# Investigations on the Foundation and Possible Modifications of the Gerasimov-Drell-Hearn Sum Rule

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**Abstract**

All derivations of the GDH sum rule are presented and discussed in detail, focussing particularly on the validity of the underlying assumptions. Several tests of the sum rule within perturbative models are reviewed. Various possible sources of modifications of the sum rule are examined. With respect to the current-algebra derivation, the crucial role of the infinite-momentum limit is pointed out. A derivation is presented that exhibits the infinite-momentum limit as its last step, opening the prospect of studying its legitimacy within perturbative models. Adopting the Weinberg-Salam model as a testing ground, it is shown that a modification due to an anomalous charge-density commutator is remedied owing to the infinite-momentum limit. This finding is confirmed upon considering $t$-channel exchange of axial-vector mesons. Several other aspects of possible sources of modifications are investigated. Evolving the GDH sum rule to non-zero photon virtualities $Q^2$, it is argued that different generalizations of the GDH integral, which coincide both at $Q^2 = 0$ and at large $Q^2$, may considerably deviate from one another at intermediate $Q^2$. The limits $Q^2 \to 0$ and $Q^2 \to \infty$ are investigated beyond leading terms. The contribution of the pion-nucleon final state to inclusive electroproduction is analyzed.

# Contents

# Chapter 1

# Introduction

## 1.1 History

The Gerasimov-Drell-Hearn (GDH) sum rule has been found in 1965 by Sergo Gerasimov [1], and then independently by Sidney Drell and Anthony Hearn [2] in 1966. The Russian original of Gerasimov's paper was published in the October 1965 issue of *Yadernaya Fizika;* the translation in *Soviet Journal of Nuclear Physics* appeared in April 1966, a month before the letter by Drell and Hearn was published in *Physical Review Letters.* This strict time ordering is why I call it the GDH sum rule and *not* DHG or even DH sum rule – names that sadly won't disappear even from the latest literature. By the way: There has been a third independent discovery of the sum rule. On May 20th 1966, the editors of *Progress of Theoretical Physics* received a letter by Hosoda and Yamamoto [3], who, besides, were the first to utilize a current-algebra technique for the derivation.

The most remarkable aspect of sum rules is that they relate particle properties that appear to be completely independent at first sight: some *low-energy* limit is expressed in terms of an integral that runs over *all energies.* In particular, the GDH sum rule

$$\boxed{-2\pi\mu_{\mathrm{a}}^2 = \int\limits_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big)} \tag{1.1}$$

relates the anomalous magnetic moment $\mu_{\mathrm{a}}$ of the nucleon (proton or neutron) to an energy-weighted integral of the difference $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ of the nucleon's polarized total photoabsorption cross sections. In Eq. (1.1), $\nu$ denotes the lab-frame energy of the photon, $\nu_0$ is the pion-production threshold, and subscripts 1/2 and 3/2 denote total helicity in the initial state, i.e., antiparallel and parallel $\gamma$N polarizations, respectively.

Albeit more than thirty years old today, the GDH sum rule still lacks a direct experimental check, since both beam and target have to be polarized and a wide range of photon energies has to be covered, which presents an enormous challenge. Presently, an experiment is in preparation at the Mainz microtron MAMI and – using the same target – at the electron stretcher facility ELSA at Bonn [4]. MAMI covers the energy range from pion-production threshold up to roughly 800 MeV, while photon energies from 500 to 3,000 MeV are generated at ELSA.

For the present, all we have experimentally are multipole analyses of unpolarized single-pion photoproduction data, from which – in an indirect fashion – estimates of the contribution

| | $-2\pi\mu_{\mathrm{a}}^2$ | $\int \frac{\mathrm{d}\nu}{\nu}\left(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\right)$ from resonance saturation |
|---|---|---|
| proton | $-204\ \mu$b | $-289\ \mu$b $(-65\ \mu$b) |
| neutron | $-233\ \mu$b | $-160\ \mu$b $(-35\ \mu$b) |
| proton-neutron difference | $29\ \mu$b | $-129\ \mu$b $(-30\ \mu$b) |

Table 1.1: Resonance saturation of the GDH integral [7]. Numbers in parentheses estimate the contribution of the $\pi\pi$N final state [5].

from low-lying resonances to the GDH integral can be extracted [5–7]. The result of the latest resonance saturation by Sandorfi, Whisnant, and Khandaker [7] is presented in Tab. 1.1. Besides resonance contributions, values for the GDH integral include non-resonant background and an estimate of the contribution of the $\pi\pi$N final state, extracted from known $\pi$N/$\pi\pi$N branching ratios of the resonances under consideration [5]. The latter contribution is indicated in parentheses in Tab. 1.1. A more detailed investigation of double-pion production has been performed by Coersmeier [8].

Hammer, Drechsel, and Mart [9] investigated the possible role of K$\Sigma$ and K$\Lambda$ final states, as well as $\phi$ and $\eta$ meson production. They found that kaon production contributes roughly $-3\ \mu$b to the GDH integral both for the proton-neutron difference and for the proton-neutron average, and that the contribution from $\phi$ and $\eta$ production is even smaller. Observe that the magnitude of these numbers is way below the discrepancy indicated in Tab. 1.1.

Although the explicit suppression of higher photon energies by the integration measure of the GDH sum rule supports the assumption that single-pion production saturates the sum rule, results of the saturations must be taken with care, since generally no errors can be attributed to the omission of resonances and final states. Nevertheless, a severe discrepancy remains for the proton-neutron difference.

The conclusion to be taken now is of course: Something is wrong, either with the saturation or with the sum rule itself! As far as the first possibility is concerned, it may be best to simply await the result of the MAMI and ELSA experiments instead of worrying too much about an interim solution. As for the second variant, there has in fact been much talk about possible modifications to the GDH sum rule ever since the late sixties. Kawarabayashi and Suzuki [10] and Khare [11] demonstrated that a modification arises if quarks possess anomalous magnetic moments. Abarbanel and Goldberger [12] pointed out that if the nucleon's Compton amplitude exhibits a $J = 1$ fixed pole in complex angular-momentum plane, then the sum rule is endowed with an additional term proportional to the residue of the pole. Owing to the lowest-order consideration of electromagnetic interactions, the fixed pole is not ruled out from $t$-channel unitarity. Chang, Liang, and Workman [13] claimed that the chiral anomaly brings about a modification by virtue of an anomalous charge-densities commutator calculated by Chang and Liang [14, 15]. Ying [16] suggested a modification due to "localized spontaneous breakdown of electromagnetic gauge symmetry".

Considering polarized inclusive electroproduction on the nucleon, the GDH sum rule can be generalized to non-zero values of the photon virtuality $Q^2$. At large $Q^2$, it has counterparts in the Bjorken [17] and Ellis-Jaffe [18] sum rules, which have been measured at SLAC [19–21] and at CERN [22, 23]. Intermediate momentum transfer ($Q^2 = 0.5$ GeV$^2$ and 1.2 GeV$^2$) was recently explored in the SLAC E143 experiment [24]. Forthcoming experiments at Jefferson Lab will considerably refine the data. The transition from photoabsorption, which is mainly driven by resonances, to the high-$Q^2$ region, being governed by perturbative QCD, is an interesting field in itself.

## 1.2 The scope of this thesis

The aim of the present thesis is a comprehensive treatment of the GDH sum rule and all possible sources of modifications suggested in the literature. A separate chapter is devoted to a discussion of the $Q^2$ evolution of the sum rule.

### Derivations of the GDH sum rule

In Ch. 2, different derivations of the GDH sum rule are presented and discussed in detail: the dispersion-theoretic approach given in the original publications by Gerasimov [1] and by Drell and Hearn [2], the derivation from the equal-times commutator of electric-charge-density operators [3, 25], and, finally, the light-cone current-algebra approach [26]. Important relations and distinguishing marks between different derivations are pointed out.

Particular attention is paid to the infinite-momentum limit [27], which enters the derivation from equal-times current algebra as a mere conjecture. In contrast to the existing literature, the infinite-momentum limit is postponed to the very end of the derivation presented in Sect. 2.2 in order to clarify its physical and mathematical meaning. Moreover, this procedure opens the prospect of thoroughly investigating the legitimacy of the infinite-momentum limit. The *finite-momentum GDH sum rule*, i.e., the form that the sum rule takes *before* the infinite-momentum limit is taken, is discussed in detail. Understanding the finite-momentum GDH sum rule is vital for the comprehension of the essentials of equal-times current algebra and the infinite-momentum limit.

### The GDH sum rule within perturbative models

Ch. 3 is devoted to an inspection of theoretical tests of the GDH sum rule within perturbative models. In Sect. 3.1, I consider the simplest model of "pointlike" pions and nucleons with pseudoscalar coupling among themselves and minimal coupling to the electromagnetic field [28]. In Sect. 3.2, I discuss QED [29, 30], while Sect. 3.3 presents the Weinberg-Salam model of electro-weak interactions of leptons [29]. This ordering implies increasing degree of complexity.

The Weinberg-Salam model is particularly well suited for investigating the legitimacy of the infinite-momentum limit in presence of an anomalous-commutator correction to the finite-momentum GDH sum rule.

### Possible sources of modifications

In Ch. 4, I review the proposed sources of modifications of the GDH sum rule. Sect. 4.1 is devoted to a discussion of a possible $J = 1$ fixed pole in angular-momentum plane [12]. In Sect. 4.2, I discuss the claimed modification coming from an anomalous charge-density algebra [13], while Sect. 4.3 aims at illustrating the significance of the infinite-momentum limit. In

Sect. 4.4, the Weinberg-Salam model is adopted to simultaneously calculate the anomalous charge-density commutator *and* the effect of the infinite-momentum limit [27]. It is shown that both of these points yield a non-trivial contribution to the GDH sum rule, in such a way that the individual modifications cancel exactly. Sect. 4.5 presents a short treatise on $t$-channel exchange of axial-vector mesons, in order to further illustrate that the conclusion of Sect. 4.4 is by no means restricted to the Weinberg-Salam model, but applies to hadrons, too.

The remaining sections of Ch. 4 are devoted to a critical discussion of several other claims of possible modifications that can be found in the literature.

### $Q^2$ evolution of the GDH sum rule

In Ch. 5, I investigate the transition from polarized photoabsorption to polarized inclusive electroproduction. After an introduction to the notion of virtual photoabsorption, I explain the physical meaning of different linear combinations of polarized nucleon structure functions. Sect. 5.2 is devoted to diverse generalizations of the GDH integral to non-zero photon virtuality. The significance of structure function $G_2(\nu, Q^2)$ is pointed out. Since all generalizations coincide both for real photons and in the scaling limit, non-leading terms in the limits $Q^2 \to 0$ and $Q^2 \to \infty$ are analyzed and discussed.

In view of resonance saturation of generalized GDH integrals, the contribution from single-pion production to structure functions and inclusive cross sections is treated in Sect. 5.3. It is expressed in terms of helicity amplitudes, Chew-Goldberger-Low-Nambu amplitudes, and helicity multipoles. The isospin decomposition is sketched.

### Tools

Some quite extensive calculations presented in this thesis have been performed with the aid of the computer algebra system *Mathematica* [31]. Feynman graphs were drawn using the LaTeX package `feynMF` [32] available from any CTAN host (e.g., ftp.dante.de).

## 1.3 Notation and conventions

Throughout this thesis, units are chosen such that Planck's constant $\hbar$, the vacuum speed of light $c$, and the permittivity of the vacuum $\varepsilon_0$ equal unity. Then the fine-structure constant is $\alpha = e^2/4\pi$.

Spacetime indices are represented by greek letters $\mu, \nu, \ldots$, whereas latin letters $i, j, \ldots$ are used for purely spatial indices. The metric is $g_{\mu\nu} = \mathrm{diag}(+---)$. Three-vectors are denoted by boldface symbols, e.g. $\boldsymbol{x}$. I use a Euclidian dreibein $\{\boldsymbol{e}_k\}$ such that $x^k = \boldsymbol{x}\cdot\boldsymbol{e}_k$. Partial derivatives are abbreviated as

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \nabla^i = \frac{\partial}{\partial x^i} = -\partial^i. \tag{1.2}$$

The four-dimensional Fourier transform reads

$$\tilde{f}(k) = \int \mathrm{d}^4 x \, e^{ik\cdot x} f(x), \tag{1.3}$$

with inverse

$$f(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, e^{-ik\cdot x} \tilde{f}(k). \tag{1.4}$$

The Fourier transform of the gradient $i\partial^\mu f(x)$ is $\tilde{f}(k)k^\mu$. For a translationally invariant, local Hilbert-space operator $A(x)$, this relation can be expressed in the form

$$i\partial^\mu A(x) = \big[A(x), \hat{p}^\mu\big], \tag{1.5}$$

where $\hat{p}^\mu$ denotes the energy-momentum operator. Eq. (1.5) corresponds to the convention that the Fourier transformed operator $\tilde{A}(k)$ maps a *bra* state $\langle \mathrm{X}|$ of given four-momentum $p_{\mathrm{X}}$ onto a state $\langle \mathrm{X}|\tilde{A}(k)$ having four-momentum $p_{\mathrm{X}} + k$. The integral form of Eq. (1.5) reads

$$A(x) = e^{i\hat{p}\cdot x} A(0) e^{-i\hat{p}\cdot x}. \tag{1.6}$$

Sandwiched between states $|\mathrm{X}\rangle$ and $|\mathrm{Y}\rangle$ with definite momenta $p_{\mathrm{X}}$ and $p_{\mathrm{Y}}$, Eq. (1.6) reduces to

$$\big\langle \mathrm{Y}\big|A(x)\big|\mathrm{X}\big\rangle = \big\langle \mathrm{Y}\big|A(0)\big|\mathrm{X}\big\rangle\, e^{i(p_{\mathrm{Y}}-p_{\mathrm{X}})\cdot x}. \tag{1.7}$$

In the theory of current commutators, I frequently deal with derivatives of delta functions, particularly the gradient of a three-dimensional delta function, which is defined by

$$\int \mathrm{d}^3 y\, \boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y})\, \Phi(\boldsymbol{y}) = \boldsymbol{\nabla}\Phi(\boldsymbol{x}), \tag{1.8}$$

where $\Phi(\boldsymbol{x})$ is a test function. That is to say, $\boldsymbol{\nabla}\delta$ may be regarded as a conventional function capable of being partially integrated and obeying

$$\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}) = \frac{\partial}{\partial \boldsymbol{x}}\delta(\boldsymbol{x}-\boldsymbol{y}) = -\frac{\partial}{\partial \boldsymbol{y}}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{1.9}$$

The Heaviside step function is defined by

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases} \tag{1.10}$$

Its derivative is the delta function, $\theta'(t) = \delta(t)$, and its Fourier transform equals $i/(E+i\varepsilon)$, i.e.,

$$\theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}E\, \frac{e^{-iEt}}{E+i\varepsilon} \tag{1.11}$$

and, inversely,

$$\frac{i}{E+i\varepsilon} = \int_0^\infty \mathrm{d}t\, e^{iEt}. \tag{1.12}$$

The totally antisymmetric tensors are normalized to $\varepsilon^{123} = \varepsilon^{0123} = +1$. For any four-vector $r$, one has the identity

$$r^\mu \varepsilon^{\nu\lambda\rho\sigma} = r^\nu \varepsilon^{\mu\lambda\rho\sigma} + r^\lambda \varepsilon^{\nu\mu\rho\sigma} + r^\rho \varepsilon^{\nu\lambda\mu\sigma} + r^\sigma \varepsilon^{\nu\lambda\rho\mu}, \tag{1.13}$$

which follows from the fact that a totally antisymmetric tensor of fifth or higher rank vanishes in four dimensions. Furthermore, I adopt the convention $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which gives rise to the trace theorem $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\varepsilon^{\mu\nu\rho\sigma}$. Dirac sigma matrices are defined by

$$\sigma^{\mu\nu} = \tfrac{i}{2}\,[\gamma^\mu, \gamma^\nu]. \tag{1.14}$$

The nucleon's mass, charge (in units of $e$), and anomalous magnetic moment (in units of the nuclear magneton $e/2M_\mathrm{p}$) are denoted by $M$, $Z$, and $\kappa$, respectively. Thus,

$$M_\mathrm{p} = 938 \text{ MeV} \qquad Z_\mathrm{p} = 1 \qquad \kappa_\mathrm{p} = 1.793 \tag{1.15a}$$

$$M_\mathrm{n} = 940 \text{ MeV} \qquad Z_\mathrm{n} = 0 \qquad \kappa_\mathrm{n} = -1.913 \tag{1.15b}$$

where the small mass difference is usually neglected. One-nucleon states $|p,\lambda\rangle$ with four-momentum $p$ and helicity $\lambda = \pm\frac{1}{2}$ are normalized covariantly,

$$\langle p', \lambda'|p, \lambda\rangle = (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}' - \boldsymbol{p})\, \delta_{\lambda'\lambda}. \tag{1.16}$$

Relative phases between different-helicity states are fixed according to Jacob and Wick [33]. Spinors $u(p,\lambda)$ are normalized as

$$\bar{u}(p,\lambda')u(p,\lambda) = 2M\,\delta_{\lambda'\lambda}. \tag{1.17}$$

Besides this helicity basis, I occasionally use states $|p,s\rangle$ and spinors $u(p,s)$ with arbitrary spin four-vector $s$ obeying

$$s^2 = -1, \tag{1.18}$$

$$p{\cdot}s = 0, \tag{1.19}$$

and

$$\bar{u}(p,s)\gamma^\mu\gamma_5 u(p,s) = 2Ms^\mu. \tag{1.20}$$

Moreover, one has that

$$u(p,s)\bar{u}(p,s) = (\not{p} + M)\,\frac{1+\gamma_5\not{s}}{2}. \tag{1.21}$$

The matrix $u(p,s)\bar{u}(p,s)/2M$ projects onto spin $s$ and positive energy. Helicity eigenstates are those for which the three-vectors $\boldsymbol{p}$ and $\boldsymbol{s}$ are collinear.

For the electron, which is considered in Sects. 3.2, 3.3, and 4.4, the nucleon mass $M$ has to be replaced by the electron mass $m$ in above formulae.

One-pion states $|p_\pi\rangle$, occurring in Ch. 5, are normalized as

$$\langle p'_\pi|p_\pi\rangle = (2\pi)^3\, 2p^0_\pi\, \delta(\boldsymbol{p}'_\pi - \boldsymbol{p}_\pi). \tag{1.22}$$

Pion-nucleon states $|p_\pi; p,\lambda\rangle$ are of course simply products of pion states $|p_\pi\rangle$ and nucleon states $|p,\lambda\rangle$. Their normalization is therefore given by the product of Eqs. (1.16) and (1.22).

Current densities $J^\mu(x)$ are defined in units of elementary charges per volume, i.e., for a single charged Dirac field $\psi(x)$ one has $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, and the physical current density (in proper units) is $eJ^\mu(x)$.

# Chapter 2

# Derivations of the GDH sum rule

In this chapter, I present three different derivations of the GDH sum rule, each of which having its advantages as well as its drawbacks. It is advisable to comprehend all of them, particularly in view of possible modifications to the GDH sum rule. The dispersion-theoretic derivation was given in the original publications by Gerasimov [1] and by Drell and Hearn [2], and it is comparatively simple. The derivation from equal-times current algebra has been presented by Hosoda and Yamamoto [3], Kawarabayashi and Suzuki [25], and, six years later, by Pradhan and Khare [34]. (All of these papers are extremely concise. Also, I must warn the reader of Ref. 34, which contains a vast number of errors on less than two pages.) Finally, there is a single and not very widely-known publication by Dicus and Palmer [26] that presents a derivation starting from a light-cone commutator of currents.

The current-algebra approaches appear much more involved than the dispersion-theoretic one, but partly this is due to the fact that in a way they re-derive the low-energy theorem, which is taken as a premise to the dispersion-theoretic approach. The general ideas of current algebra are treated in much detail in the textbook of Adler and Dashen [35], which is supplemented by a nice collection of reprints of classical papers on this topic.

## 2.1 Dispersion theory

### 2.1.1 Forward Compton amplitude

Consider the forward Compton-scattering process as depicted in Fig. 2.1. The corresponding

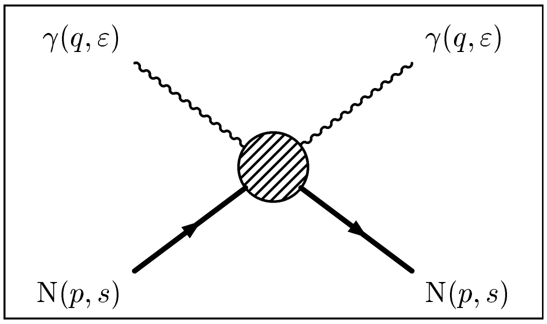



Figure 2.1: Forward Compton scattering off the nucleon

amplitude may be written

$$T^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq\cdot x}\big\langle p,s\big|\mathrm{T}J^\mu(x)J^\nu(0)\big|p,s\big\rangle, \tag{2.1}$$

where T denotes time ordering. This definition has no regard to the actual value of the photon virtuality $q^2$. In Ch. 5, it will be utilized for the generalization to virtual Compton scattering, but presently only the case $q^2 = 0$ (real photons) is of interest.

I further define the amplitude

$$T = \frac{e^2}{8\pi M}\,\varepsilon^*_\mu T^{\mu\nu}\varepsilon_\nu, \tag{2.2}$$

where the factor $e^2$ is due to the fact that the physical current density (in units of charge per volume) is $eJ^\mu(x)$, and $1/8\pi M$ absorbs a kinematic factor from the relation of this amplitude to the forward differential Compton cross section in the lab frame:

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^{\mathrm{lab}}}\right|_{\mathrm{forward}} = |T|^2. \tag{2.3}$$

The complex conjugation of the polarization vector of the outgoing photon in Eq. (2.2) is needed because circularly polarized photons will be considered. Adopting the Coulomb gauge $\varepsilon^0 = 0$, this amplitude has the decomposition

$$T(\nu) = \chi^\dagger\big[f_1(\nu)\,\boldsymbol{\varepsilon}^*\!\cdot\boldsymbol{\varepsilon} + i\nu f_2(\nu)\,\boldsymbol{\sigma}\!\cdot\!(\boldsymbol{\varepsilon}^*\!\times\boldsymbol{\varepsilon})\big]\chi, \tag{2.4}$$

where $\chi$ denotes the unit-normalized ($\chi^\dagger\chi = 1$) nucleon Pauli spinor and $\nu = p\!\cdot\! q/M$ is the photon lab-frame energy. Simultaneously flipping the photon helicity and the nucleon spin, which corresponds to the replacements $\boldsymbol{\varepsilon}^* \leftrightarrow \boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, leaves the amplitude (2.4) unaltered. This reflects the conservation of parity in strong and electromagnetic processes. Note that in the lab frame, the nucleon spin four-vector $s$ is given by $s^0 = 0$, $\boldsymbol{s} = \chi^\dagger\boldsymbol{\sigma}\chi$.

Decomposition (2.4) follows the notation of Drell and Hearn [2].[1] Observe that an explicit factor of $\nu$ renders the amplitude $f_2(\nu)$ even under crossing.

### Polarization

If one lets the photon travel along the third coordinate axis, $\boldsymbol{q} = q^0\boldsymbol{e}_3$, then the left-circularly polarized state, corresponding to photon helicity +1, is represented by $\boldsymbol{\varepsilon} = (-\boldsymbol{e}_1 - i\boldsymbol{e}_2)/\sqrt{2}$, which implies $\boldsymbol{\varepsilon}^*\!\cdot\boldsymbol{\varepsilon} = 1$ and $\boldsymbol{\varepsilon}^*\!\times\boldsymbol{\varepsilon} = i\boldsymbol{e}_3$. Thus, $T(\nu) = f_1(\nu) - \nu f_2(\nu)s^3$. If nucleon and photon have antiparallel spin, then $s^3 = -1$ and the pertinent amplitude reads

$$T_{1/2}(\nu) = f_1(\nu) + \nu f_2(\nu), \tag{2.5a}$$

where subscript 1/2 stands for "total helicity 1/2". (Note, however, that this is the difference – not the sum – of photon and nucleon helicities in the center-of-mass frame, and that the nucleon helicity in the lab frame is not even defined.) Analogously, if nucleon and photon spins are parallel, then $s^3 = +1$ and

$$T_{3/2}(\nu) = f_1(\nu) - \nu f_2(\nu). \tag{2.5b}$$

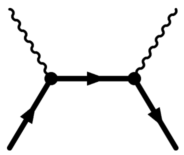
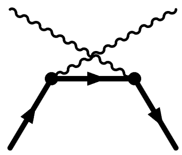
[1] The notation of Gerasimov [1] is related to the one of Drell and Hearn by $G_1(\nu) = f_1(\nu)$, $G_2(\nu) = \nu f_2(\nu)$.

**Lowest-order electromagnetic coupling**

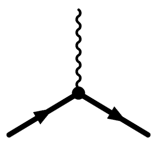
As usual in the theory of electromagnetic interactions of hadrons, all amplitudes are considered to lowest order in the electromagnetic coupling, i.e., only the order-$\alpha$ pieces of the amplitudes $f_{1,2}(\nu)$ are taken into account. Pictorially, this means that the blob in Fig. 2.1 containes only strongly interacting particles (quarks and gluons, or mesons and baryons, which do you prefer?), and no photons and leptons. As a consequence, all quantities under consideration (cross sections, masses, magnetic moments) are approximated by their lowest-order pieces. This ought to be kept in mind. I will further comment on it in the following subsections.

### 2.1.2 Low-energy theorem

In 1954, Low [36] and Gell-Mann and Goldberger [37] showed that to first order in the photon energy $\nu$, the Compton amplitude $T(\nu)$ equals its Born contribution

$T^{\text{Born}}(\nu) =$  $+$  (2.6)

where the vertices incorporate the nucleon's charge $Ze$ and anomalous magnetic moment $\mu_{\text{a}} = e\kappa/2M$,

 $= Ze\gamma^{\mu} + i\mu_{\text{a}} q_{\rho} \sigma^{\mu\rho}.$ (2.7)

The Born parts of amplitudes $f_{1,2}(\nu)$ read

$$f_1^{\text{Born}}(\nu) \equiv -\frac{Z^2\alpha}{M}, \tag{2.8a}$$

$$f_2^{\text{Born}}(\nu) \equiv -\frac{\mu_{\text{a}}^2}{2\pi}. \tag{2.8b}$$

Thus one has the zeroth-order low-energy theorem

$$f_1(0) = -\frac{Z^2\alpha}{M}, \tag{2.9a}$$

tracing back to Thirring [38], and on account of the explicit factor of $\nu$ accompanying amplitude $f_2(\nu)$, the equality of the first orders within $T(\nu)$ and $T^{\text{Born}}(\nu)$ results into the relation

$$\boxed{f_2(0) = -\frac{\mu_{\text{a}}^2}{2\pi} = -\frac{\alpha\kappa^2}{2M^2}} \tag{2.9b}$$

which was the novel finding of Low, Gell-Mann, and Goldberger.

Abarbanel and Goldberger [12] re-derived the low-energy theorem by means of dispersion relations. The methods employed in Refs. 36, 37 and 12 have been summarized by Bardakci and Pagels [39], who also emphasized the now-following point in a footnote.[2]

[2] A great deal of wisdom is frequently concealed in footnotes.

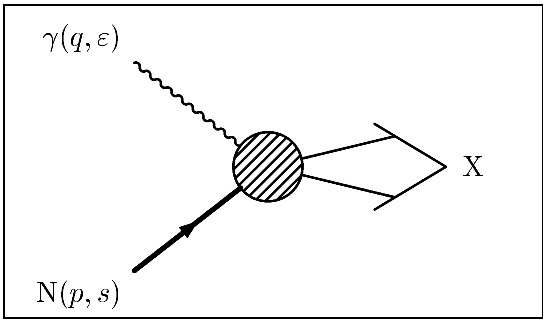



Figure 2.2: Polarized total photoabsorption on the nucleon, related to forward Compton scattering by means of the optical theorem. Cross sections for all final states X are summed over.

**Lowest-order electromagnetic coupling**

It may be worth mentioning that the quantity $\mu_{\mathrm{a}}$ in Eq. (2.9b) is actually not the true, measureable anomalous magnetic moment of the nucleon, but rather a fictitious lowest-order (in $\alpha$) contribution to it. This is due to the fact that Compton amplitudes are considered only to lowest order in the electromagnetic coupling. Pictorially, the absence of virtual photons inside the blob in Fig. 2.1 is reflected by the absence of virtual photons from the vertices of the Born contribution (2.6). We do not know this ficticious lowest-order anomalous magnetic moment since we cannot switch off electromagnetism inside hadrons. But we believe that it is a good approximation. This belief is inspired by the smallness of the Schwinger anomalous magnetic moment $\alpha/2\pi$ of the fundamental charged leptons. For hadrons, however, the approximate equality of lowest-order and all-orders anomalous magnetic moments is by no means necessary. On the other hand, it is not clear whether the low-energy theorem (2.9b) would survive the inclusion of all orders of $\alpha$. Only for the next order, it has been proved by Cheng [40, 41] and by Roy and Singh [42], that Eq. (2.9b) remains unchanged. As far as the Thirring theorem (2.9a) is concerned, the situation is completely different. This low-energy theorem is exact to all orders of $\alpha$, and the charge $Ze$ does, of course, not suffer radiative corrections, as does the magnetic moment.

### 2.1.3 Optical theorem

**Total photoabsorption**

Generally, the optical theorem, which rests upon unitarity of the $S$-matrix, relates the absorptive part of a forward amplitude to the corresponding total absorption cross section. In case of forward Compton scattering off the nucleon, the optical theorem reads

$$\mathrm{Im}\, f_1(\nu) = \frac{\nu}{8\pi}\big(\sigma_{1/2}(\nu) + \sigma_{3/2}(\nu)\big), \tag{2.10a}$$

$$\boxed{\mathrm{Im}\, f_2(\nu) = \frac{1}{8\pi}\big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big)} \tag{2.10b}$$

where $\sigma_{1/2}(\nu)$ and $\sigma_{3/2}(\nu)$ denote the photoabsorption cross sections of the nucleon for total photon-nucleon helicities $1/2$ and $3/2$, respectively. The photoabsorption process is depicted in Fig. 2.2.

**Derivation**

The optical theorem (2.10) is formally derived as follows. The absorptive part of the forward Compton amplitude (2.1) reads

$$2\pi W^{\mu\nu} := \operatorname{Abs} T^{\mu\nu} = \frac{1}{2}\int \mathrm{d}^4x\, e^{iq\cdot x} \big\langle p,s\big|[J^\mu(x), J^\nu(0)]\big|p,s\big\rangle. \tag{2.11}$$

In this expression, the second part of the commutator, $-J^\nu(0)J^\mu(x)$, can be omitted for positive photon energies $q^0$, since no intermediate state coupling to $\gamma$N is lighter than the nucleon itself:

$$\begin{aligned}
&\int \mathrm{d}^4x\, e^{iq\cdot x} \big\langle p,s\big|J^\nu(0)J^\mu(x)\big|p,s\big\rangle \\
&\qquad = \sum_{\mathrm{X}} \int \mathrm{d}^4x\, e^{iq\cdot x} \big\langle p,s\big|J^\nu(0)\big|\mathrm{X}\big\rangle\big\langle \mathrm{X}\big|J^\mu(x)\big|p,s\big\rangle \\
&\qquad = \sum_{\mathrm{X}} \int \mathrm{d}^4x\, e^{i(p_{\mathrm{X}}-p+q)\cdot x} \big\langle p,s\big|J^\nu(0)\big|\mathrm{X}\big\rangle\big\langle \mathrm{X}\big|J^\mu(0)\big|p,s\big\rangle \\
&\qquad = \sum_{\mathrm{X}} (2\pi)^4 \delta(p_{\mathrm{X}}-p+q) \big\langle p,s\big|J^\nu(0)\big|\mathrm{X}\big\rangle\big\langle \mathrm{X}\big|J^\mu(0)\big|p,s\big\rangle.
\end{aligned} \tag{2.12}$$

The delta function picks out intermediate states $|\mathrm{X}\rangle$ with four-momentum $p_{\mathrm{X}} = p - q$, which lies below the nucleon-mass shell for $q^0 > 0$. Hence, expression (2.12) vanishes, as claimed above.

The absorptive part of the decomposition (2.4) is obtained by simply replacing the amplitudes $f_{1,2}(\nu)$ by their imaginary parts,

$$\begin{aligned}
\operatorname{Abs} T(\nu) &= \frac{e^2}{8\pi M}\, \varepsilon^*_\mu \varepsilon_\nu \operatorname{Abs} T^{\mu\nu} \\
&= \chi^\dagger \big[\operatorname{Im} f_1(\nu)\, \boldsymbol{\varepsilon}^*\!\cdot\boldsymbol{\varepsilon} + i\nu \operatorname{Im} f_2(\nu)\, \boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}^*\times\boldsymbol{\varepsilon})\big]\chi.
\end{aligned} \tag{2.13}$$

The Coulomb-gauge photoabsorption cross section can be written (see, e.g., Itzykson and Zuber [43, App. A])

$$\sigma(\nu) = \frac{e^2}{4M\nu} \sum_{\mathrm{X}} (2\pi)^4 \delta(p+q-p_{\mathrm{X}}) \left|\langle \mathrm{X}|\boldsymbol{\varepsilon}\cdot\boldsymbol{J}(0)|p,s\rangle\right|^2, \tag{2.14}$$

where the sum runs over all possible final states $\mathrm{X} = \pi\mathrm{N}, \pi\pi\mathrm{N}, \ldots$, summing over spins and integrating over phase space. By virtue of the Fourier transform of the unity, $\int \mathrm{d}^4x\, e^{ik\cdot x} = (2\pi)^4\delta(k)$, and by employing the translational-invariance condition (1.7), the photoabsorption cross section can now be expressed in terms of the absorptive amplitudes,

$$\begin{aligned}
\sigma(\nu) &= \frac{e^2}{4M\nu} \int \mathrm{d}^4x\, e^{iq\cdot x} \sum_{\mathrm{X}} \langle p,s|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{J}(x)|\mathrm{X}\rangle\langle \mathrm{X}|\boldsymbol{\varepsilon}\cdot\boldsymbol{J}(0)|p,s\rangle \\
&= \frac{e^2}{2M\nu}\, \varepsilon^*_\mu \varepsilon_\nu \operatorname{Abs} T^{\mu\nu} = \frac{4\pi}{\nu} \operatorname{Abs} T(\nu).
\end{aligned} \tag{2.15}$$

Using relations (2.5), one obtains

$$\sigma_{1/2}(\nu) = \frac{4\pi}{\nu}\big(\operatorname{Im} f_1(\nu) + \nu \operatorname{Im} f_2(\nu)\big) \tag{2.16a}$$

and

$$\sigma_{3/2}(\nu) = \frac{4\pi}{\nu}\big(\mathrm{Im}\, f_1(\nu) - \nu\, \mathrm{Im}\, f_2(\nu)\big) \tag{2.16b}$$

for respective total helicities 1/2 and 3/2, from which Eqs. (2.10) follow immediately.

**Lowest-order electromagnetic coupling**

As in the case of the low-energy theorem, a few words on the omission of radiative corrections may be in order. The optical theorem relates the *true* (all orders in $\alpha$) forward Compton amplitude to the *true* photoabsorption cross section. What kind of cross section is the *lowest-order* amplitude (the one without virtual photons inside the blob of Fig. 2.1) related to? Again, this is a ficticious cross section, depicted by Fig. 2.2 if one removes all photons and lepton pairs from the final state X *and* from the blob. As far as the absence of photons and leptons from the final state is concerned, this is actually desired, since experimentally, a huge background of these particles is produced via Compton scattering off electrons and Bethe-Heitler pair production on nuclei. This background has to be disentangled from the signal anyway, which is one of the main reasons for considering only the lowest order in $\alpha$. However, neglecting virtual photons inside the interaction zone in Fig. 2.2 is an approximation. It means that effects like electromagnetic final-state interactions are disregarded. Again, it is believed that this approximation is a comparatively good one. Finally, it should be noted that the inclusion of higher orders of $\alpha$ would lead to some other non-trivial complications in the mere *definition* of amplitudes and cross sections on account of the inevitable radiation of soft photons in *any* scattering process.

