row 1} \\ (m-1)(q-1)/4 & \text{for row 2} \\ 3(q-1)/2 & \text{for row 3} \\ q-1 & \text{for row 4} \end{cases}.$$

For row 1, 3 and 4 the right hand side equals the cardinality of H_c respectively. In these cases we thus conclude that $\widetilde{\rho} = \pi$ is irreducible, that it is equal to a product of nontrivial level p^λ representations with minimal dimensions, and, in particular, that $H = H_c$.

For row 2 the right hand side is smaller than the cardinality of H_c. However, here we can sharpen the above inequality: First we note that the level p representations of dimension $(p-1)/2$ have parity $(-1)^{(p+1)/2}$, whence the product of the corresponding level m and q representations has parity $(-1)^{(mq-1)/2}$. On the other hand any irreducible subrepresentation has the same parity $\widetilde{\rho}$, i.e. the parity $(-1)^{k'} = (-1)^{(mq+1)/2}$. Hence π cannot equal a product of two nontrivial level m and q representations of minimal dimension. The dimension of the second smallest nontrivial irreducible level p representations is $(p+1)/2$. Under each of these representations T affords eigenvalue 1. Since T under $\widetilde{\rho}$ affords no m-th root of unity as eigenvalue, we conclude that π cannot be equal to a product of a $(q+1)/2$ dimensional level q and a $(m-1)/2$ dimensional level m representation. Thus,

$$\dim \pi \geq (m+1)(q-1)/4.$$

The right hand side equals $|H_c|$, and we conclude as above that $H = H_c$, that ρ is irreducible, and that $\widetilde{\rho}$ equals a product of an irreducible $(q-1)/2$ dimensional level q and an irreducible $(m+1)/2$ dimensional level m representation.

To identify ρ it thus remains to examine the nontrivial level p^λ representations with small dimensions (cf. §3 or [NW, p. 521ff]).

Let $p^\lambda = p$ be an odd prime. There exist exactly two irreducible level p representations with dimension $(p-1)/2$. The image of T under these representations has the eigenvalues $\exp(2\pi i\varepsilon x^2/p)$ $(1 \leq x \leq (p-1)/2)$ where for one of them ε is a quadratic residue modulo p, and a quadratic non-residue for the other one. Call these representations accordingly $R_1^p(\varepsilon, \chi_{-1})$. Similarly there exist exactly 2 irreducible level p representations with dimension $(p+1)/2$, denoted by $R_1^p(\varepsilon, \chi_1)$ (with ε being a quadratic residue or non-residue modulo p). The eigenvalues of $R_1^p(\varepsilon, \chi_1)$ are $\exp(2\pi i\varepsilon x^2/p)$ $(0 \leq x \leq (p-1)/2)$.

Let $p^\lambda = 8$. There exist exactly 4 irreducible two dimensional level 8 representations which we denote by D_8^x (x being an integer modulo 4). The eigenvalues of the image of T under the representation D_8^x are $\exp(2\pi i(1+2x)/8)$ and $\exp(2\pi i(7+2x)/8)$.

Let $p^\lambda = 16$. There are 16 irreducible three dimensional level 16 representations. These can be distinguished by their eigenvalues of the image of T. In particular, there are four of these representations, denoted by D_{16}^x ($x \bmod 4$), where the image of T has the eigenvalues $\exp(2\pi i(2x+3)/8), \exp(2\pi i(3x-6)/16), \exp(2\pi i(3x+2)/16)$.

Summarizing we find $\widetilde{\rho} = R_1^p(n_q, \chi_{-1}) \otimes R_1^q(n_p, \chi_{-1}) \otimes D_8^{n_8}$, $= R_1^q(n_q, \chi_{-1}) \otimes R_1^m(n_m, \chi_1)$, $= R_1^p(n_q, \chi_{-1}) \otimes D_{16}^{n_{16}}$ or $= R_1^p(n_q, \chi_{-1}) \otimes R_1^5(n_5, \chi_{-1})$, respectively, with suitable numbers $n_p, \ldots$. The latter can be easily determined using the Lemma and the description of H_c in Table 1. The resulting values are given in Table 4.4.

The representation ρ in line 5 to 9 of Table 4.4. We now consider the rational models corresponding to row 5 to 9 of Table 3. Here the level of $\widetilde{\rho}$ is a prime l, the dimension of ρ is $\leq l-1$, and the eigenvalues of $\rho(T)$ are pairwise different primitive l-th roots of unity.

We show that $\widetilde{\rho}$ is irreducible with dimension $l-1$. Assume that $\widetilde{\rho}$ is reducible or has dimension $< (l-1)$. The only irreducible level l representations with dimension $< (l-1)$ for which the image of T does not afford eigenvalue 1 are $R_1^l(\varepsilon, \chi_{-1})$. Thus there are only two possibilities: (a) $\widetilde{\rho} = R_1^l(\varepsilon, \chi_{-1})$ or (b) $\widetilde{\rho} = R_1^l(\varepsilon, \chi_{-1}) \otimes R_1^l(\varepsilon', \chi_{-1})$. For $l = 5, 17$ the representations $R_1^l(\varepsilon, \chi_{-1})$ have parity -1, whereas $\widetilde{\rho}$ has parity $+1$, a contradiction. For $l = 11, 23$ we note that $\xi\eta^2$ is an element of $M_1(\widetilde{\rho} \otimes \theta^{2-2k'})$. We shall show in moment that the dimension of $M_1(R_1^l(\varepsilon, \chi_{-1}) \otimes \theta^{2-2k'})$ is 0, which gives the desired contradiction (to recognize the contradiction in case (b) note that the 'functor' $\rho \mapsto M_k(\rho)$ respects direct sums).

Since the dimension formula gives explicit dimensions only for $k \neq 1$ we cannot apply it directly for calculating the dimension of $M = M_1(R_1^l(\varepsilon, \chi_{-1}) \otimes \theta^{2-2k'})$. For $l = 11$ we note that $\eta^2 M$ is a subspace of $M_2(R_1^l(\varepsilon, \chi_{-1}) \otimes \theta^{4-2k'})$. To the latter we can apply the dimension formula, and find (using $\operatorname{tr} R_1^l(\varepsilon, \chi_{-1})(S) = 0$, $\operatorname{tr} R_1^l(\varepsilon, \chi_{-1})(ST) = -1$) that its dimension is 0. For $l = 23$ and $\varepsilon = 1$ we consider $M_{3/2}(R_1^l(1, \chi_{-1}) \otimes \theta^{3-2k'})$ which contains ηM. We find that its dimension equals

$$\dim S_{1/2}(R_1^l(-1, \chi_{-1}) \otimes \theta^{-(3-2k')}) \leq \dim M_{1/2}(R_1^l(-1, \chi_{-1}) \otimes \theta^{-(3-2k')}),$$

which equals 0 by the supplement in §4.2 (for applying the supplement note that $(R_1^l(-1, \chi_{-1}) \otimes \theta^{-(3-2k')}$ has a kernel containing $\Gamma(23 \cdot 24)^\sharp$ and represents T with eigenvalues $\exp(2\pi i(-24x^2 + 17 \cdot 23)/23 \cdot 24)$). Finally, by the dimension formula we find

$$\dim M_1(R_1^l(-1, \chi_{-1}) \otimes \theta^{2-2k'}) = \dim S_1(R_1^l(1, \chi_{-1}) \otimes \theta^{-(2-2k')}),$$

and the right hand side equals 0 since $\dim S_{3/2}(R_1^l(1, \chi_{-1}) \otimes \theta^{-(1-2k')}) = 0$ by the supplement.

Thus, $\widetilde{\rho}$ is irreducible of dimension $l-1$, which implies in particular $H = H_c$. There exist exactly $(l-1)/2$ irreducible level l representations of dimension $l-1$ (cf. Table 3.5a). We now use property (5) of the main theorem, which implies that the Fourier coefficients of $\xi \cdot \eta^{2k'}$ are rational. Hence, by the corollary in §4.3 we find that $\widetilde{\rho}$ takes values in $\mathrm{GL}(l-1, K)$ with K being the field of l-th roots of unity. There is exactly one irreducible level l representations of dimension $l-1$ whose character takes values in K (Lemma 8 in §3.5 or [Do, p. 228]); denote it by ρ_l. Then $\widetilde{\rho} = \rho_l$.

Computation of dimensions. It remains to show $d = \dim M_k(\widetilde{\rho} \otimes \theta^{2k-2k'}) \leq 1$. For the first 4 rows of Table 4.4 this follows from the supplement in §4.2 and the irreducibility of ρ (in fact it can be shown that $d = 1$ [Sk]). For row 5 and 6 we find $d = 1$ by the dimension formula and using $\operatorname{tr} \rho_l(S) = 0$, $\operatorname{tr} \rho_l(ST) = 1$ (valid for arbitrary primes l). For the remaining cases (where $k = 1$) we multiply $M_1(\widetilde{\rho} \otimes \theta^{2-2k'})$ by η for obtaining $d' = \dim M_{3/2}(\widetilde{\rho} \otimes \theta^{3-2k'})$ as upper bound. Again, using the dimension formula and its supplement we find $d' = 1$.

This concludes the proof of the main theorem 3. $\square$

5. Construction of conformal characters

In §4 we formulated a list of five axioms which are satisfied for all known sets of conformal characters of rational models of $\mathcal{W}$-algebras. The only data from an underlying rational model which occurs in these axioms is its central charge and its conformal dimensions. We showed that, for several rational models, these axioms uniquely determine the conformal characters belonging to a given central charge and set of conformal dimensions.

Thus, once the central charge and conformal dimensions of a rational model are known, the computation of its conformal characters can be viewed as a problem which is completely independent from the theory of $\mathcal{W}$-algebras, i.e. for this computation one is left with a construction problem, namely, the problem of finding, by whatever means, a set of functions fulfilling the indicated list of axioms.

The purpose of this section is to describe such a mean which can solve in many cases this construction problem. In particular, we shall apply our method to the case of five special rational models related to Table 3.2c. The reason for the choice of these models is that the representation theory of the $\mathrm{SL}(2,\mathbb{Z})$-representation on their conformal characters can be treated homogeneously in some generality, and that the conformal characters of one of these models (of type $\mathcal{W}(2,8)$ with central charge $c=-\frac{3164}{23}$) could not be computed explicitly by the so far known methods.

This section is organized as follows: In section 5.1 we describe a general procedure for the construction of vector valued modular forms transforming under a given matrix representation of $\mathrm{SL}(2,\mathbb{Z})$ (main theorem 4 on realization by theta series). As already mentioned, this procedure is useful in general for finding explicit and easily computable formulas for conformal characters. In §5.2 we apply this general setup to the case of the five special rational models, and we derive explicit formulas for their conformal characters (main theorem 5 on theta formulas for conformal characters). Finally, in §5.3 we compare our results with those formulas for the conformal characters of the five models which can be obtained (assuming certain conjectures) from the representation theory of Casimir $\mathcal{W}$-algebras.

5.1 The general construction: Realization of modular representations by theta series.

In this section we show how one can, under certain hypothesis, construct systematically vector valued modular forms in $M_k(\rho)$ for a given matrix representation ρ of Γ and given weight k.

The first step is a realization of ρ as subrepresentation of a Weil representation.

Recall from section 3.4 that one has the following theorem.

THEOREM [NW]. *Each irreducible right-representation of* Γ *whose kernel contains a principal congruence subgroup is isomorphic to a subrepresentation of a suitable Weil representation.*

We call two quadratic modules $(M,\mathcal{Q})$ and $(M',\mathcal{Q}')$ isomorphic if there exists an isomorphism (of abelian groups) $\pi\colon M\to M'$ such that $\mathcal{Q}'\circ\pi=\mathcal{Q}$, and we denote such an isomorphism by

$$\pi\colon (M,\mathcal{Q})\xrightarrow{\approx}(M',\mathcal{Q}').$$

It is easy to show that isomorphic quadratic modules yield isomorphic Weil representations: an isomorphism of (projective or proper) Γ-representations is given by the map

$$\pi^*\colon \mathbb{C}^{M'} \to \mathbb{C}^M, \qquad f \mapsto \pi^* f = f \circ \pi.$$

As the next step for constructing elements of spaces $M_k(\rho)$ we connect Weil representations and theta series by lifting quadratic modules to lattices and quadratic forms on them.

More precisely, let $(M, \mathcal{Q})$ be a quadratic module. Assume that L is a complete lattice in some rational finite-dimensional vector space V and Q a positive definite non-degenerate quadratic form on V which takes on integral values on L, and such that there exists an isomorphism of quadratic modules

$$\pi\colon (L^\sharp/L, \widetilde{Q}) \xrightarrow{\approx} (M, \mathcal{Q}).$$

Here we use $L^\sharp$ for the dual lattice of L with respect to Q, i.e. $L^\sharp$ is the set of all $y \in V$ such that $B(L, y) \subset \mathbb{Z}$ with $B(x, y) = Q(x + y) - Q(x) - Q(y)$, and we use $\widetilde{Q}$ for the induced quadratic form

$$\widetilde{Q} : L^\sharp/L \to \mathbb{Q}/\mathbb{Z}\,, \qquad x + L \mapsto Q(x) + \mathbb{Z}.$$

We shall call such a pair (L, Q) a lift of the quadratic module $(M, \mathcal{Q})$.

Let p a homogeneous spherical polynomial on V with respect to Q of degree ν, i.e. if we choose a basis b_j of V, then $p\left(\sum b_j \xi_j\right)$ becomes a complex homogeneous polynomial in the variables ξ_j of degree ν satisfying

$$\nabla G^{-1} \nabla' \, p\Big(\sum_j b_j \xi_j\Big) = 0,$$

where $\nabla = (\frac{\partial}{\partial \xi_1}, \dots)$ and $G = (B(b_j, b_k))_{j,k}$ is the Gram matrix of B.

Finally, for $f \in \mathbb{C}^M$, set

$$\theta_f = \sum_{x \in L^\sharp} (\pi^* f)(x)\, p(x)\, q^{Q(x)}.$$

Here we view $\pi^* f$ as function on $L^\sharp$ which is periodic with period lattice L.

We assume that V has even dimension $2r$. Then the Weil representation $\omega = \omega_{(M,\mathcal{Q})}$ is proper as shown in ref. [ES2]. One has

Theorem (Representation by theta series). *The map $\mathbb{C}^M \ni f \mapsto \theta_f$ has the property $\theta_f|_{r+\nu} A = \theta_{f|\omega(A)}$ for all $A \in \Gamma$, i.e. it defines a homomorphism of Γ-modules.*

This is, in various different formulations, a well-known theorem. For the reader's convenience we shall sketch the proof in the Appendix at the end of this subsection.

Let now $\rho\colon \Gamma \to \mathrm{GL}(n, \mathbb{C})$ be a congruence matrix representation, and assume that we have determined a quadratic module $(M, \mathcal{Q})$ such that the associated Weil representation is proper and contains a subrepresentation which is isomorphic to the (right-)representation $\mathbb{C}^n \times \Gamma \ni (z, A) \mapsto z\rho(A)'$, where the prime denotes transposition. The existence of such an $(M, \mathcal{Q})$ is guaranteed by the first theorem. Thus, there exists a Γ-invariant subspace of $\mathbb{C}^M$ with basis f_j such that

$$\Phi|\omega(A) = \rho(A)\Phi \qquad (A \in \Gamma),$$

where Φ denotes the column vector build from the f_j. Assume furthermore that there exists a lift (L, Q) of $(M, \mathcal{Q})$, i.e. an isomorphism

$$\pi\colon (L^\sharp/L, \widetilde{Q}) \overset{\approx}{\to} (M, \mathcal{Q})$$

with a lattice L of even rank $2r$. Let p be a homogeneous spherical polynomial w.r.t. Q of degree ν. From the last theorem it is then clear that we have the following

MAIN THEOREM 4 (REALIZATION BY THETA SERIES). *The function*

$$\theta = \sum_{x \in L^\sharp} \Phi(\pi(x))\, p(x)\, q^{Q(x)}$$

is an element of $M_{r+\nu}(\rho)$.

Appendix. We proof the theorem on representation by theta series. It is consequence of the following

LEMMA (BASIC TRANSFORMATION FORMULA). *Let L be a lattice in a rational vector space V of dimension $2r$, let Q be a positive definite quadratic form on V which takes on integral values on L, let $L^\sharp$ and B be defined as above, let $w \in V \otimes \mathbb{C}$ with $Q(w) = 0$, let ν a non-negative integer, and let $z \in V$. Then one has*

$$\tau^{-r-\nu} \sum_{x \in L} [B(w, x+z)]^\nu \ \mathrm{e}(-Q(x+z)/\tau)$$
$$= \frac{i^{-r}}{\sqrt{[L^\sharp : L]}} \sum_{y \in L^\sharp} [B(w, y)]^\nu\, \mathrm{e}(\tau Q(y)^t - B(y, z)),$$

where τ is a variable in the complex upper half plane.

The lemma is a well-known consequence of the Poisson summation formula; for a proof cf. [Scho, p. 206]. (For verifying that our formula is equivalent to the one given loc. cit. identify L with $\mathbb{Z}^{2r}$ by choosing a $\mathbb{Z}$-basis b_j of L, and note that then $L^\sharp = G^{-1}\mathbb{Z}^{2r}$ and $\det(G) = [L^\sharp : L]$ where $G = (B(b_j, b_k))$ is the Gram matrix of L. Moreover, the transformation formula loc. cit. is only stated for $\tau = it$ (t real); the general formula follows by analytic continuation.)

PROOF OF THE THEOREM ON REPRESENTATION BY THETA SERIES. Since any homogeneous spherical polynomial of degree ν can be written as linear combination of the special ones $B(x, w)^\nu$ (where $w \in \mathbb{C}$, $Q(w) = 0$) we can assume that p is of this special form. Since S and T generate Γ it suffices to prove the asserted formula for these elements. For $A = T$ the formula is obvious. For proving the case $A = S$ let in the basic transformation formula z be an element of $L^\sharp$, multiply by $f(z)$ and sum over a set of representatives z for $L^\sharp/L$. Using

$$\sum_{x \in L^\sharp/L} \mathrm{e}(Q(x)) = i^r \sqrt{[L^\sharp : L]}$$

(Milgram's theorem, e.g. [MH, p. 127]) we realize the claimed formula. $\square$

5.2 An example (I): Theta series associated to quaternion algebras and the conformal characters of the five special models.

We shall use the notation introduced in §4 and construct the conformal characters related to the rational models in Table 3.2c. To this end we follow the procedure outlined in the foregoing section to construct elements of $M_k(\rho_l)$ where l denotes an odd prime $l \equiv -1 \bmod 3$ and ρ_l is the matrix representation introduced in §4.4, i.e. ρ_l is the (up to equivalence) unique irreducible representation whose kernel contains $\Gamma(l)$, and it takes its values in $\mathrm{GL}(l-1, \mathbb{Q}(\mathrm{e}^{2\pi i/l}))$. We shall give an explicit description of ρ_l below.

From property (3) in the main theorem 3 on uniqueness of conformal characters we have $\eta^{2k}\xi_c = \mathcal{O}(q^\delta)$ for $q \to 0$, where $\delta = -\tilde{c} + k/12$, and, in particular, that $\eta^{2k}\xi_c$ is an element of $M_k(\rho_l)$ (here ξ_c is the vector whose components are the functions $\xi_{c,h}$ ordered with increasing h). The dimensions of these spaces can be computed using the dimension formula in §4.2. The resulting dimensions and the values of δ are listed in Table 5.2.

Let $M_k^{(\delta)}(\rho_l)$ be the subspace of all $F \in M_k(\rho_l)$ satisfying $F = \mathcal{O}(q^\delta)$. In §4.4 it was shown that this subspace is one-dimensional, which, by obvious arguments, implies that ξ_c is unique up to multiplication by diagonal matrices. (Actually, it was shown that $M_h(\rho_l \otimes \theta^{2h-2k})$ is one-dimensional, where $\theta^2(A) = (\eta^2|_1 A)/\eta^2$. However, this latter space is obviously isomorphic to $M_k^{(\delta)}(\rho_l)$ via multiplication by η^{2k-2h}.)