### 2.1.4 Unsubtracted dispersion relation

**Analytical continuation**

From causality, the forward Compton amplitudes $f_{1,2}(\nu)$ can be analytically continued into the upper half of the complex energy plane. The continuation into the lower half plane is done by demanding the validity of the Schwarz reflection principle,

$$f_i(\nu^*) = f_i(\nu)^*, \quad i = 1, 2, \quad \nu \notin \mathbb{R}. \tag{2.17}$$

Approaching the real axis from below, this reduces to

$$f_i(\nu - i\varepsilon) = f_i(\nu)^*, \quad \nu \in \mathbb{R}. \tag{2.18}$$

Thus, the real parts of the amplitudes $f_i(\nu)$ are continuous, while their imaginary parts change sign when crossing the real axis and hence are discontinuous, leading to a *cut* if $f_i(\nu)$ is not real. Since one works to lowest order in $\alpha$, the lowest possible intermediate state in the Compton scattering process illustrated by Fig. 2.1 is the $\pi$N state with invariant mass $(p+q)^2 \geq (M + m_\pi)^2$. In the lab frame, this inequality reads $(M+\nu)^2 - \nu^2 \geq (M + m_\pi)^2$, or $\nu \geq \nu_0$ with

$$\nu_0 := m_\pi + \frac{m_\pi^2}{2M}, \tag{2.19}$$

which is the inelastic *threshold* of the photoproduction reaction. The least-mass state in the $u$ channel of the same reaction is again $\pi$N, opening at $(p-q)^2 \geq (M + m_\pi)^2$, or $\nu \leq -\nu_0$.

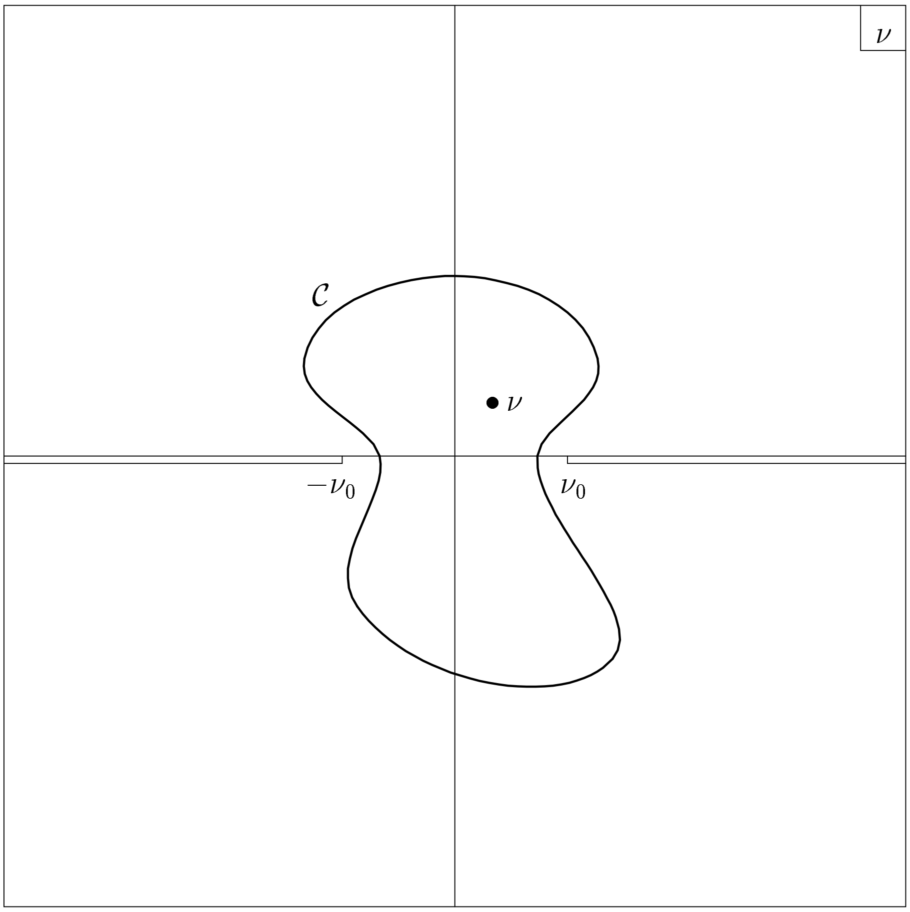



Figure 2.3: Complex energy plane of the forward Compton scattering process: cut structure, finite Cauchy-integration contour

Thus the above-mentioned cuts run from $-\infty$ to $-\nu_0$ and from $\nu_0$ to $\infty$ along the real axis of the complex $\nu$ plane. On the real axis itself, the values of functions $f_i(\nu)$ fit to the upper half plane. This structure of the complex $\nu$ plane is indicated in Fig. 2.3. Observe that crossing symmetry reads $f_i(-\nu) = f_i(\nu)$ for complex $\nu \notin \mathbb{R}$, so that for real energies, $f_i(-\nu) = f_i(-\nu + i\varepsilon) = f_i(\nu - i\varepsilon) = f_i(\nu)^*$ due to the reflection principle. Hence the real parts of the amplitudes are even functions of real $\nu$, while the imaginary parts are odd.

### Cauchy's theorem

One may now apply Cauchy's theorem,

$$f_i(\nu) = \frac{1}{2\pi i} \oint\limits_{\mathcal{C}} \frac{\mathrm{d}\nu'}{\nu' - \nu}\, f_i(\nu'), \tag{2.20}$$

where the point $\nu$ is kept away from the cuts, see Fig. 2.3. Of course, the integration contour $\mathcal{C}$ must not cross the cuts.

Now the contour is enlarged to the infinite one depicted in Fig. 2.4, assuming for the moment that the integral stays finite on every segment of the path. Later we will see whether

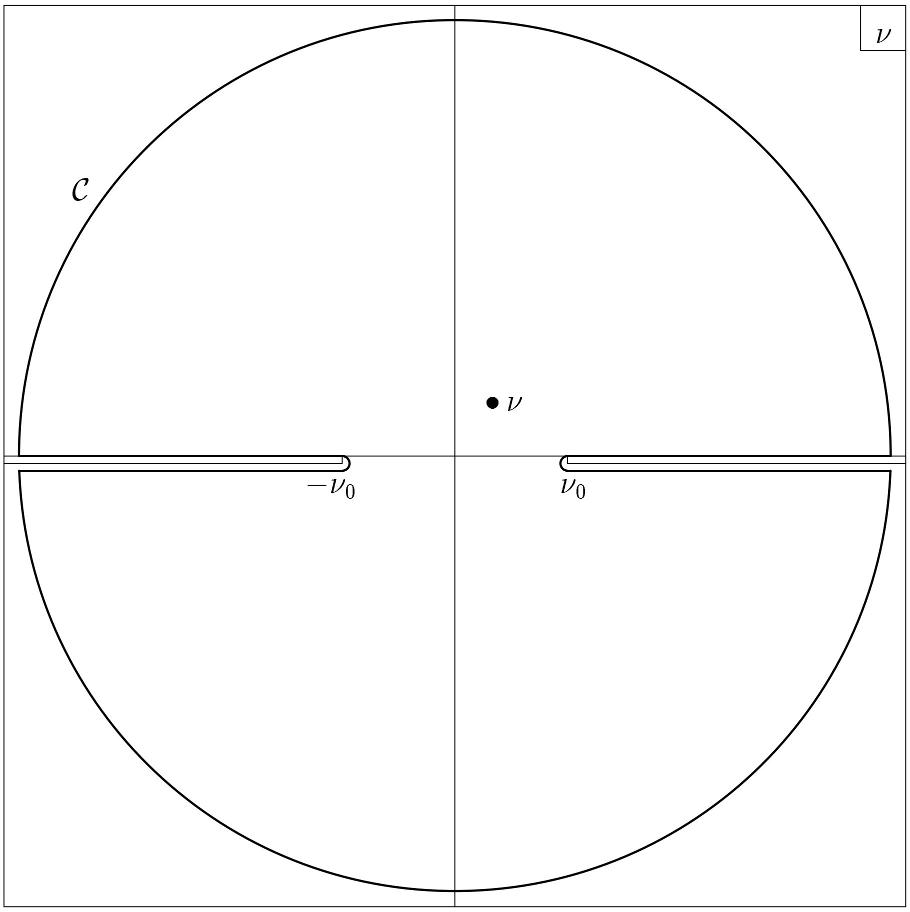



Figure 2.4: Blown-up Cauchy-integration contour, the half circles running at infinite photon energies

this assumption can be justified from Regge phenomenology. The contribution from the piece on the right-hand cut is just

$$\frac{1}{2\pi i}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'-\nu}\,f_i(\nu'). \tag{2.21a}$$

The contribution from the piece *below* the right-hand cut reads

$$\begin{aligned}\frac{1}{2\pi i}\int_{\infty-i\varepsilon}^{\nu_0-i\varepsilon}\frac{\mathrm{d}\nu'}{\nu'-\nu}\,f_i(\nu') &= -\frac{1}{2\pi i}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'-i\varepsilon-\nu}\,f_i(\nu'-i\varepsilon)\\ &= -\frac{1}{2\pi i}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'-\nu}\,f_i(\nu')^*,\end{aligned} \tag{2.21b}$$

where the first equality is a simple substitution of the integration variable and the second one is due to the continuity of the function $\nu'-\nu$ and the fact that functions $f_i(\nu)$ get complex-conjugated when crossing the real axis, Eq. (2.18). Putting Eqs. (2.21) together, one obtains a contribution

$$\frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'-\nu}\,\mathrm{Im}\,f_i(\nu') \tag{2.22a}$$

from the right-hand cut. Similarly, the pieces of the integration contour enclosing the left-hand cut contribute

$$\frac{1}{\pi}\int_{-\infty}^{-\nu_0}\frac{\mathrm{d}\nu'}{\nu'-\nu}\,\mathrm{Im}\,f_i(\nu') = \frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'+\nu}\,\mathrm{Im}\,f_i(\nu'), \tag{2.22b}$$

where the crossing properties of functions $f_i(\nu)$ have been employed.

### No-subtraction hypothesis

Now one makes the crucial assumption that the half circles at infinite complex photon energies, indicated in Fig. 2.4, do *not* contribute to the Cauchy integral (2.20). This requirement is fulfilled if functions $f_i(\nu)$ vanish asymptotically,

$$f_i(\nu) \xrightarrow[\nu\to\infty]{} 0. \tag{2.23}$$

If, on the other hand, function $f_2(\nu)$ approaches a finite constant $f_2(\infty)$ at $\nu \to \infty$, then the half circles contribute just this constant. I will revert to this possibility in Sect. 4.1. For the moment, let's stick to the assumption $f_i(\infty) = 0$. Then, the Cauchy integral (2.20) is the sum of expressions (2.22),

$$f_i(\nu) = \frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'\,\nu'}{\nu'^2-\nu^2}\,\mathrm{Im}\,f_i(\nu'). \tag{2.24}$$

Letting $\nu \to 0$ on both sides of this *unsubtracted dispersion relation* gives

$$f_i(0) = \frac{2}{\pi}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu'}{\nu'}\,\mathrm{Im}\,f_i(\nu'). \tag{2.25}$$

Together with the low-energy theorems (2.9) and the optical theorem (2.10) one arrives at

$$-\frac{4\pi^2 Z^2\alpha}{M} = \int_{\nu_0}^{\infty}\mathrm{d}\nu\big(\sigma_{1/2}(\nu)+\sigma_{3/2}(\nu)\big) \tag{2.26}$$

and

$$\boxed{-\frac{2\pi^2\alpha\kappa^2}{M^2} = \int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\big(\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\big)} \tag{2.27}$$

#### Divergence of the unpolarized integral

It can be seen at first sight that Eq. (2.26) is incorrect, because the integrand is a *positive* function of the photon energy, while the constant on the left-hand side is *negative* for the proton and *zero* for the neutron. What went wrong? Evidently, the Cauchy integration formula is valid for any *bounded* contour like the one in Fig. 2.3. But in blowing up the contour to the infinitely large shape depicted in Fig. 2.4, the contribution from the pieces enclosing the cuts might *diverge*, say, to $+\infty$. At the same time, the contribution from the large half circles will go to $-\infty$, since the sum of both is still the finite number $f(\nu)$. In this scenario, requirement (2.23) will be violated. In fact, as far as the unpolarized cross section within the integral of Eq. (2.26) is concerned, Regge theory tells us that its high-energy behavior is at least that of a constant. In case of the proton, for which one has very-high-energy data (up to photon lab-frame energies of roughly 20,000 GeV), there is even a significant *rise* of the total cross section [44, 45] (see also the *Review of Particle Physics* [46]). So this integral is indeed badly divergent.

#### Convergence of the GDH integral

Eq. (2.27) is the GDH sum rule. Mueller and Trueman [47] showed under the general assumptions of Regge theory, that a "worst-case" high-energy behavior of $\sigma_{1/2} - \sigma_{3/2}$ is caused by a cut in complex angular momentum plane running from $-\infty$ to 1 (possibly due to two-pomeron exchange [48]), leading to

$$\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu) \xrightarrow[\nu\to\infty]{} \frac{\text{const}}{\ln^2 \nu}, \tag{2.28}$$

which renders the integral finite,[3]

$$\int \frac{\mathrm{d}\nu}{\nu} \frac{\text{const}}{\ln^2 \nu} = \text{const} \int \frac{\mathrm{d} \ln \nu}{\ln^2 \nu} < \infty. \tag{2.29}$$

## 2.2 Equal-times current algebra

In this section, I present the current-algebra[4] derivation of the GDH sum rule, which is based essentially on two premises. Firstly, the operators of electric charge densities are assumed to commute at equal times. I will demonstrate below how the validity of this assumption is *naively* obtained from canonical anticommutation relations among fundamental Dirac fields, e.g., quarks. Sect. 4.2 is devoted to a more thorough investigation of this point. Secondly, one assumes that taking the infinite-momentum limit is legitimate. At the end of this section, the reader will hopefully have a solid notion of what this really means – both mathematically and physically. In the literature, one can find a couple of ansätze [10, 11, 13] that weaken the former assumption (equal-times commuting charge-density operators), but carelessly, the legitimacy of the infinite-momentum limit has never been questioned seriously. I will argue

[3] Observe that an asymptotically vanishing $\mathrm{Im}\, f_2(\nu)$ alone does *not* ensure convergence. For instance, a high-energy behavior of $\text{const}/\ln \nu$ would result in

$$\int \frac{\mathrm{d}\nu}{\nu} \frac{\text{const}}{\ln \nu} = \text{const} \int \frac{\mathrm{d} \ln \nu}{\ln \nu} \to \infty.$$

[4] For historical reasons, "current-algebra" means "equal-times current-algebra" here and in the following.

in Sect. 4.4 that in view of possible modifications to the GDH sum rule, the validity of the named assumptions is highly correlated.

I follow the idea of Hosoda and Yamamoto [3], but I postpone the infinite-momentum limit to the very end of the calculation in order to shed some light on its meaning.

### Timelike photons

For reasons that will become transparent later, the forward Compton amplitude

$$T^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq\cdot x}\langle p,s\big|\mathrm{T}J^\mu(x)J^\nu(0)\big|p,s\rangle \tag{2.30}$$

must be generalized to timelike photon virtualities $q^2 > 0$ in the current-algebra approach to sum rules. Since a virtual photon – in contrast to a real photon – can have longitudinal polarization, the number of invariant amplitudes increases from two (cf. Eq. (2.4)) to four. An appropriate decomposition of amplitude (2.30) runs

$$\begin{aligned} T^{\mu\nu} = &\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) S_1(\nu,q^2) \\ &+ \frac{1}{M^2}\left(p^\mu - \frac{M\nu}{q^2}q^\mu\right)\left(p^\nu - \frac{M\nu}{q^2}q^\nu\right) S_2(\nu,q^2) \\ &- \frac{i}{M}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\nu,q^2) \\ &- \frac{i}{M^3}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho\big((M\nu)s_\sigma - (q\cdot s)p_\sigma\big)A_2(\nu,q^2), \end{aligned} \tag{2.31}$$

where the multiplying powers of $M$ are such that all invariant amplitudes $S_{1,2}(\nu,q^2)$ and $A_{1,2}(\nu,q^2)$ are dimensionless. Observe that amplitudes $S_{1,2}(\nu,q^2)$ multiply tensors that are *symmetric* with respect to $\mu \leftrightarrow \nu$ and *independent* of the nucleon spin $s$, whereas amplitudes $A_{1,2}(\nu,q^2)$ multiply tensors *antisymmetric* w.r.t. $\mu \leftrightarrow \nu$ and *linear* in $s$. This coincidence reflects the conservation of parity in strong and electromagnetic interactions, analogously to the real-photon case (cf. the discussion following Eq. (2.4)).

At $q^2 = 0$, the above amplitudes are related to the amplitudes $f_{1,2}(\nu)$ of the previous section by

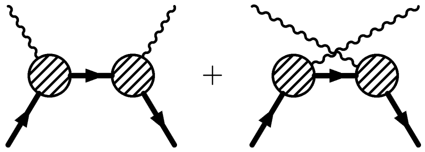
$$f_1(\nu) = \frac{\alpha}{2M}\, S_1(\nu,0) \tag{2.32a}$$

and

$$f_2(\nu) = \frac{\alpha}{2M^2}\, A_1(\nu,0). \tag{2.32b}$$

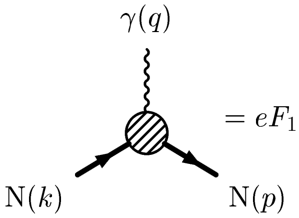
### Born contribution to forward virtual Compton amplitude

For later use, I report the Born contibution to amplitude (2.31), originating from the diagrams



(2.33)

The vertices incorporate the nucleon's Dirac and Pauli form factors, normalized by $F_1(0) = Z$ and $F_2(0) = \kappa$,




$$= eF_1(q^2)\,\gamma^\mu + \frac{ie}{2M}\,F_2(q^2)\,q_\rho\sigma^{\mu\rho}, \tag{2.34}$$

where $q = k - p$. The Born contribution to the invariant amplitudes $S_{1,2}(\nu, q^2)$ and $A_{1,2}(\nu, q^2)$ reads[5]

$$S_1^{\text{Born}}(\nu, q^2) = -2F_1^2(q^2) - \frac{2(q^2)^2\,G_{\text{M}}^2(q^2)}{(2M\nu)^2 - (q^2)^2}, \tag{2.35a}$$

$$S_2^{\text{Born}}(\nu, q^2) = 2\,\frac{4M^2q^2\,F_1^2(q^2) - (q^2)^2\,F_2^2(q^2)}{(2M\nu)^2 - (q^2)^2}, \tag{2.35b}$$

$$A_1^{\text{Born}}(\nu, q^2) = -F_2^2(q^2) + \frac{4M^2q^2\,F_1(q^2)G_{\text{M}}(q^2)}{(2M\nu)^2 - (q^2)^2}, \tag{2.35c}$$

$$A_2^{\text{Born}}(\nu, q^2) = \frac{4M^3\nu\,F_2(q^2)G_{\text{M}}(q^2)}{(2M\nu)^2 - (q^2)^2}, \tag{2.35d}$$

with the magnetic Sachs form factor $G_{\text{M}}(q^2) = F_1(q^2) + F_2(q^2)$. Observe that amplitudes $S_1(\nu, q^2)$ and $A_1(\nu, q^2)$ involve non-pole parts $-2F_1^2(q^2)$ and $-F_2^2(q^2)$, which do not vanish at $\nu \to \infty$.

In what follows, I will only need the "polarized" amplitudes $A_{1,2}(\nu, q^2)$, in particular the linear combination

$$f_2(\nu, q^2) := \frac{\alpha}{2M^2}\left(A_1(\nu, q^2) + \frac{q^2}{M\nu}A_2(\nu, q^2)\right), \tag{2.36}$$

which reduces to the familiar amplitude $f_2(\nu)$ at $q^2 = 0$. From Eqs. (2.35), its Born contribution is obtained as

$$f_2^{\text{Born}}(\nu, q^2) = -\frac{\alpha\,F_2^2(q^2)}{2M^2} + \frac{2\alpha q^2\,G_{\text{M}}^2(q^2)}{(2M\nu)^2 - (q^2)^2}. \tag{2.37}$$

At non-zero $q^2$, the linear combination (2.36) represents polarized forward scattering of a *purely transverse* virtual photon off a nucleon. This will be demonstrated in Ch. 5, where the notion of forward virtual Compton scattering recurs in the context of inclusive electroproduction. Observe, however, that presently I deal with timelike photon virtualities, while the photon being exchanged in electroproduction processes is always spacelike.

### 2.2.1 Equal-times commutator of electric charge densities

#### Current density of a Dirac field

The current density originating from a Dirac field $\psi(x)$ has the form

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) = \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x). \tag{2.38}$$

[5]Eqs. (2.35) remain unchanged if one supplements an on-shell vanishing portion proportional to $q^\mu \not{q}$ to the vertex (2.34).

In particular, the charge density reads

$$J^0(x) = \psi^\dagger(x)\psi(x). \tag{2.39}$$

For the present, a single Dirac field is considered. Multiple flavors and colors will be included below.

### Naive current-density commutator

In quantum field theory, the fields $\psi(x)$ and $\psi^\dagger(x)$ are taken to be operators. Then the definition (2.38) has its deficiencies, owing to the equality of the spacetime arguments of $\psi(x)$ and $\psi^\dagger(x)$. I postpone the discussion of this point to Sect. 4.2. For the moment, I want to stick to the definition (2.38) and work out the *naive* equal-times commutator of current densities, following from the canonical anticommutation relations

$$\big\{\psi_\alpha(x), \psi^\dagger_\beta(y)\big\}_{\rm et} = \delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{2.40a}$$

$$\big\{\psi^\dagger_\alpha(x), \psi^\dagger_\beta(y)\big\}_{\rm et} = \big\{\psi_\alpha(x), \psi_\beta(y)\big\}_{\rm et} = 0 \tag{2.40b}$$

among Dirac fields. Here, indices $\alpha$, $\beta$ denote spinor components, and the subscript "et" stands for "equal times", i.e., $x^0 = y^0$. The anticommutation relations (2.40) are employed repeatedly in the following calculation,

$$\begin{aligned}
\big[J^\mu(x), J^\nu(y)\big]_{\rm et} &= (\gamma^0\gamma^\mu)_{\alpha\beta}(\gamma^0\gamma^\nu)_{\gamma\delta}\big[\psi^\dagger_\alpha(x)\psi_\beta(x)\psi^\dagger_\gamma(y)\psi_\delta(y) \\
&\qquad - \psi^\dagger_\gamma(y)\psi_\delta(y)\psi^\dagger_\alpha(x)\psi_\beta(x)\big] \\
&= (\gamma^0\gamma^\mu)_{\alpha\beta}(\gamma^0\gamma^\nu)_{\gamma\delta}\big[\psi^\dagger_\alpha(x)\big(\delta_{\beta\gamma}\delta(\boldsymbol{x}-\boldsymbol{y}) - \psi^\dagger_\gamma(y)\psi_\beta(x)\big)\psi_\delta(y) \\
&\qquad - \psi^\dagger_\gamma(y)\big(\delta_{\delta\alpha}\delta(\boldsymbol{x}-\boldsymbol{y}) - \psi^\dagger_\alpha(x)\psi_\delta(y)\big)\psi_\beta(x)\big] \\
&= \psi^\dagger(x)[\gamma^0\gamma^\mu, \gamma^0\gamma^\nu]\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned} \tag{2.41}$$

Evidently, the commutator vanishes if $\mu = 0$ or $\nu = 0$, since $(\gamma^0)^2 = 1$ commutes with any matrix. This means that at equal times, the charge density commutes with each component of the current density,

$$\big[J^0(x), J^\mu(y)\big]_{\rm et} = 0. \tag{2.42}$$

For later use, I also work out the naive commutator of spatial components. One has

$$\gamma^0[\gamma^0\gamma^i, \gamma^0\gamma^j] = -\gamma^0[\gamma^i, \gamma^j] = 2i\,\varepsilon^{ijk}\gamma^k\gamma_5. \tag{2.43}$$

Thus,

$$\big[J^i(x), J^j(y)\big]_{\rm et} = 2i\,\varepsilon^{ijk}\,\bar\psi(x)\gamma^k\gamma_5\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{2.44}$$

which can be merged with Eq. (2.42), giving

$$\big[J^\mu(x), J^\nu(y)\big]_{\rm et} = 2i\,\varepsilon^{0\mu\nu\lambda}\,\bar\psi(x)\gamma_\lambda\gamma_5\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{2.45}$$

### Quarks

I now want to introduce quarks of different flavors and colors. This amounts to endowing the spinors with two additional indices and supplementing appropriate Kronecker symbols to the canonical anticommutation relations (2.40). In matrix notation, the electromagnetic current now reads

$$J^\mu(x) = \sum_{q=u,d(,s)} Z_q\, \bar{q}(x)\gamma^\mu q(x) = \bar{\psi}(x)\gamma^\mu Z^{(\mathrm{q})}\psi(x), \tag{2.46}$$

where $\psi(x)$ henceforth embodies diverse quark spinors and $Z^{(\mathrm{q})}$ denotes the diagonal quark charge matrix:

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix} \qquad \text{and} \qquad Z^{(\mathrm{q})} = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \tag{2.47a}$$

for two flavors u and d, or

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix} \qquad \text{and} \qquad Z^{(\mathrm{q})} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{2.47b}$$

for three flavors u, d, and s. The notion $\gamma^\mu Z^{(\mathrm{q})}$ in Eq. (2.46) actually denotes the tensor product $\gamma^\mu_{\text{Dirac}} \otimes Z^{(\mathrm{q})}_{\text{flavor}} \otimes 1_{\text{color}}$. Re-doing the above calculation of the naive current commutator, one arrives at

$$\big[J^\mu(x), J^\nu(y)\big]_{\text{et}} = 2i\,\varepsilon^{0\mu\nu\lambda}\,\bar{\psi}(x)\gamma_\lambda\gamma_5\big(Z^{(\mathrm{q})}\big)^2\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{2.48}$$

Again, the charge density commutes with each component of the current. (This fact is independent of flavors and colors.) In particular, the naive equal-times commutator of electric charge densities, which is the one required by the current-algebra approach to the GDH sum rule, reads

$$\boxed{\big[J^0(x), J^0(y)\big]_{\text{et}} = 0} \tag{2.49}$$

### Electric dipole-moment operator

Most of the information contained in the charge-density commutator (2.49) is redundant with respect to the GDH sum rule. In fact, only a certain *moment* of it is needed, namely the commutator of electric dipole moments, defined in the usual way as

$$\boldsymbol{D}(x^0) = e\int \mathrm{d}^3x\, \boldsymbol{x} J^0(x). \tag{2.50}$$

Relation (2.49) implies

$$\big[D^i(0), D^j(0)\big] = 0 \tag{2.51}$$

for any pair of spatial indices $i, j$. The dipole moments are evaluated at time 0, but of course any other (equal) time would do. Still, the derivation of the GDH sum rule requires by far not the full information contained in Eq. (2.51), but only the *nucleon matrix element* thereof. I take the incoming nucleon to be traveling along $\boldsymbol{e}_3$,

$$p^\mu = \big(p^0, 0, 0, \sqrt{(p^0)^2 - M^2}\big), \tag{2.52}$$

and define definite-chirality components of the dipole operator,

$$D^{\mathrm{L,R}}(0) = \frac{1}{\sqrt{2}}\big(D^1(0) \pm i D^2(0)\big), \tag{2.53}$$

which correspond to circularly polarized photons. Finally, both nucleon states are taken to have positive helicity,

$$\big\langle p', \tfrac{1}{2}\big|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\big|p, \tfrac{1}{2}\big\rangle = 0. \tag{2.54}$$

### Complete set of intermediate states

As is standard in the current-algebra derivation of sum rules, one now inserts a complete set of intermediate states and separates the one-nucleon states from the continuum:

$$1 = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3 p''}{(2\pi)^3 2p''^0}\, |p'', \lambda\rangle\langle p'', \lambda| + \sum_{\mathrm{X}}{}' |\mathrm{X}\rangle\langle\mathrm{X}|. \tag{2.55}$$

The primed sum runs over all hadronic intermediate states except the one-nucleon states. Non-hadronic particles are excluded from the sum, corresponding to the fact that all quantities are consided to lowest order in the electromagnetic coupling, as discussed in the previous section.

### One-nucleon contribution

To work out the one-nucleon intermediate-state contribution to the matrix element (2.54),

$$\begin{aligned}
&\big\langle p', \tfrac{1}{2}\big|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\big|p, \tfrac{1}{2}\big\rangle_{\text{one-nucleon}} \\
&\quad = \sum_{\lambda=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3 p''}{(2\pi)^3 2p''^0}\, \big\langle p', \tfrac{1}{2}\big|D^{\mathrm{L}}(0)\big|p'', \lambda\big\rangle\big\langle p'', \lambda\big|D^{\mathrm{R}}(0)\big|p, \tfrac{1}{2}\big\rangle - \{\mathrm{L} \leftrightarrow \mathrm{R}\},
\end{aligned} \tag{2.56}$$

I need the nucleon matrix element of the electromagnetic current:

$$\big\langle p_1\big|J^\mu(0)\big|p_2\big\rangle = \bar{u}(p_1)\,\Gamma^\mu(p_1 - p_2)\,u(p_2), \tag{2.57}$$

where

$$\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} q_\nu \frac{F_2(q^2)}{2M} \tag{2.58a}$$

$$= \gamma^\mu F_1(q^2) - \frac{1}{2}\,[\gamma^\mu, \not{q}]\,\frac{F_2(q^2)}{2M}. \tag{2.58b}$$

In particular,

$$\gamma^0 \Gamma^0(q) = F_1(q^2) + \boldsymbol{\gamma}\cdot\boldsymbol{q}\frac{F_2(q^2)}{2M}. \tag{2.59}$$

Eventually, Dirac and Pauli form factors $F_{1,2}(q^2)$ will be evaluated only at zero momentum transfer, where they are normalized to $F_1(0) = Z$ and $F_2(0) = \kappa$. The matrix element of the dipole-moment operator (2.50) can now be written[6]

$$\langle p_1 | D^i(0) | p_2 \rangle = ie\,(2\pi)^3\, \nabla^i \delta(\boldsymbol{p}_1 - \boldsymbol{p}_2)\, \bar{u}(p_1)\, \Gamma^0(p_1 - p_2)\, u(p_2), \tag{2.60}$$

and the one-nucleon intermediate-state contribution to the matrix element of the commutator of dipole moments runs

$$\begin{aligned} &\langle p', \tfrac{1}{2} | [D^i(0), D^j(0)] | p, \tfrac{1}{2} \rangle_{\text{one-nucleon}} \\ &\quad = (2\pi)^3 \int \frac{\mathrm{d}^3 p''}{2p''^0}\, \nabla^i \delta(\boldsymbol{p}' - \boldsymbol{p}'')\, \nabla^j \delta(\boldsymbol{p} - \boldsymbol{p}'') \\ &\quad\quad \times \bar{u}(p', \tfrac{1}{2})\, \Gamma^0(p' - p'')\, (\not{p}'' + M)\, \Gamma^0(p'' - p)\, u(p, \tfrac{1}{2}) - \{i \leftrightarrow j\}. \end{aligned} \tag{2.61}$$

Further manipulation of this expression is quite intricate.[7] One has to integrate by parts to get rid of the $\boldsymbol{p}''$ integral together with one of the delta functions. Up to second derivatives of $\delta(\boldsymbol{p}' - \boldsymbol{p})$ will be left then. The second derivative $\nabla^i \nabla^j$ does not contribute, owing to the explicit antisymmetrization in indices $i, j$ due to the commutator. Subsequently, one has to repeatedly employ Leibnitz rule in the form

$$\frac{\partial}{\partial \boldsymbol{p}'} \delta(\boldsymbol{p}' - \boldsymbol{p})\, \Phi(\boldsymbol{p}, \boldsymbol{p}') + \delta(\boldsymbol{p}' - \boldsymbol{p})\, \frac{\partial}{\partial \boldsymbol{p}'} \Phi(\boldsymbol{p}, \boldsymbol{p}') = \frac{\partial}{\partial \boldsymbol{p}'} \big[ \delta(\boldsymbol{p}' - \boldsymbol{p})\, \Phi(\boldsymbol{p}, \boldsymbol{p}') \big], \tag{2.62}$$

which implies

$$\boldsymbol{\nabla} \delta(\boldsymbol{p}' - \boldsymbol{p})\, \Phi(\boldsymbol{p}, \boldsymbol{p}') = \boldsymbol{\nabla} \delta(\boldsymbol{p}' - \boldsymbol{p})\, \Phi(\boldsymbol{p}, \boldsymbol{p}) - \delta(\boldsymbol{p}' - \boldsymbol{p})\, \frac{\partial}{\partial \boldsymbol{p}'} \Phi(\boldsymbol{p}, \boldsymbol{p}'), \tag{2.63}$$

and it turns out that no derivative of $\delta(\boldsymbol{p}' - \boldsymbol{p})$ remains at all. Introducing L and R components again, the result reads

$$\langle p', \tfrac{1}{2} | [D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)] | p, \tfrac{1}{2} \rangle_{\text{one-nucleon}} = (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}' - \boldsymbol{p}) \left( \frac{2\pi\alpha\kappa^2}{M^2} - \frac{2\pi\alpha(Z + \kappa)^2}{(p^0)^2} \right). \tag{2.64}$$

Here I stress the presence of the second term, which vanishes in the infinite-momentum limit $p^0 \to \infty$. Hitherto, its only appearance in the literature was in Ref. 34, where it has an incorrect form.

### Continuum contribution

To obtain the continuum contribution to matrix element (2.54), i.e., the sum over all intermediate states $|\mathrm{X}\rangle$ except the one-nucleon state,

$$\langle p', \tfrac{1}{2} | [D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)] | p, \tfrac{1}{2} \rangle_{\text{cont}} = {\sum_{\mathrm{X}}}' \langle p', \tfrac{1}{2} | D^{\mathrm{L}}(0) | \mathrm{X} \rangle \langle \mathrm{X} | D^{\mathrm{R}}(0) | p, \tfrac{1}{2} \rangle - \{\mathrm{L} \leftrightarrow \mathrm{R}\}, \tag{2.65}$$

[6] The multiplication with coordinate $\boldsymbol{x}$ in Eq. (2.50) corresponds to a derivative in momentum space, while the $\boldsymbol{x}$ integration gives rise to a delta function. This leads to the expression $\nabla^i \delta(\boldsymbol{p}_1 - \boldsymbol{p}_2)$ in Eq. (2.60). Due to CP conservation, the nucleon itself does of course *not* possess an electric dipole moment. If it did, expression (2.58a) would contain a third term of the form $\gamma_5 \sigma^{\mu\nu} q_\nu F_3(q^2)/2M$ (see, e.g., Itzykson and Zuber [43, Eq. (3-203)]).

[7] I proceeded by introducing a concrete representation for spinors and gamma matrices in Eq. (2.61), letting computer-algebra system *Mathematica* handle the resulting large expression. Particular caution is advisable on account of the momentum dependence of form factors and spinors. I mention in passing that if one forgets the latter, one arrives essentially at the erroneous result of Pradhan and Khare [34].

I employ current conservation $\partial_\mu J^\mu(x) = 0$ to get

$$\dot{D}^i(x^0) = e\int \mathrm{d}^3x\, x^i \partial_0 J^0(x) = -e\int \mathrm{d}^3x\, x^i \boldsymbol{\nabla}\cdot\boldsymbol{J}(x) = e\int \mathrm{d}^3x\, J^i(x). \tag{2.66}$$

Since the time derivative of any operator is related to the commutator of that operator with the Hamiltonian (cf. Eq. (1.5)), one has

$$\begin{aligned}\langle \mathrm{X}|D^{\mathrm{R}}(0)|p,\tfrac{1}{2}\rangle &= \frac{i}{p^0 - p^0_{\mathrm{X}}}\,\langle \mathrm{X}|\dot{D}^{\mathrm{R}}(0)|p,\tfrac{1}{2}\rangle \\ &= \frac{ie}{p^0 - p^0_{\mathrm{X}}}\int \mathrm{d}^3x\,\langle \mathrm{X}|J^{\mathrm{R}}(\boldsymbol{x},0)|p,\tfrac{1}{2}\rangle \\ &= \frac{ie}{p^0 - p^0_{\mathrm{X}}}\,(2\pi)^3\delta(\boldsymbol{p}_{\mathrm{X}} - \boldsymbol{p})\,\langle \mathrm{X}|J^{\mathrm{R}}(0)|p,\tfrac{1}{2}\rangle, \end{aligned} \tag{2.67}$$

where definite-chirality components of the current are defined by $J^{\mathrm{L,R}}(x) = (J^1(x) \pm iJ^2(x))/\sqrt{2}$. The translational-invariance condition (1.7) has been used to carry out the spatial integrations. Inserting this into Eq. (2.65), one obtains

$$\begin{aligned}&\langle p',\tfrac{1}{2}|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]|p,\tfrac{1}{2}\rangle_{\mathrm{cont}} \\ &\quad = (2\pi)^3\,\delta(\boldsymbol{p}' - \boldsymbol{p})\sum_{\mathrm{X}}{}'\,(2\pi)^3\,\delta(\boldsymbol{p}_{\mathrm{X}} - \boldsymbol{p})\,\frac{4\pi\alpha|\langle p,\frac{1}{2}|J^{\mathrm{L}}(0)|\mathrm{X}\rangle|^2}{(p^0 - p^0_{\mathrm{X}})^2} - \{\mathrm{L} \to \mathrm{R}\}. \end{aligned} \tag{2.68}$$

Owing to the factor of $\delta(\boldsymbol{p}' - \boldsymbol{p})$ and since $J^{\mathrm{L}}(x)$ is the adjoint of $J^{\mathrm{R}}(x)$, the right-hand side of Eq. (2.68) could be written in terms of the square of a current matrix element between a one-nucleon initial state and a hadronic final state being summed over. This looks very much like a photoabsorption cross section, but there's some work left to do.