Table 5.2: Certain data related to five rational models

$\mathcal{W}$-algebra	c	l	k	δ	$\dim M_k(\rho_l)$
$\mathcal{W}_{G_2}(2,1^{14})$	$-\frac{8}{5}$	5	4	$\frac{1}{5}$	1
$\mathcal{W}_{F_4}(2,1^{26})$	$\frac{4}{5}$	5	10	$\frac{3}{5}$	3
$\mathcal{W}(2,4)$	$-\frac{444}{11}$	11	6	$\frac{5}{11}$	5
$\mathcal{W}(2,6)$	$-\frac{1420}{17}$	17	2	$\frac{2}{17}$	2
$\mathcal{W}(2,8)$	$-\frac{3164}{23}$	23	10	$\frac{18}{23}$	17

We first describe how to obtain ρ_l from a proper Weil representation. Let ω be the Weil representation associated to the quadratic module $(\mathbb{F}(l^2), \mathrm{n}(x)/l)$. Here $\mathbb{F}(l^2)$ is the field with l^2 elements, and $\mathrm{n}(x) = x \cdot \overline{x}$ with $x \mapsto \overline{x} = x^l$ denoting the non-trivial automorphism of $\mathbb{F}(l^2)$. Note that $\mathrm{tr}(x\overline{y})/l$ where $\mathrm{tr}(x) = x + \overline{x}$ is the bilinear form associated to $n(x)/l$. The Weil representation ω associated is thus a (right-)representation of Γ on the space of functions $f\colon \mathbb{F}(l^2) \to \mathbb{C}$, and it is given by

$$f|\omega(T)(x) = \mathrm{e}(\mathrm{n}(x)/l)\, f(x), \qquad f|\omega(S)(x) = \frac{-1}{l} \sum_{y \in \mathbb{F}(l^2)} \mathrm{e}(-\mathrm{tr}(\overline{x}y)/l)\, f(y).$$

Here we used

$$\sum_{x \in \mathbb{F}(l^2)} \mathrm{e}(\mathrm{n}(x)/l) = -l,$$

as follows for instance from Milgram's theorem and the considerations below where we shall obtain $\mathbb{F}(l^2) = L^\sharp/L$ with a lattice of rank 4. Note that this identity implies in particular that ω is a proper representation (cf. the discussion in section 3.4).

Let χ be one of the two characters of order 3 of the multiplicative group of nonzero elements in $\mathbb{F}(l^2)$, and let G be the subgroup of elements with $\mathrm{n}(x) = 1$. Note that the existence of χ follows from the assumption $l \equiv -1 \bmod 3$. Let $X(\chi)$ be the subspace of all $\phi \in X$ which satisfy $\phi(gx) = \chi(g)\phi(x)$ for all $g \in G$. It is easily checked that $X(\chi)$ is a Γ-submodule of X. In fact, it is even an irreducible one [NW, Satz 2]. As basis for $X(\chi)$ we may pick the functions χ_r $(1 \le r \le l-1)$ which are defined by $\chi_r(x) = \chi(x)$ if $\mathrm{n}(x) = r$ and $\chi_r(x) = 0$ otherwise. Let Φ_χ be the complex column vector valued function on $\mathbb{F}(l^2)$ whose r-th component equals χ_r. We then have $\Phi_\chi|A = \rho(A)\Phi_\chi$ with a unique matrix representation $\rho\colon \Gamma \to \mathrm{GL}(l-1, \mathbb{C})$. It is an easy exercise to verify the identities

$$\rho(T) = \mathrm{diag}(\mathrm{e}^{2\pi i 1/l}, \cdots, \mathrm{e}^{2\pi i (l-1)/l}), \quad \rho(S) = (\lambda(rs))_{1 \le r,s \le l-1},$$

where we use

$$\lambda(r) = \frac{-1}{l} \sum_{\substack{x \in \mathbb{F}(l^2) \\ \mathrm{n}(x) = r}} \chi(x)\, \mathrm{e}(\mathrm{tr}(x)/l).$$

(In the identity $\mathrm{n}(x) = r$ the r has to be viewed as an element of $\mathbb{F}(l^2)$.)

Note that $\lambda(r)$ does not depend on the choice of χ, as is easily deduced by replacing in its defining sum x by $\overline{x}$ and by using $\chi(\overline{x}) = \overline{\chi}(x)$ and $\mathrm{tr}(\overline{x}) = \mathrm{tr}(x)$. The independence of the choice of χ implies that $\lambda(r)$, for any r, is contained in the field of l-th roots of unities (actually, $\lambda(r)$ is even real as follows from the easily proved facts that $\rho(S)$ is unitary, symmetric and satisfies $\rho(S)^2 = 1$.) Thus ρ satisfies the properties listed at the begin of the section, and hence is equivalent to ρ_l. Indeed, by permuting the components of the vector valued function ξ_c occurring in the definition of ρ_l and by multiplying by a suitable diagonal matrix we can even assume that $\rho = \rho_l$.

We now set $\Phi = \Phi_\chi + \Phi_{\overline{\chi}}$. The independence of the matrices $\rho(A)$ $(A \in \Gamma)$ of the choice of χ then implies that the subspace spanned by the components of Φ is invariant under Γ, and that $\Phi|\omega(A) = \rho_l(A)\Phi$ for all $A \in \Gamma$. It is easily verified that for all $x \in \mathbb{F}(l^2)$ one has

$$\Phi(x) \in \{0, -1, 2\}^{l-1}, \; (r\text{-th entry of } \Phi(x)) \bmod l = \begin{cases} \mathrm{tr}(x^{(l^2-1)/3}) & \text{if } \mathrm{n}(x) = r, \\ 0 & \text{otherwise,} \end{cases}$$

where r runs from 1 to $l-1$.

Next we describe lifts of $(\mathbb{F}(l^2), \mathrm{n}(x)/l)$. Let V be the quaternion algebra over $\mathbb{Q}$ ramified at l and ∞. If we set $K = \mathbb{Q}(\sqrt{-l})$ then V can be described as $V = K + Ku$, where $u^2 = -1/3$ and $\alpha u = u\overline{\alpha}$ for all $\alpha \in K$. The map $c = \alpha + \beta u \mapsto \overline{c} := \alpha - u\overline{\beta}$ defines an anti-involution of V. The reduced norm $\mathrm{n}(c)$ and reduced trace $\mathrm{tr}(c)$ of a $c \in V$ are given by

$$\mathrm{n}(c) = c\overline{c} = |\alpha|^2 + \frac{1}{3}|\beta|^2, \qquad \mathrm{tr}(c) = c + \overline{c} = \alpha + \overline{\alpha}.$$

Let $\mathfrak{o}$ be the ring of integers in K. Note that the rational prime 3 splits in K since $l \equiv -1 \bmod 3$. i.e. $3 = \mathfrak{p}\overline{\mathfrak{p}}$ with a prime ideal $\mathfrak{p}$ in K. (Indeed, one can take $\mathfrak{p} = 3\mathfrak{o} + (1 + \sqrt{-l})\mathfrak{o}$.) We set

$$\mathfrak{O} = \mathfrak{o} + \mathfrak{p}v, \qquad v = \begin{cases} u & \text{for } l \equiv 3 \bmod 4 \\ \frac{1+u}{2} & \text{for } l \equiv 1 \bmod 4 \end{cases}.$$

It can be easily checked that $\mathfrak{O}$ is an order in V (i.e. a subring which, viewed as $\mathbb{Z}$-module, is free of rank 4). In fact, $\mathfrak{O}$ is even a maximal order since the determinant of the Gram matrix $(\operatorname{tr}(e_j\overline{e}_k))$, for any $\mathbb{Z}$-basis e_j of $\mathfrak{O}$, equals l^2 (cf. [Vi, Chap. III, Corollaire 5.3].).

We now have

LEMMA. *(1) The dual lattice of $\sqrt{-l}\,\mathfrak{O}$ w.r.t. the quadratic form* $\mathrm{n}(c)/l$ *is $\mathfrak{O}$. The quotient ring $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$ is the field with l^2 elements, and the anti-involution $c \mapsto \overline{c}$ on $\mathfrak{O}$ induces the Frobenius automorphism $x \mapsto x^l$ on $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$.*

(2) Let $I \subset \mathfrak{O}$ be an $\mathfrak{O}$-left ideal, and let $n = \mathrm{n}(I)$ be the reduced norm of I (i.e. the g.c.d. of the integers $\mathrm{n}(x)$ where x runs through I). Then the dual lattice of $\sqrt{-l}I$ with respect to $\mathrm{n}(c)/ln$ *is I. There exists a $c_0 \in I$ such that $n(c_0)/n \equiv 1 \bmod l$, and for any such c_0 the map $c \mapsto c_0 \cdot c$ defines an isomorphism of quadratic modules $(\mathfrak{O}/\sqrt{-l}\mathfrak{O}, \mathrm{n}(x)/l) \xrightarrow{\approx} (I/\sqrt{-l}I, \mathrm{n}(x)/ln)$.*

Here, for convenience, we use the same symbols $\overline{n}(x)/nl$ for the quadratic form on I as well as for the quadratic form induced by it on $I/\sqrt{-l}I$. The Lemma follows easily from standard facts in the theory of quaternion algebras; for the reader's convenience we sketch the proof in the Appendix to this section.

The Lemma provides us with lifts $(I, n(x)/nl)$ of $(\mathbb{F}(l^2), \mathrm{n}(x)/l)$, and we now can write down explicitly elements of $M_k(\rho_l)$.

To this end let $\Phi\colon \mathbb{F}(l^2) = \mathfrak{O}/\sqrt{-l}\mathfrak{O} \to \{-1, 0, 2\}^{l-1}$ be defined as above. Let I be an $\mathfrak{O}$-left ideal, choose c_0 as in the Lemma, and let

$$\pi : I \to \mathfrak{O}/\sqrt{-l}\mathfrak{O}, \qquad \pi(c) = \lambda + \sqrt{-l}\mathfrak{O} \quad \text{with } c \equiv \lambda c_0 \bmod \sqrt{-l}\mathfrak{O}.$$

Finally, let p be a homogeneous spherical polynomial function on V. If we write polynomial functions on V as polynomials p in α, $\overline{\alpha}$, β, $\overline{\beta}$, then it is spherical of degree ν (with respect to any nonzero multiple of $n(c)$) if and only if p is homogeneous of degree ν and satisfies

$$\left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}} + 3\frac{\partial^2}{\partial\beta\partial\overline{\beta}}\right)p = 0.$$

Set

$$\theta(\tau; I, p) = \sum_{c \in I} \Phi(\pi(c))\, p(c)\, q^{n(c)/n(I)l}.$$

We suppress the dependence of this function on c_0 since a different choice results only in multiplying $\theta(\tau; I, p)$ by a scalar. By the theorem on realization by theta series we then have

$$\theta(\tau; I, p) \in M_{2+\deg(p)}(\rho_l).$$

It is easy to compute these functions using a computer. In fact, by a computer calculation we found

THEOREM. *Let l, k be as in Table 5.2. Then the space $M_k(\rho_l)$ is spanned by the series $\theta(\tau; I, p)$, where $I = \mathfrak{O}$ for $l \neq 17$, and $I = \mathfrak{O}, \mathfrak{O}\mathfrak{p}$ for $l = 17$, and where p runs through the homogeneous polynomial functions on the quaternion algebra V of degree $k-2$ which are spherical with respect to the quadratic form $\mathrm{n}(c)$.*

It is an open question whether the spaces $M_k(\rho_l)$, for arbitrary k or primes l ($\equiv -1 \bmod l$), are always spanned by theta series of the form $\theta(\tau; I, p)$, or, more generally, which spaces $M_k(\rho)$ of vector valued modular forms at all can be generated by theta series.