Introducing the timelike virtual photon momentum $q$ with $\boldsymbol{q} = 0$, I can substitute

$$\delta(\boldsymbol{p}_{\mathrm{X}} - \boldsymbol{p}) = \int_{q^0_{\mathrm{thr}}}^{\infty} \mathrm{d}q^0\,\delta(p_{\mathrm{X}} - p - q) = (2\pi)^{-4}\int_{q^0_{\mathrm{thr}}}^{\infty} \mathrm{d}q^0 \int \mathrm{d}^4x\, e^{-i(p_{\mathrm{X}} - p - q)\cdot x}, \tag{2.69}$$

where the pion-production threshold $q^0_{\mathrm{thr}} \equiv M\nu_{\mathrm{thr}}/p^0$ is determined by $(p + q_{\mathrm{thr}})^2 = (M + m_\pi)^2$, or explicitly,

$$\nu_{\mathrm{thr}} = \frac{p^0 q^0_{\mathrm{thr}}}{M} = \left(m_\pi + \frac{m_\pi^2}{2M}\right)\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2Mm_\pi + m_\pi^2}{(p^0)^2}}\right)^{-1}. \tag{2.70}$$

For $p^0 \to \infty$, $\nu_{\mathrm{thr}}$ approaches the familiar pion-photoproduction threshold $\nu_0 = m_\pi + m_\pi^2/2M$.

Utilizing translational invariance once more, Eq. (2.68) can be cast into the form

$$\begin{aligned}\langle p',\tfrac{1}{2}|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]|p,\tfrac{1}{2}\rangle_{\mathrm{cont}} &= (2\pi)^3\, 2p^0\,\delta(\boldsymbol{p}' - \boldsymbol{p})\int_{q^0_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}q^0}{q^0}\,\frac{\alpha}{p^0 q^0} \\ &\times \int \mathrm{d}^4x\, e^{iq\cdot x}\sum_{\mathrm{X}}{}'\,\langle p,\tfrac{1}{2}|J^{\mathrm{L}}(x)|\mathrm{X}\rangle\langle \mathrm{X}|J^{\mathrm{R}}(0)|p,\tfrac{1}{2}\rangle - \{\mathrm{L} \to \mathrm{R}\}. \end{aligned} \tag{2.71}$$

Observe now that the prime at the sum over intermediate states can be dropped, i.e., one-nucleon states can be added without altering the result, as can most easily be seen by considering the four-dimensional delta function within Eq. (2.69): For an on-shell nucleon and a timelike (or even lightlike) photon with positive energy $q^0$, the four-vector $p_{\mathrm{X}} = p + q$ will always represent a state having a mass strictly greater than that of the nucleon.[8] In physical terms, this simply means that a nucleon cannot absorb a photon without also emitting something. Upon using the completeness relation $\sum_{\mathrm{X}} |\mathrm{X}\rangle\langle\mathrm{X}| = 1$, one then has that

$$\begin{aligned}\langle p', \tfrac{1}{2}|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]|p, \tfrac{1}{2}\rangle_{\mathrm{cont}} &= (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}' - \boldsymbol{p}) \int\limits_{q^0_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}q^0}{q^0} \frac{2\alpha}{M\nu} \\ &\times \frac{1}{2}\int \mathrm{d}^4x\, e^{iq\cdot x} \langle p, \tfrac{1}{2}|J^{\mathrm{L}}(x) J^{\mathrm{R}}(0)|p, \tfrac{1}{2}\rangle - \{\mathrm{L} \leftrightarrow \mathrm{R}\},\end{aligned} \tag{2.72}$$

where $\nu = p^0 q^0/M = p{\cdot}q/M$ is the photon's lab-frame energy. The last line is recognized as the antisymmetrized LR component of the absorptive part (2.11) of the forward virtual Compton amplitude. In view of the decomposition (2.31) and using $\boldsymbol{q} = 0$, $s^3 = p^0/M$, and $\varepsilon^{\mathrm{LR}03} = -i$, one has

$$\mathrm{Abs}\, T^{\mathrm{LR}} - \mathrm{Abs}\, T^{\mathrm{RL}} = \frac{\nu}{\pi M}\, \mathrm{Im}\left( A_1(\nu, q^2) + \frac{q^2}{M\nu} A_2(\nu, q^2)\right), \tag{2.73}$$

with $q^2 = (q^0)^2$. Now the continuum contribution (2.72) can be written in terms of the imaginary part of the forward virtual Compton amplitude $f_2(\nu, q^2)$, Eq. (2.36),

$$\langle p', \tfrac{1}{2}|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]|p, \tfrac{1}{2}\rangle_{\mathrm{cont}} = (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}' - \boldsymbol{p})\, 8 \int\limits_{q^0_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}q^0}{q^0}\, \mathrm{Im}\, f_2(\nu, q^2). \tag{2.74}$$

### The finite-momentum GDH sum rule

Since the one-nucleon part (2.64) and the continuum part (2.74) sum up to give the commutator matrix element (2.54), I conclude

$$\boxed{-\frac{2\pi^2\alpha\kappa^2}{M^2} + \frac{2\pi^2\alpha(Z+\kappa)^2}{(p^0)^2} = \int\limits_{\nu_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, 8\pi\, \mathrm{Im}\, f_2\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right)} \tag{2.75}$$

where I have performed a linear substitution $q^0 \mapsto \nu = p^0 q^0/M$ of the integration variable. I call this equation the *finite-momentum* GDH sum rule. It is based solely on the naive charge-density commutator (2.49), or on the weaker assumption presented by Eq. (2.54). In particular, the integral on the right-hand side of Eq. (2.75) *converges*, since it relies only on the validity of the completeness relation for the physical intermediate states. This is irrespective of the convergence of the genuine GDH integral with its integrand $8\pi\, \mathrm{Im}\, f_2(\nu, 0)/\nu$. The integration path in the $(\nu, q^2)$ plane for various values of the energy $p^0$ is depicted in Fig. 2.5. Note that for any finite value of $p^0$, this path is a parabola that extends to arbitrarily high timelike photon virtualities.

[8] Note that this is not in contradiction to the fact that the one-nucleon state yields a certain finite contribution to the matrix element of the dipole-moment commutator. In fact, the exclusion of the one-nucleon state is crucial for the manipulation presented in Eq. (2.67) not to produce an artificial pole, and the exclusion of the point $q^0 = 0$ from the $q^0$ integral of Eq. (2.71) enables one to re-include the one-nucleon state.

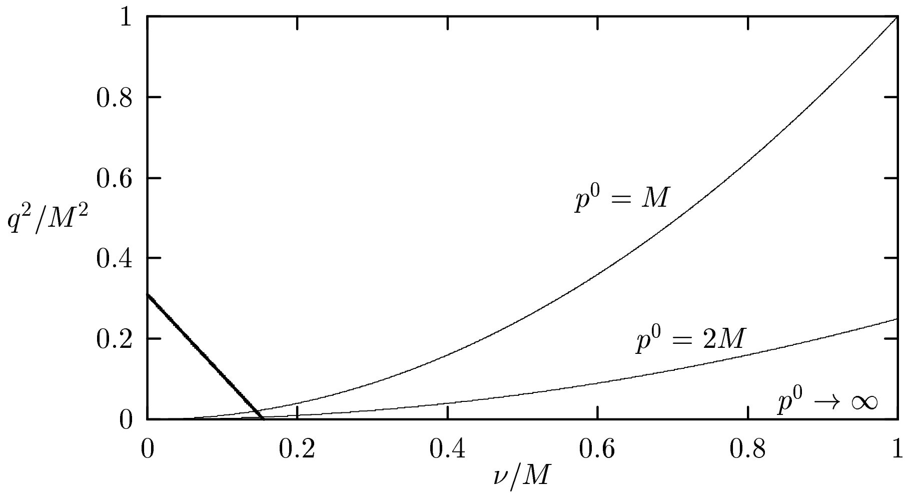



Figure 2.5: Integration path of the finite-momentum GDH sum rule (2.75) in the $(\nu, q^2)$ plane for nucleon energies $p^0 = M$ and $p^0 = 2M$. The heavy line represents the pion-production threshold.

### Correspondence between dispersion theory and current algebra

At this point, a remark on the connection between dispersion-theoretic and current-algebra derivation of the GDH sum rule is appropriate. In a way, the one-nucleon intermediate-state contribution to the commutator of electric charge densities corresponds to the Born portion of the forward Compton amplitude, which in turn determines the low-energy limit of the true amplitude due to the Low-Gell-Mann-Goldberger theorem, as discussed in the previous section. Therefore it is not surprising that the left-hand side of Eq. (2.75) has a part very similar to the constant on the left-hand side of the GDH sum rule. Furthermore, the continuum contribution gives rise to an integral like the one on the right-hand side of the GDH sum rule, whose integrand is "almost" fit to apply the optical theorem. By virtue of the Bjorken-Johnson-Low limit, there is a third correspondence, namely the relation between the *high-energy* limit of the forward Compton amplitude and the current commutator itself. This will be illustrated in the Ch. 4.

The only thing that spoils exact one-to-one correspondence in all three instances is the finiteness of the nucleon energy $p^0$, resulting in a non-vanishing photon virtuality $q^2$. Re-inspecting the dipole-moment commutator matrix element (2.54), this could actually have been evident from the very beginning of the current-algebra derivation, because in momentum space, fixed time corresponds to an *integral over all photon energies* $q^0$, while the space integral implicit in the definition of the dipole moment corresponds to *fixed photon three-momentum* $\boldsymbol{q} = 0$. Therefore the photon virtuality will *inevitably* run along with the energy, and the derivation has to be supplemented by the infinite-momentum limit, which will turn out to be most critical, although least considered in the past.

### The finite-momentum GDH sum rule as a $q^0$ dispersion relation

The finite-momentum GDH sum rule (2.75) can be derived in a quite different fashion, namely as an unsubtracted dispersion relation in variable $q^0$ for the function $f_2(\nu, q^2)$ at $\boldsymbol{q} = 0$, i.e., letting

$$\nu = \frac{p^0 q^0}{M} \quad \text{and} \quad q^2 = (q^0)^2. \tag{2.76}$$

Analogously to the fixed-$q^2$ dispersion relation (2.25), I write

$$\lim_{q^0 \to 0} f_2(\nu, q^2) = \frac{2}{\pi} \int_{q^0_{\rm thr}}^{\infty} \frac{\mathrm{d}q^0}{q^0} \, \mathrm{Im}\, f_2(\nu, q^2). \tag{2.77}$$

As illustrated in Sect. 2.1.2, the Low-Gell-Mann-Goldberger theorem states that to first order in the photon energy $\nu$, the real Compton amplitude equals its Born contribution. It can be shown [49] that this also holds for the virtual Compton amplitude. Thus,

$$\lim_{q^0 \to 0} f_2(\nu, q^2) = \lim_{q^0 \to 0} f_2^{\rm Born}(\nu, q^2), \tag{2.78}$$

where the right-hand side can be calculated from Eq. (2.37). Note that it would be meaningless to simply write $f_2(0,0)$, because function $f_2(\nu, q^2)$ is discontinuous at the point (0,0), which lets the limit depend upon the way this point is approached. For instance, observe that

$$\lim_{\nu \to 0} f_2^{\rm Born}(\nu, 0) = f_2(0) = -\frac{\alpha\kappa^2}{2M^2}, \tag{2.79a}$$

whereas

$$\lim_{q^2 \to 0} f_2^{\rm Born}(0, q^2) = \infty. \tag{2.79b}$$

In terms of Fig. 2.5, the limit is finite as long as one runs horizontally into the point (0,0), as does the parabola $q^0 \to 0$ taken in Eq. (2.77). Introducing the kinematics (2.76) into Eq. (2.37) results into

$$\lim_{q^0 \to 0} f_2(\nu, q^2) = -\frac{\alpha\kappa^2}{2M^2} + \frac{\alpha(Z+\kappa)^2}{2(p^0)^2}. \tag{2.80}$$

The finite-momentum GDH sum rule (2.75) is obtained by putting together Eqs. (2.77) and (2.80). I remark that this result confirms the correctness of Eq. (2.64) as opposed to the result of Pradhan and Khare [34].

### 2.2.2 Infinite-momentum limit

Taking the infinite-momentum limit now constitutes the last step of the derivation of the GDH sum rule. I take the limit $p^0 \to \infty$ on both sides of the finite-momentum GDH sum rule (2.75):

$$-\frac{2\pi^2\alpha\kappa^2}{M^2} = \lim_{p^0 \to \infty} \int_{\nu_{\rm thr}}^{\infty} \frac{\mathrm{d}\nu}{\nu} \, 8\pi \, \mathrm{Im}\, f_2\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right). \tag{2.81}$$

To get the accustomed form of the sum rule, one now has to interchange the limit $p^0 \to \infty$ with the $\nu$ integration, because then the integrand is the imaginary part of forward Compton amplitude $f_2(\nu, q^2)$ at the *real* photon point $q^2 = 0$, where it is related to the observable photoabsorption cross section via the optical theorem

$$8\pi \operatorname{Im} f_2(\nu, 0) = \sigma_{1/2}(\nu) - \sigma_{3/2}(\nu). \tag{2.82}$$

As it stands, Eq. (2.81) is actually worthless, since no observable is related to the timelike virtual Compton amplitude. Pictorially, the infinite-momentum limit corresponds to stretching the parabolae indicated in Fig. 2.5 down to the straight line running at $q^2 = 0$.

**Legitimacy of the infinite-momentum limit**

Will the properties of function $\operatorname{Im} f_2(\nu, q^2)$ allow the limit to be dragged into the integral? I stress that bare current algebra is exhausted as regards this problem. That is to say, no statement on the permutation of $p^0 \to \infty$ limit and $\nu$ integration can be inferred from any kind of equal-times commutators. In a way, current algebra tells us something about the parabolae shown in Fig. 2.5 and nothing about the straight line $q^2 = 0$. The legitimacy of the infinite-momentum limit enters the current-algebra derivation of the GDH sum rule as a mere conjecture!

Sect. 4.3 of this thesis is devoted to a discussion of possible modifications due to the infinite-momentum limit.

## 2.3 Light-cone current algebra

In 1972, Dicus and Palmer [26] presented a derivation of the GDH sum rule from the algebra of currents on the light-cone, which circumvents the infinite-momentum limit necessary in the usual current-algebra approach. In this section, I want to review the basics of this technique.

### 2.3.1 Light-cone coordinates

For any four-vector $r$, one defines $\pm$ components by

$$r^{\pm} = \frac{1}{\sqrt{2}}\,(r^0 \pm r^3). \tag{2.83}$$

Remaining components are subsummed as

$$\boldsymbol{r}_\perp = (r^1, r^2). \tag{2.84}$$

The scalar product then reads

$$q{\cdot}x = q^+x^- + q^-x^+ - \boldsymbol{q}_\perp{\cdot}\boldsymbol{x}_\perp. \tag{2.85}$$

The derivation of the sum rule essentially takes the following course: each instance of *time* $x^0$ is replaced by *light-cone time* $x^+$; Ordinary space components $\boldsymbol{x}$ are replaced by $(x^-, \boldsymbol{x}_\perp)$; *light-cone energy* $q^-$ – naturally defined as the $q$ component accompanying variable $x^+$ in the scalar product (2.85) – is substituted for ordinary energy $q^0$; etc.

Nevertheless, the reader shall beware of mistaking the light-cone technique for a simple transform of the coordinate system. The vital foundation of the whole story is that the underlying field theory is *quantized on the light cone*. For instance, the canonical anticommutation relations of a fermionic theory are defined at equal light-cone times[9] rather than equal times as in Eqs. (2.40).

### 2.3.2 Light-cone charge-density commutator

Defining the fermionic current in the usual way, the naive current commutator at equal light-cone times can be derived from the canonical anticommutation relations analogously to the derivation of the naive equal-times commutator presented on p. 19 of this thesis. The result for the commutator of + components of the current (the light-cone charge density) reads

$$\left[J^+(x), J^+(y)\right]_{\mathrm{lc}} = 0, \tag{2.86}$$

where subscript "lc" stands for $x^+ = y^+$.[10]

#### Light-cone electric dipole moment

One now defines the first moment of $J^+(x)$ [26],

$$\boldsymbol{D}_\perp(x^+) = e\int \mathrm{d}x^-\ \mathrm{d}^2x_\perp\ \boldsymbol{x}_\perp J^+(x), \tag{2.87}$$

the light-cone analogue of the electric dipole moment (2.50). In particular, one defines definite-chirality components $D^{\mathrm{L,R}}(0)$ as in Eq. (2.53) and sandwiches the naive commutator

$$\left[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)\right] = 0, \tag{2.88}$$

which follows from Eq. (2.86), between one-nucleon states of positive helicity (taking the incoming nucleon to be propagating along $\boldsymbol{e}_3$),

$$\left\langle p', \tfrac{1}{2}\right|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\left|p, \tfrac{1}{2}\right\rangle = 0. \tag{2.89}$$

Inserting a complete set of intermediate states and separating the one-nucleon states from the continuum, one finds [26]

$$\left\langle p', \tfrac{1}{2}\right|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\left|p, \tfrac{1}{2}\right\rangle_{\text{one-nucleon}} = (2\pi)^3\, 2p^+\, \delta(p'^+ - p^+)\, \delta(\boldsymbol{p}'_\perp)\, \frac{2\pi\alpha\kappa^2}{M^2} \tag{2.90}$$

and

$$\begin{aligned}\left\langle p', \tfrac{1}{2}\right|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\left|p, \tfrac{1}{2}\right\rangle_{\text{cont}} &= (2\pi)^3\, 2p^+\, \delta(p'^+ - p^+)\, \delta(\boldsymbol{p}'_\perp) \\ &\quad\times 8\int_{q^-_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}q^-}{q^-}\ \mathrm{Im}\, f_2(\nu, 0),\end{aligned} \tag{2.91}$$

where $\nu = p{\cdot}q/M = p^+q^-/M$ and $q^-_{\mathrm{thr}} = M\nu_0/p^+$. The GDH sum rule follows immediately upon application of the optical theorem (2.82).

[9] The fact that canonical (anti)commutation relations are defined at equal + components of spacetime is the actual reason why these components are called light-cone *time*.

[10] In Minkowski space, condition $x^+ = \text{const}$ does actually not define a cone, but rather a plane tangential to the light-cone, i.e., a light-*front*. Thus, "light-cone formalism" should properly be called "light-front formalism", and indeed a wee minority of theorists does so.

### 2.3.3 Light cone vs. infinite momentum

Most remarkably, this derivation works without employing the infinite-momentum technique. The one-nucleon contribution (2.90) to the commutator matrix element is not contaminated by an extra term like in Eq. (2.64), and the continuum part (2.91), in contrast to Eq. (2.74), exhibits just the wanted integrand without further manipulation. In fact, there is actually not even the need for defining amplitude $f_2(\nu)$ off the real-photon domain $q^2 = 0$. Recalling the discussion on p. 25, this is simply due to the fact that the photon virtuality

$$q^2 = (q^0)^2 - \boldsymbol{q}^2 = 2q^+q^- - \boldsymbol{q}_\perp^2 \tag{2.92}$$

is *linear* in the light-cone energy $q^-$ and vanishes identically upon letting $q^+ = 0$ and $\boldsymbol{q}_\perp = 0$, whereas it is quadratic in the ordinary energy $q^0$ and runs irresistibly with the $q^0$ integral (2.74). In terms of Fig. 2.5, we have the contour of the $q^-$ integration of Eq. (2.91) right where we wanted it: on the abscissa $q^2 = 0$. In a way, the light-cone current-algebra method incorporates the infinite-momentum limit from the very beginning. This is a great advantage of the method. A disadvantage is that little is known about possible non-naive forms of the light-cone charge-density commutator (2.86) – even less than in the case of the equal-times commutator (2.49). I mention in advance that Dicus and Palmer already suggest a non-naive form, namely the one induced by a possible anomalous magnetic moment of quarks. This point will be presented in detail in Sect. 4.6.

# Chapter 3

# The GDH sum rule within perturbative models

In this chapter, I present some important tests of the GDH sum rule, namely its investigation within perturbative models. That is to say, the polarized total photoabsorption cross section $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ (or, equivalently, the imaginary part $\mathrm{Im}\, f_2(\nu)$ of the polarized forward Compton amplitude) of a given fermion, as well as its anomalous magnetic moment $\kappa$, are calculated to lowest non-trivial order in a specific perturbation theory, and it is checked whether both sides of the sum rule coincide:

$$-\frac{2\pi^2\alpha\kappa^2}{M^2} \stackrel{?}{=} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu}\big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big) \qquad \text{(to specific order)} \tag{3.1}$$

I mention in advance that all tests are concluded positively. This signals that any particular gauge invariant class of Feynman diagrams obeys an unsubtracted dispersion relation,

$$f_2(0) = \frac{2}{\pi}\int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu'}{\nu'}\, \mathrm{Im}\, f_2(\nu'), \tag{3.2}$$

and that the low-energy theorem

$$f_2(0) = -\frac{\alpha\kappa^2}{2M^2} \tag{3.3}$$

holds order by order. If you think about it, both of these points are not *too* surprizing, owing to the following reasons. Firstly, once you have a class of Feynman diagrams that gives rise to a *convergent* dispersion integral, it will generally not necessitate a finite subtraction. Secondly, the order-by-order validity of the low-energy theorem gets plausible if you recall its derivation by Gell-Mann and Goldberger [37], which in fact *is* a perturbative one. Nevertheless, all of the results presented here are certainly non-trivial.

In Sect. 3.1, I present the investigation of the sum rule within a model involving nucleons and pions, using pseudoscalar coupling [28]. To lowest non-trivial order, the only final state of the photoabsorption process is $\pi$N. This study is also quite useful with respect to resonance saturation of the sum rule. In Sect. 3.2, the fermion under consideration is taken to be an electron, utilizing quantum electrodynamics to calculate cross sections [29, 30]. The final state then is $\gamma e^-$, i.e., Compton scattering.[1] The Weinberg-Salam model [29] (see also Brodsky and Schmidt [50]), being most involved, is considered in the last section of this chapter. In a sense, this model is an extension of QED, since it incorporates weak gauge bosons $Z^0$ and $W^\pm$, the Higgs boson H, and the neutrino $\nu_e$ in addition to $\gamma$ and $e^-$. Final states are $\gamma e^-$, $Z^0 e^-$, $W^- \nu_e$, and $H e^-$.

**What does the inspection of perturbative models aim at?**

Perturbative models are not directly related to the experimentally accessible GDH sum rule. As far as the nucleon is concerned, this can be attributed to the fact that effective perturbative descriptions (like chiral perturbation theory) work at low energies only, while perturbative QCD requires the photon to be highly virtual. At $q^2 = 0$ and large $\nu$, the photon-nucleon interaction is conspicuously non-perturbative. The electron GDH sum rule, on the other hand, is not measureable at all. Nevertheless, investigating a sum rule perturbatively is surely very illustrative also in view of the "real world".

However, the main purpose of presenting these considerations here is that the effect of anomalous commutators and of the infinite-momentum limit can be studied within the Weinberg-Salam model. This will be done in Sect. 4.4, but I cannot resist anticipating what the result will be: There is *no* anomalous-commutator modification of the GDH sum rule!

## 3.1 Pseudoscalar pion-nucleon model

In 1975, Gerasimov and Moulin [28] investigated the GDH sum rule within the simplest model involving electromagnetic interactions of pions and nucleons as well as strong interactions of the named hadrons themselves. Interaction vertices of this model are shown in Fig. 3.1. Photons are coupled to charged hadrons by means of minimal coupling. (The $\gamma\gamma\pi\pi$ contact vertex of Fig. 3.1(b) is not needed in the present context.) In particular, the nucleon has no anomalous magnetic moment on the vertex level. Interactions of pions with nucleons are described by the Lagrangian

$$\mathcal{L}_{\pi\mathrm{NN}}(x) = g\,\bar{\psi}(x)\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)\psi(x), \tag{3.4}$$

where $\psi(x)$ denotes the isodoublet nucleon spinor and $\boldsymbol{\phi}(x)$ is the isotriplet of pion fields. The interaction (3.4) is called *pseudoscalar* as opposed to *pseudovector* interaction, for which $\gamma_5\boldsymbol{\phi}$ is replaced by $i\gamma_5 \not{\partial}\boldsymbol{\phi}$. With pseudoscalar coupling, the coupling constant $g$ is dimensionless. In Ref. 28, the integrand $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ of the GDH sum rule is calculated within this model to lowest non-trivial order in electromagnetic and strong coupling constants, i.e., to order $\alpha g^2$. The pertinent Feynman graphs are depicted in Fig. 3.2. To this order, the only possible final state of the photoabsorption reaction is $\pi$N. For the $\gamma$p initial state, there are final-state isospin channels $\pi^+$n and $\pi^0$p, whereas for $\gamma$n, there is only $\pi^-$p. The cross section for process $\gamma n \to \pi^0 n$ vanishes identically to the order considered, since the photon couples to neither of the neutral particles.

[1] Don't be puzzled by the double appearance of the term Compton scattering. The integrand of the GDH integral is *always* the imaginary part of the forward Compton amplitude, being related, via the optical theorem, to the photoabsorption cross section. In terms of Feynman diagrams, the optical theorem relates one-loop diagrams of the Compton process to sqares of tree-order diagrams for photoabsorption, where intermediate states of the former are final states of the latter. These states differ for different models, and in lowest-order QED they happen to coincide with the initial state. (In the next order, one would have $\gamma\gamma e^-$ and $e^+e^-e^-$.) Of course, "photoabsorption" actually becomes a somewhat inappropriate notion then.

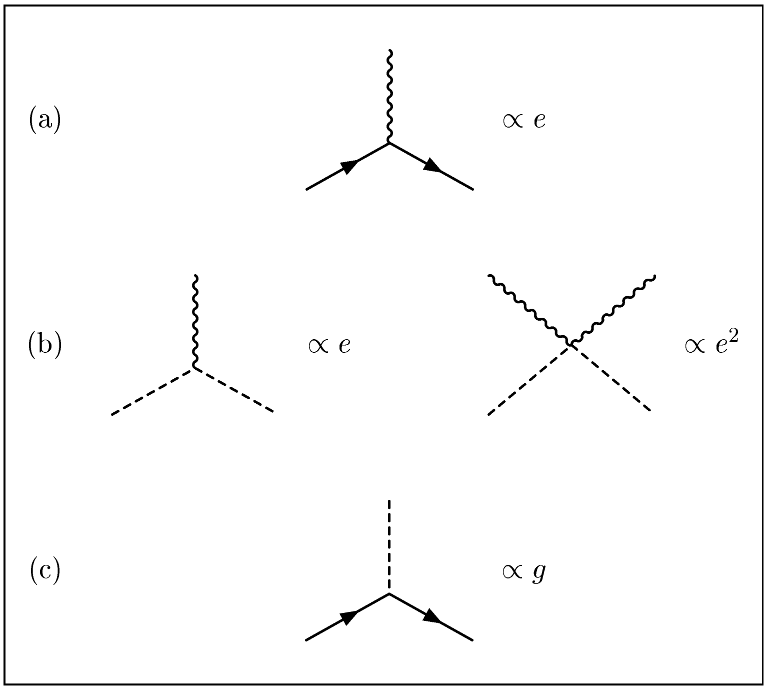
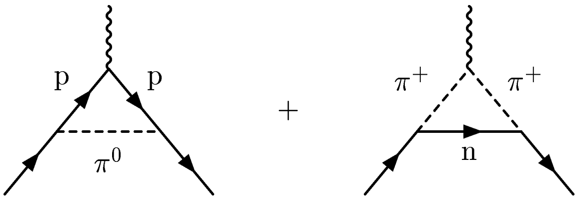



Figure 3.1: Interaction vertices of the pseudoscalar pion-nucleon model. Solid, dashed, and wavy lines represent nucleons, pions, and photons, respectively.
(a) $\gamma$p coupling
(b) $\gamma\pi^{\pm}$ coupling
(c) pseudoscalar $\pi$N coupling

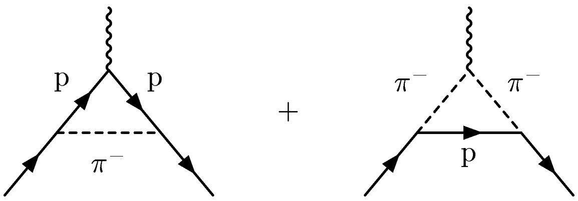
### Anomalous magnetic moment

To lowest order, the anomalous magnetic moment $\kappa_\mathrm{p}$ of the proton is determined by the Feynman graphs




(3.5a)

while $\kappa_\mathrm{n}$ is determined by




(3.5b)

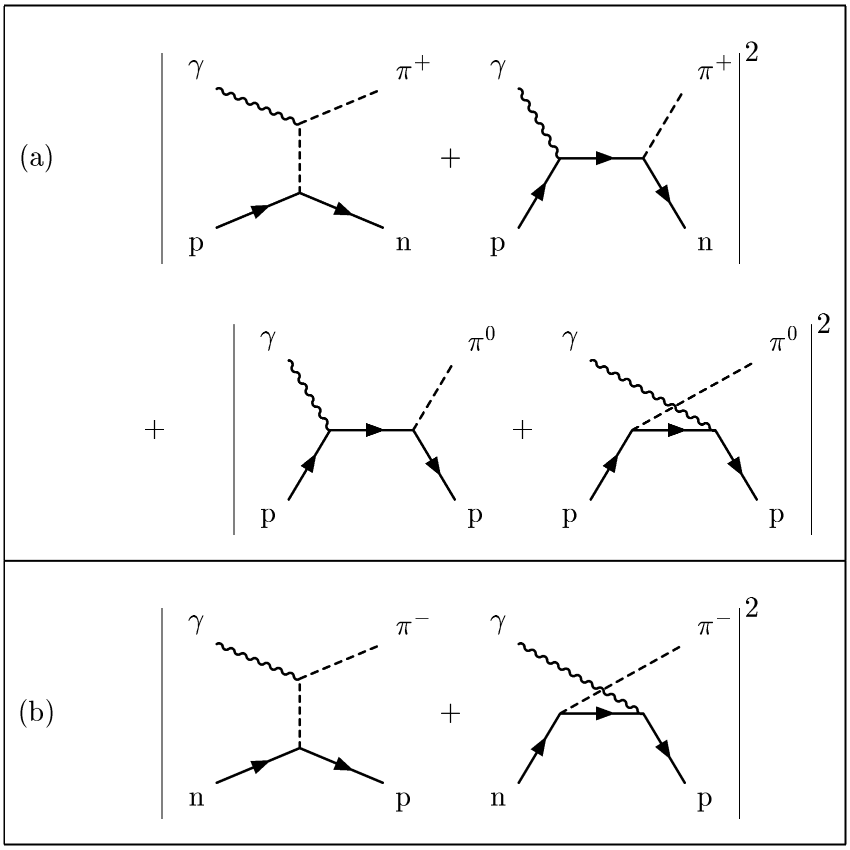



Figure 3.2: Feynman graphs determining the nucleon photoabsorption cross section to order $\alpha g^2$.
(a) proton
(b) neutron

Thus, both quantities are of order $g^2$. Consequently, the left-hand side of the GDH sum rule (3.1) is of order $\alpha g^4$, so that to order $\alpha g^2$ it reads

$$0 = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \left(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\right). \tag{3.6}$$

**Polarized photoabsorption cross sections**

Gerasimov and Moulin [28] calculated The integrand of Eq. (3.6) and showed that the integral indeed vanishes. Since $\mathrm{d}\nu/\nu = \mathrm{d}(\ln \nu)$, this is reflected by the fact that the curves in Fig. 3.3 change sign and that the areas enclosed with the abscissa to the left and to the right of the sign change are equal. To plot the cross sections, I have taken the experimental values

$$\frac{g^2}{4\pi} = 14.6 \tag{3.7}$$

and $m_\pi = 140$ MeV (the charged pion's mass, since $t$-channel pion exchange is the dominant process).

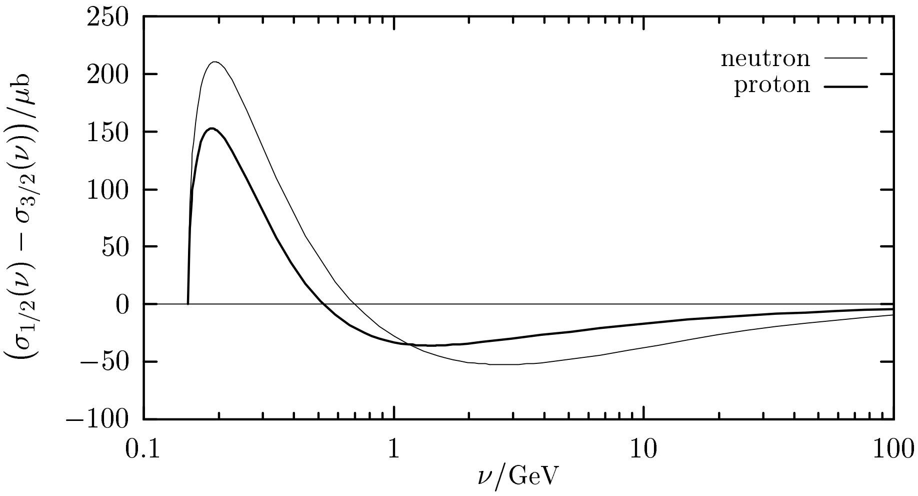
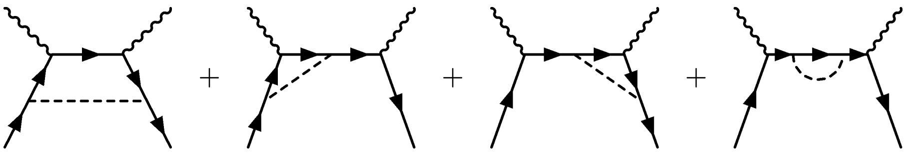



Figure 3.3: Polarized total photoabsorption cross section $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ of proton (heavy line) and neutron (light line) to order $\alpha g^2$, calculated from the Feynman graphs of Fig. 3.2. Since $\mathrm{d}\nu/\nu = \mathrm{d}(\ln \nu)$, the vanishing of the integral (3.6) is reflected by the equality of the areas enclosed by the curves and the abscissa. Cross sections vanish below pion-production threshold $\nu_0 = m_\pi + m_\pi^2/2M = 0.15$ GeV.

Of course, the value (3.7) does not exactly suggest a perturbative treatment. The curves drawn in Fig. 3.3 will be far off the experimental photoabsorption cross sections, especially at resonances and above two-pion threshold. But numerical accuracy is presently not on target. Rather, inspection of perturbative models aims at testing and illustrating simultaneously all assumptions on which the derivation of the sum rule is based.

**Optical theorem**

At this point, a few words on the optical theorem may be appropriate. As mentioned above, instead of calculating the cross section difference $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ by squaring the tree-order amplitudes of, say, the $\gamma\mathrm{p} \to \pi^0\mathrm{p}$ process depicted in the second line of Fig. 3.2(a), and by subsequently integrating over final state configurations, i.e., pion scattering angles, one can equally well calculate $8\pi \,\mathrm{Im}\, f_2(\nu)$, where forward Compton amplitude $f_2(\nu)$ is obtained from the one-loop graphs



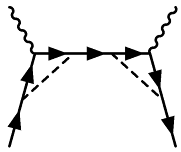

(3.8)

(plus the same graphs with their photon legs crossed). In other words: The optical theorem

$$\mathrm{Im}\, f_2(\nu) = \frac{1}{8\pi}\big[\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big] \tag{3.9}$$

holds perturbatively. Considering the graphs (3.8) together with the low-energy theorem (3.3) and the graphs (3.5) responsible for the anomalous magnetic moment, it is clear that the left-hand side of the GDH sum rule will indeed vanish to the given order, since a contribution to

$f_2(0)$ proportional to $\kappa^2$ will not arise until two-loop graphs like

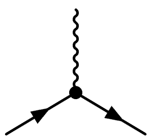


(3.10)

are taken into account.

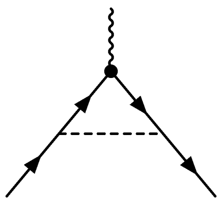
**Anomalous magnetic moment on the vertex level**

In Fig. 3.1, the photon couples to the nucleon minimally, i.e., to its charge only. I now investigate the case of non-minimal coupling



$$= Ze\gamma^\mu + \frac{ie\kappa_0}{2M}\, q_\rho \sigma^{\mu\rho}. \tag{3.11}$$

That is to say, the nucleon adopts an anomalous magnetic moment $\kappa_0$ on the vertex level. In this scenario, the left-hand side of the GDH sum rule is in fact divergent, due to the (non-renormalizable) divergence of one-loop graphs of the form



(3.12)

What happens to the right-hand side of the sum rule? I calculated its integrand $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ by inserting the pion-electroproduction magnetic Born terms of von Gehlen [51] into Eq. (5.92a) below, specializing to $Q^2 = 0$. The result is depicted in Fig. 3.4. Observe that the polarized cross section difference approaches a non-zero constant as $\nu \to \infty$, indicating that the GDH integral will diverge. (This is of course consistent with the divergence of the left-hand side of the sum rule.) More precisely, one has that

$$\sigma^{\mathrm{p}}_{1/2}(\nu) - \sigma^{\mathrm{p}}_{3/2}(\nu) \xrightarrow[\nu\to\infty]{} \frac{\alpha g^2}{8M^2}\big((\kappa_0^{\mathrm{p}})^2 - (\kappa_0^{\mathrm{n}})^2\big) = -33\ \mu\mathrm{b} \tag{3.13a}$$

and

$$\sigma^{\mathrm{n}}_{1/2}(\nu) - \sigma^{\mathrm{n}}_{3/2}(\nu) \xrightarrow[\nu\to\infty]{} \frac{\alpha g^2}{8M^2}\big((\kappa_0^{\mathrm{n}})^2 - (\kappa_0^{\mathrm{p}})^2\big) = 33\ \mu\mathrm{b}, \tag{3.13b}$$

where the physical values $\kappa_0^{\mathrm{p}} = 1.79$, $\kappa_0^{\mathrm{n}} = -1.91$ have been inserted. Letting $\kappa_0^{\mathrm{p}} = \kappa_0^{\mathrm{n}} = 0$ leads back to the Gerasimov-Moulin result depicted in Fig. 3.3.

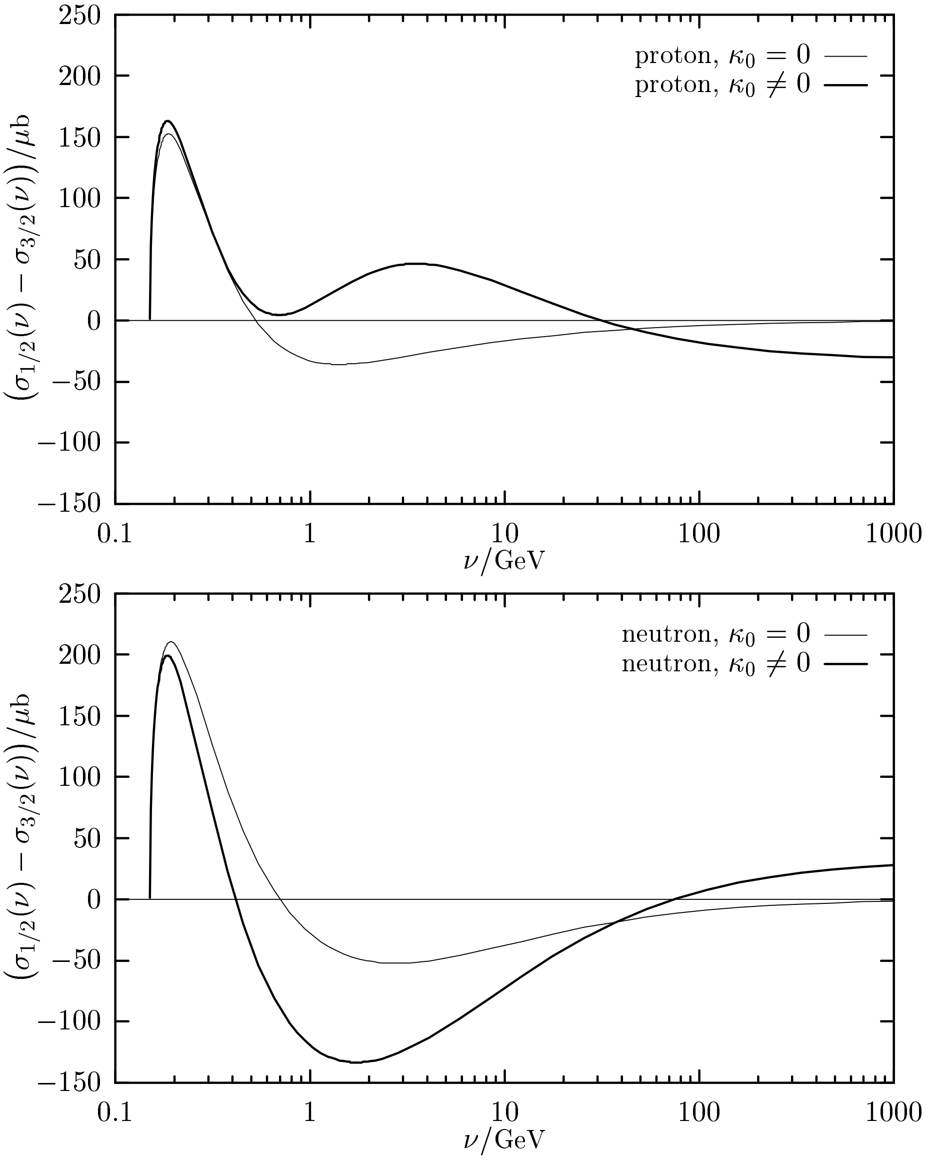


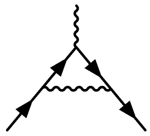

Figure 3.4: Polarized total photoabsorption cross section $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ of proton (upper plot) and neutron (lower plot) to order $\alpha g^2$, with (heavy lines) and without (light lines) anomalous magnetic moment $\kappa_0$ on the vertex level. As for the former case, where the physical values $\kappa_0^{\mathrm{p}} = 1.79$, $\kappa_0^{\mathrm{n}} = -1.91$ have been adopted for definiteness, the cross section difference approaches a non-vanishing constant at $\nu \to \infty$. The $\kappa_0 = 0$ curves are the same as in Fig. 3.3. Cross sections vanish below pion-production threshold $\nu_0 = m_\pi + m_\pi^2/2M = 0.15$ GeV.

## 3.2 Quantum electrodynamics

For the remainder of this chapter, the fermion under consideration is taken to be an electron rather than a nucleon. In this case, quantum electrodynamics (QED) as well as the Weinberg-Salam model of electro-weak interactions suggest themselves as testing grounds for the GDH sum rule. QED – certainly the most prominent perturbative quantum field theory – is actually *part of* the Weinberg-Salam model. (As a matter of fact, the first study of the GDH sum rule within QED, performed by Altarelli, Cabibbo, and Maiani [29] in 1972, incorporated weak interactions, too. Tsai, DeRaad, and Milton [30] investigated QED only, but were additionally concerned with the Burkhardt-Cottingham sum rule [52].) Nevertheless, for pedagogical reasons I want to present both models seperately. This section is devoted to the less involved QED case.

**Anomalous magnetic moment**

The procedure is analogous to the pion-nucleon model presented in the preceding section, except that the pion is replaced by a photon. To order $\alpha$, the anomalous magnetic moment of the electron is determined by the graph



(3.14)

which gives rise to the famous Schwinger moment

$$\kappa = \frac{\alpha}{2\pi} = 1.16\times 10^{-3}. \tag{3.15}$$

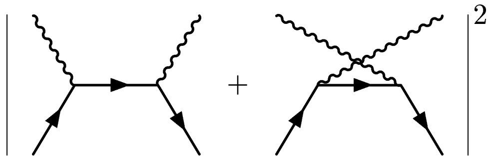
(For the calculation, see, e.g., Itzykson and Zuber [43].) Consequently, the left-hand side of the GDH sum rule (3.1) is of order $\alpha^3$, so that to order $\alpha^2$ it reads

$$0 = \int_0^\infty \frac{\mathrm{d}\nu}{\nu}\left(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\right). \tag{3.16}$$

**Polarized photoabsorption cross sections**

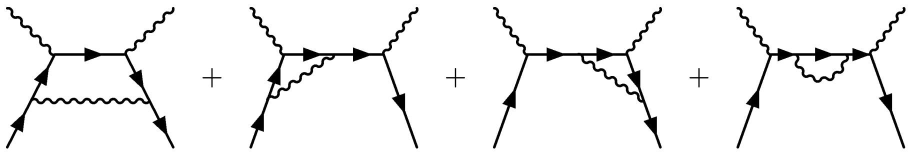
Of course, the integrand on the right-hand side is of order $\alpha^2$. There is no order-$\alpha$ piece, since at least one additional photon is needed to make up the final state. (There is no such thing as a “purely hadronic final state” here.) Observe that the threshold $\nu_0$, i.e., the lower bound of the GDH integral (3.16), vanishes, since the final-state particles have the same mass than the particles in the initial state. This opens the question for infrared convergence of integral (3.16), in addition to the usual question for ultraviolet convergence. I mention in advance that both of these questions will be affirmed.

Opposed to the graphs shown in Fig. 3.2, cross sections $\sigma_{1/2}(\nu)$ and $\sigma_{3/2}(\nu)$ are now calculated from



(3.17)

Again, the perturbative validity of the optical theorem can be checked explicitly by computing $\mathrm{Im}\, f_2(\nu)$ from the one-loop graphs



(3.18)

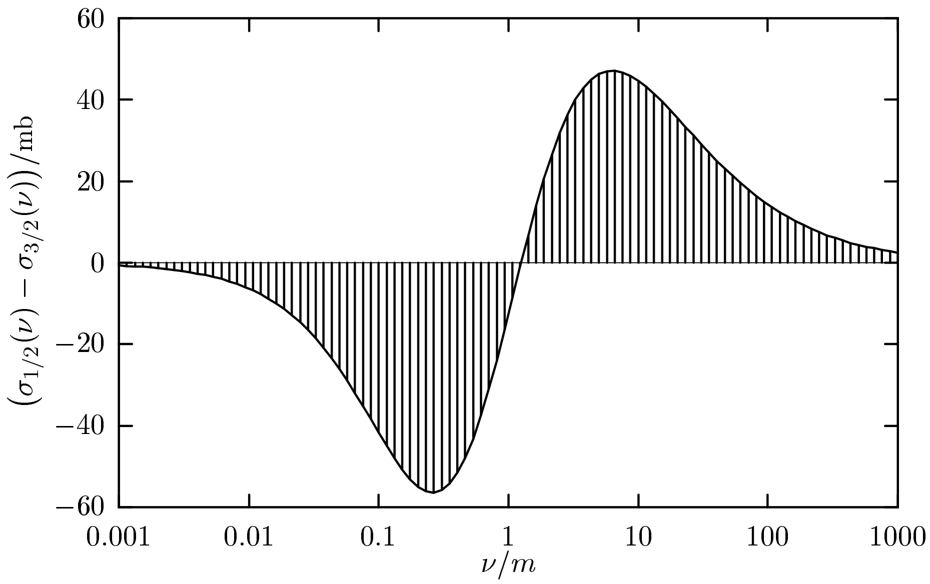



Figure 3.5: Polarized total photoabsorption cross section $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$ of the electron in order-$\alpha^2$ QED. Since $\mathrm{d}\nu/\nu = \mathrm{d}(\ln \nu)$, the vanishing of the integral (3.16) is reflected by the equality of the shaded areas.

(plus the same graphs with their photon legs crossed). The result reads [29]

$$\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu) = \frac{2\pi\alpha^2}{m\nu}\left[\left(1 + \frac{m}{\nu}\right)\ln\left(1 + \frac{2\nu}{m}\right) - 2\left(1 + \frac{\nu^2}{(m+2\nu)^2}\right)\right], \tag{3.19}$$

which is depicted in Fig. 3.5.

**Convergence of the GDH integral**

Expanding expression (3.19) about $\nu = 0$ and $\nu \to \infty$, one finds[2]

$$\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu) \xrightarrow[\nu \to 0]{} -\frac{8\pi\alpha^2\nu}{3m^3} \tag{3.20}$$

and

$$\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu) \xrightarrow[\nu \to \infty]{} \frac{2\pi\alpha^2}{m\nu}\ln\frac{\nu}{m}, \tag{3.21}$$

respectively, so that integral (3.16) converges. An explicit analytical integration[3] shows that it indeed vanishes, as indicated by the shaded areas in Fig. 3.5.

## 3.3 Weinberg-Salam model

As noted above, the Weinberg-Salam model (for an introduction, I recommend the textbook of Aitchison and Hey [53]) is an extension of QED. Neutrino, $W^{\pm}$, $Z^0$, and Higgs come into

[2]Note that the *average* of $\sigma_{1/2}(\nu)$ and $\sigma_{3/2}(\nu)$ does *not* vanish as $\nu \to 0$. Rather, it approaches the Thomson limit $8\pi\alpha^2/3m^2 = 665$ mb.

[3]*Mathematica* copes with this task.

play in addition to electron and photon. Since these particles implicate parity violation, some caution is in order concerning the definition of cross sections $\sigma_{1/2}$ and $\sigma_{3/2}$. For instance, the cross section $\sigma_{+-}$ corresponding to photon helicity $\lambda_\gamma = +1$ and electron helicity $\lambda_\mathrm{e} = -\frac{1}{2}$ is no longer equal to the cross section $\sigma_{-+}$ for $\lambda_\gamma = -1$, $\lambda_\mathrm{e} = +\frac{1}{2}$. Both were formerly called $\sigma_{3/2}$ (parallel helicities). Now one must properly define what is meant by parallel helicities. An obvious choice is $\sigma_{3/2} := \frac{1}{2}(\sigma_{+-} + \sigma_{-+})$ and likewise for $\sigma_{1/2}$. I note that Bacé and Hari Dass [54] claimed that the parity violating portion of the Compton amplitude vanishes in the one-loop order of the Weinberg-Salam model.

However, I intend to employ the model for learning something about anomalous commutators and the infinite-momentum limit, and I am not concerned with parity violation. Therefore, I make life a bit easier by letting the Weinberg angle $\theta_\mathrm{W}$ adopt its ideal value, i.e.,

$$\sin\theta_\mathrm{W} = \frac{1}{2}, \tag{3.22}$$

which, besides, is not far from what is observed experimentally, viz. $\sin\theta_\mathrm{W} = 0.48$ [46]. Therewith, the $\mathrm{Z}^0$ boson has a purely axial-vector coupling to electrons, and graphs involving the $\mathrm{Z}^0$ boson are parity conserving. Fixing the Weinberg angle implies that the number of coupling constants is reduced from two ($g = e/\sin\theta_\mathrm{W}$ and $g' = e/\cos\theta_\mathrm{W}$) to one, which can be chosen as the electric charge $e$. Consequently, all quantities will be expanded in the fine-structure constant $\alpha$, just as in QED. The Fermi constant is then given by

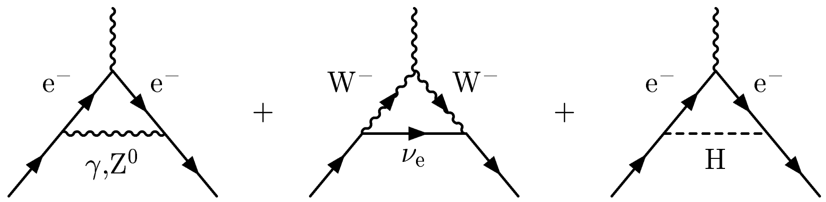
$$\frac{G_\mathrm{F}}{\sqrt{2}} = \frac{2\pi\alpha}{M_\mathrm{W}^2} = \frac{8\pi\alpha}{3M_\mathrm{Z}^2}. \tag{3.23}$$

For a compilation of Feynman rules, see App. F.3 of Aitchison and Hey [53]. I adopt the unitary gauge, which has the advantage that the particle content of the theory is manifest. That is to say, Feynman graphs contain the above-mentioned physical particles only, and no ficticious particles (so-called would-be-Goldstone bosons) are needed. In unitary gauge, the propagators of the massive gauge bosons $\mathrm{W}^\pm$, $\mathrm{Z}^0$ read

$$\mu \overset{k}{\sim\!\sim\!\sim\!\sim} \nu \quad = \frac{i}{k^2 - M_\mathrm{W,Z}^2 + i\varepsilon}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_\mathrm{W,Z}^2}\right). \tag{3.24}$$

### Anomalous magnetic moment

To order $\alpha$, the anomalous magnetic moment of the electron is determined by the graphs



(3.25)

the first of which also appears in QED, Eq. (3.14). The graph involving the W boson has first been calculated by Brodsky and Sullivan [55]. Its contribution to the electron's anomalous

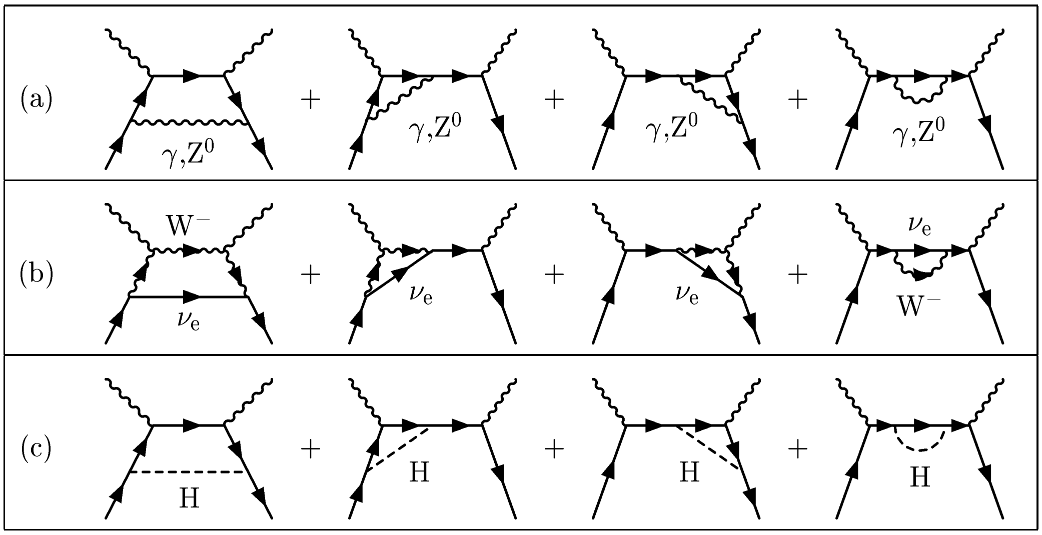



Figure 3.6: Feynman graphs determining the order-$\alpha^2$ forward Compton amplitude of the electron in the Weinberg-Salam model. Addition of crossed graphs is understood.
(a) $\gamma e^-$ and $Z^0 e^-$ intermediate states
(b) $W^-\nu_e$ intermediate state
(c) $H e^-$ intermediate state
External-line insertions like vacuum polarization are not shown in this figure.

magnetic moment reads[4]

$$\Delta\kappa = \frac{10m^2}{3M_W^2 \sin^2\theta_W}\,\frac{\alpha}{2\pi} = 6.83{\times}10^{-13}. \tag{3.26}$$

Similar expressions are obtained for the $Z^0$- and Higgs-exchange graphs of Eq. (3.25) [29].

Thus, again, the left-hand side of the GDH sum rule (3.1) is of order $\alpha^3$, so that to order $\alpha^2$ it reads

$$0 = \int_0^\infty \frac{d\nu}{\nu}\,\big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big). \tag{3.27}$$

**Polarized photoabsorption cross sections**

The imaginary part of forward Compton amplitude $f_2(\nu)$, and therewith, via the optical theorem, the cross section difference $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$, is calculated from the Feynman graphs

[4]Note that the value reported by Altarelli et al. [29] is a factor of eight too small. Observe that, numerically, the value (3.26) lies some *nine orders of magnitude* below the Schwinger moment (3.15), due to the smallness of the e-W mass ratio. Yet, it is taken to be of order $\alpha$, since one is not concerned with numerics, but rather with the order-by-order consistency of the model, which has no regard to the phenomenological masses of the particles involved. This state of affairs is similar to the case of the $\pi$N coupling constant discussed on p. 34 of this thesis.

of Fig. 3.6. It is found that the integral converges[5], and Eq. (3.27) is explicitly affirmed [29].[6]

[5]As a matter of fact, Altarelli et al. [29] employed the GDH sum rule to fix the anomalous magnetic moment $\kappa_{\mathrm{W}}$ of the W boson. They showed that the GDH integral converges *if and only if* $\kappa_{\mathrm{W}} = 1$. This state of affairs is similar to the case of the vertex-level anomalous magnetic moment of the nucleon discussed on p. 36 of this thesis. The value $\kappa_{\mathrm{W}} = 1$ naturally emerges from what is nowadays called the standard model, so no-one really doubts it anymore.

[6]For a plot of the $\mathrm{W}^{-}\nu_{\mathrm{e}}$ contribution to $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$, determined by the Feynman graphs of Fig. 3.6(b), see Brodsky and Schmidt [50]. Numerically, this contribution does not exceed a few pb, compared to tens of mb found as the QED contribution, Fig. 3.5. See also footnote 4.

# Chapter 4

# Possible sources of modifications

In this chapter, I review the proposed sources of modifications of the GDH sum rule.

Sect. 4.1 is devoted to a discussion of a possible $J = 1$ fixed pole in angular-momentum plane [12]. After a short paragraph on the relevant ingredients of Regge theory, I emphasize the crucial role that spin plays with respect to the particular Regge singularity under consideration. Next, I comment on the question, which Compton-scattering observable the residue of the fixed pole is related to. Finally, I want to present a plausibility argument *against* the presence of a fixed pole.

In Sect. 4.2, I discuss the claimed modification coming from an anomalous charge-density algebra [13]. After an introduction to anomalous commutators, seagull amplitudes, and the Bjorken-Johnson-Low technique [17, 56], I comprehensively discuss this theory, focussing particularly on Ref. 13. It is stressed that the commutator anomaly modifies the finite-momentum GDH sum rule only, providing no information on the legitimacy of the infinite-momentum limit.

Sect. 4.3 aims at illustrating the significance of the infinite-momentum limit itself, without regard to an anomalous charge-density algebra. I give an example to shed some light on the way a modification of the GDH sum rule might in principle be brought about by the infinite-momentum limit.

In Sect. 4.4, the Weinberg-Salam model is adopted to simultaneously calculate the anomalous charge-density commutator *and* the effect of the infinite-momentum limit [27]. It is shown that both of these points yield a non-trivial contribution to the GDH sum rule, in such a way that the individual modifications cancel exactly.

Sect. 4.5 presents a short treatise on $t$-channel exchange of axial-vector mesons, which in fact generally cannot contribute to the GDH sum rule owing to fundamental symmetries. This section shall further illustrate what is actually done in anomalous-commutator calculations.

Sect. 4.6 is devoted to the discussion of a possible modification of the GDH sum rule that arises as soon as one permits quarks to possess anomalous magnetic moments. The nature of the apparent modification obtained in constituent-quark models is discussed. In Sect. 4.7, I discuss the "spontaneous breakdown of electromagnetic gauge symmetry" [16], while in Sect. 4.8, I comment on the question whether the Low-Gell-Mann-Goldberger low-energy theorem [36, 37] gets modified by a non-naive current commutator.

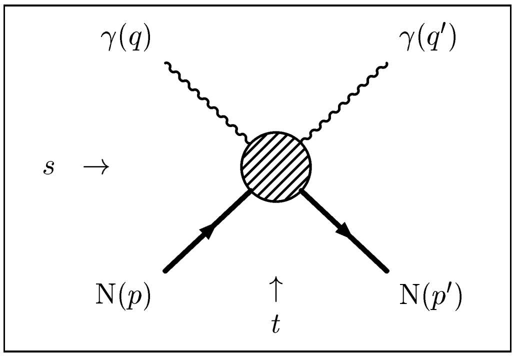



Figure 4.1: Nucleon Compton scattering in the $s$ channel corresponds to nucleon-antinucleon annihilation into two photons in the $t$ channel. Both processes are described by the same amplitudes. Regge theory relates the high-energy behavior of the $s$-channel amplitudes to the partial-wave expansions of the $t$-channel amplitudes, continued to complex angular momenta. The amplitudes relevant to the GDH sum rule correspond to total helicity 2 in the $t$ channel. Hence their partial-wave expansions start at $J = 2$, which renders possible a singularity at $J = 1$.

## 4.1 Fixed pole in angular momentum plane

In 1968, Abarbanel and Goldberger [12] pointed out that the GDH sum rule is modified if the nucleon Compton amplitude exhibits a $J = 1$ *fixed pole* in complex angular-momentum plane in the $t$ channel. Within the scope of this thesis, I cannot go into the details of Regge theory (for reviews, see, e.g., Collins and Squires [57] or Drechsler [58]). Nevertheless, I would like to emphasize a few points relevant to the GDH sum rule that might be somewhat concealed in the existing literature.

#### Loss of bilinear unitarity in the $t$ channel

Fixed poles cannot occur in purely hadronic processes owing to bilinear unitarity in the $t$ channel [59]. However, in electromagnetic interactions, bilinear unitarity is lost, because the amplitude is considered to lowest order in the electromagnetic coupling. Therefore, fixed poles are not ruled out from first principles. On the other hand, there is also no a priori evidence for their *presence*, particularly as regards the $J = 1$ case relevant to the GDH sum rule.

#### Regge theory

Reggeization of a scattering amplitude is constituted of three steps. Firstly, the amplitude $A^{\mathrm{s}}(s, \cos\theta_{\mathrm{s}})$ of the $s$-channel process under consideration (in the present case: Compton scattering $\gamma(q)\mathrm{N}(p) \to \gamma(q')N(p')$, $s = (p+q)^2 = (p'+q')^2$, cf. Fig. 4.1) is identified with the amplitude $A^{\mathrm{t}}(t, \cos\theta_{\mathrm{t}})$ of the corresponding $t$-channel process ($\mathrm{N}(p)\bar{\mathrm{N}}(-p') \to \gamma(-q)\gamma(q')$, $t = (p-p')^2 = (q'-q)^2$) by continuation to all values of $s$ and $t$. (The scattering angles $\theta_{\mathrm{s}}$ and $\theta_{\mathrm{t}}$ are specifically related to $s$ and $t$.) If spin is involved, as in the present case, then the relation between $s$-channel and $t$-channel amplitudes might be intricate, but still it is trivial. Secondly, the $t$-channel amplitude is partial-wave expanded, thereby transforming the dependence on the continuous variable $\cos\theta_{\mathrm{t}}$ (or $s$) into the discrete dependence on angular momentum $J$. The partial-wave amplitude $A^{\mathrm{t}}(t, J)$ is analytically continued to complex $J$, and the partial-wave series is written as a Cauchy integral, whose contour must enclose all physical values of $J$, but none of the poles and cuts that came in by analytic continuation (Sommerfeld-Watson transform). The integration contour is deformed to a vertical line at $J = -\frac{1}{2}$, closed by a half circle at infinity, keeping trace of every pole and cut that is encountered. At large anergies, both the line at $J = -\frac{1}{2}$ and the infinitely distant half circle do not contribute [57]. The scattering amplitude is then expressed as the sum of the residues times some explicit function of the position of all poles and cuts. From this expression, the large-$s$ behavior of the scattering amplitude is inferred in the third step of the procedure.

### Spin

Most surveys on Regge theory mainly treat the scattering of spin-0 particles to explain the basic concepts, because explicit formulae become very lengthy if spin is involved. However, in the present context spin plays a vital role, since the sort of fixed poles considered here (arising from continuation of Legendre functions) always occurs at $J = 0$ or at $J = 1$ [12], while generally, Regge singularities can only occur at angular momenta that are not included in the partial-wave series. The possibility of a $J = 1$ fixed pole is thus intimitely related to the occurrence of $t$-channel total helicity *two*.

In the present context, one wants to obtain a statement on the high-energy behavior of the forward Compton amplitude with equal helicities in the initial and final states (cf. Sect. 2.1), i.e., $s$-channel helicity amplitudes [12]

$$A^{\mathrm{s}}_{\frac{1}{2}1,\frac{1}{2}1}(s,t) \quad \text{and} \quad A^{\mathrm{s}}_{\frac{1}{2}-1,\frac{1}{2}-1}(s,t) \tag{4.1a}$$

at $t = 0$. These amplitudes correspond to $t$-channel helicity amplitudes [12]

$$A^{\mathrm{t}}_{\frac{1}{2}-\frac{1}{2},1-1}(t,s) \quad \text{and} \quad A^{\mathrm{t}}_{-\frac{1}{2}\frac{1}{2},1-1}(t,s). \tag{4.1b}$$

Observe that the final-state photons have *parallel* helicity. Hence, the total helicity in the $t$ channel is *two*. Correspondingly, the partial-wave expansion starts at $J = 2$, S and P waves are absent.[1] After the Sommerfeld-Watson transform, the complex-$J$ integration must not enclose the entire positive real axis, but only the values $J = 2, 3, \ldots$. In deforming the contour as described above, it is therefore possible to encounter a pole at $J = 1$. Indeed, it is shown in Ref. 12 that such a pole might occur due to the properties of the analytic continuation of second-kind Legendre functions. It has to be stressed, however, that there is no model independent statement on the residue of that pole. Moreover, there is unfortunately not even a reasonable model prediction on the residue.

The fixed pole resides at $J = 1$ and contributes to an even-parity combination of $t$-channel helicity amplitudes. Hence it can be assigned the quantum numbers $J^P = 1^+$ of the $\mathrm{a}_1$ meson (more precicely: $\mathrm{a}_1$ and $\mathrm{f}_1$ mesons). Observe that the pole can*not* be attributed to the $t$-channel exchange of a physical $\mathrm{a}_1$ meson.

### Effect on the GDH sum rule

A $J = 1$ fixed pole results [12] in an asymptotically non-vanishing real part of forward Compton amplitude $f_2(\nu)$,

$$\mathrm{Re}\, f_2(\infty) \neq 0, \tag{4.2a}$$

[1] This point is somewhat concealed in Ref. 12. It is reflected in the appearance of the symbol $e^{J\pm}_{\lambda\mu}(\cos\theta_{\mathrm{t}})$ with indices $\lambda = 2, \mu = 1$, which involves rotation matrix elements $d^J_{\lambda\pm\mu}(\cos\theta_{\mathrm{t}})$ that are defined for $-J \leq \lambda, \mu \leq J$ only.

while the imaginary part still vanishes:

$$\operatorname{Im} f_2(\infty) = 0. \tag{4.2b}$$

(This is consistent with the convergence of the GDH integral.) The effect of the non-vanishing real constant $f_2(\infty)$, which essentially represents the residue[2] of the fixed pole, can be seen in two equivalent ways. Firstly, one may go back to the Cauchy integration formula (2.20) applied to function $f_2(\nu)$,

$$f_2(\nu) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}\nu'}{\nu' - \nu} f_2(\nu'), \tag{4.3}$$

where the contour $\mathcal{C}$ shall be the one depicted in Fig. 2.4. Now, since function $f_2(\nu)$ does not fall off at $\nu \to \infty$, there is a contribution to integral (4.3) coming from the half circles at infinite photon energies:

$$\frac{1}{2\pi i} \int_{\substack{\text{circle}\\ \text{at } \infty}} \frac{\mathrm{d}\nu'}{\nu' - \nu} f_2(\nu') = \frac{1}{2\pi i} \oint \frac{\mathrm{d}\nu'}{\nu' - \nu} f_2(\infty) = f_2(\infty), \tag{4.4a}$$

where the last equality follows simply by application of Cauchy's theorem to a constant function. As demonstrated in Sect. 2.1.4, the part of the integration contour that encloses the cuts reads

$$\frac{1}{2\pi i} \int_{\text{cuts}} \frac{\mathrm{d}\nu'}{\nu' - \nu} f_2(\nu') = \frac{2}{\pi} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu'\, \nu'}{\nu'^2 - \nu^2} \operatorname{Im} f_2(\nu'). \tag{4.4b}$$

Thus, letting $\nu \to 0$ and employing optical theorem (2.10b) and low-energy theorem (2.9b), one has

$$\boxed{-\frac{2\pi^2 \alpha \kappa^2}{M^2} = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big) + 4\pi^2 f_2(\infty)} \tag{4.5}$$

An equivalent way to see that this modification of the GDH sum rule is brought about by a non-vanishing $f_2(\infty)$ is to write down a *subtracted* dispersion relation, i.e., one considers the function

$$\frac{f_2(\nu) - f_2(0)}{\nu^2} \tag{4.6}$$

instead of $f_2(\nu)$ itself. Due to the denominator $\nu^2$, function (4.6) has an improved high-energy behavior. In particular, it vanishes at $\nu \to \infty$. (Observe that it does *not* exhibit an artificial

[2] In the notation of Ref. 12, the residue of the fixed pole is given by

$$R_{\mathrm{b}}(t{=}0) = \frac{4\pi}{M} f_2(\infty).$$

pole at $\nu = 0$, because $f_2(\nu)$ is even under crossing.) The dispersion relation for function (4.6) reads

$$\frac{f_2(\nu) - f_2(0)}{\nu^2} = \frac{2}{\pi} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu' \, \nu'}{\nu'^2 - \nu^2} \frac{\mathrm{Im}\, f_2(\nu') - \mathrm{Im}\, f_2(0)}{\nu'^2}, \tag{4.7}$$

where, as before, the point $\nu$ must not lie on one of the cuts. Taking into account that $\mathrm{Im}\, f_2(0) = 0$, this becomes

$$f_2(0) = \frac{2}{\pi} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu' (-\nu^2)}{\nu'(\nu'^2 - \nu^2)} \, \mathrm{Im}\, f_2(\nu') + f_2(\nu). \tag{4.8}$$

Letting $\nu$ approach *infinity* in Eq. (4.8), one gets

$$f_2(0) = \frac{2}{\pi} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu'}{\nu'} \, \mathrm{Im}\, f_2(\nu') + f_2(\infty), \tag{4.9}$$

which gives Eq. (4.5) upon application of optical theorem and low-energy theorem. Note that the subtraction here was *not* enforced by a divergent integral, in which case it would have been impossible to drag the limit $\nu \to \infty$ inside the $\nu'$ integral of Eq. (4.8). This type of subtraction is conveniently called a "subtraction at infinity".

**High-energy polarized forward Compton scattering**

Considering the modified GDH sum rule (4.5), the question might arise, which Compton-scattering observable the magnitude of quantity $f_2(\infty)$ is related to. In view of Sect. 2.1.1, it is clear that the answer will be some polarized forward differential Compton cross section at large photon energy. Nevertheless, one has to go a little further than Sect. 2.1.1, where only helicity eigenstates of photon and nucleon were considered, and initial and final particles were taken to be in the same spin state. Analogously to Eq. (2.4), I define the Coulomb-gauge forward Compton amplitude

$$T(\nu) = \chi'^{\dagger} \left[ f_1(\nu) \, \boldsymbol{\varepsilon}'^{*} \!\cdot\! \boldsymbol{\varepsilon} + i\nu f_2(\nu) \, \boldsymbol{\sigma} \!\cdot\! (\boldsymbol{\varepsilon}'^{*} \!\times\! \boldsymbol{\varepsilon}) \right] \chi, \tag{4.10}$$

where the polarization vectors $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$ of incident and scattered photon, as well as Pauli spinors $\chi, \chi'$ of initial and final nucleon are allowed to differ. As in Sect. 2.1.1, I take the photon to be travelling along $\boldsymbol{e}_3$. I use the term "nucleon helicity" synonymously with center-of-mass nucleon helicity, although all quantities are considered in the lab frame. Positive nucleon helicity corresponds to polarization towards the incident photon beam. Of course, if all particles are in a helicity eigenstate, then owing to the forward direction, the amplitude (4.10) vanishes unless $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon}$ and $\chi' = \chi$. According to Eqs. (2.5), one obtaines the two amplitudes $T_{1/2,3/2}(\nu) = f_1(\nu) \pm \nu f_2(\nu)$ for antiparallel and parallel helicity, respectively. To project out $f_2(\nu)$ alone, one has to either polarize the nucleon transversely (instead of longitudinally) or polarize the photon linearly (instead of circularly). Adopting the latter alternative, I assume the incident photon to be linearly polarized in the direction of the first coordinate axis, i.e., $\boldsymbol{\varepsilon} = \boldsymbol{e}_1$. The nucleon shall have helicity $+\frac{1}{2}$. Then, Eq. (4.10) yields

$$T_{\mathrm{cons}}(\nu) = f_1(\nu) \tag{4.11}$$

if the photon polarization is conserved ($\boldsymbol{\varepsilon}' = \boldsymbol{e}_1$), and

$$T_{\text{flip}}(\nu) = i\nu f_2(\nu) \tag{4.12}$$

if it is flipped by 90 degrees ($\boldsymbol{\varepsilon}' = \boldsymbol{e}_2$). (It vanishes if the final-state nucleon has negative helicity.) Falling back on Eq. (2.3), I conclude that at large photon energies, the forward differential cross section for 90-degrees spin-flip scattering of a linearly polarized photon off a longitudinally polarized nucleon,

$$\left.\frac{\mathrm{d}\sigma_{\text{flip}}}{\mathrm{d}\Omega^{\text{lab}}}\right|_{\text{forward}} = \nu^2 |f_2(\nu)|^2, \tag{4.13}$$

grows quadratically with energy if $f_2(\infty) \neq 0$. Practically, however, this is certainly not measureable directly. Moreover, it is perhaps worth noting that no lower bound on the *total* Compton cross section can be read off from Eq. (4.13), because there is no statement on the scattering-angle dependence.