As explained above we are especially interested in the one-dimensional subspace $M_k^{(\delta)}(\rho_l)$ of functions in $M_k(\rho_l)$ which are $\mathcal{O}(q^\delta)$ with δ as in Table 5.2. Here we have

MAIN THEOREM 5 (THETA FORMULAS FOR CONFORMAL CHARACTERS).

(1) Let c, l, k and δ be as in Table 5.2, and let $I = \mathfrak{O}$ for $l \neq 17$ and $I = \mathfrak{O}\mathfrak{p}$ for $l = 17$. Then there exists a homogeneous spherical polynomial function p of degree $k-2$ such that the $\theta(\tau; I, p)$ is nonzero and satisfies $\theta(\tau; I, p) = \mathcal{O}(q^\delta)$.

(2) Moreover, for any p with this property, there exists a nonzero constant κ such that the components of the Fourier coefficients of $\kappa\theta(\tau; I, p)$ are rational integers. In particular, the components of $\kappa\eta(\tau)^{-2k}\theta(\tau; I, p)$ satisfy the properties (1) to (5) in the main theorem 3 on the uniqueness of conformal characters in §4.1.

PROOF. (1) The existence of a p with Fourier development starting at q^δ follows from the preceding theorem and the fact that the subspace $M_k^{(\delta)}(\rho_l)$ contains nonzero elements. For the latter cf. the discussion at the beginning of §5.2; of course, it can also be checked by a straight forward calculation using the $\theta(\tau; I, p)$ that $M_k^{(\delta)}(\rho_l)$ is one-dimensional.

(2) This last fact also shows that for proving the second statement of the theorem, it suffices to prove that, for at least one p satisfying the condition $\theta(\tau; I, p) = \mathcal{O}(q^\delta)$, the function $\theta(\tau; I, p)$ has rational Fourier coefficients.

For proving this let $P_\nu(F)$ be the set of spherical homogeneous functions p on V of degree ν which are defined over the subfield $F \subset \mathbb{C}$. By the latter we mean that the coefficients of $p(c)$, when written as polynomial in the coefficients of c with respect to a fixed basis of V, are in F. Note that this property does not depend on the choice of the $\mathbb{Q}$-basis of V. Since $P_\nu(F)$ is the kernel of a differential operator which has constant rational coefficients, when written with respect to any $\mathbb{Q}$-basis of V, it is clear that $P_\nu(\mathbb{C}) = P_\nu(\mathbb{Q}) \otimes \mathbb{C}$, i.e. we can find a basis of $P_\nu(\mathbb{C})$ which is contained in $P_\nu(\mathbb{Q})$. But then we deduce, using the preceding theorem, that $M_k(\rho_l)$ has a basis θ_j $(1 \leq j \leq d)$ whose Fourier coefficients $a_{\theta_j}(r)$ $(r = 1, 2, \ldots)$ are elements of $\mathbb{Q}^{l-1}$. For deducing this note that $\theta(\tau; I, p)$ for $p \in P_\nu(\mathbb{Q})$ has rational Fourier coefficients since $\Phi(x)$ is rational. The elements of $M_k^{(\delta)}(\rho_l)$ are now the linear combinations $\sum_j c_j\theta_j$ such that $\sum_j c_j a_{\theta_j}(r) = 0$ for all $1 \leq r < l\delta$. Since the latter system of linear equations is defined over $\mathbb{Q}$ and has a nonzero solution by part (1) we conclude the existence of rational nonzero solution, i.e. the existence of a linear combination of the θ_j with Fourier coefficients in $\mathbb{Q}$. $\square$

If we pick a p as described in the theorem, and if we denote by $\xi_{c,h}$ the r-th component of $\eta^{-2k}\theta(\tau; I, p)$, where $\frac{r}{l} - \frac{k}{12} \equiv h - \frac{c}{24} \bmod \mathbb{Z}$ then it is clear that these functions satisfy properties (1) to (5) in the main theorem 3 in §4.1 (after multiplied by a constant, if necessary). Hence, by the uniqueness result proven in section 4.1, they are up to a constant the conformal characters of the $\mathcal{W}$-algebras introduced in the same section. In fact, the $\xi_{c,h}$ ($l \neq 5$) have interesting product expansions, which we shall discuss elsewhere; from these product expansions it can immediately read off that they can be normalized such that their Fourier coefficients are even non-negative integers, as its should be for conformal characters.

Appendix.

PROOF OF THE LEMMA. (1) Let $l \equiv -1 \bmod 4$. For $c = \alpha + u\beta \in V$ we have

$$\mathrm{tr}(c\overline{\mathfrak{O}}\sqrt{-l})/l = \mathrm{tr}(\alpha\,\mathfrak{o}/\sqrt{-l}) + \mathrm{tr}(\beta\,\overline{\mathfrak{p}}/3\sqrt{-l}).$$

Thus the left hand side is in $\mathbb{Z}$ if and only if each of the two terms on the right are in $\mathbb{Z}$. The latter is easily checked to be equivalent to $\alpha \in \mathfrak{o}$ and $\beta \in \mathfrak{p}$, i.e. to $c \in \mathfrak{O}$. The case $l \equiv 1 \bmod 4$ can be treated similarly, and is left to the reader.

It is clear that $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$ is a ring of characteristic l with l^2 elements. Hence it is isomorphic to a ring extension of $\mathbb{F}(l) = \mathbb{Z}/l\mathbb{Z}$ with l^2 elements. Moreover, it contains a root of X^2+3, namely $3u + \sqrt{-l}\mathfrak{O}$. Since -3 is not a quadratic residue modulo l the polynomial X^2+3 is irreducible over $\mathbb{F}(l)$, hence $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$ is a field. The anti-involution $c \mapsto \overline{c}$ induces an automorphism of the field $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$ which is nontrivial since it maps u to $-u$, and which hence is the Frobenius automorphism.

(2) If I is an $\mathfrak{O}$-left ideal then $I^* = \overline{I}\cdot\mathfrak{O}^*/\,\mathrm{n}(I)$, where, for any left ideal I, we use

$$I^* = \{c \in V \mid \mathrm{tr}(Ic) \in \mathbb{Z}\}.$$

(We were not able to find a reference for this basic formula: it can easily be proved using adelic methods. However, we shall need it only for $I = \mathfrak{O}$ or $I = \mathfrak{O}\mathfrak{p}$ (cf. the two theorems of the preceding section), and here it can be easily verified by direct computation. We omit the details for the general case.) Thus we find $\mathrm{tr}(\sqrt{-l}I\overline{c})/\,\mathrm{n}(I)l \in \mathbb{Z}$ if and only if $\overline{c} \in \overline{I}\cdot\mathfrak{O}^*\sqrt{-l}$. Using $\mathfrak{O}^* = \mathfrak{O}/\sqrt{-l}$, as follows from part (1), we find that the latter statement is indeed equivalent to $c \in I$.

Left-multiplication in the quaternion algebra induces on $I/\sqrt{-l}I$ a structure of a one-dimensional $\mathfrak{O}/\sqrt{-l}\mathfrak{O}$-vector space. Let $c_0 + \sqrt{-l}I$ be a basis element. Clearly $\mathrm{n}(c_0)/\,\mathrm{n}(I)$ is not divisible by l since otherwise $\mathrm{n}(c)/\,\mathrm{n}(I)$ would be divisible by l for any $c \in I$ contradicting the definition of $\mathrm{n}(I)$ as g.c.d. of all $\mathrm{n}(c)$ ($c \in I$). Thus we can choose a $\lambda \in \mathfrak{O}$ with $\mathrm{n}(\lambda)\,\mathrm{n}(c_0)/\,\mathrm{n}(I) \bmod l$. Replacing c_0 by λc_0 it is then clear that $c \mapsto cc_0$ induces the isomorphism claimed to exist. □

5.3 An example (II): Comparison to formulas derivable from the representation theory of Kac-Moody and Casimir $\mathcal{W}$-algebras.

In this section we compare our explicit formulas for the conformal characters with the ones obtained from the representation theory of Casimir $\mathcal{W}$-algebras [FKW], the Virasoro algebra [RC] and Kac-Moody algebras [Ka].

The last three rational models related to Table 5.2 are minimal models of so-called Casimir $\mathcal{W}$-algebras. For this kind of algebras the minimal models have been determined (assuming a certain conjecture) in [FKW]. The representation theory of the two composite rational models ($\mathcal{W}_{\mathcal{G}_2}(2,1^{14})$ and $\mathcal{W}_{\mathcal{F}_4}(2,1^{26})$) is well-known [RC,Ka].

In order to give the explicit formulas for the conformal characters of the minimal models of the Casimir $\mathcal{W}$-algebras, the Virasoro algebra and Kac-Moody algebras we have to fix some notation first.

Let $\mathcal{K}$ be a simple complex Lie algebra of rank l and dimension n, h ($h^\vee$) its (dual) Coxeter number, ρ ($\rho^\vee$) the sum of its (dual) fundamental weights, W the Weyl group and Λ ($\Lambda^\vee$) the (dual) weight lattice of $\mathcal{K}$. For $\lambda \in \Lambda$ denote by π_λ the highest weight representation with highest weight λ.

Firstly, consider the case of the three rational models of the Casimir $\mathcal{W}$-algebras. Formulas for the central charge, the conformal dimensions and the conformal characters of rational models of Casimir $\mathcal{W}$-algebras have been derived assuming a certain conjecture [FKW, p. 320] and are collected in Appendix 8.4.

Using these formulas for $\mathcal{B}_2$ with $c(p,q) = c(11,6) = -\frac{444}{11}$ and for $\mathcal{G}_2$ with $c(p,q) = c(17,12) = -\frac{1420}{17}$ for $\mathcal{W}(2,4)$ and $\mathcal{W}(2,6)$, respectively, one obtains the conformal characters given in the last section (as can be checked by simply comparing a sufficient number of Fourier coefficients). The last rational model, of type $\mathcal{W}(2,8)$, is a rational model of $\mathcal{W}E_7$ with $c(p,q) = c(18,23)$. However, in this case the above formula for the corresponding conformal characters contains a sum over a rank 7 lattice (the dual weight lattice) and a sum over the Weyl group of $\mathcal{E}_7$ which has order 2.903.040. Therefore, this formula is of no practical use for explicit calculations in this case. However, our formula in the foregoing section involves only a sum over a rank 4 lattice which is easy to implement on a computer.

Secondly, consider the rational models $\mathcal{W}_{\mathcal{G}_2}(2,1^{14})$ and $\mathcal{W}_{\mathcal{F}_4}(2,1^{26})$. These rational models are 'tensor products' of the Virasoro minimal model with $c = -\frac{22}{5}$ and the rational model associated to the level 1 Kac-Moody algebra of $\mathcal{G}_2$ or $\mathcal{F}_4$, respectively. The two conformal characters of the Virasoro minimal model with central charge $c = -\frac{22}{5}$ are given by [RC]

$$\chi_0^{Vir}(q) = q^{\frac{11}{60}} \prod_{n\equiv\pm2 \bmod 5} (1-q^n)^{-1}, \qquad \chi_{-1/5}^{Vir}(q) = q^{-\frac{1}{60}} \prod_{n\equiv\pm1 \bmod 5} (1-q^n)^{-1}.$$

The characters of rational models associated to the level 1 Kac-Moody algebras are well known from the Kac-Weyl formula [Ka, p. 173]. The rational model associated to the level k Kac-Moody algebra of $\mathcal{K}$ has the following central charge and conformal dimensions

$$c^{\mathcal{K}}(k) = \frac{12k}{h^\vee(h^\vee+k)}\rho^2, \qquad h_\lambda^{\mathcal{K}} = \frac{(\rho+\lambda)^2-\rho^2}{2(h^\vee+k)} \quad ((\lambda,\psi)\le k)$$

where ψ is the highest root of $\mathcal{K}$. The corresponding characters read

$$\chi^{\mathcal{K},\lambda}(q) = \eta(q)^{-n} q^{\frac{n-c^{\mathcal{K}}(k)}{24}} \sum_{t\in\Lambda^\vee} \dim(\pi_{\rho+\lambda+(h^\vee+k)t}) q^{\frac{(\rho+\lambda+(h^\vee+k)t)^2-\rho^2}{2(h^\vee+k)}}.$$

The two conformal characters associated to the level 1 Kac-Moody algebras of $\mathcal{G}_2$ and $\mathcal{F}_4$ are given by:

$$\chi_0^{\mathcal{K}} = \chi^{\mathcal{K},0} \quad \chi_h^{\mathcal{K}} = \chi^{\mathcal{K},\lambda_1}$$

where $h = h^{\mathcal{K}}_{\lambda_1} = 2/5$ or $3/5$ and λ_1 is the fundamental weight of $\mathcal{G}_2$ or $\mathcal{F}_4$ with $\dim(\pi_\lambda) = 7$ or 26, respectively.

Using these formulas one obtains exactly the four conformal characters of the models $\mathcal{W}_{\mathcal{G}_2}(2, 1^{14})$ and $\mathcal{W}_{\mathcal{F}_4}(2, 1^{26})$:

$$\chi_0 = \chi_0^{Vir} \cdot \chi_0^{\mathcal{K}}, \quad \chi_{-1/5} = \chi_{-1/5}^{Vir} \cdot \chi_0^{\mathcal{K}}, \quad \chi_h = \chi_0^{Vir} \cdot \chi_h^{\mathcal{K}}, \quad \chi_{h-1/5} = \chi_{-1/5}^{Vir} \cdot \chi_h^{\mathcal{K}}$$

with $\mathcal{K} = \mathcal{G}_2, \mathcal{F}_4$ and $h = 2/5, 3/5$ respectively. The product formulas for the Virasoro characters and the formula for the conformal characters associated to the Kac-Moody algebras show that the Fourier coefficients of the two rational models are positive integers. Indeed, as one can show by comparing a sufficient number of Fourier coefficients, these conformal characters are equal to the ones computed in the last section.

6. Conclusion and outlook

Finally, we summarize and comment on the main results in this thesis.

Firstly, we have classified all strongly-modular fusion algebras of dimension less than or equal to four and all nondegenerate strongly-modular fusion algebras of dimension less than 24. In order to obtain our results we have used the classification of the irreducible representations of the groups $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$. Not all modular fusion algebras in our classification show up in known RCFTs. However, all corresponding fusion algebras are realized in known RCFTs apart from the fusion algebra of type B_9. This fusion algebra can formally be related to the Casimir $\mathcal{W}$-algebra $\mathcal{W}B_2$ at $c = -24$ and seems to be an analogue of the fusion algebra formally associated to the Virasoro algebra with central charge $c = c(3, 9)$.

Unfortunately, the methods used in this thesis seem to be not sufficient for obtaining a complete classification of strongly-modular fusion algebras. For those strongly-modular fusion algebras which are degenerate the corresponding representation of the modular group is in general reducible and therefore there are a lot of possible choices of the distinguished basis in the representation space. In the proof of the main theorem 1 on the classification of the strongly-modular fusion algebras of dimension less than or equal to four we have shown how one can deal with this problem in the case of two, three and four dimensional fusion algebras. However, we do not know a general method to overcome this problem for arbitrary dimension.

We would like to stress that the main assumption for obtaining our classifications, namely that fusion algebras are induced by representations of $\mathrm{SL}(2, \mathbb{Z}_N)$, is valid for all known examples of rational conformal field theories. Nevertheless, the question whether all fusion algebras associated to RCFTs are strongly-modular is not yet answered (cf. the conjecture at the end of section 2.2).

Secondly, we have shown that the conformal characters of certain rational models are uniquely determined by the central charge and the set of conformal dimensions of the model.

This result has several implications. It shows that the simple constraints imposed on modular functions by the five axioms stated in the main theorem 3 are surprisingly restrictive. Apart from giving an aesthetical satisfaction this observation gives further evidence that conformal characters are modular functions of a rather special nature, which may deserve further studies, even independent from the theory of $\mathcal{W}$-algebras.

Furthermore, it implies that, in the case of the rational models considered in §4, the conformal characters a priori do not encode more information about the underlying rational model than the central charge and the conformal dimensions. This is in perfect accordance with the more general belief that these data already determine completely the rational models of $\mathcal{W}$-algebras which do not contain currents (currents are nonzero elements of dimension 1). In general one expects that a unique characterization of rational models can be obtained if one takes into account certain additional quantum numbers which can be defined in terms of the zero modes of the currents.

Finally, our result has a useful practical consequence for the computation of conformal characters. Apart from several well-understood rational models where one has simple closed formulas for the conformal characters, it is in general difficult to compute them directly. Any attempt to obtain the first few Fourier coefficients by the so-called direct calculations in the $\mathcal{W}$-algebra, the so far only known method in the case where no closed formulas are available, requires considerable computer power. Our result indicates a way to avoid the direct calculations: Once the central charge and conformal dimensions are determined, the computation of the conformal characters can be viewed as a problem which belongs solely to the theory of modular forms, i.e. a problem whose solution affords no further data of the rational model in question.

Of course, one of the important open questions is whether a uniqueness result like the main theorem 3 holds for more or even for all rational models. For rational models with effective central charge less than 26 there is at least some hope that the central charge and the set of conformal dimensions already determine the conformal characters: Looking at the dimension formula in §4.2 we see that 'main' contribution to the dimension of the space of vector valued modular forms of weight k transforming under a representation ρ is given by $\frac{k-1}{12}\dim(\rho)$. For rational models with $\tilde{c} < 26$ this contribution is less than the number of conformal dimensions of the rational model. Therefore, one might hope that the $\dim(\rho)$ conditions on the pole orders of the conformal characters at $i\infty$ imposed by fixing the central charge and the conformal dimensions already determine the conformal characters uniquely. Note, however, that these $\dim(\rho)$ conditions will in general not be independent and that one also has to take into account the 'correction' terms in the dimension formula.. To obtain more general results one would like to have a dimension formula for vector valued modular forms having a prescribed vanishing order at $i\infty$. One might speculate that such a formula should be related to the Atiyah-Singer index theorem since the dimension formula in §4.2 is related to the Riemann-Roch theorem.

Thirdly, we have shown that one can reconstruct the conformal characters of certain rational models merely from the knowledge of the central charge and the set of conformal dimensions of the model by using theta series, and, in particular, how one obtains in this way explicit closed formulas for the conformal characters of certain nontrivial rational models which could not be computed using known methods.

The main unsolved question concerning the construction procedure described in section 5 is whether all spaces of vector valued modular forms transforming under a congruence representation are generated by theta series.

Finally, I would like to stress that the methods and results developed in this thesis have lead to a better understanding of the structure of RCFTs. However, the classification program of rational conformal field theories is still a fascinating open problem and deserves future effort.

I would like to end with the following quote [M]:

" ... Our general approach follows the philosophy of nahmism (called "nahmsense" by its detractors) in which one begins with modular forms and then proceeds to try to deduce some interesting physics from them. We will show that some of the interesting forms do indeed arise in physical theories. ..."

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7. Appendix

7.1 The irreducible level p^λ representations of dimension ≤ 4.

Using the results in §3 one obtains as a complete list of two dimensional irreducible level p^λ representations

$$\begin{array}{ll} p^\lambda = 2^1, & N_1(\chi_1) \\ p^\lambda = 3^1, & N_1(\chi_1) \otimes B_i \\ p^\lambda = 5^1, & R_1(1,\chi_{-1}),\ R_1(2,\chi_{-1}) \\ p^\lambda = 2^2, & N_1(\chi_1) \otimes C_3 \\ p^\lambda = 2^3, & N_3(\chi)_+ \otimes C_j \\ \text{where } i = 1,2,3;\ j = 1,\ldots,4. & \end{array}$$

The explicit form of the representations which are not related by tensor products with B_i or C_j is given in Table 7.1a.