Very recently, L'vov, Scopetta, Drechsel, and Scherer [60] claimed that the forward Compton amplitudes $f_{1,2}(\nu)$ can be determined in a certain energy range by measuring small-angle photoproduction of electron-positron pairs.

### Discussion

As noted before, in case of the nucleon no sensible model prediction for the magnitude of $f_2(\infty)$, i.e., the residue of the $J = 1$ fixed pole, can be found in the literature. This is the state of the art three decades after Abarbanel and Goldberger suggested the fixed pole. It can be traced back to the fact that all models of hadrons fail in the particular kinematical domain under consideration, namely $Q^2 = 0$ and $\nu \to \infty$ (high-energy real Compton scattering). Perturbative QCD, i.e., the parton model and its refinements, require high spacelike photon virtualities $Q^2$, while effective models work at low or intermediate energies only.

Nevertheless, as illustrated in the previous chapter, the GDH sum rule can be studied within perturbative models, if the nucleon is replaced by a fundamental fermion like the electron (Sects. 3.2 and 3.3), or if one employs a toy model of the nucleon interacting perturbatively with some fundamental pseudoscalar meson, called a pion for simplicity (Sect. 3.1). It has to be stressed that all of these models yield the unmodified GDH sum rule, i.e., $f_2(\infty) = 0$.

I emphasize that in principle, simple perturbative models are by all means capable of yielding fixed poles in electro-weak amplitudes. Consider, for instance, the amplitude for the scattering of isovector photons off pions. Naive current algebra gives rise to the Fubini[3] sum rule [61], which in fact *requires* a certain non-vanishing residue of a fixed pole at $J = 1$, proportional to the pion form factor [62–64]. (This is analogous to the fact that, as regards Compton scattering off nucleons, naive current algebra gives rise to the unmodified GDH sum rule, which implies a *vanishing* fixed-pole residue.) Indeed, the correct residue is obtained within a sigma model [62, 63].

Of all arguments against the presence of the fixed pole that I can offer you, here is my personal favourite (although I have to admit that there is some handwaving in it): Inspired by the successful tests of the GDH sum rule within QED and the Weinberg-Salam model,

[3] In Refs. 62 and 63, the Fubini sum rule is called Fubini-Dashen-Gell-Mann sum rule. Occasionally, it is also called Fubini-Furlan-Rossetti sum rule.

which are *per se* of higher order in $\alpha$, it is likely that the nucleon's *full* forward Compton amplitude $\bar{f}_2(\nu)$, involving all orders of $\alpha$,

$$\bar{f}_2(\nu) = \sum_{n=1}^{\infty} \alpha^n f_2^{(n)}(\nu), \tag{4.14}$$

is well-defined, which I want to assume in the following. Observe that the first term in Eq. (4.14), $\alpha f_2^{(1)}(\nu)$, is the quantity otherwise called $f_2(\nu)$. Since all orders of $\alpha$ are considered, bilinear unitarity is restored. As noted above, this rules out the presence of a fixed pole. Hence, function (4.14) obeys an unsubtracted dispersion relation

$$\bar{f}_2(0) = \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}\nu}{\nu} \operatorname{Im} \bar{f}_2(\nu), \tag{4.15}$$

i.e., $\bar{f}_2(\infty) = 0$. Now, insert expansion (4.14) into Eq. (4.15):

$$\sum_{n=1}^{\infty} \alpha^n f_2^{(n)}(0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \alpha^n \int_0^{\infty} \frac{\mathrm{d}\nu}{\nu} \operatorname{Im} f_2^{(n)}(\nu). \tag{4.16}$$

Since, by definition, all of the functions $f_2^{(n)}(\nu)$ are independent of $\alpha$, I can seperately equate the coefficients on both sides of Eq. (4.16). In particular,

$$f_2^{(1)}(0) = \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}\nu}{\nu} \operatorname{Im} f_2^{(1)}(\nu), \tag{4.17}$$

and that's it!

Yet, I would like to re-formulate this point somewhat differently. Assume that *there are* orders $n$ that need a subtraction at infinity:

$$f_2^{(n)}(0) = \frac{2}{\pi} \int_0^{\infty} \frac{\mathrm{d}\nu}{\nu} \operatorname{Im} f_2^{(n)}(\nu) + f_2^{(n)}(\infty). \tag{4.18}$$

Introducing these relations into Eq. (4.16), one obtains

$$\sum_{n=1}^{\infty} \alpha^n f_2^{(n)}(\infty) = 0. \tag{4.19}$$

If not all coefficients $f_2^{(n)}(\infty)$ were zero, then this relation (in fact, there is one such relation for each nucleon) would implicitly determine one of the fundamental parameters of the standard model (the unit charge) from some of the others, since functions $f_2^{(n)}(\nu)$ embody "pure QCD". This, however, would be crazy.

## 4.2 Anomalous charge-density commutator

In view of the current-algebra derivation of the GDH sum rule presented in Sect. 2.2, a modification may arise if the charge-density commutator does not have the naive form

$$\left[J^0(x), J^0(y)\right]_{\text{et}} = 0. \tag{4.20}$$

Any deviation from this form I call "non-naive" commutator. However, in this thesis I pay particular attention to modifications of (4.20) stemming from the chiral anomaly, which I call "anomalous commutators".

This subject already has a quite long history: Seminal work was done in 1969 by Jackiw and Johnson [65] (see also Jackiw [66]) and by Adler and Boulware [67], who were concerned with spinor electrodynamics and the decay of the neutral pion, and who considered the commutator of a vector current with an axial-vector current. Charge-density commutators within the non-linear sigma model were considered by Kramer, Palmer, and Meetz [68, 69]. Fadeev [70] investigated a chiral gauge theory, i.e., a theory of chiral fermions interacting by means of quantized gauge fields with quantum numbers that couple to both the fermionic current *and* its commutator. These studies were elaborated by Jo [71–73], who employed a Bjorken-Johnson-Low (BJL) technique [17, 56], and by Ghosh and Banerjee [74, 75] using a point-splitting method.

Note that in the present instance, we are concerned with the electromagnetic current, whose commutator – as can be read off, e.g., from Eq. (2.48) – involves the quantum numbers of an *axial*-vector current, which is *not* coupled to any gauge field, at least in the standard model. (Weak bosons are out of the question with regard to the hadronic GDH sum rule.) Moreover, since all amplitudes are considered to lowest non-trivial order in $\alpha$, the photon is presently regarded as an external field, i.e., it is not quantized. Hence the above-mentioned methods do not seem to be applicable to the problem under consideration. To overcome this deficiency, Chang and Liang [14, 15] proposed an anomalous charge-density algebra that had no apparent regard to any gauge field. (As you will see below, this is not exactly true, because a field newly introduced ad hoc in Refs. 14 and 15 turns out to be just the gauge field of chiral transformations of quarks.) The proposed charge-density algebra was applied to the GDH sum rule by Chang, Liang, and Workman [13]. This work has recently caused some attention, since the result sounds quite promising. Ref. 13 *appeared* to provide a theoretical solution to the puzzle initiated by the multipole analyses mentioned in the introduction. However, I am able to show [27] that the proposed modification is actually a delusion! This is because anomalous commutators modify the *finite-momentum* GDH sum rule only, while the infinite-momentum limit *takes back* the modification!

Explaining this fact is the purpose of the present and the following sections. First, I try to shed some light on questions that might arise about non-naive charge-density commutators. What renders them possible and what are they good for? Then, I want to motivate the use of anomalous commutators by presenting their "classical" application, namely $\pi^0$ decay. In the two following short subsections, I explain the difference between physical scattering amplitudes and T-product amplitudes, as well as the BJL limit. Next, I illustrate the work of Chang and Liang [14, 15] and show what modification of the finite-momentum GDH sum rule arises. Sect. 4.3 will be devoted to a detailed discussion of the infinite-momentum limit. Finally, in Sect. 4.4, I compute the anomalous commutator within the Weinberg-Salam model, where the legitimacy of the infinite-momentum limit can explicitly be checked.

### 4.2.1 Why non-naive charge-density commutators?

For the moment, let us consider a single charged Dirac field $\psi(x)$ with current-density four-vector

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x), \tag{4.21}$$

interacting with a gauge field $A^\mu(x)$.

#### Gauge invariance and commutators

The local gauge transformation of fields $\psi(x)$ and $A^\mu(x)$ reads

$$\begin{aligned} eA^\mu(x) \mapsto eA'^\mu(x) &= eA^\mu(x) + \partial^\mu\chi(x), \\ \psi(x) \mapsto \psi'(x) &= e^{-i\chi(x)}\psi(x), \end{aligned} \tag{4.22}$$

where $\chi(x)$ is an arbitrary function. The part of the transformation acting on the matter fields $\psi(x)$, $\bar{\psi}(x)$ is generated by the charge-density operator $J^0(x) = \bar{\psi}(x)\gamma^0\psi(x) = \psi^\dagger(x)\psi(x)$. This means that the action of an infinitesimal transformation $\delta\psi(x) = -i\delta\chi(x)\psi(x)$ on any local operator $O_\mathrm{m}(x)$ *containing only matter fields* is given by

$$\delta O_\mathrm{m}(x) = -i\int \mathrm{d}^3y \left[J^0(y), O_\mathrm{m}(x)\right]_\mathrm{et}\delta\chi(y), \tag{4.23}$$

where subscript "et" denotes equal times, i.e., $x^0 = y^0$. Local gauge invariance of $O_\mathrm{m}(x)$ means that this vanishes for every function $\delta\chi(x)$. Consequently,

$$[J^0(y), O_\mathrm{m}(x)]_\mathrm{et} = 0. \tag{4.24}$$

Now, as a matter of course, the current density itself is a gauge invariant observable. Thus, naively,

$$\left[J^0(x), J^\mu(y)\right]_\mathrm{et} = 0. \tag{4.25}$$

#### Conventional c-number Schwinger term

On the other hand, as Julian Schwinger showed [76] in the 50's, for the spatial current density ($\mu = 1, 2, 3$) the above equal-times commutator is not only *allowed* to be non-vanishing, it in fact *has* to be non-vanishing. This can be understood (see also Ref. 43, Sect. 11-3-1) by considering the divergence of the space part of the naive commutator (4.25),

$$\left[J^0(x), \boldsymbol{\nabla}\cdot\boldsymbol{J}(y)\right]_\mathrm{et} = 0, \tag{4.26}$$

employing current conservation,

$$\left[J^0(x), \partial_0 J^0(y)\right]_\mathrm{et} = 0, \tag{4.27}$$

taking the vacuum expectation value, and inserting a sum over a complete set of energy eigenstates, $1 = \sum_n |n\rangle\langle n|$, which gives in the limit $\boldsymbol{y} \to \boldsymbol{x}$

$$\sum_n E_n \left|\langle n|J^0(x)|0\rangle\right|^2 = 0. \tag{4.28}$$

From the positivity of energy this would imply that $\langle n|J^0(x)|0\rangle = 0$ for all excited states $|n\rangle$. Thus, the vacuum would be an eigenvector of the charge density, which is "excluded physically for the example of the charge flux vector" [76]. The simplest non-naive form of the commutator reads

$$\left[J^0(x), \boldsymbol{J}(y)\right]_{\mathrm{et}} = iK\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.29}$$

with some (c-number) constant $K$. A specfic instance of a c-number Schwinger term, motivated by vector-meson dominance, will be presented in Sect. 4.2.5 of this thesis.

So, what about gauge invariance then?

### Point splitting

As Schwinger pointed out [76], the current (4.21), given as a fermion bilinear with *equal* spacetime coordinates within $\bar{\psi}$ and $\psi$, is in fact ill-defined. To make it a well-defined quantity, the spacetime points have to be *separated* by a small spacelike distance $\varepsilon$, letting $\varepsilon \to 0$ eventually. Now, the quantity $\bar{\psi}(x-\varepsilon/2)\gamma^\mu\psi(x+\varepsilon/2)$ is indeed no longer locally gauge invariant on account of the different action of transformation (4.22) at points $x+\varepsilon/2$ and $x-\varepsilon/2$. To restore gauge invariance of the so-defined current, a factor involving the gauge field $A^\mu(x)$ has to be supplemented,

$$J^\mu(x;\varepsilon) := \bar{\psi}(x-\varepsilon/2)\gamma^\mu\psi(x+\varepsilon/2)(1+ie\,\boldsymbol{\varepsilon}\cdot\boldsymbol{A}(x)). \tag{4.30}$$

The crucial thing to be noted is that by means of this so-called point-splitting method, the current becomes gauge-field dependent! Then, however, Eq. (4.23) must be generalized. Considering a local operator $O(x)$ that containes both matter and gauge fields, the action of an infinitesimal gauge transformation $\delta A^\mu(x) = \partial^\mu\delta\chi(x)$, $\delta\psi(x) = -i\delta\chi(x)\psi(x)$ reads

$$\delta O(x) = -i\int \mathrm{d}^3y\left[G(y), O(x)\right]_{\mathrm{et}}\delta\chi(y), \tag{4.31}$$

where

$$G(x) = J^0(x) + \frac{i}{e}\nabla^k\frac{\delta}{\delta A^k(x)} \tag{4.32}$$

represents the *full* generator of gauge transformations, whose second component acts on the gauge field.

Consequently, gauge invariance manifests itself as

$$\left[G(x), J^\mu(y)\right]_{\mathrm{et}} = 0, \tag{4.33}$$

while a non-vanishing commutator of $J^\mu(y)$ with the charge density, which acts on the matter field only, is by no means prohibited.

### Caution!

As a conclusion, one ought to keep in mind that regarding the matter current density as a purely matter-field dependent quantity, which would have to commute with the charge-density operator (being the matter part of the generator of the gauge transformation), is too naive an idea to yield the correct commutation relations. Some caution is in order!

### 4.2.2 Decay of the neutral pion

To motivate the use of anomalous commutators in particle physics, I now sketch the current-algebra calculation of the $\pi^0 \to \gamma\gamma$ decay width as given by Jackiw and Johnson in 1969 [65] (see also Jackiw [66]).

The neutral pion is observed to decay into two photons with a width of roughly 8 eV (branching ratio 98.8%). As you will see below, this rather large value is naively unexpected. Indeed, among the few experimentally observed manifestations of the chiral anomaly [77], $\pi^0$ decay is the clearest one.[4] Its width measures the $S$-matrix element

$$S_{\mathrm{fi}} = \big\langle \gamma(k_1, \varepsilon_1)\, \gamma(k_2, \varepsilon_2) \big| S \big| \pi^0(p) \big\rangle, \tag{4.34}$$

where $p = k_1 + k_2$. I study the limit $p \to 0$, which is not too far from the pion mass shell due to the smallness of $m_\pi^2$. (This is a rather lax phrase for an actually much deeper meaning, which has to do with the Goldstone-boson character of the pion.) Applying the Lehmann-Symanzik-Zimmermann reduction formulae (see, e.g., Bjorken and Drell [78]) to the pion and one of the photons, one gets

$$S_{\mathrm{fi}} \propto m_\pi^2\, \varepsilon_{1\mu} \int \mathrm{d}^4x\, e^{ik_1 \cdot x} \int \mathrm{d}^4y\, \big\langle 0 \big| \mathrm{T}\, J^\mu(x) \phi(y) \big| \gamma(k_2, \varepsilon_2) \big\rangle, \tag{4.35}$$

where $\phi(x)$ is the $\pi^0$ field operator and T, as usual, represents time ordering. I omit any constant factor except the pion mass, which I keep trace of in order to show that the final result will not exhibit a zero or a pole at $m_\pi^2 = 0$. The pion field can be expressed in terms of an axial-vector current by virtue of the PCAC ("Partially Conserved Axial-vector Current") relation [79–81]

$$\partial_\mu J_5^\mu(x) = F_\pi m_\pi^2 \phi(x), \tag{4.36}$$

where $F_\pi = 93$ MeV is the pion decay constant. The axial-vector current $J_5^\mu(x)$ has the same quantum numbers than the pion. In QCD it reads

$$J_5^\mu(x) = \tfrac{1}{2}\big(\bar{u}(x)\, \gamma^\mu \gamma_5\, u(x) - \bar{d}(x)\, \gamma^\mu \gamma_5\, d(x)\big). \tag{4.37}$$

By means of the step function, the time-ordered product can be written

$$\begin{aligned} \mathrm{T}\, J^\mu(x)\, \partial_\nu J_5^\nu(y) &= \theta(x^0 - y^0)\, J^\mu(x)\, \partial_\nu J_5^\nu(y) + \theta(y^0 - x^0)\, \partial_\nu J_5^\nu(y)\, J^\mu(x) \\ &= \theta(x^0 - y^0)\, \big[J^\mu(x), \partial_\nu J_5^\nu(y)\big] + \partial_\nu J_5^\nu(y)\, J^\mu(x). \end{aligned} \tag{4.38}$$

The second term does not contribute to the amplitude (4.35), since it is a total divergence with respect to $y$. As for the first term, integrate partially and observe that

$$\frac{\partial}{\partial y^0}\, \theta(x^0 - y^0) = -\delta(x^0 - y^0), \tag{4.39a}$$

$$\frac{\partial}{\partial y^k}\, \theta(x^0 - y^0) = 0. \tag{4.39b}$$

[4] The so-called proton spin crisis is nowadays widely accepted to be a manifestation of the $\mathrm{U}(1)_\mathrm{A}$ anomaly, which has much in common with the chiral anomaly but is not the same!

Therewith, amplitude (4.35) runs

$$S_{\mathrm{fi}} \propto \varepsilon_{1\mu} \int \mathrm{d}^4x\, e^{ik_1\cdot x} \int \mathrm{d}^3y\, \langle 0 | \big[J^\mu(x), J_5^0(y)\big]_{\mathrm{et}} | \gamma(k_2, \varepsilon_2)\rangle. \tag{4.40}$$

Observe how the time-ordered product in Eq. (4.35) translates into the equal-times commutator in Eq. (4.40) due to the divergence appearing in the PCAC relation (4.36) and owing to the fact that the derivative of the step function is the delta function. It is always for this little trick that commutators emerge in current-algebra treatment of soft-pion processes.

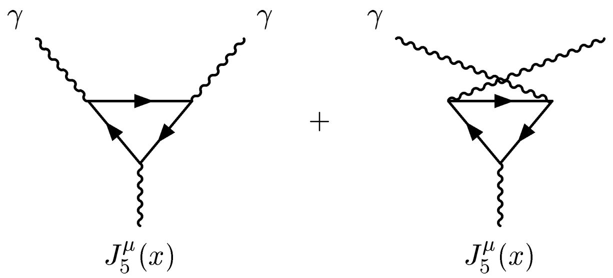
### Sutherland-Veltman theorem

Analogously to the naive commutator (2.42) of a *vector* charge density with a vector current, the commutator $[J^\mu(x), J_5^0(y)]_{\mathrm{et}}$ involving an "axial charge density" vanishes on employing canonical anticommutation relations among quark fields. Thus, naively the decay amplitude (4.34) vanishes at $p^2 = 0$. This misbelief is called *Sutherland-Veltman theorem* [82, 83]. If the decay amplitude is not dramatically varying from $p^2 = 0$ to $p^2 = m_\pi^2$, then, according to the Sutherland-Veltman theorem, the decay width of the neutral pion ought to be much smaller than 8 eV.

After Sutherland and Veltman pointed out this "experimental failure of PCAC" [66], the most widely accepted explanation was that the decay amplitude *was* in fact rapidly varying with $p^2$, for unknown reasons. Although this is principally not impossible, no reason could be found for the putative rapid variation, and it became clear that either PCAC or the naive commutator $[J^\mu(x), J_5^0(y)]_{\mathrm{et}} = 0$ has to be sacrificed. As it turns out, one can indeed *choose* which of these sacrifices to make [66]. Since I am concerned with anomalous commutators, I of course focus on the latter one.

### Anomalous commutator

Anomalous commutators like the one under consideration are conveniently calculated by virtue of the Bjorken-Johnson-Low (BJL) technique [17, 56]. More precicely, this technique fetches *matrix elements* – not commutators themselves, but after all, we are only *interested* in matrix elements. The BJL method will be explained in Sect. 4.2.4. It reduces the commutator problem to the calculation of a certain high-energy limit of the according amplitude, which is obtained from the Feynman graphs




(4.41)

Such fermion triangle (or higher polygon) graphs with an odd number of axial-vector currents attached to the vertices always signalize appearance of the anomaly. I stress that the result does not depend on the kind of fermions that make up the triangle. Since the current $J_5^\mu(x)$ couples to the third component of "axial isospin", the graphs (4.41) are proportional to

$$3\big[\big(\tfrac{2}{3}\big)^2 \times \tfrac{1}{2} + \big(-\tfrac{1}{3}\big)^2 \times \big(-\tfrac{1}{2}\big)\big] = \tfrac{1}{2} \tag{4.42a}$$

upon letting u and d quarks – one of each color – circulate within the loop, and

$$\big(1\big)^2\times\tfrac{1}{2}+\big(0\big)^2\times\big(-\tfrac{1}{2}\big)=\tfrac{1}{2} \tag{4.42b}$$

if one takes "bare" nucleons with axial charge $g_{\mathrm{A}}=1$, which has been the historical version. The equality of expressions (4.42) implies independence on the kind of fermions used.

I will comprehensively illustrate the method in Sect. 4.4, where I am concerned with the anomalous *vector*-charge-density commutator – the one underlying the GDH sum rule. Here I only cite the result of Jackiw and Johnson [65]:

$$\big[J^0(x),J_5^0(y)\big]_{\mathrm{et}}=\frac{ie}{2\pi^2}\,\tilde{F}^{0j}(y)\,\nabla^j\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.43a}$$

$$\big[J^i(x),J_5^0(y)\big]_{\mathrm{et}}=\frac{ie}{4\pi^2}\,\tilde{F}^{ij}(x)\,\nabla^j\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.43b}$$

where $\tilde{F}^{\mu\nu}(x)$ denotes the dual electromagnetic tensor,

$$\tilde{F}^{\mu\nu}=\tfrac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta-\partial_\beta A_\alpha). \tag{4.44}$$

For later use, I re-write the charge-charge commutator (4.43a) in terms of the electromagnetic field $A^\mu(x)$, employing Eq. (4.44):

$$\big[J^0(x),J_5^0(y)\big]_{\mathrm{et}}=-\frac{ie}{2\pi^2}\,\big(\boldsymbol{\nabla}\times\boldsymbol{A}(x)\big)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{4.45}$$

Since $A^\mu(x)$ is the field operator of the photon, it has a non-vanishing matrix element between vacuum and the one-photon state. Therefore, commutators (4.43) lead to a non-zero decay amplitude (4.40). Performing the calculation thoroughly, keeping all factors, one obtaines a $\pi^0$ decay width that is indeed amazingly close to the experimental value. Taking fractional-charge quarks to make up the triangles in Eq. (4.41), this result lends support to the idea that quarks carry color degrees of freedom.

### 4.2.3 Time ordering and seagulls

Up to now, I throughout treated physical amplitudes of electromagnetic processes as matrix elements of *time-ordered products* of electromagnetic currents. As I will demonstrate shortly, this is not necessarily correct. The difference between a physical amplitude and its associated T-product amplitude, the so-called seagull amplitude, did not play any role in this thesis so far, but it is relevant to the discussion of anomalous charge-density commutators, particularly as regards my own investigation within the Weinberg-Salam model, which I will present in Sect. 4.4. Therefore, I now give an introduction into the origin and properties of seagulls. (In Sect. 4.4.1, you will also learn where their funny name comes from.)

By means of the BJL technique, the matrix element of the charge-density commutator is given by a certain high-energy limit of the (generally non-forward) Compton amplitude of the fermion under consideration. Therefore I am concerned with the $S$-matrix element

$$S_{\mathrm{fi}}=\big\langle q',\varepsilon';p',\lambda'\big|S\big|q,\varepsilon;p,\lambda\big\rangle \tag{4.46}$$

for the scattering of a photon with momentum $q$ and polarization $\varepsilon$ off a fermion (either nucleon or electron) with momentum $p$ and helicity $\lambda$. In terms of the $S$-matrix element (4.46), the (physical) Compton amplitude $T^{\mu\nu}$ is defined by the relation

$$S_{\mathrm{fi}}=-ie^2\,(2\pi)^4\delta(p+q-p'-q')\,\varepsilon'^*_\mu\,\varepsilon_\nu T^{\mu\nu}. \tag{4.47}$$

For definiteness, the T-product Compton amplitude is now endowed with a subscript 0:

$$T_0^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq'\cdot x}\big\langle p',\lambda'\big|\mathrm{T}J^\mu(x)J^\nu(0)\big|p,\lambda\big\rangle, \tag{4.48}$$

To find the difference $T^{\mu\nu} - T_0^{\mu\nu}$, one has to recall the way relation (4.47) emerges by virtue of the Lehmann-Symanzik-Zimmermann reduction technique (see, e.g., Bjorken and Drell [78]). The crucial point is that the reduction formula is applied *twice*, since there are two external photons.

Applying the reduction formula to the incident photon yields

$$S_{\mathrm{fi}} = i\,\varepsilon_\nu \int \mathrm{d}^4y\, e^{-iq\cdot y}\,\big\langle p',\lambda';q',\varepsilon'\big|\Box A^\nu(y)\big|p,\lambda\big\rangle, \tag{4.49}$$

where $A^\mu(x)$ is the operator of the electromagnetic field, and the d'Alembert differential operator is $\Box = \partial_\mu\partial^\mu = \partial_0^2 - \boldsymbol{\nabla}^2$. Employing Maxwell's equations

$$\Box A^\mu(x) = J^\mu(x), \tag{4.50}$$

as well as translational invariance (1.7), one has

$$S_{\mathrm{fi}} = ie\,\varepsilon_\nu \int \mathrm{d}^4y\, e^{-i(p+q-p'-q')\cdot y}\,\big\langle p',\lambda';q',\varepsilon'\big|J^\nu(0)\big|p,\lambda\big\rangle. \tag{4.51}$$

The $y$ integration gives rise to a four-dimensional delta function:

$$S_{\mathrm{fi}} = ie\,(2\pi)^4\delta(p+q-p'-q')\,\varepsilon_\nu\,\big\langle p',\lambda';q',\varepsilon'\big|J^\nu(0)\big|p,\lambda\big\rangle. \tag{4.52}$$

Now the final-state photon is reduced out:

$$\big\langle p',\lambda';q',\varepsilon'\big|J^\nu(0)\big|p,\lambda\big\rangle = -i\,\varepsilon_\mu'^{*}\int \mathrm{d}^4x\, e^{iq'\cdot x}\,\big\langle p',\lambda'\big|\Box\big(\mathrm{T}A^\mu(x)J^\nu(0)\big)\big|p,\lambda\big\rangle. \tag{4.53}$$

By comparison with Eqs. (4.47) and (4.52), one identifies[5]

$$e\,T^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq'\cdot x}\,\big\langle p',\lambda'\big|\Box\big(\mathrm{T}A^\mu(x)J^\nu(0)\big)\big|p,\lambda\big\rangle. \tag{4.54}$$

Observe that the d'Alembert operator acts on the full time-ordered product and not directly on the field $A^\mu(x)$. Writing out the T product,

$$\begin{aligned}\mathrm{T}A^\mu(x)J^\nu(0) &= \theta(x^0)A^\mu(x)J^\nu(0) + \theta(-x^0)J^\nu(0)A^\mu(x) &\text{(4.55a)}\\ &= \theta(x^0)\big[A^\mu(x),J^\nu(0)\big] + J^\nu(0)A^\mu(x), &\text{(4.55b)}\end{aligned}$$

one realizes that the time-derivative part of $\Box = \partial_0^2 - \boldsymbol{\nabla}^2$ is caught on the step function,

$$\partial_0^2\big(\mathrm{T}A^\mu(x)J^\nu(0)\big) = \partial_0\Big(\delta(x^0)\big[A^\mu(x),J^\nu(0)\big] + \theta(x^0)\big[\partial_0A^\mu(x),J^\nu(0)\big]\Big) + J^\nu(0)\partial_0^2A^\mu(x)$$

[5] Occasionally, the expression $\Box(\mathrm{T}A^\mu(x)J^\nu(0))$ is abbreviated as $e\,\mathrm{T}^*J^\mu(x)J^\nu(0)$. Note, however, that this is only a notation, because strictly speaking, there is no operator $\mathrm{T}^*$ acting on the currents which yields such an expression.

$$= \partial_0\Big(\delta(x^0)\left[A^\mu(x), J^\nu(0)\right]\Big) + \delta(x^0)\left[\partial_0 A^\mu(x), J^\nu(0)\right] + \mathrm{T}\,\partial_0^2 A^\mu(x) J^\nu(0), \tag{4.56}$$

whereas the spatial derivatives go through,

$$\boldsymbol{\nabla}^2\big(\mathrm{T}A^\mu(x)J^\nu(0)\big) = \mathrm{T}\,\boldsymbol{\nabla}^2 A^\mu(x)J^\nu(0). \tag{4.57}$$

Hence, the d'Alembert operator does generally not commute with the time ordering:

$$\Box\big(\mathrm{T}A^\mu(x)J^\nu(0)\big) = \mathrm{T}\,\Box A^\mu(x)J^\nu(0) + \partial_0\Big(\delta(x^0)\left[A^\mu(x), J^\nu(0)\right]\Big) + \delta(x^0)\left[\partial_0 A^\mu(x), J^\nu(0)\right]. \tag{4.58}$$

Inserting Eq. (4.58) into Eq. (4.54) and employing Maxwell's equations again, one arrives at

$$\boxed{T^{\mu\nu} = T_0^{\mu\nu} + S^{\mu\nu}} \tag{4.59}$$

where the seagull amplitude $S^{\mu\nu}$ is given by

$$\begin{aligned} e\,S^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq'\cdot x}\Big[&\partial_0\Big(\delta(x^0)\langle p',\lambda'\big|[A^\mu(x), J^\nu(0)]\big|p,\lambda\rangle\Big) \\ &+ \delta(x^0)\langle p',\lambda'\big|[\partial_0 A^\mu(x), J^\nu(0)]\big|p,\lambda\rangle\Big]. \end{aligned} \tag{4.60}$$

After a partial integration, one can immediately carry out the $x^0$ integration:

$$\begin{aligned} e\,S^{\mu\nu} = {}& q'^0\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}}\,\langle p',\lambda'\big|[A^\mu(x), J^\nu(0)]_{\mathrm{et}}\big|p,\lambda\rangle \\ &+ i\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}}\,\langle p',\lambda'\big|[\partial_0 A^\mu(x), J^\nu(0)]_{\mathrm{et}}\big|p,\lambda\rangle. \end{aligned} \tag{4.61}$$

Observe that the seagull (4.61) is a first-order[6] polynomial in the energy $q'^0$. In all cases I am concerned with, it even turns out to be independent of $q'^0$, i.e., the linear term in Eq. (4.61) vanishes.[7] Nevertheless, presently the only important thing to be kept in mind is the fact that $S^{\mu\nu}$ is polynomial in $q'^0$.

### 4.2.4 Bjorken-Johnson-Low limit

In 1966, Bjorken [17] and Johnson and Low [56] proposed a method for computing an equal-times commutator from the pertinent T-product amplitude. I now want to illustrate this techique, following Bjorken's original presentation [17] (see also Jackiw [66]).

[6]More generally, it can be shown [66] that the seagull is always a (possibly higher-order) polynomial, even if one deals with an amplitude that does not involve conserved currents coupled to a gauge field as in the present context.

[7]This circumstance gets plausible by considering the point-splitting definition (4.30) of the electromagnetic current. As illustrated in Sect. 4.2.1, one can generally work with canonical commutators among fundamental fields as long as one recalls that the current must not be defined as a fermion bilinear at coincident spacetime points. The explicit dependence of the point-splitting current (4.30) on the gauge field $A^\mu(x)$ leaves its equal-times commutator with the gauge field itself canonically vanishing, whereas it will generally not commute with the canonically conjugated field, i.e., the electric field, which occurs in the constant portion of expression (4.61).

Let $A(x)$ and $B(x)$ be any two local operators (eventually, I will of course take currents), and sandwich the time-ordered product between any two hadronic or leptonic states $|\mathrm{X}\rangle$ and $|\mathrm{Y}\rangle$:

$$T := i\int \mathrm{d}^4x\, e^{iq'\cdot x}\,\big\langle \mathrm{Y}\big|\mathrm{T}A(x)B(0)\big|\mathrm{X}\big\rangle. \tag{4.62}$$

Writing out the T product occurring in Eq. (4.62) as in Eq. (4.55a) and introducing the Fourier transform (1.11) of the step function, one has

$$\begin{aligned} T &= -\frac{1}{2\pi}\int \mathrm{d}^4x\, e^{iq'\cdot x}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{E+i\varepsilon}\Big(\big\langle \mathrm{Y}\big|A(x)B(0)\big|\mathrm{X}\big\rangle\, e^{-iEx^0} + \big\langle \mathrm{Y}\big|B(0)A(x)\big|\mathrm{X}\big\rangle\, e^{iEx^0}\Big) \\ &= -\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{E+i\varepsilon}\big(\rho(q'^0-E)+\bar\rho(q'^0+E)\big), \end{aligned} \tag{4.63}$$

where functions $\rho(q'^0)$ and $\bar\rho(q'^0)$ are defined by

$$\rho(q'^0) = \frac{1}{2\pi}\int \mathrm{d}^4x\, e^{iq'\cdot x}\,\big\langle \mathrm{Y}\big|A(x)B(0)\big|\mathrm{X}\big\rangle \tag{4.64a}$$

and

$$\bar\rho(q'^0) = \frac{1}{2\pi}\int \mathrm{d}^4x\, e^{iq'\cdot x}\,\big\langle \mathrm{Y}\big|B(0)A(x)\big|\mathrm{X}\big\rangle \tag{4.64b}$$

respectively. Linear substitutions of the integration variable transform Eq. (4.63) into

$$T = -\int_{-\infty}^{\infty}\mathrm{d}E'\left(\frac{\rho(E')}{q'^0-E'+i\varepsilon}+\frac{\bar\rho(E')}{-q'^0+E'+i\varepsilon}\right). \tag{4.65}$$

Multiplying with $q'^0$ and letting $q'^0$ approach infinity, this reduces to

$$\begin{aligned} \lim_{q'^0\to\infty} q'^0\,T &= -\int \mathrm{d}E'\big(\rho(E')-\bar\rho(E')\big) \\ &= -\frac{1}{2\pi}\int \mathrm{d}E'\int \mathrm{d}^4x\, e^{i(E'x^0-\boldsymbol{q}'\cdot\boldsymbol{x})}\,\big\langle \mathrm{Y}\big|[A(x),B(0)]\big|\mathrm{X}\big\rangle. \end{aligned} \tag{4.66}$$

The $E'$ and $x^0$ integrations are performed straightforwardly:

$$\begin{aligned} \lim_{q'^0\to\infty} q'^0\,T &= -\int \mathrm{d}^4x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}}\,\delta(x^0)\,\big\langle \mathrm{Y}\big|[A(x),B(0)]\big|\mathrm{X}\big\rangle \\ &= -\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}}\,\big\langle \mathrm{Y}\big|[A(x),B(0)]_{\mathrm{et}}\big|\mathrm{X}\big\rangle. \end{aligned} \tag{4.67}$$

This is the BJL formula for the matrix element of an equal-times commutator. Re-introducing current operators and one-fermion states, I conclude

$$\boxed{\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}}\,\big\langle p',\lambda'\big|[J^\mu(x),J^\nu(0)]_{\mathrm{et}}\big|p,\lambda\big\rangle = -\lim_{q'^0\to\infty} q'^0\,T_0^{\mu\nu}} \tag{4.68}$$

**Identifying seagulls**

In Sect. 4.4, I aim at a perturbative calculation of the anomalous charge-density commutator within the Weinberg-Salam model, i.e., from (one-loop) Feynman graphs, which do not yield the T-product amplitude $T_0^{\mu\nu}$, but rather the physical amplitude $T^{\mu\nu}$. However, it is clear from the above considerations that the difference $S^{\mu\nu} = T^{\mu\nu} - T_0^{\mu\nu}$, i.e., the seagull amplitude, is readily identified by its polynomial behavior in energy $q'^0$. Thus, the procedure simply amounts to calculating $T^{\mu\nu}$ and subsequently dropping all polynomials in $q'^0$. In other words, the above limit picks out the linear term of the Laurent series of amplitude $T^{\mu\nu}$ in the variable $1/q'^0$.