Table 7.1a: Two dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
2	$N_1(\chi_1)$	$\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$	$\mathrm{diag}(0, \frac{1}{2})$
3	$N_1(\chi)$	$-\frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{3}, \frac{2}{3})$
5	$R_1(1,\chi_{-1})$	$\frac{2i}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \end{pmatrix}$	$\mathrm{diag}(\frac{1}{5}, \frac{4}{5})$
	$R_1(2,\chi_{-1})$	$\frac{2i}{\sqrt{5}}\begin{pmatrix} -\sin(\frac{2\pi}{5}) & -\sin(\frac{\pi}{5}) \\ -\sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5}) \end{pmatrix}$	$\mathrm{diag}(\frac{2}{5}, \frac{3}{5})$
2^3	$N_3(\chi)_+$	$\frac{i}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{3}{8}, \frac{5}{8})$

Similarly, one obtains as a complete list of three dimensional irreducible level p^λ representations

$$\begin{array}{ll} p^\lambda = 3^1, & N_1(\chi_1) \\ p^\lambda = 5^1, & R_1(1,\chi_1),\ R_1(2,\chi_1) \\ p^\lambda = 7^1, & R_1(1,\chi_{-1}),\ R_1(2,\chi_{-1}) \\ p^\lambda = 2^2, & D_2(\chi)_+ \otimes C_j \\ p^\lambda = 2^3, & R_3^0(1,3,\chi)_+ \otimes C_j,\ R_3^0(1,3,\chi)_- \otimes C_j \\ p^\lambda = 2^4, & R_4^0(1,1,\chi)_+ \otimes C_j,\ R_4^0(1,1,\chi)_- \otimes C_j, \\ & R_4^0(3,1,\chi)_+ \otimes C_j,\ R_4^0(3,1,\chi)_- \otimes C_j \\ \text{where } j = 1,\ldots,4. & \end{array}$$

The explicit form of the representations which are not related by tensor products with C_j is given in Table 7.1b.

Table 7.1b: Three dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
3	$N_1(1,\chi_1)$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{3},\frac{2}{3},0)$
5	$R_1(1,\chi_1)$	$\frac{2}{\sqrt{5}}\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -s_1 & s_2 \\ \frac{1}{\sqrt{2}} & s_2 & -s_1 \end{pmatrix}$	$\mathrm{diag}(0,\frac{1}{5},\frac{4}{5})$
	$R_1(2,\chi_1)$	$\frac{2}{\sqrt{5}}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -s_2 & s_1 \\ -\frac{1}{\sqrt{2}} & s_1 & -s_2 \end{pmatrix}$ $s_j=\cos(\frac{j\pi}{5})$	$\mathrm{diag}(0,\frac{2}{5},\frac{3}{5})$
7	$R_1(1,\chi_{-1})$	$\frac{2}{\sqrt{7}}\begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & -s_3 & s_1 \\ s_3 & s_1 & -s_2 \end{pmatrix}$	$\mathrm{diag}(\frac{2}{7},\frac{1}{7},\frac{4}{7})$
	$R_1(2,\chi_{-1})$	– – " – – $s_j=\sin(\frac{j\pi}{7})$	$\mathrm{diag}(\frac{5}{7},\frac{6}{7},\frac{3}{7})$
2^2	$D_2(\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{4},\frac{1}{2},0)$
2^3	$R_3^0(1,3,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{1}{2},\frac{5}{8},\frac{1}{8})$
	$R_3^0(1,3,\chi)_-$	$(-1)\ \cdot$ (– – " – –)	$\mathrm{diag}(\frac{1}{2},\frac{7}{8},\frac{3}{8})$
2^4	$R_4^0(1,1,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \end{pmatrix}$	$\mathrm{diag}(\frac{5}{8},\frac{1}{16},\frac{9}{16})$
	$R_4^0(1,1,\chi)_-$	– – " – –	$\mathrm{diag}(\frac{1}{8},\frac{5}{16},\frac{13}{8})$
	$R_4^0(3,1,\chi)_+$	$\frac{i}{2}\begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}$	$\mathrm{diag}(\frac{7}{8},\frac{3}{16},\frac{11}{16})$
	$R_4^0(3,1,\chi)_-$	– – " – –	$\mathrm{diag}(\frac{3}{8},\frac{15}{16},\frac{7}{16})$

Table 7.1c: Four dimensional irreducible level p^λ representations

level	type of rep.	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T))$
5	$N_1(\chi),\ \chi^3 \not\equiv 1$	$\frac{2i}{5}\begin{pmatrix} \eta_- & \sqrt{3}s_2 & \eta_+ & \sqrt{3}s_4 \\ \sqrt{3}s_2 & -\eta_+ & \sqrt{3}s_4 & \eta_- \\ \eta_+ & \sqrt{3}s_4 & -\eta_- & -\sqrt{3}s_2 \\ \sqrt{3}s_4 & \eta_- & -\sqrt{3}s_2 & \eta_+ \end{pmatrix}$ $s_j = \sin(\frac{j\pi}{5}),\ \eta_\pm = s_2 \pm s_4$	$\mathrm{diag}(\frac{3}{5},\frac{4}{5},\frac{2}{5},\frac{1}{5})$
	$N_1(\chi),\ \chi^3 \equiv 1$	$-\frac{2}{5}\begin{pmatrix} \xi_1 & -\xi_2 & \xi_1 & -\xi_3 \\ -\xi_2 & -\xi_1 & \xi_3 & \xi_1 \\ \xi_1 & \xi_3 & \xi_1 & \xi_2 \\ -\xi_3 & \xi_1 & \xi_2 & -\xi_1 \end{pmatrix}$ $r_j = \cos(\frac{j\pi}{5}),\ \xi_1 = r_1 - r_4 - \frac{1}{2},$ $\xi_2 = 3r_2 + 2r_4,\ \xi_3 = 2r_2 + 3r_4$	$\mathrm{diag}(\frac{3}{5},\frac{4}{5},\frac{2}{5},\frac{1}{5})$
7	$R_1(1,\chi_1)$	$\sqrt{\frac{2}{7}}i\begin{pmatrix} -\frac{1}{\sqrt{2}} & -1 & -1 & -1 \\ -1 & \xi_1 & \xi_2 & \xi_3 \\ -1 & \xi_2 & \xi_3 & \xi_1 \\ -1 & \xi_3 & \xi_1 & \xi_2 \end{pmatrix}$	$\mathrm{diag}(0,\frac{1}{7},\frac{4}{7},\frac{2}{7})$
	$R_1(2,\chi_1)$	$(-1)\cdot\quad(--\text{"}--)$ $s_j = \sqrt{\frac{2}{7}}\sin(\frac{j\pi}{7}),$ $\xi_1 = 2s_2 - s_4,\ \xi_2 = 2s_4 + s_6$ $\xi_2 = 2s_4 + s_6,\ \xi_3 = -2s_6 - s_2$	$\mathrm{diag}(0,\frac{6}{7},\frac{3}{7},\frac{5}{7})$
2^3	$N_3(\chi),\ \chi^3 \not\equiv 1$	$\frac{i}{\sqrt{8}}\begin{pmatrix} 1 & 1 & \sqrt{3}i & -s_1\sqrt{3}i \\ 1 & -1 & -\sqrt{3}i & -s_1\sqrt{3}i \\ -\sqrt{3}i & \sqrt{3}i & 1 & s_1 \\ s_2\sqrt{3}i & s_2\sqrt{3}i & s_2 & -1 \end{pmatrix}$ $s_j = e^{2\pi i\frac{j}{3}}$	$\mathrm{diag}(\frac{3}{8},\frac{5}{8},\frac{1}{8},\frac{7}{8})$
3^2	$R_2^1(1,1,\chi),\ \chi^3 \equiv 1$	$\frac{2i}{3}\begin{pmatrix} -s_8 & -s_4 & -s_2 & -s_6 \\ -s_4 & s_2 & -s_8 & s_6 \\ -s_2 & -s_8 & s_4 & s_6 \\ -s_6 & s_6 & s_6 & 0 \end{pmatrix}$	$\mathrm{diag}(\frac{4}{9},\frac{1}{9},\frac{7}{9},\frac{1}{3})$
	$R_2^1(2,1,\chi),\ \chi^3 \equiv 1$	$(-1)\ \cdot\ (--\text{"}--)$	$\mathrm{diag}(\frac{2}{9},\frac{5}{9},\frac{8}{9},\frac{2}{3})$
	$R_2^1(1,1,\chi),\ \chi^3 \not\equiv 1$	$\frac{2}{3}\begin{pmatrix} s_1 & s_5 & s_7 & s_6 \\ s_5 & -s_7 & -s_1 & s_6 \\ s_7 & -s_1 & s_5 & -s_6 \\ s_6 & s_6 & -s_6 & 0 \end{pmatrix}$	$\mathrm{diag}(\frac{4}{9},\frac{1}{9},\frac{7}{9},\frac{1}{3})$
	$R_2^1(2,1,\chi),\ \chi^3 \not\equiv 1$	$--\text{"}--$ $s_j = \sin(\frac{\pi j}{18})$	$\mathrm{diag}(\frac{5}{9},\frac{8}{9},\frac{2}{9},\frac{2}{3})$

Finally, one obtains as a complete list of four dimensional irreducible level p^λ representations

$$\begin{aligned}
p^\lambda &= 5^1, && N_1(\chi)\ (\chi^3 \not\equiv 1), N_1(\chi)\ (\chi^3 \equiv 1),\\
p^\lambda &= 7^1, && R_1(1,\chi_1),\ R_1(2,\chi_1)\\
p^\lambda &= 2^3, && N_3(\chi),\ C_4 \otimes N_3(\chi)\\
p^\lambda &= 3^2, && B_i \otimes R_2^1(1,1,\chi), B_i \otimes R_2^1(2,1,\chi)
\end{aligned}$$

where $i = 1,2,3$ and for $p^\lambda = 3^2$ the character χ is a primitive character of order 3 or 6 (so there are 12 four dimensional irreducible level 3^2 representations).

The explicit form of the representations which are not related by tensor products with C_j or B_i is given in Table 7.1c.

7.2 The strongly-modular fusion algebras of dimension ≤ 4.

In this appendix we give complete lists the simple strongly-modular fusion algebras of dimension less than or equal to four.

Table 7.2a: Two and three dimensional strongly-modular fusion algebras

$\mathcal{F}$	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T)) \bmod \mathbb{Z}$
$\Phi_1\cdot\Phi_1=\Phi_0$ ($\mathbb{Z}_2$)	$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{1}{8},\frac{3}{8})\\ \mathrm{diag}(\frac{7}{8},\frac{5}{8})\end{cases}$
$\Phi_1\cdot\Phi_1=\Phi_0+\Phi_1$ ("(2,5)")	$\frac{2}{\sqrt{5}}\begin{pmatrix}-\sin(\frac{\pi}{5}) & -\sin(\frac{2\pi}{5})\\ -\sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5})\end{pmatrix}$ $\frac{2}{\sqrt{5}}\begin{pmatrix}-\sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5})\\ \sin(\frac{\pi}{5}) & \sin(\frac{2\pi}{5})\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{19}{20},\frac{11}{20})\\ \mathrm{diag}(\frac{1}{20},\frac{9}{20})\end{cases}$ $\begin{cases}\mathrm{diag}(\frac{3}{20},\frac{7}{20})\\ \mathrm{diag}(\frac{17}{20},\frac{13}{20})\end{cases}$
$\Phi_1\cdot\Phi_1=\Phi_2$ $\Phi_1\cdot\Phi_2=\Phi_0$ $\Phi_2\cdot\Phi_2=\Phi_1$ ($\mathbb{Z}_3$)	$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{2\pi i\frac{1}{3}} & e^{2\pi i\frac{2}{3}}\\ 1 & e^{2\pi i\frac{2}{3}} & e^{2\pi i\frac{1}{3}}\end{pmatrix}$	$\mathrm{diag}(\frac{1}{4},\frac{7}{12},\frac{7}{12})$
$\Phi_1\cdot\Phi_1=\Phi_0+\Phi_2$ $\Phi_1\cdot\Phi_2=\Phi_1+\Phi_2$ $\Phi_2\cdot\Phi_2=\Phi_0+\Phi_1+\Phi_2$ ("(2,7)")	$\frac{2}{\sqrt{7}}\begin{pmatrix}-s_2 & -s_1 & s_3\\ -s_1 & -s_3 & -s_2\\ s_3 & -s_2 & s_1\end{pmatrix}$ $\frac{2}{\sqrt{7}}\begin{pmatrix}-s_3 & -s_1 & s_2\\ -s_1 & -s_2 & -s_3\\ s_2 & -s_3 & s_1\end{pmatrix}$ $\frac{2}{\sqrt{7}}\begin{pmatrix}s_1 & s_2 & s_3\\ s_2 & -s_3 & s_1\\ s_3 & s_1 & -s_2\end{pmatrix}$ $s_j=\sin(\frac{j\pi}{7})$	$\begin{cases}\mathrm{diag}(\frac{4}{7},\frac{1}{7},\frac{2}{7})\\ \mathrm{diag}(\frac{3}{7},\frac{6}{7},\frac{5}{7})\end{cases}$ $\begin{cases}\mathrm{diag}(\frac{1}{7},\frac{4}{7},\frac{2}{7})\\ \mathrm{diag}(\frac{6}{7},\frac{3}{7},\frac{5}{7})\end{cases}$ $\begin{cases}\mathrm{diag}(\frac{2}{7},\frac{1}{7},\frac{4}{7})\\ \mathrm{diag}(\frac{5}{7},\frac{6}{7},\frac{3}{7})\end{cases}$
$\Phi_1\cdot\Phi_1=\Phi_0$ $\Phi_1\cdot\Phi_2=\Phi_2$ $\Phi_2\cdot\Phi_2=\Phi_0+\Phi_1$ ("(3,4)")	$\frac{1}{2}\begin{pmatrix}1 & 1 & \sqrt{2}\\ 1 & 1 & -\sqrt{2}\\ \sqrt{2} & -\sqrt{2} & 0\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{8-n}{16},\frac{16-n}{16},\frac{n}{8})\\ \mathrm{diag}(\frac{16-n}{16},\frac{8-n}{16},\frac{n}{8})\\ n=0,\ldots,7\end{cases}$

Table 7.2b: Four dimensional simple strongly-modular fusion algebras

$\mathcal{F}$	$\rho(S)$	$\frac{1}{2\pi i}\log(\rho(T)) \bmod \mathbb{Z}$
$\Phi_1^2=\Phi_2,\quad \Phi_1\cdot\Phi_2=\Phi_3,$ $\Phi_2^2=\Phi_0,\quad \Phi_1\cdot\Phi_3=\Phi_0,$	$\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&-1&-1\\1&-i&-1&i\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{7}{8},\frac{1}{4},\frac{3}{8},\frac{1}{4})\\\mathrm{diag}(\frac{3}{8},\frac{1}{4},\frac{7}{8},\frac{1}{4})\end{cases}$
$\Phi_3^2=\Phi_2,\quad \Phi_2\cdot\Phi_3=\Phi_1,$ $(\ \mathbb{Z}_4\ )$	$\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-i&-1&i\\1&-1&-1&-1\\1&i&-1&-i\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{5}{8},\frac{3}{4},\frac{1}{8},\frac{3}{4})\\\mathrm{diag}(\frac{1}{8},\frac{3}{4},\frac{5}{8},\frac{3}{4})\end{cases}$
$\Phi_1^2=\Phi_0,\quad \Phi_1\cdot\Phi_2=\Phi_3,$ $\Phi_2^2=\Phi_0,\quad \Phi_1\cdot\Phi_3=\Phi_2,$ $\Phi_3^2=\Phi_0,\quad \Phi_2\cdot\Phi_3=\Phi_1$ $(\ \mathbb{Z}_2\otimes\mathbb{Z}_2\ )$	$\frac{1}{2}\begin{pmatrix}1&-1&-1&-1\\-1&1&-1&-1\\-1&-1&1&-1\\-1&-1&-1&1\end{pmatrix}$	$\begin{cases}\mathrm{diag}(0,0,0,\frac{1}{2})\\\mathrm{diag}(\frac{1}{2},0,0,0)\end{cases}$
$\Phi_1^2=\Phi_0+\Phi_2$ $\Phi_1\cdot\Phi_2=\Phi_1+\Phi_3$	$\frac{2}{3}\begin{pmatrix}-s_4&s_1&s_3&-s_2\\s_1&s_2&s_3&s_4\\s_3&s_3&0&-s_3\\-s_2&s_4&-s_3&s_1\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{7}{36},\frac{19}{36},\frac{1}{12},\frac{31}{36})\\\mathrm{diag}(\frac{29}{36},\frac{17}{36},\frac{11}{12},\frac{5}{36})\end{cases}$
$\Phi_1\cdot\Phi_3=\Phi_2+\Phi_3$ $\Phi_2^2=\Phi_0+\Phi_2+\Phi_3$	$\frac{2}{3}\begin{pmatrix}s_1&s_2&s_3&s_4\\s_2&-s_4&s_3&-s1\\s_3&s_3&0&-s_3\\s_4&-s_1&-s_3&s_2\end{pmatrix}$	$\begin{cases}\mathrm{diag}(\frac{31}{36},\frac{7}{36},\frac{1}{12},\frac{19}{36})\\\mathrm{diag}(\frac{5}{36},\frac{29}{36},\frac{11}{12},\frac{17}{36})\end{cases}$
$\Phi_2\cdot\Phi_3=\Phi_1+\Phi_2+\Phi_3$ $\Phi_3^2=\Phi_0+\Phi_1+\Phi_2+\Phi_3$ $(\ "(2,9)"\ )$	$\frac{2}{3}\begin{pmatrix}s_2&-s_4&s_3&-s_1\\-s_4&s_1&s_3&-s_2\\s_3&s_3&0&-s_3\\-s_1&-s_2&-s_3&-s4\end{pmatrix}$ $s_j=\sin(\frac{j\pi}{9})$	$\begin{cases}\mathrm{diag}(\frac{19}{36},\frac{31}{36},\frac{1}{12},\frac{7}{36})\\\mathrm{diag}(\frac{17}{36},\frac{5}{36},\frac{11}{12},\frac{29}{12})\end{cases}$

7.3 The strongly-modular fusion algebras of dimension less than 24: Representations ρ, fusion matrices and graphs.

In this appendix we present the representations ρ of the modular group, the fusion matrices and the fusion graphs related to the nondegenerate strongly-modular fusion algebras of dimension less than 24.

Table 7.3: Simple nondegenerate strongly-modular fusion of dimension less than 24 (q is a prime satisfying $q < 47$)

fusion	dim	ρ
$\mathbb{Z}_2$	2	$C_4 \otimes N_3(\chi)_\pm, \ (p^\lambda = 2^3)$
"$c(3,4)$" Ising	3	$C_4 \otimes D_2(\chi)_+, \quad (p^\lambda = 2^2)$ $C_4 \otimes R_3^0(1,3,\chi)_\pm, \quad (p^\lambda = 2^3)$ $C_4 \otimes R_4^0(r,3,\chi)_\pm, \ (r = 1,2; p^\lambda = 2^4)$
"$(2,q)$"	$\frac{1}{2}(q-1)$	$C_4^{\frac{q+1}{2}} \otimes R_1(r,\chi_{-1}), \ ((\frac{r}{p}) = \pm 1; p^\lambda = q)$
"$(2,9)$"	4	$C_4 \otimes R_2^1(r,1,\chi), \quad (r = 1,2; \chi^3 \equiv 1; p^\lambda = 3^2)$
B_9	6	$N_2(\chi), \quad (\chi^3 \equiv 1; p^\lambda = 3^2)$
B_{11}	10	$N_1(\chi), \quad (\chi^3 \equiv 1; p^\lambda = 11)$
G_9	12	$C_4 \otimes R_3^1(r,1,\chi), \quad (r = 1,2; \chi^3 \equiv 1; p^\lambda = 3^3)$
G_{17}	16	$N_1(\chi), \quad (\chi^3 \equiv 1; p^\lambda = 17)$
E_{23}	22	$N_1(\chi), \quad (\chi^3 \equiv 1; p^\lambda = 23)$

The fusion matrices $\mathcal{N}_1$ which define the distinguished basis of the simple nondegenerate strongly-modular fusion algebras of dimension less than 24 are given by:

$$\mathbb{Z}_2 : \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{"}(3,4)\text{"} : \quad \mathcal{N}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\text{"}(2,q)\text{"} : \quad \mathcal{N}_1 = \left.\begin{pmatrix} 0 & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 \\ & & 1 & 1 \end{pmatrix}\right\} \tfrac{q-1}{2}$$

$$\mathrm{B}_9 : \qquad \mathcal{N}_1 = \begin{pmatrix} 0 & 1 & 0 & & & \\ 1 & 0 & 1 & 1 & & \\ 0 & 1 & 0 & 0 & 1 & \\ & 1 & 0 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 1 \\ & & & 0 & 1 & 1 \end{pmatrix}$$

$$\mathrm{B}_{11}: \qquad \mathcal{N}_1 = \begin{pmatrix}
0&1&0&0&&&&&& \\
1&0&1&0&0&&&&& \\
0&1&0&0&1&0&&&& \\
0&1&0&0&1&0&1&&& \\
&0&1&1&0&1&0&1&& \\
&&0&0&1&0&0&0&1& \\
&&&1&0&0&0&1&0&0 \\
&&&&1&0&1&1&1&0 \\
&&&&&1&0&1&1&1 \\
&&&&&&0&0&1&1
\end{pmatrix}$$