### 4.2.5 Anomalous commutator of Chang and Liang

To illustrate the work of Chang and Liang [14, 15], I need some preliminaries. The electromagnetic current (2.46) has to be generalized to a U(3) nonet of currents

$$J_a^\mu(x) = \bar\psi(x)\,\gamma^\mu \frac{\lambda_a}{2}\,\psi(x) \tag{4.69a}$$

and a nonet of axial-vector currents

$$J_{5a}^\mu(x) = \bar\psi(x)\,\gamma^\mu\gamma_5 \frac{\lambda_a}{2}\,\psi(x), \tag{4.69b}$$

where, as usual,

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \tag{4.70}$$

and $\lambda_{0\ldots8}$ denote Gell-Mann matrices, which are normalized by

$$\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}. \tag{4.71}$$

Of particular interest are the three diagonal matrices

$$\lambda_0 = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{4.72}$$

You have met a certain combination of $\lambda_3$ and $\lambda_8$ before:

$$\frac{\lambda_3}{2} + \frac{1}{\sqrt3}\frac{\lambda_8}{2} = \begin{pmatrix} \frac23 & 0 & 0 \\ 0 & -\frac13 & 0 \\ 0 & 0 & -\frac13 \end{pmatrix} = Z^{(\mathrm{q})}, \tag{4.73}$$

the quark charge matrix. So, the familiar electromagnetic current is given by

$$J^\mu(x) = J_3^\mu(x) + \frac{1}{\sqrt3}\,J_8^\mu(x). \tag{4.74}$$

Gell-Mann matrices $\lambda_{0\ldots8}$ are generators of the group U(3) in the sense that any unitary $3{\times}3$ matrix $U$ can be written $U = \exp(i\alpha_a\lambda_a/2)$. Matrix $\lambda_0$ is special in that it has a non-vanishing trace, i.e., it generates the baryon-number U(1) subgroup of flavor U(3). Matrices

$\lambda_{1\ldots8}$ accordingly generate the conventional flavor SU(3) subgroup. Since quarks carry baryon number 1/3, the baryon-number density reads

$$\frac{1}{3}\,\psi^\dagger(x)\,\psi(x) = \sqrt{\frac{2}{3}}\,J_0^0(x). \tag{4.75}$$

Commutators and anticommutators of Gell-Mann matrices satisfy

$$\left[\frac{\lambda_a}{2},\frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2}, \qquad \left\{\frac{\lambda_a}{2},\frac{\lambda_b}{2}\right\} = d_{abc}\frac{\lambda_c}{2}, \tag{4.76}$$

where $f_{abc}$ is totally antisymmetric and $d_{abc}$ is totally symmetric. Both, $f$ and $d$ symbols are real numbers. Since the matrix $\lambda_0$ commutes with all others, $f_{abc}$ vanishes upon letting any of its indices equal zero, while the $d$ symbol obeys $d_{ab0} = \sqrt{2/3}\,\delta_{ab}$.

### Naive current commutators

From definitions (4.69) of vector and axial-vector currents, naive commutation relations can be derived utilizing canonical anticommutation relations among quark fields as in Sect. 2.2.1. The result reflects the fact that the charge densities $J_a^0(x)$ and $J_{5a}^0(x)$ generate local U(3)×U(3) transforms:

$$\begin{aligned}
\left[J_a^0(x), J_b^\mu(y)\right]_{\text{et}} &= \left[J_{5a}^0(x), J_{5b}^\mu(y)\right]_{\text{et}} = if_{abc}\,J_c^\mu(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}),\\
\left[J_a^0(x), J_{5b}^\mu(y)\right]_{\text{et}} &= \left[J_{5a}^0(x), J_b^\mu(y)\right]_{\text{et}} = if_{abc}\,J_{5c}^\mu(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}).
\end{aligned} \tag{4.77}$$

Observe that these relations incorporate the algebra of charge densities ($\mu = 0$) as well as charge-current commutators ($\mu = 1,2,3$). Both are free of anomalies and Schwinger terms.

### Vector-meson dominance

For later use, I give the form of the charge-current algebra involving finite c-number Schwinger terms as inspired by vector-meson dominance (VMD, occasionally also called "current-field identity") [84, 85]:

$$\left[J_a^0(x), \boldsymbol{J}_b(y)\right]_{\text{et}} = \left[J_{5a}^0(x), \boldsymbol{J}_{5b}(y)\right]_{\text{et}} = if_{abc}\,\boldsymbol{J}_c\,\delta(\boldsymbol{x}-\boldsymbol{y}) + 2iF_\pi^2\,\delta_{ab}\,\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.78a}$$

$$\left[J_a^0(x), \boldsymbol{J}_{5b}(y)\right]_{\text{et}} = \left[J_{5a}^0(x), \boldsymbol{J}_b(y)\right]_{\text{et}} = if_{abc}\,\boldsymbol{J}_{5c}\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{4.78b}$$

The idea of VMD is easy to explain. Since the currents $J_a^\mu(x)$ and $J_{5a}^\mu(x)$ have the same quantum numbers as vector and axial-vector mesons, respectively, they can couple directly to these mesons. As far as the $\rho$ meson and the isovector part of the photon are concerned, hadronic form factors are observed to be *dominated* by this process at low momentum transfer. If vector mesons are regarded as fundamental particles with field operators $v_a^\mu(x)$, then this dominance can be expressed as [86, 87]

$$J_a^\mu(x) = \frac{m_{\text{V}}^2}{f_{\text{V}}}\,v_a^\mu(x) \tag{4.79}$$

where $m_{\text{V}}$ and $f_{\text{V}}$ denote universal mass and coupling constant of fields $v_a^\mu(x)$. On the other hand, these fields obey standard canonical commutation relations of massive spin-1 fields [84, 85]:

$$m_{\text{V}}^2\left[v_a^0(x), \boldsymbol{v}_b(y)\right]_{\text{et}} = if_{\text{V}}f_{abc}\,\boldsymbol{v}_c(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}) + i\delta_{ab}\,\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.80}$$

and likewise for the axial-vector mesons. If one substitutes currents for the meson fields by virtue of the VMD relation (4.79), Eqs. (4.78) follow upon using the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation [88, 89]

$$\frac{m_{\mathrm{V}}^2}{f_{\mathrm{V}}^2} = 2F_\pi^2. \tag{4.81}$$

By the way: The equality of vector and axial-vector Schwinger terms in Eq. (4.78a) can actually be proved from much weaker assumptions than those demanded by VMD. The proof, given by Weinberg [90], is recapitulated nicely and comprehensibly in a footnote[8] in the textbook of de Alfaro, Fubini, Furlan, and Rossetti [85, p. 365].

### Triple-commutator vacuum expectation value

With these formulae at hand, I can write down the quantity that has actually been calculated by Chang and Liang, namely the vacuum expectation value of a triple commutator of four charge densities, incorporating an odd number of axial charge densities, e.g. [14, 15][9]

$$\begin{aligned}&\langle 0|\big[J^0_{5a}(x),\big[J^0_b(y),\big[J^0_c(w),J^0_d(z)\big]\big]\big]_{\mathrm{et}}|0\rangle\\ &\quad=\frac{iN_{\mathrm{c}}}{24\pi^2}\,f_{abe}\,d_{cde}\,\big(\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y})\times\boldsymbol{\nabla}\delta(\boldsymbol{y}-\boldsymbol{z})\big)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{z}-\boldsymbol{w}),\end{aligned} \tag{4.82}$$

where $N_{\mathrm{c}} = 3$ is the number of colors. All time components are equal: $x^0 = y^0 = z^0 = w^0$. Expression (4.82) is the same if the axial charge density is at any other position or if there are three axial charge densities. It vanishes if their number is zero, two, or four. Observe that the symmetry of the $d$ symbol does not contradict the antisymmetry of the innermost commutator, since $\boldsymbol{\nabla}\delta(\boldsymbol{z}-\boldsymbol{w})$ changes sign upon interchanging $\boldsymbol{w}$ and $\boldsymbol{z}$.

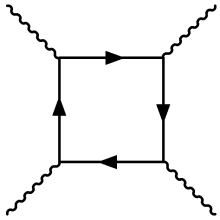
The anomalous charge-density commutator is then *proposed* as the simplest one reproducing the triple-commutator (4.82) upon iterative application, together with the standard conventional Schwinger term (4.78) inferred from VMD. I postpone the presentation of these ideas and first comment on the calculation of Eq. (4.82).

Two different methods have been employed to yield the triple-commutator vacuum expectation value (4.82): firstly an anomaly-functional method [14], which utilizes a triple Bjorken-Johnson-Low limit to calculate the triple commutator from a four-point function; secondly a rather direct computation [15], performing all possible Wick contractions among properly normal-ordered products of currents. The anomaly is caused by a subtle type of divergence, which can be remedied by a shift of the integration variable. Both calculations start from a theory of massless, non-interacting quarks. The vanishing of the quark mass has the effect that right- and left-handed quark fields $\frac{1}{2}(1\pm\gamma_5)\psi(x)$ are entirely independent, so that right- and left-handed charge densities $\frac{1}{2}(J^0(x)\pm J^0_5(x))$ commute. The fact that quarks are taken to be non-interacting has the effect that the lowest-order result is exact.

[8]See footnote 2 on p. 9 of this thesis.

[9]I remark that the pertinent Eq. (11) in Ref. 15 has an evident misprint since its right-hand side is real while the left-hand side is purely imaginary. Eq. (23) of Ref. 14 does not suffer from this kind of inconsistency, but its sign is opposite to the one in my Eq. (4.82), which I obtained by re-doing the calculation of Ref. 15. So, the anomalous charge-density algebra I gained has opposite sign compared to Refs. 13–15, but there is a compensating second sign change in the inferred modification of the finite-momentum GDH sum rule (not the *genuine* sum rule, of course).

Here, "lowest order" means the following: In a way, either of these approaches is equivalent to calculating box diagrams



$$+ \text{ permutations,} \tag{4.83}$$

with an odd number of axial-vector currents attached. If quarks interact, say, via gluon exchange, then there is an infinite number of gluon-radiative corrections to these diagrams.

Without any mass scale in the theory, it is clear from dimensional considerations why a product of a minimum number of *four* charge densities is needed to gain a non-vanishing vacuum expectation value (4.82): In 3+1-dimensional spacetime, charge density has dimension three, i.e., $\text{GeV}^3$. Thus, the innermost commutator in Eq. (4.82) has dimension six, and every further charge density adds three. On the other hand, there is clearly one delta function of dimension three per commutator, so three gradients altogether have to be supplied to balance dimensions. For a single or double commutator, no scalar expression incorporating three gradients can be found, but for a triple commutator, there is the exterior product displayed in Eq. (4.82). Moreover, this expression is unique up to a constant factor. Note that in 1+1-dimensional spacetime, matters stand differently: Since currents and delta functions have dimension one, a *single* charge-density commutator can exhibit a non-vanishing vacuum expectation value without introducing any mass scale, namely a multiple of the derivative of $\delta(x^1 - y^1)$. This commutator is known as *Kac-Moody algebra.*

### The number of colors

Considering the box diagrams (4.83), it is also clear why the number of colors occurs in the result (4.82), since of course one quark of each color can circulate in the box.

### Anomaly cancellation

In Refs. 14 and 15, quarks are the only fermionic degrees of freedom. If leptons were added consistently, i.e., if the currents (4.69) were complemented for leptonic pieces so that leptons circulated in the box diagrams (4.83), too, then the famous mechanism of anomaly cancellation would cause the triple commutator (4.82) to vanish. As you shall see below, hadrons (more precicely: spin-1 mesons) will be coupled to the external currents when the whole thing is applied to the GDH sum rule. This is the only argument for the omission of leptons. That is to say, in view of anomaly cancellation it is clear from the beginning that additional particles other than quarks will have to be introduced and that these particles will have to be hadrons.

### A first-guess charge-density algebra

Now comes the most critical step of the procedure: An algebra of charge densities is *proposed* which reproduces the triple-commutator vacuum expectation value (4.82) in an iterative computation. That is to say: one simply *guesses* a charge-density commutator that – upon commuting with two further charge densities and taking the vacuum expectation value – fetches expression (4.82). To this end, two new nonets of fields $b^{\mu}_{\mathrm{V}a}(x)$, $b^{\mu}_{\mathrm{A}a}(x)$ have to be introduced ad hoc. The simplest such guess reads [13–15][10]

$$\begin{aligned}
\big[J^0_a(x), J^0_b(y)\big]_{\mathrm{et}} = \big[J^0_{5a}(x), J^0_{5b}(y)\big]_{\mathrm{et}} = \\
i f_{abc}\, J^0_c(x)\, \delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{iN_{\mathrm{c}}}{48\pi^2}\, d_{abc}\big(\boldsymbol{\nabla}\times\boldsymbol{b}_{\mathrm{A}c}(x)\big)\cdot\boldsymbol{\nabla}\,\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (4.84\mathrm{a})\\
\big[J^0_a(x), J^0_{5b}(y)\big]_{\mathrm{et}} = \big[J^0_a(x), J^0_{5b}(y)\big]_{\mathrm{et}} = \\
i f_{abc}\, J^0_{5c}(x)\, \delta(\boldsymbol{x}-\boldsymbol{y}) - \frac{iN_{\mathrm{c}}}{48\pi^2}\, d_{abc}\big(\boldsymbol{\nabla}\times\boldsymbol{b}_{\mathrm{V}c}(x)\big)\cdot\boldsymbol{\nabla}\,\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (4.84\mathrm{b})
\end{aligned}$$

Furthermore, commutators between charge densities and the newly introduced $b$ fields have to be supplied:

$$\big[J^0_a(x), \boldsymbol{b}_{\mathrm{V}b}(y)\big]_{\mathrm{et}} = \big[J^0_{5a}(x), \boldsymbol{b}_{\mathrm{A}b}(y)\big]_{\mathrm{et}} = i f_{abc}\, \boldsymbol{b}_{\mathrm{V}c}\, \delta(\boldsymbol{x}-\boldsymbol{y}) + 2i\,\delta_{ab}\,\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (4.84\mathrm{c})$$

$$\big[J^0_a(x), \boldsymbol{b}_{\mathrm{A}b}(y)\big]_{\mathrm{et}} = \big[J^0_{5a}(x), \boldsymbol{b}_{\mathrm{V}b}(y)\big]_{\mathrm{et}} = i f_{abc}\, \boldsymbol{b}_{\mathrm{A}c}\, \delta(\boldsymbol{x}-\boldsymbol{y}). \qquad (4.84\mathrm{d})$$

Eq. (4.82) can indeed be derived from Eqs. (4.84): one needs (4.84a) for the innermost commutator, next the canonical form (4.84d), and finally the c-number Schwinger term (4.84c) – plus a bit of vector algebra, of course.

Considering that algebra (4.84) is claimed to be the simplest possible one, the reader may find that it looks pretty complex. So, let me try to motivate its form. From causality, each of the given commutators must vanish unless $\boldsymbol{x}=\boldsymbol{y}$, i.e., it must be a finite sum of derivatives of $\delta(\boldsymbol{x}-\boldsymbol{y})$. (In momentum space, this corresponds to a polynomial in the according three-momentum.) A *single* derivative is of course the simplest ansatz. Thus, the additional term in the charge-density commutator $[J^0_a(x), J^0_b(y)]_{\mathrm{et}}$ is proposed as $i\boldsymbol{B}_{ab}(x)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y})$. Due to antisymmetry of the commutator, this expression has to change sign upon simultaneously interchanging $a$ with $b$ and $\boldsymbol{x}$ with $\boldsymbol{y}$, i.e., $\boldsymbol{B}_{ab}(x)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}) \overset{!}{=} -\boldsymbol{B}_{ba}(y)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{y}-\boldsymbol{x}) = \boldsymbol{B}_{ba}(x)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}) + \boldsymbol{\nabla}\cdot\boldsymbol{B}_{ba}(x)\,\delta(\boldsymbol{x}-\boldsymbol{y})$, which is most easily achieved by choosing $\boldsymbol{B}_{ab}(x)$ symmetric in its indices and source free. Recalling that we intend to reproduce the triple-commutator (4.82), the $d$ symbol suggests itself: $\boldsymbol{B}_{ab}(x) = d_{abc}\boldsymbol{B}_c(x)$, where the source free field $\boldsymbol{B}_c(x)$ can be expressed as the curl of another field. Finally, parity considerations reveal that the new field is an axial vector in Eq. (4.84a) and a vector in Eq. (4.84b).

As for relations (4.84c) and (4.84d), these are just in the style of the VMD-inspired charge-current commutators (4.78).

So far, no qualitative statement has been given concerning matrix elements of the fields $b^{\mu}_{\mathrm{V}a}(x)$ and $b^{\mu}_{\mathrm{A}a}(x)$, which is of course our desire in view of the GDH sum rule. To proceed further, yet another guess is made in Ref. 13, namely that the fields are *proportional* to the currents, $b^{\mu}_{\mathrm{V}a}(x) \propto J^{\mu}_a(x)$, $b^{\mu}_{\mathrm{A}a}(x) \propto J^{\mu}_{5a}(x)$, at least as far as low-lying matrix elements are concerned. If one now confronts Eq. (4.84c) with Eq. (4.78a), it is clear that

$$b^{\mu}_{\mathrm{V}a}(x) = \frac{1}{F_\pi^2}\, J^{\mu}_a(x) \qquad \text{and} \qquad b^{\mu}_{\mathrm{A}a}(x) = \frac{1}{F_\pi^2}\, J^{\mu}_{5a}(x), \qquad (4.85)$$

so that the anomalous charge-density commutator (4.84a), for electric charge-densities $J^0 = J^0_3 + J^0_8/\sqrt{3}$, becomes

$$\boxed{\big[J^0(x), J^0(y)\big]_{\mathrm{et}} = i\,\big(\boldsymbol{\nabla}\times\boldsymbol{a}(x)\big)\cdot\boldsymbol{\nabla}\,\delta(\boldsymbol{x}-\boldsymbol{y})} \qquad (4.86)$$

[10]Note that I have corrected for sign errors, and that I have taken a factor of $N_{\mathrm{c}}$ out of the definition of the $b$ fields, so that the Schwinger term in Eq. (4.84c) has the simplest possible form. This normalization appeares most natural to me.

with[11]

$$
\begin{aligned}
a^{\mu}(x) &= -\frac{1}{8\pi^2 F_\pi^2}\,\bar{\psi}(x)\,\gamma^{\mu}\gamma_5\,\big(Z^{(\mathrm{q})}\big)^2\,\psi(x) \\
&= -\frac{1}{24\pi^2 F_\pi^2}\left(\sqrt{\tfrac{8}{3}}\,J^{\mu}_{5\,0}(x) + J^{\mu}_{5\,3}(x) + \tfrac{1}{\sqrt{3}}\,J^{\mu}_{5\,8}(x)\right).
\end{aligned} \tag{4.87}
$$

Axial-vector currents $J^{\mu}_{5a}(x)$ are defined in Eq. (4.69b).

The anomalous commutator (4.86) implies a modification of the *finite-momentum* GDH sum rule (2.75). If the infinite-momentum limit were legitimate, then the (genuine) GDH sum rule would be modified, too. This legitimacy is *assumed* in Ref. 13. (More precisely, the possibility of its failure is not taken into account.) As explained in Sect. 2.2, the problem of the infinite-momentum limit cannot be addressed within the present context, since the charge-density commutator alone does not tell us anything about the validity of the infinite-momentum limit. This can only be achieved by considering a specific field theoretic model and calculating the forward virtual Compton amplitude in the whole $(\nu, q^2)$ plane [27]. This check is presented in Sect. 4.4, and it reveales that there is *no modification of the GDH sum rule due to anomalous commutators!*

### The Emperor's New Clothes

The "revocation" of the modification by virtue of the infinite-momentum limit is of course the central point of my criticism of Ref. 13, because it obviously invalidates its result. For completeness, I would nonetheless like to remark on the nature of the somewhat obscure fields $b^{\mu}_{\mathrm{V}a}(x)$ and $b^{\mu}_{\mathrm{A}a}(x)$. Eqs. (4.85) may be misleading in that they suggest that these fields are directly related to pions, which is not true. It is instructive to recall that the factors $1/F_\pi^2$ in Eqs. (4.85) originated from the KSRF relation (4.81). Taking back the manipulation performed by applying KSRF, and in view of the VMD relations (4.79), Eqs. (4.85) run

$$
b^{\mu}_{\mathrm{V}a}(x) = 2f_{\mathrm{V}}\,v^{\mu}_{a}(x) \qquad \text{and} \qquad b^{\mu}_{\mathrm{A}a}(x) = 2f_{\mathrm{A}}\,a^{\mu}_{a}(x). \tag{4.88}
$$

Hence, up to a factor of 2, fields $b^{\mu}_{\mathrm{V}a}(x)$ and $b^{\mu}_{\mathrm{A}a}(x)$ are simply the respective field operators $v^{\mu}_{a}(x)$ and $a^{\mu}_{a}(x)$ of vector and axial-vector mesons (with the coupling constants absorbed), and what was done by Chang and Liang was effectively the calculation of the commutator anomaly of quark currents in the presence of U(3)×U(3) gauge fields represented by spin-1 mesons. (In view of this new insight, also observe the similarity of Eq. (4.84b) with the Jackiw-Johnson commutator (4.45).) Thus, contrary to their claim, Chang and Liang did not really do something new. Their result reproduces a lowest-order approximation to, e.g., the results of Jo [71].[12]

[11] I note in passing that the combination of axial-vector currents appearing in Eq. (4.87) is the same as that occurring in the naive commutator (2.48) of spatial electromagnetic current densities.

[12] More precisely: If one considers only the left-handed current commutators of Refs. 13–15, or if one suitably generalizes Jo's result [71, Eq. (2.20)] to vector currents, then the latter turns out to be a factor of three larger. This inconsistency may be related to the problem of inferring a single commutator from the triple-commutator vacuum expectation value (see discussion "Iterative vs. simultaneous triple commutator" below). Also note the domain dependence of regularized linearly divergent integrals, which is characteristic for the anomaly. Chang and Liang integrated a linearly divergent three-dimensional integral [15, Eq. (5)] over a solid ball of radius $R$, subsequently letting $R \to \infty$. If instead, the domain of integration is taken to be the space between two parallel planes with distance $H \to \infty$, then one obtains a supplementary factor of three.

### Spin-1 mesons as gauge fields

The idea that chiral transforms are gauged – with spin-1 mesons emerging as gauge fields and a mass term that breaks the gauge invariance locally but preserves it globally – is essentially the idea of VMD and current algebra [84, 85], and it has a couple of deficiencies, at least in its primitive form as employed in Ref. 13. For instance, it demands that all spin-1 mesons couple to all other hadrons with one universal coupling constant $f_\mathrm{A} = f_\mathrm{V}$, and that the masses within the vector-meson nonet ($\rho$, $\omega$, $\phi$, K$^*$) are degenerate. The masses within the axial-vector-meson nonet (a$_1$, f$_1$(1285), f$_1$(1420), K$_1$) must also be degenerate, and mutually, the nonets are related by $m_\mathrm{A}^2 = 2m_\mathrm{V}^2$ [90]. Experimentally, all this is quite far from truth.[13] Moreover, the chiral anomaly is known to yield much worse numerical results for spin-1 mesons than for the pion [77].

### Iterative vs. simultaneous triple commutator

Finally, I want to draw your attention to a discussion on the risks of iterative computation of multiple commutators. As has been pointed out in Refs. 91–93, the iterative procedure need not necessarily yield the same result as a simultaneous calculation like the one underlying Eq. (4.82). Moreover, the Jacobi identity might fail. Eq. (4.82), however, can be shown to be consistent with the Jacobi identity.

### Nonetheless...

In spite of all that, one may adopt the following point of view. Anomalous charge-density commutators are not prohibited. And just in case someone calculated the commutator in a more reasonable model than the one just described, it is by all means desireable to learn more about the possibility of a modification of the GDH sum rule and about the role of the infinite-momentum limit.

## 4.2.6 Modification of the finite-momentum GDH sum rule

Consider the general non-naive charge-density commutator (4.86). (In view of other applications, I temporarily do not assume a specific form of the axial-vector field $a^\mu(x)$.) With this commutator at hand, I re-inspect the current-algebra derivation of the GDH sum rule as presented in Sect. 2.2.

Instead of the naive dipole-moment commutator (2.51), one now has

$$\begin{aligned}\left[D^i(0), D^j(0)\right] &= e^2 \int \mathrm{d}^3x\, x^i \int \mathrm{d}^3y\, y^j \left[J^0(x), J^0(y)\right]_\mathrm{et} \\ &= ie^2 \int \mathrm{d}^3x\, x^i \int \mathrm{d}^3y\, y^j \left(\boldsymbol{\nabla}\times\boldsymbol{a}(x)\right)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{4.89}$$

(Time $x^0$ is understood to be 0 throughout.) Observe that

$$\int \mathrm{d}^3y\, y^j\, \boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}) = \boldsymbol{\nabla}x^j = \boldsymbol{e}_j, \tag{4.90}$$

[13] In historical applications of this idea, neither strange nor flavor-singlet mesons were involved, but only the isovector fields $\rho$ and a$_1$, i.e., SU(2)×SU(2) symmetry was considered instead of U(3)×U(3). The Weinberg relation $m_\mathrm{A}^2 = 2m_\mathrm{V}^2$ implied a mass of 1090 MeV for the a$_1$, and indeed there was a bump in the $\rho$-$\pi$ spectrum at 1080 MeV. Nowadays, however, the mass of the a$_1$ meson is known to be $1230 \pm 40$ MeV [46].

leaving

$$\left[D^i(0), D^j(0)\right] = ie^2\, \varepsilon^{jlk} \int \mathrm{d}^3x\, x^i\, \nabla^l a^k(x). \tag{4.91}$$

Partial integration yields

$$\left[D^i(0), D^j(0)\right] = ie^2\, \varepsilon^{ijk} \int \mathrm{d}^3x\, a^k(x). \tag{4.92}$$

The integral in this expression is simply the total "current" of the axial-vector field $a^\mu(x)$. Proceeding along the same lines as in Sect. 2.2, I define left- and right-handed components $D^{\mathrm{L,R}}(0)$ as in Eq. (2.53), and sandwich the resulting operator equation

$$\left[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)\right] = e^2 \int \mathrm{d}^3x\, a^3(x) \tag{4.93}$$

between one-nucleon states of positive helicity. Utilizing translational invariance (1.7), this leads to

$$\begin{aligned}\left\langle p', \tfrac{1}{2}\middle|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\middle|p, \tfrac{1}{2}\right\rangle &= e^2 \int \mathrm{d}^3x\, e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot x} \left\langle p', \tfrac{1}{2}\middle|a^3(0)\middle|p, \tfrac{1}{2}\right\rangle \\ &= (2\pi)^3 \delta(\boldsymbol{p}'-\boldsymbol{p})\, e^2 \left\langle p, \tfrac{1}{2}\middle|a^3(0)\middle|p, \tfrac{1}{2}\right\rangle. \end{aligned} \tag{4.94}$$

By Lorentz covariance and the quantum numbers of field $a^\mu(x)$, the matrix element emerging here is fixed up to a constant $\tilde{g}_{\mathrm{A}}$, which I define by

$$\boxed{\tilde{g}_{\mathrm{A}}\, \bar{u}(p, \tfrac{1}{2})\, \gamma^\mu \gamma_5\, u(p, \tfrac{1}{2}) = \left\langle p, \tfrac{1}{2}\middle|a^\mu(0)\middle|p, \tfrac{1}{2}\right\rangle} \tag{4.95}$$

Since $\boldsymbol{p} \propto \boldsymbol{e}_3$, the third component of the spinor expression obeys

$$\bar{u}(p, \tfrac{1}{2})\, \gamma^3 \gamma_5\, u(p, \tfrac{1}{2}) = 2Ms^3 = 2p^0, \tag{4.96}$$

so that finally

$$\left\langle p', \tfrac{1}{2}\middle|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\middle|p, \tfrac{1}{2}\right\rangle = (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}'-\boldsymbol{p})\, 4\pi\alpha\, \tilde{g}_{\mathrm{A}}. \tag{4.97}$$

On the other hand, one directly evaluates the commutator matrix element by inserting a complete set of intermediate states and separating the one-nucleon states from the continuum, which has been done in Sect. 2.2:

$$\begin{aligned}&\left\langle p', \tfrac{1}{2}\middle|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\middle|p, \tfrac{1}{2}\right\rangle \\ &\quad = \left\langle p', \tfrac{1}{2}\middle|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\middle|p, \tfrac{1}{2}\right\rangle_{\text{one-nucleon}} + \left\langle p', \tfrac{1}{2}\middle|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\middle|p, \tfrac{1}{2}\right\rangle_{\text{cont}} \\ &\quad = (2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}'-\boldsymbol{p}) \left( \frac{2\pi\alpha\kappa^2}{M^2} - \frac{2\pi\alpha(Z+\kappa)^2}{(p^0)^2} + 8 \int_{q^0_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}q^0}{q^0}\, \mathrm{Im}\, f_2(\nu, q^2) \right). \end{aligned} \tag{4.98}$$

Equating (4.97) with (4.98) gives

$$-4\pi^2\alpha \left( \frac{\kappa^2}{2M^2} - \frac{(Z+\kappa)^2}{2(p^0)^2} - \tilde{g}_{\mathrm{A}} \right) = \int_{\nu_{\mathrm{thr}}}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, 8\pi\, \mathrm{Im}\, f_2\!\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right). \tag{4.99}$$

### Infinite-momentum limit

Like the unmodified finite-momentum GDH sum rule (2.75), the modified version (4.99), as it stands, is useless, because for any finite value of the nucleon energy $p^0$, the integration on the right-hand side runs over arbitrarily high timelike photon virtualities (see Fig. 2.5), where the Compton amplitude is not related to observables. Upon taking the limit $p^0 \to \infty$ on both sides of Eq. (4.99),

$$-4\pi^2\alpha\left(\frac{\kappa^2}{2M^2} - \tilde{g}_\mathrm{A}\right) = \lim_{p^0\to\infty}\int_{\nu_\mathrm{thr}}^{\infty}\frac{\mathrm{d}\nu}{\nu}\,8\pi\,\mathrm{Im}\,f_2\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right), \tag{4.100}$$

one arrives at a sensible relation only if one knows (or assumes) what modification is brought about by dragging the limit into the integral. If there was *no* additional modification (as conjectured in Ref. 13), then one would arrive at

$$\boxed{-4\pi^2\alpha\left(\frac{\kappa^2}{2M^2} - \tilde{g}_\mathrm{A}\right) = \int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\big(\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)\big)} \tag{4.101}$$

upon using the optical theorem (2.10b).

I now report the explicit numbers that have been found by Chang, Liang, and Workman [13]. Eqs. (4.87) and (4.95) determine the constant $\tilde{g}_\mathrm{A}$ in terms of axial-vector currents $J^\mu_{5a}(x)$:

$$\tilde{g}_\mathrm{A}\,\bar{u}(p,\tfrac12)\,\gamma^\mu\gamma_5\,u(p,\tfrac12) = -\frac{1}{24\pi^2F_\pi^2}\,\langle p,\tfrac12|\Big(\sqrt{\tfrac83}\,J^\mu_{5\,0}(0) + J^\mu_{5\,3}(0) + \tfrac{1}{\sqrt3}\,J^\mu_{5\,8}(0)\Big)|p,\tfrac12\rangle. \tag{4.102}$$

On the other hand, the forward matrix elements emerging in Eq. (4.102) define axial charges $g_{\mathrm{A}a}$ of the nucleon by means of

$$g_{\mathrm{A}a}\,\bar{u}(p,\tfrac12)\,\gamma^\mu\gamma_5\,u(p,\tfrac12) = \langle p,\tfrac12|J^\mu_{5a}(0)|p,\tfrac12\rangle, \tag{4.103}$$

which implies

$$4\pi^2\alpha\,\tilde{g}_\mathrm{A} = -\frac{\alpha}{6F_\pi^2}\left(\sqrt{\tfrac83}\,g_{\mathrm{A}0} + g_{\mathrm{A}3} + \tfrac{1}{\sqrt3}\,g_{\mathrm{A}8}\right). \tag{4.104}$$

Assuming isospin symmetry, the proton and neutron isosinglet axial charges are degenerate,

$$g^\mathrm{p}_{\mathrm{A}0} = g^\mathrm{n}_{\mathrm{A}0} \quad\text{and}\quad g^\mathrm{p}_{\mathrm{A}8} = g^\mathrm{n}_{\mathrm{A}8}, \tag{4.105a}$$

whereas the isotriplet axial charges have opposite sign and equal magnitude:

$$g^\mathrm{p}_{\mathrm{A}3} = -g^\mathrm{n}_{\mathrm{A}3} = g_\mathrm{A} = 1.26 \tag{4.105b}$$

(the neutron-beta-decay constant). Values for $g_{\mathrm{A}0}$ and $g_{\mathrm{A}8}$ are difficult to fit experimentally. They can be estimated from hyperon beta decay under the (severely restrictive) additional assumption of SU(3) symmetry [13]. The proton-neutron difference, however, involves $g_\mathrm{A}$ only. In this channel, the modification constant occurring in Eq. (4.101) is given by

$$4\pi^2\alpha\,(\tilde{g}^\mathrm{p}_\mathrm{A} - \tilde{g}^\mathrm{n}_\mathrm{A}) = -\frac{\alpha\,g_\mathrm{A}}{3F_\pi^2} = -137\ \mu\mathrm{b}, \tag{4.106}$$

where $F_\pi = 93$ MeV has been used. Accidentally, the value (4.106) is relatively close to the discrepancy that was estimated from pion-photoproduction multipole analyses. For instance, the latest such estimate, given by Sandorfi, Whisnant, and Khandaker, is $-158\ \mu$b [7], while Karliner [5] found $-107\ \mu$b.

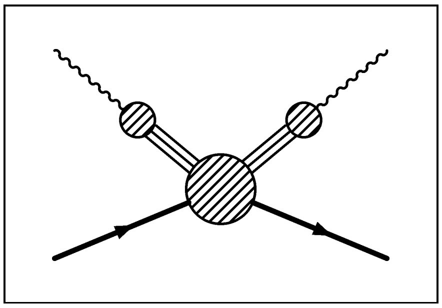


Figure 4.2: Intermediate hadron states in virtual Compton scattering, leading to singularities in the photon mass $q^2$

## 4.3 The infinite-momentum limit

In Sect. 2.2, it was shown that, provided merely that the operators of electric charge densities commute at equal times, one arrives at the following preliminary form of the GDH sum rule:

$$-\frac{2\pi^2\alpha\kappa^2}{M^2} = \lim_{p^0\to\infty} \int\limits_{\nu_{\rm thr}}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, 8\pi\, \mathrm{Im}\, f_2\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right). \tag{4.107a}$$

I stressed that current algebra alone cannot tell whether the $p^0 \to \infty$ limit can be dragged into the $\nu$ integral without bringing about a modification. Starting from the anomalous charge-density commutator (4.86), a modified form of Eq. (4.107) occurred in the previous section, viz.

$$-4\pi^2\alpha\left(\frac{\kappa^2}{2M^2} - \tilde{g}_{\rm A}\right) = \lim_{p^0\to\infty} \int\limits_{\nu_{\rm thr}}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, 8\pi\, \mathrm{Im}\, f_2\left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right). \tag{4.107b}$$

Again, consideration of current commutators will not help us getting ahead.

**Why some caution is advisable**

Physicists commonly don't make a fuss about permuting limits or integrations. They simply do it and see whether the result makes any sense.

Nevertheless, one can easily see the possible origin of difficulties on quite general grounds. Recall that the timelike virtual Compton amplitude meets singularities in the photon mass $q^2$ due to intermediate hadron states, as indicated in Fig. 4.2. Thus one expects for the amplitude $f_2(\nu, q^2)$ a spectral representation of the form

$$\mathrm{Im}\, f_2(\nu, q^2) = \frac{1}{\pi} \int\limits_{q^2_{\rm min}}^{\infty} \frac{\mathrm{d}q'^2}{q^2 - q'^2}\, \rho(\nu, q'^2), \tag{4.108}$$

where $q^2_{\rm min}$ is the mass of the lowest-lying state that couples to the photon. Inserting Eq. (4.108) into one of Eqs. (4.107), it is evident that for finite $p^0$, the $\nu$ integration also meets the $q^2$ singularities, since $q^2 = M^2\nu^2/(p^0)^2$. Only if one can drag the limit $p^0 \to \infty$ inside the integral in Eq. (4.107), these singularities play no explicit role.