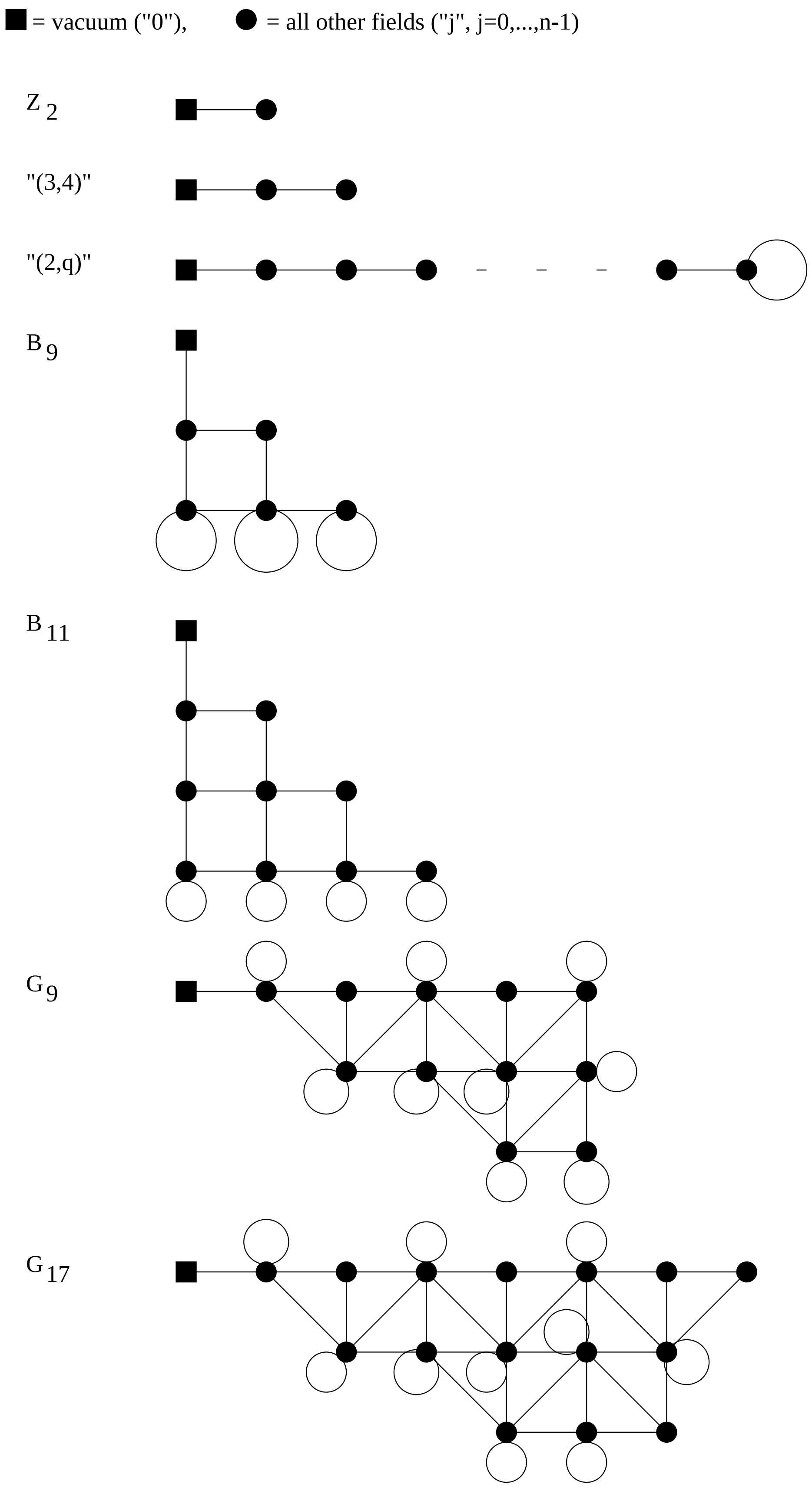
$$\mathrm{G}_{9}: \qquad \mathcal{N}_1 = \begin{pmatrix}
0&1&0&0&0&&&&&&& \\
1&1&1&1&0&0&&&&&& \\
0&1&1&1&1&1&0&&&&& \\
0&1&1&0&0&1&0&0&&&& \\
0&0&1&0&1&1&1&1&0&&& \\
&0&1&1&1&1&0&1&1&0&& \\
&&0&0&1&0&1&1&0&1&1& \\
&&&0&1&1&1&1&1&0&1&1 \\
&&&&0&1&0&1&0&0&0&1 \\
&&&&&0&1&0&0&1&1&0 \\
&&&&&&1&1&0&1&1&1 \\
&&&&&&&1&1&0&1&1
\end{pmatrix}$$

$$\mathrm{G}_{17}: \qquad \mathcal{N}_1 = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&&&&&&& \\
1&1&0&1&1&0&0&0&0&0&&&&&& \\
0&0&0&0&0&1&0&0&0&1&0&&&&& \\
0&1&0&0&1&0&0&0&0&0&1&0&&&& \\
0&1&0&1&1&0&0&0&0&0&1&1&0&&& \\
0&0&1&0&0&0&0&0&0&1&0&0&0&1&& \\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&1& \\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1 \\
0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0 \\
&0&1&0&0&1&1&0&0&1&0&0&0&1&1&0 \\
&&0&1&1&0&0&1&0&0&1&1&0&0&0&1 \\
&&&0&1&0&0&0&0&0&1&1&1&0&0&1 \\
&&&&0&0&0&0&1&0&0&1&1&0&1&1 \\
&&&&&1&0&1&0&1&0&0&0&1&1&1 \\
&&&&&&1&0&1&1&0&0&1&1&1&1 \\
&&&&&&&1&0&0&1&1&1&1&1&1
\end{pmatrix}$$

$$\mathrm{E}_{23}: \qquad \mathcal{N}_1 = \begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&&&&&&& \\
1&0&1&1&0&0&1&0&0&0&0&0&0&0&0&0&&&&&& \\
0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&&&&& \\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&&&& \\
0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&&& \\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&0&0&0&0&& \\
0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0& \\
0&0&0&0&1&0&0&0&0&1&0&0&0&1&1&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1&0 \\
0&0&1&1&0&1&1&1&0&0&0&0&0&0&0&1&0&0&1&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&0&1 \\
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&1&0 \\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&1&1&0&0&1&0 \\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&1&1&0&1 \\
0&0&0&0&0&1&0&1&1&0&1&0&0&0&0&0&1&0&1&1&1&0 \\
&0&0&1&0&0&1&0&0&1&0&1&0&0&0&1&0&0&1&1&0&1 \\
&&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&1&0&0&1&1 \\
&&&0&0&0&0&0&0&0&1&0&1&0&0&0&1&1&0&1&1&1 \\
&&&&1&0&0&0&0&1&0&0&0&1&1&1&0&0&1&0&1&1 \\
&&&&&0&0&0&0&1&0&1&0&1&1&1&0&1&0&1&1&1 \\
&&&&&&0&0&1&0&0&1&1&0&1&0&1&1&1&1&1&1 \\
&&&&&&&1&0&0&1&0&0&1&0&1&1&1&1&1&1&1
\end{pmatrix}.$$

The corresponding fusion graphs can be found on the next page.

We have omitted the fusion graph of the fusion algebra of type E_{23} since it is not possible to draw it without intersections in a plane.



7.4 Minimal models of Casimir $\mathcal{W}$-algebras.

In this last appendix we give some data related to the rational models of Casimir $\mathcal{W}$-algebras.

Let $\mathcal{K}$ be a simple Lie algebra of rank l over $\mathbb{C}$. Then the rational models of the Casimir $\mathcal{W}$-algebra related to this Lie algebra have central charge

$$c_{\mathcal{K}}(p,q) = l - \tfrac{12}{pq}(q\,\rho - p\,\rho^\vee)^2 \quad p,q \text{ coprime}, \quad h^\vee \le p \quad h \le q$$

where p and q have to be chosen minimal, h ($h^\vee$) denotes the (dual) Coxeter number of $\mathcal{K}$ and ρ ($\rho^\vee$) denotes the sum of its (dual) fundamental weights λ_i ($\lambda_i^\vee$). The conformal dimensions and conformal characters of the minimal model are given by [FKW]:

$$\begin{aligned} h_{\lambda,\nu^\vee} &= \frac{1}{2pq}\left((q\lambda - p\nu^\vee)^2 - (q\rho - p\rho^\vee)^2\right) \\ \chi_{\lambda,\nu^\vee}(q) &= \eta(q)^{-l} \sum_{w\in W}\sum_{t\in\Lambda^\vee} \epsilon(w) q^{\frac{1}{2pq}(qw(\lambda+\rho)-p(\nu^\vee+\rho^\vee)+pqt)^2} \end{aligned}$$

where λ ($\nu^\vee$) lies in the (dual) weight lattice so that $\lambda = \sum_{i=1}^l l_i\lambda_i$ and $\nu^\vee = \sum_{i=1}^l l_i^\vee \lambda_i^\vee$. λ and $\nu^\vee$ have to satisfy $\sum_{i=1}^l l_i m_i \le p-1$, $\sum_{i=1}^l l_i^\vee m_i^\vee \le q-1$ where m_i are the normalized components of the highest root ψ in the directions of the simple roots α_i, i.e. $\frac{\psi}{\psi^2} = \sum_{i=1}^l m_i \frac{\alpha_i}{\alpha_i^2}$. $m_i^\vee$ is given by $m_i^\vee = \frac{2}{\alpha_i^2} m_i$. Note that the set of conformal dimensions given by this condition has a symmetry so that all conformal dimensions of the minimal model occur with the same multiplicity in it (in the nonsimply laced cases the multiplicity is just 2). For more details see [FKW,B,GO,FL].

Table 7.4: Values of $m_i, m_i^\vee$ for all simple Lie algebras [GO].

Lie algebra	(m_i)	$(m_i^\vee)$
$\mathcal{A}_l$	$(1,\dots,1)$	$(1,\dots,1)$
$\mathcal{B}_l$	$(1,2,\dots,2,1)$	$(1,2,\dots,2)$
$\mathcal{C}_l$	$(1,\dots,1)$	$(2,\dots,2,1)$
$\mathcal{D}_l$	$(1,2,\dots,2,1,1)$	$(1,2,\dots,2,1,1)$
$\mathcal{E}_6$	$(1,2,2,3,2,1)$	$(1,2,2,3,2,1)$
$\mathcal{E}_7$	$(2,2,3,4,3,2,1)$	$(2,2,3,4,3,2,1)$
$\mathcal{E}_8$	$(2,3,4,6,5,4,3,2)$	$(2,3,4,6,5,4,3,2)$
$\mathcal{F}_4$	$(1,2,3,2)$	$(2,4,3,2)$
$\mathcal{G}_2$	$(2,1)$	$(2,3)$

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