### An example

To further illustrate the possibility of a modification due to non-interchangeability of limit and integration in Eqs. (4.107), suppose that function $f_2(\nu, q^2)$ had a piece $\tilde{f}_2(\nu, q^2)$ independent of photon lab-frame energy $\nu$, i.e.,

$$\tilde{f}_2(\nu, q^2) = \tilde{f}(q^2), \tag{4.109}$$

where the imaginary part of function $\tilde{f}(q^2)$ is assumed to be logarithmically integrable and, for simplicity, vanishing at $q^2 < \nu_0^2$. (As a matter of fact, exactly such an expression will emerge in Sect. 4.4.) The simplest non-trivial function incorporating these properties is a zero-width pole $\tilde{f}(q^2) \propto (q^2 - \mu^2 - i\varepsilon)^{-1}$ with imaginary part $\mathrm{Im}\, \tilde{f}(q^2) \propto \delta(q^2 - \mu^2)$. One has

$$\int \frac{\mathrm{d}\nu}{\nu} \,\mathrm{Im}\, \tilde{f}_2 \left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right) = \frac{1}{2} \int \frac{\mathrm{d}(q^2)}{q^2} \,\mathrm{Im}\, \tilde{f}(q^2) =: C, \tag{4.110}$$

which might be any real constant you like. Consequently, the $p^0 \to \infty$ limit of above integral equals just that constant $C$. In contrast,

$$\int \frac{\mathrm{d}\nu}{\nu} \lim_{p^0 \to \infty} \mathrm{Im}\, \tilde{f}_2 \left(\nu, \frac{M^2\nu^2}{(p^0)^2}\right) = \int \frac{\mathrm{d}\nu}{\nu} \,\mathrm{Im}\, \tilde{f}(0) = 0, \tag{4.111}$$

since I assumed $\mathrm{Im}\, \tilde{f}(0) = 0$. This scenario exemplifies the possibility of a *finite* modification brought about by the infinite-momentum limit. In principle, there might equally well be an *infinite* modification.

### The state of the art

To conclude this section, I want to stress that during several years of research, I found no instance of the *unmodified* "preliminary" GDH sum rule (4.107a) that would have been modified by virtue of the infinite-momentum limit. Yet, there is an instance of the *modified* form (4.107b) that is affected by interchanging the $p^0 \to \infty$ limit with the $\nu$ integration, in such a way that the modification disappears! This very strong evidence for the validity of the unmodified GDH sum rule is presented in the now-following section.

## 4.4 Anomalous commutator within Weinberg-Salam model

In this section, I present the calculation of the anomalous charge-density commutator by means of the Bjorken-Johnson-Low (BJL) technique [17, 56] within the order-$\alpha^2$ Weinberg-Salam model, which I introduced in Sect. 3.3. More precisely, the one-electron matrix element of that commutator is calculated. In fact, since the BJL method starts from an amplitude, i.e., a matrix element, it is only capable to yield matrix elements of commutators, not commutators themselves. But after all, the one-electron matrix element is exactly what is needed for the discussion of the GDH sum rule.

The contents of my surveys on T-product amplitudes and seagulls (Sect. 4.2.3), as well as on the BJL technique (Sect. 4.2.4) apply to the electron in just the same manner as to the nucleon.

In addition to the Feynman graphs discussed in Sect. 3.3 (Fig. 3.6), a few more one-loop graphs have to be considered, since one has to go off the real photon point $q^2 = 0$.

I employ the BJL technique to calculate the matrix element of the charge-density commutator

$$\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}} \,\langle p', \tfrac{1}{2}\big|[J^0(x), J^0(0)]_{\mathrm{et}}\big|p, \tfrac{1}{2}\rangle = -\lim_{q'^0\to\infty} q'^0\, T^{00} \tag{4.112}$$

within the Weinberg-Salam model to order $\alpha^2$, i.e., to one loop. The Weinberg angle $\theta_{\mathrm{W}}$ is assumed to adopt its ideal value, i.e.,

$$\sin\theta_{\mathrm{W}} = \frac{1}{2}. \tag{4.113}$$

The Fermi constant is then given by

$$\frac{G_{\mathrm{F}}}{\sqrt{2}} = \frac{2\pi\alpha}{M_{\mathrm{W}}^2} = \frac{8\pi\alpha}{3M_{\mathrm{Z}}^2}. \tag{4.114}$$

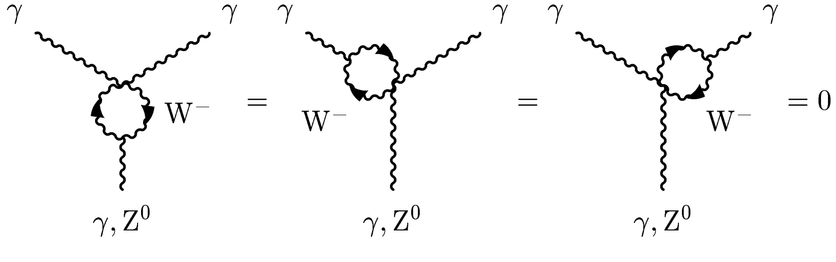
The external particles in Eq. (4.112) are electrons with positive helicity. Although the result will exhibit a factor of $\delta(\boldsymbol{p}' - \boldsymbol{p})$, it is essential to go (slightly) off the forward direction during the calculation. (One can show that this is owing to the fact that the information contained in *all* components of the *forward* Compton amplitude $T^{\mu\nu}$ can be translated into the *time-time* component $T^{00}$ of the *off-forward* amplitude.) As explained in Sect. 4.2.4, the seagull portion of amplitude $T^{\mu\nu}$ has to be dropped after it is identified on account of its polynomial dependence on $q'^0$.

This investigation has been published in Ref. 27, but it is presented here in much more detail.

### 4.4.1 Additional Feynman graphs

Since the energy $q'^0$ of the scattered photon (and with it the energy $q^0 = q'^0 + p'^0 - p^0$ of the incident photon) runs while its three-momentum $\boldsymbol{q}'$ is kept fixed, the photon necessarily gets virtual. Therefore one has to consider a couple of Feynman graphs in addition to the ones depicted in Fig. 3.6, Sect. 3.3. the newly emerging graphs are shown in Fig. 4.3. Their distinguishing mark is the fact that they cannot be cut into two parts such that both pieces represent the *absorption* of a (real) photon. The crucial graph is the $\mathrm{Z}^0$-exchange graph, because the coupling of the $\mathrm{Z}^0$ boson to the photons by virtue of an electron-triangle subgraph gives rise to the chiral anomaly.

Yet a few more one-loop diagrams can be drawn but are individually vanishing:




(4.115a)

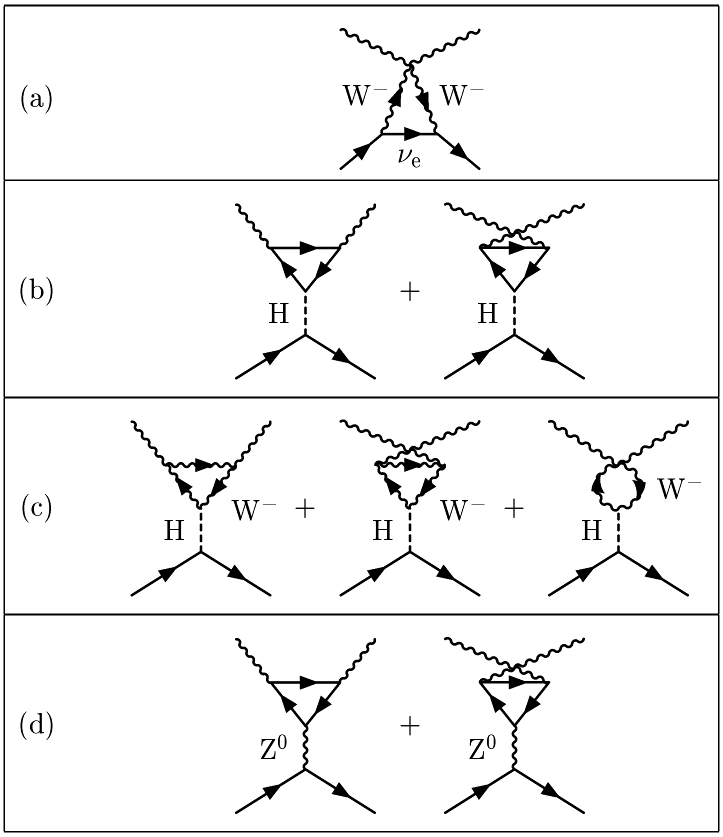
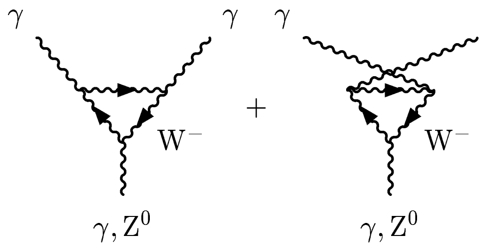
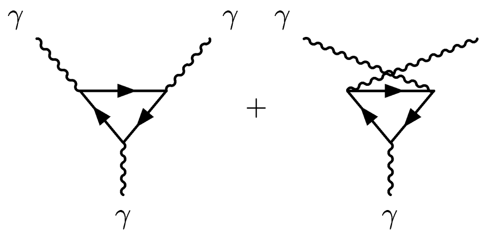



Figure 4.3: One-loop Feynman graphs contributing to the virtual Compton amplitude of the electron in the Weinberg-Salam model, in addition to those depicted if Fig. 3.6.
(a) WW$\gamma\gamma$ contact graph
(b) Higgs exchange via fermion loop
(c) Higgs exchange via W-boson loop
(d) $Z^0$ exchange

or cancel each other:




$$= 0, \tag{4.115b}$$

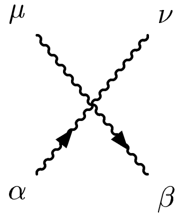



$$= 0. \tag{4.115c}$$

As far as the three-photon coupling is concerned, the vanishing of the sum of these diagrams represents the one-loop version of Furry's theorem. Then, naturally, in case of the W-boson loops, one of the photons can be replaced by a $Z^0$ boson, since up to a factor of $\tan\theta_W$, the $WW\gamma$ coupling equals the WWZ coupling.

### WW$\gamma\gamma$ contact graph

Let us first have a look at the $WW\gamma\gamma$ contact graph of Fig. 4.3(a). The Feynman rule for the quadrilinear coupling reads [53]



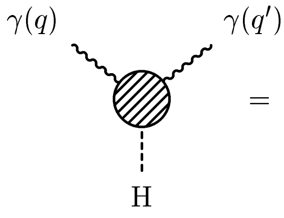

$$= -ie^2(2g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{4.116}$$

Without having to perform the loop integration, one realizes that this graph is *independent of the photon energy* $q'^0$. (This is a general feature of contact graphs.) Hence, according to Sect. 4.2.4, it contributes solely to the seagull amplitude and must be dismissed in the calculation of the BJL limit (4.112). By the way: what animal[14] (if any) does Fig. 4.3(a) look like? You guess it! This is the origin of the name "seagull".

### Higgs exchange

Next, consider the Higgs-exchange graphs depicted in Fig. 4.3(b,c). Since the Higgs boson has spin 0, it does not couple to the electron's spin. Hence, a contribution to the GDH sum rule is not expected. While this is evident in the dispersion-theoretic approach, the spin argument is somewhat concealed as regards the anomalous charge-density commutator. Nevertheless, one can show that the Higgs-exchange graphs do not contribute to the matrix element of the dipole-moment commutator in the BJL limit, independently of how the Higgs boson is coupled to the photons. I want to sketch the proof in the following.

Upon crossing symmetry and gauge invariance, the $H\gamma\gamma$ vertex can be written




$$\begin{aligned} &= \left(g^{\mu\nu} - \frac{q^\mu q'^\nu}{q\cdot q'}\right) H_1(q^2, q'^2, q\cdot q') \\ &+ \left(q'^\mu - \frac{q'^2}{q\cdot q'}\, q^\mu\right)\left(q^\nu - \frac{q^2}{q\cdot q'}\, q'^\nu\right) H_2(q^2, q'^2, q\cdot q') \\ &+ \varepsilon^{\mu\nu\alpha\beta}\, q_\alpha q'_\beta\, H_3(q^2, q'^2, q\cdot q'), \end{aligned} \tag{4.117}$$

where Lorentz index $\mu$ ($\nu$) corresponds to the scattered (incident) photon. (The parity violating piece proportional to invariant function $H_3$ emerges if graphs incorporating the W boson,

[14]Theorists love animals (seagulls, penguins, even tadpoles), parts of animals (cat ears) and parts of ex-animals (handbags).

like the ones depicted in Fig. 4.3(c), are taken into account.) Crossing symmetry imposes the additional constraints

$$H_i(q^2, q'^2, q{\cdot}q') = H_i(q'^2, q^2, q{\cdot}q'), \quad i = 1, 2, 3. \tag{4.118}$$

Letting $\mu = \nu = 0$ in Eq. (4.117) and taking the BJL limit $q'^0 \to \infty$ gives rise to terms proportional to $\boldsymbol{q}'^2$ and $\boldsymbol{q}{\cdot}\boldsymbol{q}'$. The four-momentum $q$ of the incident photon is fixed by energy- and momentum conservation, $p + q = p' + q'$. In other words, it is accounted a shorthand of expression $q' + p' - p$ when the BJL limit is taken. Hence one has terms proportional to $\boldsymbol{q}'^2$ and $(\boldsymbol{p}' - \boldsymbol{p}){\cdot}\boldsymbol{q}'$. On account of Eq. (4.112), this has to be Fourier transformed to coordinate space, yielding $\boldsymbol{\nabla}^2\delta(\boldsymbol{x})$ and $(\boldsymbol{p}' - \boldsymbol{p}){\cdot}\boldsymbol{\nabla}\delta(\boldsymbol{x})$. The crucial point now is that only a certain moment of the charge-density is considered, namely the dipole moment $D^i(x^0) = e\int \mathrm{d}^3x\, x^i J^0(x)$. Thus the second derivative $\boldsymbol{\nabla}^2\delta(\boldsymbol{x})$ gives no contribution, while the term involving the first derivative of $\delta(\boldsymbol{x})$ gives something proportional to $(p' - p)^i$. As explained in detail below, taking the dipole moment $D^j$ of the other charge-density involved in the commutator simply gives a factor of $\nabla^j\delta(\boldsymbol{p}' - \boldsymbol{p})$. Observe now that

$$(p' - p)^i\, \nabla^j\delta(\boldsymbol{p}' - \boldsymbol{p}) = g^{ij}\,\delta(\boldsymbol{p}' - \boldsymbol{p}), \tag{4.119}$$

which is multiplied by a factor of

$$\frac{\bar{u}(p', \frac{1}{2})\,\gamma_5\, u(p, \frac{1}{2})}{(p' - p)^2 - M_{\mathrm{H}}^2} \tag{4.120}$$

coming from the propagator of the Higgs boson and its coupling to the electron line. Finally, observe that Eq. (4.119) picks out the forward direction, and that

$$\bar{u}(p, \tfrac{1}{2})\,\gamma_5\, u(p, \tfrac{1}{2}) = 0. \tag{4.121}$$

So, indeed, Higgs-exchange graphs do not contribute to the commutator matrix element under consideration.

I note in passing that I have explicitly calculated the electron-loop graphs of Fig. 4.3(b), adopting a Pauli-Villars regularization scheme following Steinberger [94, Sect. III.A.1], with the result that the H$\gamma\gamma$ coupling (4.117) by virtue of the electron triangle is of order $(q'^0)^{-2}$, thus vanishing in the BJL limit, regardless of the lower parts of the Higgs-exchange diagrams, i.e., the coupling of the Higgs to the electron line.

### $\mathbf{Z^0}$ exchange

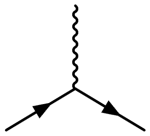
The contribution of the $\mathrm{Z}^0$-exchange graphs of Fig. 4.3(d) to the Compton amplitude $T^{\mu\nu}$ reads

$$T^{\mu\nu}_{(\mathrm{Z})} = -\frac{M_{\mathrm{Z}}^2 G_{\mathrm{F}}}{2\sqrt{2}}\, Z^{\mu\nu\rho}(q, q')\, \frac{-g_{\rho\sigma} + (p' - p)_\rho (p' - p)_\sigma / M_{\mathrm{Z}}^2}{(p' - p)^2 - M_{\mathrm{Z}}^2}\, \bar{u}(p', s')\gamma^\sigma\gamma_5 u(p, s), \tag{4.122}$$

where the integral

$$Z^{\mu\nu\rho} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{Tr}\left(\gamma^\mu \frac{i}{\not{k} - m + i\varepsilon}\gamma^\nu \frac{i}{\not{k} - \not{q} - m + i\varepsilon}\gamma^\rho\gamma_5 \frac{i}{\not{k} - \not{q}' - m + i\varepsilon}\right) + \left\{\begin{matrix} \mu \leftrightarrow \nu \\ q \leftrightarrow -q' \end{matrix}\right\} \tag{4.123}$$

embodies the electron-loops, and the constant $M_\mathrm{Z}^2 G_\mathrm{F}/2\sqrt{2} = e^2/3$ is due to the coupling of the $\mathrm{Z}^0$ boson to the electron lines:



$$= -\frac{ie}{\sqrt{3}}\,\gamma^\rho\gamma_5. \tag{4.124}$$

the $\mathrm{Z}^0$ propagator in Eq. (4.122),

$$i\,\frac{-g_{\rho\sigma} + (p'-p)_\rho(p'-p)_\sigma/M_\mathrm{Z}^2}{(p'-p)^2 - M_\mathrm{Z}^2}, \tag{4.125}$$

is written in unitary gauge. However, all results will be explicitly independent of the second term in its numerator, so that any other gauge would yield the same result. (The $i\varepsilon$ prescription is omitted, since the momentum of the $\mathrm{Z}^0$ boson is not integrated over, and since we are far off its mass shell.) Finally, Eq. (4.122) exhibits a factor of $-1$ due to the closed fermion loop.

For the moment, I consider arbitrary external-electron spins $s, s'$ and all Lorentz indices $\mu, \nu$ of the amplitude. For the calculation of the charge-density commutator, only $\mu = \nu = 0$ and positive helicities will be needed. Later on, however, I want to study the infinite-momentum limit, for which I need all Lorentz indices and spins, but only the forward direction $p = p', s = s'$.

The triangle loop integral (4.123) can be cast into the form

$$\begin{aligned} Z^{\mu\nu\rho} &= \varepsilon^{\nu\rho\alpha\beta} q'_\alpha q_\beta (q^\mu Z_1 + q'^\mu Z_2) + \varepsilon^{\mu\nu\rho\alpha} q_\alpha Z_3 \\ &\quad + \varepsilon^{\mu\rho\alpha\beta} q'_\alpha q_\beta (q'^\nu Z'_1 + q^\nu Z'_2) + \varepsilon^{\mu\nu\rho\alpha} q'_\alpha Z'_3, \end{aligned} \tag{4.126}$$

where $Z_{1,2,3}$ are functions of the Lorentz invariants $q^2$, $q'^2$, and $q{\cdot}q'$. From crossing symmetry, the primed functions are given by $Z'_{1,2,3}(q^2, q'^2, q{\cdot}q') = Z_{1,2,3}(q'^2, q^2, q{\cdot}q')$. Gauge invariance $q_\nu T^{\mu\nu}_{(\mathrm{Z})} = 0$, $q'_\mu T^{\mu\nu}_{(\mathrm{Z})} = 0$, imposes the constraint

$$q{\cdot}q' Z_1 + q'^2 Z_2 + Z_3 = q{\cdot}q' Z'_1 + q^2 Z'_2 + Z'_3 = 0. \tag{4.127}$$

Evaluation of the integral (4.123) proceeds as follows. By definition of the fermion propagators, the Dirac trace becomes

$$\begin{aligned} &\mathrm{Tr}\left(\gamma^\mu \frac{i}{\not k - m + i\varepsilon}\gamma^\nu \frac{i}{\not k - \not q - m + i\varepsilon}\gamma^\rho\gamma_5 \frac{i}{\not k - \not q' - m + i\varepsilon}\right) = \\ &- i\,\frac{\mathrm{Tr}\big(\gamma^\mu(\not k + m)\gamma^\nu(\not k - \not q + m)\gamma^\rho\gamma_5(\not k - \not q' + m)\big)}{\big(k^2 - m^2 + i\varepsilon\big)\big((k-q)^2 - m^2 + i\varepsilon\big)\big((k-q')^2 - m^2 + i\varepsilon\big)}, \end{aligned} \tag{4.128}$$

where the numerator is evaluated using trace theorems, while the denominator is made capable by means of the Feynman formula

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{1}{\big((A-C)x + (B-C)y + C\big)^3}, \tag{4.129}$$

valid for arbitrary non-zero constants $A, B, C$. The result, being linearly divergent (as, besides, is already apparent from the defining equation (4.123)), is Pauli-Villars regularized by subtracting from the integrand the same expression with a large cut-off $\Lambda$ substituted for the electron mass $m$, letting $\Lambda \to \infty$ after the loop integration is performed. Functions $Z_{1,2}(q^2, q'^2, q{\cdot}q')$ turn out to be finite in the limit $\Lambda \to \infty$, and are given by

$$Z_1 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{xy}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon} \tag{4.130a}$$

and

$$Z_2 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{-x(1-x)}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon}. \tag{4.130b}$$

On the other hand, function $Z_3(q^2, q'^2, q{\cdot}q')$ is formally divergent, but can be fixed by means of the gauge-invariance condition (4.127), giving

$$Z_3 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{x(1-x)\, q'^2 - xy\, q{\cdot}q'}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon}, \tag{4.130c}$$

This fact was first observed by Rosenberg [95], who was concerned with the same Feynman integral applied to a different process (see also Adler [96]). As noted above, the primed invariant functions $Z'_{1,2,3}$ are obtained from the unprimed functions by interchanging $q^2$ and $q'^2$. Subsequently re-naming the Feynman parameters, $x \leftrightarrow y$, one arrives at

$$Z'_1 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{xy}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon}, \tag{4.130d}$$

$$Z'_2 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{-y(1-y)}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon}, \tag{4.130e}$$

and

$$Z'_3 = \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{y(1-y)\, q^2 - xy\, q{\cdot}q'}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon}. \tag{4.130f}$$

### 4.4.2 Anomalous charge-density commutator

With the result (4.130) at hand, I am now in a position to calculate the anomalous charge-density commutator due to $Z^0$ exchange by virtue of the BJL limit (4.112):

$$\begin{aligned}
&\int \mathrm{d}^3x\, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}} \left\langle p', \tfrac{1}{2}\right| [J^0(x), J^0(0)]_{\mathrm{et}} \left| p, \tfrac{1}{2}\right\rangle = -\lim_{q'^0\to\infty} q'^0\, T^{00}_{(\mathrm{Z})} = \\
&\frac{M_{\mathrm{Z}}^2 G_{\mathrm{F}}}{2\sqrt{2}} \lim_{q'^0\to\infty} q'^0\, Z^{00\rho}(q, q')\, \frac{-g_{\rho\sigma} + (p'-p)_\rho (p'-p)_\sigma / M_{\mathrm{Z}}^2}{(p'-p)^2 - M_{\mathrm{Z}}^2}\, \bar{u}(p', \tfrac{1}{2}) \gamma^\sigma \gamma_5 u(p, \tfrac{1}{2}),
\end{aligned} \tag{4.131}$$

from Eq. (4.122). Inserting the decompostion (4.126), one has

$$Z^{00\rho} = \varepsilon^{0\rho\alpha\beta} q'_\alpha q_\beta (q^0 Z_1 + q'^0 Z_2 + q'^0 Z'_1 + q^0 Z'_2). \tag{4.132}$$

The invariant functions $Z_3^{(\prime)}$ do not occur. Recall that $q^0 = q'^0 + p'^0 - p^0$, so that $q^0$ is of the same order than $q'^0$ in the limit $q'^0 \to \infty$. Observe that from Eqs. (4.130), functions $Z_{1,2}^{(\prime)}$ are of order $(q'^0)^{-2}$. Hence, expression $q'^0 Z^{00\rho}$ is finite. For the specific combination of functions $Z_{1,2}^{(\prime)}$ appearing in Eq. (4.132), the Feynman-parameter integration gets particularly simple in the limit $q'^0 \to \infty$:

$$\begin{aligned}
&(q'^0)^2 (Z_1 + Z_2 + Z'_1 + Z'_2) \\
&\quad = \frac{i}{\pi^2} (q'^0)^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{2xy - x(1-x) - y(1-y)}{x(1-x)\, q^2 + y(1-y)\, q'^2 - 2xy\, q{\cdot}q' - m^2 + i\varepsilon} \\
&\quad \xrightarrow[q'^0 \to \infty]{} -\frac{i}{\pi^2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y = -\frac{i}{2\pi^2}.
\end{aligned} \tag{4.133}$$

Note that the dependence on the mass $m$ of the particle making up the triangle loop dropped out. This is characteristic for the anomaly. By means of Eq. (4.133), the expression occurring in the charge-density commutator (4.131) can be written

$$\lim_{q'^0 \to \infty} q'^0 Z^{00\rho} = -\frac{i}{2\pi^2} \, \varepsilon^{0\rho\alpha\beta} q'_\alpha q_\beta = -\frac{i}{2\pi^2} \, \varepsilon^{0\rho\alpha\beta} q'_\alpha (p' - p)_\beta, \tag{4.134}$$

where in the last expression, $q$ has been replaced by $q' + p' - p$, and the antisymmetry of the epsilon tensor has been employed. Observe that due to the term $\varepsilon^{0\rho\alpha\beta}$, the indices $\alpha, \beta$ are spatial only. Inserting this into Eq. (4.131), it is clear that the (gauge dependent) second part of the propagator gives no contribution, since $\varepsilon^{0\rho\alpha\beta}(p' - p)_\beta (p' - p)_\rho = 0$, while the $g_{\rho\sigma}$ part gives

$$\begin{aligned}
&\int \mathrm{d}^3x \, e^{-i\boldsymbol{q}'\cdot\boldsymbol{x}} \, \langle p', \tfrac{1}{2} | [J^0(x), J^0(0)]_{\mathrm{et}} | p, \tfrac{1}{2} \rangle \\
&\quad = i \, \frac{M_Z^2 G_F}{4\pi^2\sqrt{2}} \, \varepsilon^{0\rho\alpha\beta} \, q'_\alpha (p' - p)_\beta \, \frac{\bar{u}(p', \frac{1}{2}) \gamma_\rho \gamma_5 u(p, \frac{1}{2})}{(p' - p)^2 - M_Z^2} \\
&\quad = -i \, \frac{M_Z^2 G_F}{4\pi^2\sqrt{2}} \, \varepsilon^{kij} \, q'^i (p' - p)^j \, \frac{\bar{u}(p', \frac{1}{2}) \gamma^k \gamma_5 u(p, \frac{1}{2})}{(p' - p)^2 - M_Z^2}.
\end{aligned} \tag{4.135}$$

The $\boldsymbol{q}'$ dependence lies solely in the factor $q'^i$, whose Fourier transform is $-i\nabla^i \delta(\boldsymbol{x})$. Hence, it is a trivial matter to transform into coordinate space:

$$\langle p', \tfrac{1}{2} | [J^0(x), J^0(0)]_{\mathrm{et}} | p, \tfrac{1}{2} \rangle = -\frac{M_Z^2 G_F}{4\pi^2\sqrt{2}} \, \varepsilon^{kij} \, (p' - p)^j \, \frac{\bar{u}(p', \frac{1}{2}) \gamma^k \gamma_5 u(p, \frac{1}{2})}{(p' - p)^2 - M_Z^2} \, \nabla^i \delta(\boldsymbol{x}). \tag{4.136}$$

### Anomalous commutator in terms of gauge field

I point out that the same result is obtained upon first considering the vacuum-to-$\mathrm{Z}^0$ matrix element[15] of the commutator, which yields in terms of the $\mathrm{Z}^0$-boson field operator $Z^\mu(x)$:

$$\big[J^0(x), J^0(y)\big]_{\mathrm{et}} = \frac{-ie}{2\pi^2\sqrt{3}}\big(\boldsymbol{\nabla}\times\boldsymbol{Z}(x)\big)\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.137}$$

subsequently sandwiching this relation between one-electron states, using the coupling (4.124) and the equation of motion of the $\mathrm{Z}^0$ boson. This procedure amounts to seperately considering the upper and lower halfs of the $\mathrm{Z}^0$-exchange diagrams depicted in Fig. 4.3(d). Observe that expression (4.137) has the form of the general anomalous charge-density commutator (4.86) discussed in Sect. 4.2.

### Electric dipole moment

The reader may now recall that we are interested in the dipole-moment commutator. I proceed from charge-density $J^0(x)$ to the dipole moment $D^i(0)$ by computing the first moment of Eq. (4.136):

$$\begin{aligned}\big\langle p',\tfrac{1}{2}\big|[D^i(0), J^0(0)]\big|p,\tfrac{1}{2}\big\rangle &= e\int \mathrm{d}^3x\, x^i\, \big\langle p',\tfrac{1}{2}\big|[J^0(x), J^0(0)]_{\mathrm{et}}\big|p,\tfrac{1}{2}\big\rangle \\ &= \frac{eM_{\mathrm{Z}}^2 G_{\mathrm{F}}}{4\pi^2\sqrt{2}}\,\varepsilon^{kij}\,(p'-p)^j\,\frac{\bar{u}(p',\frac{1}{2})\gamma^k\gamma_5 u(p,\frac{1}{2})}{(p'-p)^2 - M_{\mathrm{Z}}^2}.\end{aligned} \tag{4.138}$$

Translational invariance (1.7) gives rise to a phase factor,

$$\big\langle p',\tfrac{1}{2}\big|[D^i(0), J^0(y)]\big|p,\tfrac{1}{2}\big\rangle = \big\langle p',\tfrac{1}{2}\big|[D^i(0), J^0(0)]\big|p,\tfrac{1}{2}\big\rangle\, e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{y}}, \tag{4.139}$$

which in turn fetches a gradient of $\delta(\boldsymbol{p}'-\boldsymbol{p})$ when the other dipole-moment operator is introduced:

$$\int \mathrm{d}^3y\, y^j\, e^{-i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{y}} = i(2\pi)^3\nabla^j\delta(\boldsymbol{p}'-\boldsymbol{p}). \tag{4.140}$$

Finally, the identity

$$(p'-p)^{j'}\nabla^j\delta(\boldsymbol{p}'-\boldsymbol{p}) = -\delta^{j'j}\,\delta(\boldsymbol{p}'-\boldsymbol{p}) \tag{4.141}$$

leads to

$$\big\langle p',\tfrac{1}{2}\big|[D^i(0), D^j(0)]\big|p,\tfrac{1}{2}\big\rangle = -i\,\frac{e^2M_{\mathrm{Z}}^2 G_{\mathrm{F}}}{4\pi^2\sqrt{2}}\,\varepsilon^{kij}\,(2\pi)^3\delta(\boldsymbol{p}'-\boldsymbol{p})\,\frac{\bar{u}(p,\frac{1}{2})\gamma^k\gamma_5 u(p,\frac{1}{2})}{-M_{\mathrm{Z}}^2}. \tag{4.142}$$

As usual, I take the electron to be travelling along $\boldsymbol{e}_3$ and introduce circularly polarized components

$$D^{\mathrm{L,R}}(0) = \frac{1}{\sqrt{2}}\big(D^1(0) \pm iD^2(0)\big), \tag{4.143}$$

[15] Eq. (4.137) is consistent with the lowest-order approximation to the result obtained by Jo [71, Eq. (2.20)], suitably generalized from left-handed currents and left-handed gauge fields to vector currents and axial-vector gauge fields.

which gives rise to the spinor expression

$$\bar{u}(p,\tfrac{1}{2})\,\gamma^3\gamma_5\,u(p,\tfrac{1}{2}) = 2ms^3 = 2p^0. \tag{4.144}$$

Consequently,

$$\boxed{\langle p',\tfrac{1}{2}\big|[D^{\mathrm{L}}(0),D^{\mathrm{R}}(0)]\big|p,\tfrac{1}{2}\rangle = (2\pi)^3\,2p^0\,\delta(\boldsymbol{p}'-\boldsymbol{p})\,\frac{\alpha}{\pi}\,\frac{G_{\mathrm{F}}}{\sqrt{2}}} \tag{4.145}$$

instead of the naive, i.e., vanishing, commutator, on which the conventional current-algebra derivation of the GDH sum rule is based.

### Modified finite-momentum GDH sum rule

Observe that Eq. (4.145) has the same form as Eq. (4.97). Thus, analogously to the procedure presented in Sect. 4.2.6, the finite-momentum GDH sum rule obtains a modification:

$$\boxed{\alpha\,\frac{G_{\mathrm{F}}}{\sqrt{2}} = \lim_{p^0\to\infty}\int_0^\infty \frac{\mathrm{d}\nu}{\nu}\,8\pi\,\mathrm{Im}\,f_2\left(\nu,\frac{m^2\nu^2}{(p^0)^2}\right)} \tag{4.146}$$

where the polarized forward virtual Compton amplitude $f_2(\nu,q^2)$ is defined as in Sect. 2.2:

$$f_2(\nu,q^2) := \frac{\alpha}{2m^2}\left(A_1(\nu,q^2)+\frac{q^2}{m\nu}A_2(\nu,q^2)\right). \tag{4.147}$$

### Infinite-momentum limit

For any finite value of the electron energy $p^0$, the integral on the right-hand side of Eq. (4.146) runs along a parabola in $(\nu,q^2)$ plane, as depicted in Fig. 4.4. What if the $p^0\to\infty$ limit and the $\nu$ integration were interchangeable in Eq. (4.146)? Upon application of the optical theorem, one would then of course have the familiar GDH integral

$$\int_0^\infty \frac{\mathrm{d}\nu}{\nu}\left(\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\right) \tag{4.148}$$

on the right-hand side, while the left-hand side is a specific order-$\alpha^2$ constant. (Recall that the Fermi constant (4.114) is of order $\alpha$). However, I illustrated in Sect. 3.3 that the left-hand side of the sum rule is of order $\alpha^3$ in the Weinberg-Salam model, and that Altarelli, Cabibbo, and Maiani [29] proved that the integral on the right-hand side confirms this, i.e., it vanishes to order $\alpha^2$. Hence, at this point it actually has to be strongly *expected* that the limit cannot be dragged inside the integral in Eq. (4.146) without further modification.

Of course, Altarelli et al. [29] considered only the kinematical domain of the genuine GDH sum rule, i.e., $q^2=0$ (the abscissa in Fig. 4.4). As a consequence, the $\mathrm{Z}^0$-exchange graph did simply not emerge. On the other hand, I inspect the sum rule at finite momenta and take the $p^0\to\infty$ limit eventually, which is certainly *the long way* to the sum rule. Let me remind you that I do this in order to *test* for the legitimacy of the infinite-momentum limit, particularly if there is an anomalous-commutator modification. I work in a perturbative model, because

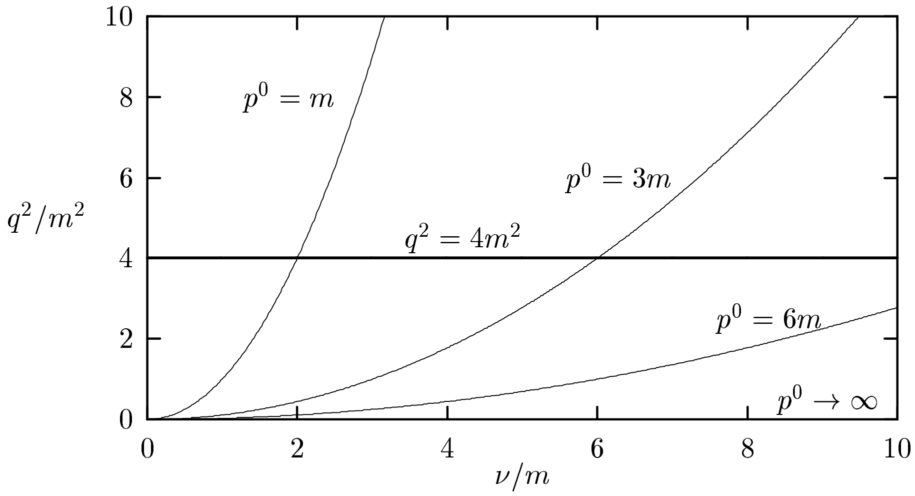



Figure 4.4: The integration path of the finite-momentum GDH sum rule (4.146) in the $(\nu, q^2)$ plane for electron energies $p^0 = m$, $3m$, and $6m$. For any finite value of $p^0$, the integration passes the two-electron threshold $q^2 = 4m^2$, picking up the constant on the left-hand side of Eq. (4.146). The genuine GDH sum rule integrates along the straight line at $q^2 = 0$.

this enables me to explicitly calculate the "limit of an integral" (4.146) *and* the "integral of a limit" (4.148), in order to confront both quantities.

To make a long story short: In the following I will show that permuting limit and integration in Eq. (4.146) gives rise to a second modification that exactly cancels the previous one. This modification is due to the same Feynman graphs that give rise to the commutator anomaly, namely the $Z^0$-exchange diagrams of Fig. 4.3(d). More specifically, the corresponding contribution to Compton amplitude $f_2(\nu, q^2)$ picks up an imaginary part if the photon mass $q^2$ exceeds the two-electron threshold $4m^2$, which of course it does for every finite energy $p^0$. At $q^2 = 0$, these graphs do not contribute to $f_2(\nu, q^2)$.

### 4.4.3 Infinite-momentum limit

I want to calculate the contribution of the $Z^0$-exchange diagrams to the forward virtual Compton amplitude $f_2(\nu, q^2)$. That is to say, Eqs. (4.122) and (4.126) are specialized to the drastically simplifying case $p = p'$, $q = q'$, and $s = s'$:

$$T^{\mu\nu}_{(\mathrm{Z})} = -\frac{M_{\mathrm{Z}}^2 G_{\mathrm{F}}}{2\sqrt{2}}\, Z^{\mu\nu\rho}(q)\, \frac{-g_{\rho\sigma}}{-M_{\mathrm{Z}}^2}\, \bar{u}(p,s)\, \gamma^\sigma \gamma_5\, u(p,s) \tag{4.149}$$

with

$$Z^{\mu\nu\rho}(q) = \varepsilon^{\mu\nu\rho\alpha} q_\alpha \left(Z_3(q^2) + Z_3'(q^2)\right). \tag{4.150}$$

By virtue of the identity

$$\bar{u}(p,s)\, \gamma^\sigma \gamma_5\, u(p,s) = 2ms^\sigma, \tag{4.151}$$

one obtains

$$T^{\mu\nu}_{(\mathrm{Z})} = m\,\frac{G_{\mathrm{F}}}{\sqrt{2}}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma \big(Z_3(q^2) + Z_3'(q^2)\big). \tag{4.152}$$

From Eqs. (4.130c) and (4.130f), I read off

$$\begin{aligned} Z_3 + Z_3' &= \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{x(1-x)\,q^2 + y(1-y)\,q^2 - 2xy\,q^2}{x(1-x)\,q^2 + y(1-y)\,q^2 - 2xy\,q^2 - m^2 + i\varepsilon} \\ &= \frac{i}{\pi^2}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{(x-y)(1-x-y)\,q^2}{(x+y)(1-x-y)\,q^2 - m^2 + i\varepsilon}, \end{aligned} \tag{4.153}$$

which involves the sum $x+y$ only. Hence, substituting $x \mapsto z = x+y$ renders the y integration trivial:

$$Z_3 + Z_3' = \frac{i}{\pi^2}\int_0^1 \mathrm{d}z \int_0^z \mathrm{d}y\, \frac{z(1-z)\,q^2}{z(1-z)\,q^2 - m^2 + i\varepsilon} = -\frac{i}{2\pi^2}\, f(q^2), \tag{4.154}$$

where

$$f(q^2) := -2\int_0^1 \mathrm{d}z\, \frac{z^2(1-z)\,q^2}{z(1-z)\,q^2 - m^2 + i\varepsilon}. \tag{4.155}$$

The integration in Eq. (4.155) can be performed explicitly. The result

$$f(q^2) = \begin{cases} -1 + \frac{4m^2}{\sqrt{(q^2-4m^2)q^2}}\,\mathrm{arcoth}\sqrt{1-\frac{4m^2}{q^2}} & \text{if } q^2 < 0 \\ -1 + \frac{4m^2}{\sqrt{(4m^2-q^2)q^2}}\,\mathrm{arccot}\sqrt{\frac{4m^2}{q^2}-1} & \text{if } 0 < q^2 < 4m^2 \\ -1 + \frac{4m^2}{\sqrt{(q^2-4m^2)q^2}}\left(\mathrm{artanh}\sqrt{1-\frac{4m^2}{q^2}} + \frac{i\pi}{2}\right) & \text{if } q^2 > 4m^2 \end{cases} \tag{4.156}$$

is depicted in Fig. 4.5. Note that function $f(q^2)$ is continuous at $q^2 = 0$.

### $\mathbf{Z^0}$-exchange contribution to forward Compton amplitude

Inserting Eq. (4.154) into Eq. (4.152), I conclude that

$$T^{\mu\nu}_{(\mathrm{Z})} = -\frac{im}{2\pi^2}\,\frac{G_{\mathrm{F}}}{\sqrt{2}}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma\, f(q^2). \tag{4.157}$$

On comparison with the invariant decomposition of the polarized, i.e., antisymmetric part of the forward virtual Compton amplitude,

$$\begin{aligned} \tfrac{1}{2}\big(T^{\mu\nu} - T^{\nu\mu}\big) = &- \frac{i}{m}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma\, A_1(\nu, q^2) \\ &- \frac{i}{m^3}\,\varepsilon^{\mu\nu\rho\sigma} q_\rho\big((M\nu)s_\sigma - (q\cdot s)p_\sigma\big) A_2(\nu, q^2), \end{aligned} \tag{4.158}$$

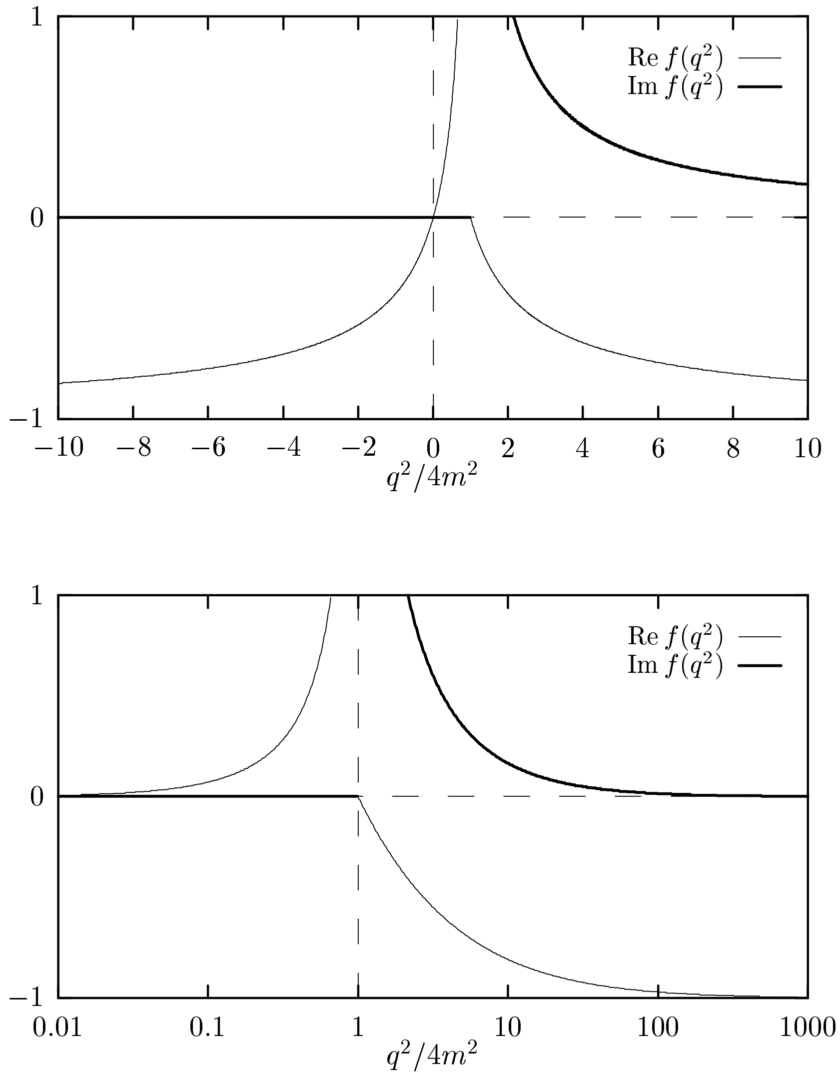



Figure 4.5: Real part (light line) and imaginary part (heavy line) of the function $f(q^2)$ occurring in the $\mathrm{Z}^0$-exchange contribution $f_2^{(\mathrm{Z})}(\nu, q^2) \propto f(q^2)$ to the forward virtual Compton amplitude of the order-$\alpha^2$ Weinberg-Salam model. Upper plot: linear $q^2$ scale; lower plot: logarithmic $q^2$ scale, only $q^2 > 0$. A branch-point singularity occurs at the two-electron threshold $q^2 = 4m^2$.

one realizes that $\mathrm{Z}^0$ exchange contributes to amplitude $A_1(\nu, q^2)$ only:

$$\begin{aligned} A_1^{(\mathrm{Z})}(\nu, q^2) &= \frac{m^2}{2\pi^2} \frac{G_\mathrm{F}}{\sqrt{2}} f(q^2), \\ A_2^{(\mathrm{Z})}(\nu, q^2) &= 0. \end{aligned} \tag{4.159}$$

By definition (4.147), this implies

$$\boxed{f_2^{(\mathrm{Z})}(\nu, q^2) = \frac{\alpha}{4\pi^2} \frac{G_\mathrm{F}}{\sqrt{2}} f(q^2)} \tag{4.160}$$

### Interchanging limit and integration

Let us now have a closer look at the result (4.160) together with Eq. (4.155). The first and most essential thing to be noted is that function $f_2^{(\mathrm{Z})}(\nu, q^2)$ depends solely on the photon's virtuality $q^2$. At $q^2 = 0$, it vanishes:

$$f_2^{(\mathrm{Z})}(\nu, 0) = 0, \tag{4.161}$$

according to the fact that $\mathrm{Z}^0$ exchange does not contribute to the real Compton process. Hence, of course,

$$\int_0^\infty \frac{\mathrm{d}\nu}{\nu} \, \mathrm{Im}\, f_2^{(\mathrm{Z})}(\nu, 0) = 0, \tag{4.162}$$

Now, what happens if $q^2$ rises? Since the expression $z(1-z)$ attaines a maximum of $\frac{1}{4}$ at $z = \frac{1}{2}$, the integrand in Eq. (4.155) exhibits poles in the vicinity of the real axis if $q^2 > 4m^2$, thereupon giving function $f(q^2)$ an imaginary part. Considering Fig. 4.3(d), this branch point singularity is owing to the fact that at $q^2 > 4m^2$, the two electrons that couple to the photon can be put on their mass shells. So, what about the $\mathrm{Z}^0$-exchange contribution to the *finite-momentum* GDH integral (4.146)? Using Eqs. (4.155) and (4.160), I infer

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\nu}{\nu} \, 8\pi \, \mathrm{Im}\, f_2^{(\mathrm{Z})}\left(\nu, \frac{m^2\nu^2}{(p^0)^2}\right) &= \frac{2\alpha}{\pi} \frac{G_\mathrm{F}}{\sqrt{2}} \int_0^\infty \frac{\mathrm{d}\nu}{\nu} \, \mathrm{Im}\, f\left(\frac{m^2\nu^2}{(p^0)^2}\right) \\ &= \frac{\alpha}{\pi} \frac{G_\mathrm{F}}{\sqrt{2}} \int_{4m^2}^\infty \frac{\mathrm{d}(q^2)}{q^2} \, \mathrm{Im}\, f(q^2).\end{aligned} \tag{4.163}$$

Observe that owing to the $\nu$ independence of function $f_2^{(\mathrm{Z})}(\nu, q^2)$, the result is independent of the electron's energy $p^0$. This means that on each parabola in the $(\nu, q^2)$ plane of Fig. 4.4, the GDH integral picks up the *same* constant from the part of the parabola that lies above the two-electron threshold, while Eq. (4.162) reflects the fact that there is no such contribution if the integration path lies entirely below the two-electron threshold. The integral (4.163) can be calculated either directly from the imaginary part of Eq. (4.156), or by recalling that it represents a $q^2$ dispersion integral, subtracted at infinity on account of the limit $f(\infty) = -2\int_0^1 \mathrm{d}z\, z = -1$. The result reads

$$\int_0^\infty \frac{\mathrm{d}\nu}{\nu} \, 8\pi \, \mathrm{Im}\, f_2^{(\mathrm{Z})}\left(\nu, \frac{m^2\nu^2}{(p^0)^2}\right) = \alpha \frac{G_\mathrm{F}}{\sqrt{2}}. \tag{4.164}$$

Note that, again, the $m$ dependence dropped out. Confrontation of Eqs. (4.162) and (4.164) reveals that interchanging $p^0 \to \infty$ limit and $\nu$ integration is non-trivial owing to the $\mathrm{Z}^0$-exchange graphs:

$$\left(\lim_{p^0\to\infty} \int_0^\infty \frac{\mathrm{d}\nu}{\nu} - \int_0^\infty \frac{\mathrm{d}\nu}{\nu} \lim_{p^0\to\infty}\right) 8\pi \, \mathrm{Im}\, f_2\left(\nu, \frac{m^2\nu^2}{(p^0)^2}\right) = \alpha \frac{G_\mathrm{F}}{\sqrt{2}}. \tag{4.165}$$

Thus, an additive constant occurs in the relation of the GDH integral (4.148) to the finite-momentum GDH integral on the right-hand side of Eq. (4.146):

$$\boxed{\int_0^\infty \frac{\mathrm{d}\nu}{\nu}\left(\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\right) = \lim_{p^0\to\infty}\int_0^\infty \frac{\mathrm{d}\nu}{\nu}\, 8\pi\, \mathrm{Im}\, f_2\left(\nu, \frac{m^2\nu^2}{(p^0)^2}\right) - \alpha\,\frac{G_\mathrm{F}}{\sqrt{2}}} \tag{4.166}$$

Inserting the finite-momentum GDH sum rule (4.146) leads back to the undisturbed sum rule (3.27).

### 4.4.4 Summary and discussion

The Weinberg-Salam model serves as an ideal testing ground for the legitimacy of the infinite-momentum limit, particularly in the presence of an anomalous-commutator modification of the finite-momentum GDH sum rule. I have shown that via the BJL limit, the $\mathrm{Z}^0$-exchange Feynman graphs of Fig. 4.3(d) lead to an anomalous charge-density commutator (4.136) (or dipole-moment commutator (4.145)), which modifies the finite-momentum GDH sum rule (4.146). By explicit calculation of the pertinent Compton amplitude at all values of $\nu$ and $q^2$, I proved that the naive infinite-momentum limit fails in the presence of this modification. Rather, letting $p^0 \to \infty$ gives rise to a second modification (4.166) that cancels the previous one. In other words, *the anomalous-commutator modification does not survive the infinite-momentum limit!* I stress that this result is consistent with the fact that no order-$\alpha^2$ modification of the GDH sum rule within the Weinberg-Salam model is observed by direct evaluation at $q^2 = 0$ [29].

#### The role of gauge invariance

It is perhaps worth noting that the exact cancellation of the two modifications of the GDH sum rule can be traced back to the gauge-invariance condition (4.127), which relates the function $Z_3$ occurring in Eq. (4.152) to functions $Z_{1,2}$ of Eq. (4.133).

#### BJL limit and Regge limit

Observe that the $\mathrm{Z}^0$-exchange contribution (4.160) to the forward virtual Compton amplitude presents an explicit instance of the non-coincidence of BJL limit ($q^0 \to \infty$, $\boldsymbol{q}$ fixed) and Regge limit ($\nu \to \infty$, $q^2$ fixed):

$$\lim_{\substack{q^0\to\infty \\ \boldsymbol{q}\text{ fixed}}} f_2^{(\mathrm{Z})}(\nu, q^2) = \frac{\alpha}{4\pi^2}\,\frac{G_\mathrm{F}}{\sqrt{2}}\, f(\infty), \tag{4.167a}$$

whereas

$$\lim_{\substack{\nu\to\infty \\ q^2\text{ fixed}}} f_2^{(\mathrm{Z})}(\nu, q^2) = \frac{\alpha}{4\pi^2}\,\frac{G_\mathrm{F}}{\sqrt{2}}\, f(q^2). \tag{4.167b}$$

Function $f(q^2)$ (see Eq. (4.156), Fig. 4.5) obeys $f(0) = 0$ and $f(\pm\infty) = -1$. Thus, the limits (4.167) coincide at large $q^2$. This also holds for large *spacelike* $q^2$, i.e., $Q^2 := -q^2 \to \infty$, indicating the equivalence of BJL and Regge limits in deep-inelastic scattering.

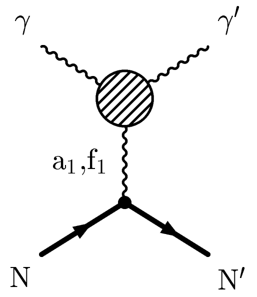
Observe, further, that Eq. (4.167b) presents a $J = 1$ fixed pole in complex angular-momentum plane (cf. Sect. 4.1), whose residue is non-polynomial in the photon virtuality $q^2$, contrary to the common sense of early studies on fixed poles (see, e.g., Cheng and Tung [97]). At $q^2 = 0$, the residue vanishes.

#### Anomaly cancellation

Finally, I remark that if quarks were included into the model, i.e., if not only electrons, but also u and d quarks were taken to circulate in the loop of Fig. 4.3(d), then the customary effect of anomaly cancellation would remove the modification of the charge-density commutator as well as the one due to the infinite-momentum limit. This can be traced back to the $m$ independence of Eqs. (4.133) and (4.164).

## 4.5 $t$-channel $a_1$ exchange

The possible fixed pole in angular-momentum plane, discussed in Sect. 4.1, possesses the quantum numbers $J^P = 1^+$ of the $a_1$ meson (more precisely: $a_1$ and $f_1$ mesons). This fact is misunderstood once in a while. I stress that fixed poles of given spin $J$ can generally not be attributed to $t$-channel exchange of a physical spin-$J$ particle. To shed some light on this point, the present section is devoted to a discussion of the process represented by the following Feynman graph:




(4.168)

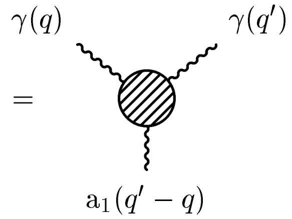
Apart from doing away with the above-mentioned misbelief, I aim at a further discussion of the alleged anomalous-commutator modification of the GDH sum rule, claimed by Chang, Liang, and Workman [13].

#### Landau-Yang theorem

Half a century ago, Landau [98] and Yang [99] inferred from symmetry considerations, that two real photons never are in a state of total angular momentum one.[16] Hence, one expects

[16]More precisely, a two-photon state cannot have total angular momentum $J = 1$ or $J = 3, 5, \ldots$ and odd parity [98, 99].

the upper vertex

$V^{\mu\nu\rho}(q, q') =$ 

(4.169)

of diagram (4.168) to vanish identically if real photons are attached. In Eq. (4.169), Lorentz indices $\mu$, $\nu$, and $\rho$ represent scattered and incident photon and the meson, respectively. I let $q^2 = q'^2 = 0$. By parity conservation and crossing symmetry, the general invariant decomposition of vertex (4.169) runs

$$\begin{aligned} V^{\mu\nu\rho} &= \varepsilon^{\mu\nu\alpha\beta}(q'-q)^\rho q'_\alpha q_\beta \, V_1(q\cdot q') \\ &+ (\varepsilon^{\nu\rho\alpha\beta} q'^\mu + \varepsilon^{\mu\rho\alpha\beta} q^\nu) q'_\alpha q_\beta \, V_2(q\cdot q') \\ &+ \varepsilon^{\mu\nu\rho\alpha}(q+q')_\alpha \, V_3(q\cdot q'). \end{aligned} \tag{4.170}$$

(A fourth tensor $(\varepsilon^{\nu\rho\alpha\beta} q^\mu + \varepsilon^{\mu\rho\alpha\beta} q'^\nu) q'_\alpha q_\beta$ is proven to be linearly dependent on the others by virtue of identity (1.13).) Gauge invariance, $q'_\mu V^{\mu\nu\rho} = 0$, implies

$$V_3(q\cdot q') = 0, \tag{4.171}$$

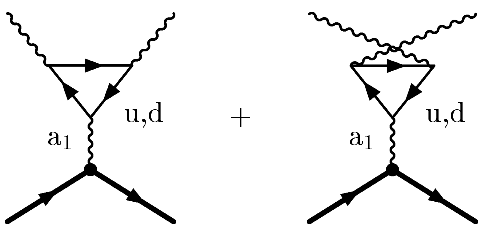
while due to the condition $\varepsilon\cdot q = \varepsilon'\cdot q' = 0$, function $V_2(q\cdot q')$ does not contribute when the vertex function (4.170) is contracted with real photon polarization vectors:

$$\varepsilon'_\mu \varepsilon_\nu V^{\mu\nu\rho} = \varepsilon^{\mu\nu\alpha\beta} \varepsilon'_\mu \varepsilon_\nu q'_\alpha q_\beta (q'-q)^\rho \, V_1(q\cdot q'). \tag{4.172}$$

In accordance with the Landau-Yang theorem, this expression vanishes upon contraction with a third polarization vector $\varepsilon''_\rho$ representing the axial-vector meson and satisfying $\varepsilon''\cdot(q'-q) = 0$. Contracted with a propagator, expression (4.172) does not necessarily vanish, due to the fact that an off-shell spin-1 propagator has a spin-0 part, unless it is written in Landau gauge. Nevertheless, taking $q = q'$ in Eq. (4.172) explicitly reveals that the graph (4.168) does not contribute to real forward Compton scattering. Hence it does not affect the GDH sum rule. However, it *does* affect the *finite-momentum* GDH sum rule, as shown in the following.

**Anomalous charge-density commutator**

A striking analogy can be revealed between $\mathrm{a}_1$ exchange and $\mathrm{Z}^0$ exchange as discussed in Sect. 4.4, if the axial-vector meson is coupled to the photons by means of quark triangle-loop diagrams:



(4.173)

This means that spin-1 mesons are regarded as chiral gauge fields with pointlike coupling to quarks. (For a discussion, I refer the reader to p. 65 of this thesis.) For simplicity, I consider the proton-neutron difference in order to project out isovector exchange, i.e., $a_1$ exchange only, rather than $a_1$ and $f_1$ exchange. Falling back on Eqs. (4.122)–(4.125), I can re-use the results of Sect. 4.4 by translating the lower $Z^0e^-e^-$ vertex into the $a_1NN$ vertex of graphs (4.173) according to

$$-\frac{ie}{\sqrt{3}}\gamma^\rho\gamma_5 \mapsto ig_A f_A\,\gamma^\rho\gamma_5, \tag{4.174a}$$

and the upper $Z^0e^-e^-$ vertex into the $a_1qq$ vertex according to

$$-\frac{ie}{\sqrt{3}}\gamma^\sigma\gamma_5 \mapsto \frac{i}{2}f_A\,\gamma^\sigma\gamma_5, \tag{4.174b}$$

where the constant $f_A$ couples the physical meson to the axial-vector current (cf. Eq. (4.88)), $g_A = 1.26$ is the neutron beta-decay constant (the nucleon matrix element of the axial-vector current), and a factor of two in Eq. (4.174a) accounts for taking the proton-neutron difference. The triangle loops couple to the axial charge times the square of the electric charge. Moreover, there are $N_c = 3$ quarks of each flavor. Thus, one has to substitute[17]

$$-1 = -\big(-1\big)^2 \mapsto N_c\big[\big(\tfrac{2}{3}\big)^2 - \big(-\tfrac{1}{3}\big)\big] = 1. \tag{4.174c}$$

Combining Eqs. (4.174), one encounters the substitution

$$\frac{e^2}{3} = \frac{M_Z^2 G_F}{2\sqrt{2}} \mapsto \frac{1}{2}\,g_A f_A^2. \tag{4.175}$$

The $Z^0$ propagator (4.125) has to be replaced by the $a_1$-meson propagator

$$i\,\frac{-g_{\rho\sigma} + (p'-p)_\rho (p'-p)_\sigma/m_A^2}{(p'-p)^2 - m_A^2}. \tag{4.176}$$

Putting these pieces together, the calculation of anomalous charge-density and dipole-moment commutators proceeds in perfect analogy to Sect. 4.4. One obtains

$$\begin{aligned}&\big\langle p(p',\tfrac{1}{2})\big|[J^0(x),J^0(0)]_{\text{et}}\big|p(p,\tfrac{1}{2})\big\rangle - \big\langle n(p',\tfrac{1}{2})\big|[J^0(x),J^0(0)]_{\text{et}}\big|n(p,\tfrac{1}{2})\big\rangle \\ &\quad = \frac{g_A f_A^2}{4\pi^2}\,\varepsilon^{kij}\,(p'-p)^j\,\frac{\bar{u}(p',\tfrac{1}{2})\gamma^k\gamma_5 u(p,\tfrac{1}{2})}{(p'-p)^2 - m_A^2}\,\nabla^i\delta(\boldsymbol{x})\end{aligned} \tag{4.177}$$

in place of Eq. (4.136), and

$$\begin{aligned}&\big\langle p(p',\tfrac{1}{2})\big|[D^L(0),D^R(0)]\big|p(p,\tfrac{1}{2})\big\rangle - \big\langle n(p',\tfrac{1}{2})\big|[D^L(0),D^R(0)]\big|n(p,\tfrac{1}{2})\big\rangle \\ &\quad = -(2\pi)^3\,2p^0\,\delta(\boldsymbol{p}'-\boldsymbol{p})\,\frac{\alpha}{\pi}\,\frac{g_A f_A^2}{m_A^2}\end{aligned} \tag{4.178}$$

in place of Eq. (4.145). Proceeding as in Ref. 13, I employ the KSRF relation (4.81) together with Weinberg's identity $m_A^2 = 2m_V^2$ [90], as well as the fact that within the present context,

[17] Readers familiar with the chiral anomaly may recognize Eq. (4.174c) as being the reason for the anomaly-cancellation mechanism.

vector- and axial-vector-meson couplings necessarily are degenerate, $f_{\mathrm{V}} = f_{\mathrm{A}}$ [84, 85]. I arrive at

$$\begin{aligned}&\big\langle \mathrm{p}(p',\tfrac{1}{2})\big|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\big|\mathrm{p}(p,\tfrac{1}{2})\big\rangle - \big\langle \mathrm{n}(p',\tfrac{1}{2})\big|[D^{\mathrm{L}}(0), D^{\mathrm{R}}(0)]\big|\mathrm{n}(p,\tfrac{1}{2})\big\rangle \\ &\quad = -(2\pi)^3\, 2p^0\, \delta(\boldsymbol{p}' - \boldsymbol{p})\, \frac{\alpha g_{\mathrm{A}}}{\pi F_\pi^2}. \end{aligned} \tag{4.179}$$

Up to a factor of three,[18] Eq. (4.179) reproduces the result of Chang, Liang, and Workman [13] as summarized in Eqs. (4.97) and (4.106) of this thesis. In other words: If you look hard at the investigation presented in Refs. 13–15, you realize that actually the $t$-channel exchange of axial-vector mesons is studied.[19] Hence, it is equivalent to the investigation of the anomalous commutator within the Weinberg-Salam model, as long as all particles and couplings are substituted appropriately. This again shows that indeed the commutator anomaly modifies merely the *finite-momentum* GDH sum rule, being remedied upon taking the infinite-momentum limit.

## 4.6 Anomalous magnetic moments of quarks

This section is devoted to the discussion of possible modifications of the GDH sum rule that arise in a non-naive, yet canonical manner if one assumes quarks to possess anomalous magnetic moments $\kappa_q$ [10, 11]. The reader may wonder why this is a field of investigation, because in the standard model quarks are pointlike, thus having $\kappa_q = 0$. (As a matter of fact, some of the most vital foundations of the standard model – like renormalizability – depend upon the exact vanishing of $\kappa_q$.) But "quarks" can also mean "constituent quarks" – those nice objects having masses on hadronic scales and quantum numbers of bare quarks. One can fancy a constituent quark as a bare quark being surrounded by a cloud of mesons or quark-antiquark pairs and gluons. Such an object clearly has an internal structure. Moreover, constituent-quark models are non-perturbative models, so there is no reason – apart from aesthetics, if you want – for the anomalous magnetic moment of a constituent quark to vanish. And as the GDH sum rule is occasionally investigated in the context of constituent-quark models (see, e.g., Refs. 100–104), it is by all means worth while considering the case $\kappa_q \neq 0$.

#### Modified charge-density algebra

To derive the GDH sum rule within the current-algebra approach, one needs the equal-times commutator $[J^0(x), J^0(y)]_{\mathrm{et}}$ of electric charge densities. In Sect. 2.2.1, I presented a derivation of this commutator from canonical anticommutation relations (2.40) among quark fields, assuming the electromagnetic current to have the form

$$J^\mu(x) = \sum_{q=u,d,s} Z_q\, \bar{q}(x)\, \gamma^\mu\, q(x) = \bar{\psi}(x)\, \gamma^\mu Z^{(\mathrm{q})}\, \psi(x), \tag{4.180}$$

where

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix} \qquad \text{and} \qquad Z^{(\mathrm{q})} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{4.181}$$

[18] See footnote 12 on page 64.

[19] Cf. discussion "The Emperor's New Clothes" on page 64 of this thesis.

for three flavors u, d, and s.[20] The charge-density algebra then reads

$$\left[J^0(x), J^0(y)\right]_{\mathrm{et}} = 0. \tag{4.182}$$

However, if the current does not have the minimal form (4.180) – that is to say, if it does not arise from a Lagrangian density $\bar{\psi}(i\not{\partial} - m)\psi$ by virtue of minimal coupling – then the algebra (4.182) may be modified. A particularly interesting example is the addition of a Pauli term to the Lagrangian, which adds to the current a derivative of a tensor current [10, 11]:

$$J^\mu(x) = \bar{\psi}(x)\,\gamma^\mu Z^{(\mathrm{q})}\,\psi(x) + \frac{\partial}{\partial x^\nu}\left(\bar{\psi}(x)\,\sigma^{\mu\nu}\frac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\,\psi(x)\right). \tag{4.183}$$

Here, $\kappa^{(\mathrm{q})}$ and $m^{(\mathrm{q})}$ are diagonal 3×3 matrices of quark anomalous magnetic moments and quark masses, respectively:

$$\kappa^{(\mathrm{q})} = \begin{pmatrix} \kappa_{\mathrm{u}} & 0 & 0 \\ 0 & \kappa_{\mathrm{d}} & 0 \\ 0 & 0 & \kappa_{\mathrm{s}} \end{pmatrix}, \qquad m^{(\mathrm{q})} = \begin{pmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & m_{\mathrm{d}} & 0 \\ 0 & 0 & m_{\mathrm{s}} \end{pmatrix}. \tag{4.184}$$

Thus,

$$\frac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}} = \tfrac{1}{2}\,\kappa^{(\mathrm{q})}\left(m^{(\mathrm{q})}\right)^{-1} = \begin{pmatrix} \frac{\kappa_{\mathrm{u}}}{2m_{\mathrm{u}}} & 0 & 0 \\ 0 & \frac{\kappa_{\mathrm{d}}}{2m_{\mathrm{d}}} & 0 \\ 0 & 0 & \frac{\kappa_{\mathrm{s}}}{2m_{\mathrm{s}}} \end{pmatrix}. \tag{4.185}$$

From $\gamma^0\sigma^{0k} = i\gamma^k$, and taking $\mu = 0$ in Eq. (4.183), the charge density is obtained as

$$J^0(x) = \psi^\dagger(x)\,Z^{(\mathrm{q})}\,\psi(x) + i\boldsymbol{\nabla}\cdot\left(\psi^\dagger(x)\,\boldsymbol{\gamma}\frac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\,\psi(x)\right). \tag{4.186}$$

The equal-times commutator $[J^0(x), J^0(y)]_{\mathrm{et}}$ is worked out by repeatedly employing the canonical anticommutation relations (2.40) analogously to the procedure presented in Sect. 2.2.1, p. 19 of this thesis. Most generally, the commutator of two fermion bilinears is obtained as

$$\begin{aligned}&\left[\psi^\dagger(x)\,\Gamma\Lambda\,\psi(x), \psi^\dagger(y)\,\Gamma'\Lambda'\,\psi(y)\right]_{\mathrm{et}} \\ &\quad= \frac{1}{2}\,\psi^\dagger(x)\left([\Gamma,\Gamma']\{\Lambda,\Lambda'\} + \{\Gamma,\Gamma'\}[\Lambda,\Lambda']\right)\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{4.187}$$

where $\Gamma, \Gamma'$ act in Dirac space and $\Lambda, \Lambda'$ act in flavor space. That is to say, $\Gamma, \Gamma'$ are any linear combinations of the matrices 1, $\gamma^\mu$, $\sigma^{\mu\nu}$, $\gamma^\mu\gamma_5$, and $\gamma_5$, while $\Lambda, \Lambda'$ are linear combinations of Gell-Mann matrices $\lambda_{0\ldots8}$. Presently, I consider only the diagonal flavor matrices $Z^{(\mathrm{q})}$ and $\kappa^{(\mathrm{q})}/2m^{(\mathrm{q})}$, which mutually commute. Thus, the second term of the sum in Eq. (4.187) can be omitted, leaving

$$\left[\psi^\dagger(x)\,\Gamma\Lambda\,\psi(x), \psi^\dagger(y)\,\Gamma'\Lambda'\,\psi(y)\right]_{\mathrm{et}} = \psi^\dagger(x)\,[\Gamma,\Gamma']\Lambda\Lambda'\,\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{4.188}$$

[20] In this section, I only consider the case of three flavors, but all formulae can be applied to the two-flavor case by letting $Z_{\mathrm{s}} = \kappa_{\mathrm{s}} = 0$ throughout.

From this relation it can be seen immediately that the canonical piece $\psi^\dagger(x)\, Z^{(\mathrm{q})}\, \psi(x)$ of the charge density (4.186) commutes not only with itself, but also with the additional piece. Thus one is left with

$$\begin{aligned}\left[J^0(x), J^0(y)\right]_{\mathrm{et}} &= -\frac{\partial^2}{\partial x^i \partial y^j}\left[\psi^\dagger(x)\,\gamma^i \tfrac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\,\psi(x), \psi^\dagger(y)\,\gamma^j \tfrac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\,\psi(y)\right]_{\mathrm{et}} \\ &= -\frac{\partial^2}{\partial x^i \partial y^j}\Big(\psi^\dagger(x)\,[\gamma^i,\gamma^j]\big(\tfrac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\big)^2\,\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y})\Big).\end{aligned} \tag{4.189}$$

Using the relation

$$[\gamma^i,\gamma^j] = -2i\,\varepsilon^{ijk}\gamma^0\gamma^k\gamma_5, \tag{4.190}$$

one arrives at

$$\boxed{\left[J^0(x), J^0(y)\right]_{\mathrm{et}} = 2i\left[\boldsymbol{\nabla}\times\Big(\bar\psi(x)\,\boldsymbol{\gamma}\gamma_5\big(\tfrac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\big)^2\,\psi(x)\Big)\right]\cdot\boldsymbol{\nabla}\delta(\boldsymbol{x}-\boldsymbol{y})} \tag{4.191}$$

### Modified finite-momentum GDH sum rule

Observe that the charge-density algebra (4.191) has exactly the form (4.86) with the axial-vector field $a^\mu(x)$ given by

$$\begin{aligned}a^\mu(x) &= 2\,\bar\psi(x)\,\gamma^\mu\gamma_5\big(\tfrac{\kappa^{(\mathrm{q})}}{2m^{(\mathrm{q})}}\big)^2\,\psi(x) = \sum_{q=u,d,s}\frac{\kappa_q^2}{2m_q^2}\,\bar q(x)\,\gamma^\mu\gamma_5\,q(x) \\ &= \sqrt{\frac{2}{3}}\left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}+\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}+\frac{\kappa_{\mathrm{s}}^2}{2m_{\mathrm{s}}^2}\right)J^\mu_{5\,0}(x) + \left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}-\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}\right)J^\mu_{5\,3}(x) \\ &\quad + \frac{1}{\sqrt{3}}\left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}+\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}-\frac{\kappa_{\mathrm{s}}^2}{m_{\mathrm{s}}^2}\right)J^\mu_{5\,8}(x).\end{aligned} \tag{4.192}$$

Axial-vector currents $J^\mu_{5a}(x)$ are defined in Eq. (4.69b). Their coefficients in Eq. (4.192) are easily obtained upon using orthogonality of Gell-Mann matrices, Eq. (4.71). According to Sect. 4.2.6, Eq. (4.192) leads to the following modification of the finite-momentum GDH sum rule:

$$-4\pi^2\alpha\left(\frac{\kappa^2}{2M^2}-\frac{(Z+\kappa)^2}{2(p^0)^2}-\tilde g_{\mathrm{A}}\right) = \int\limits_{\nu_{\mathrm{thr}}}^{\infty}\frac{\mathrm{d}\nu}{\nu}\,8\pi\,\mathrm{Im}\,f_2\left(\nu,\frac{M^2\nu^2}{(p^0)^2}\right), \tag{4.193}$$

where the constant $\tilde g_{\mathrm{A}}$ is determined by

$$\tilde g_{\mathrm{A}}\,\bar u(p,\tfrac12)\,\gamma^\mu\gamma_5\,u(p,\tfrac12) = \langle p,\tfrac12|a^\mu(0)|p,\tfrac12\rangle. \tag{4.194}$$

Introducing Eq. (4.192), as well as axial charges $g_{\mathrm{A}a}$ of the nucleon, defined by means of

$$g_{\mathrm{A}a}\,\bar u(p,\tfrac12)\,\gamma^\mu\gamma_5\,u(p,\tfrac12) = \langle p,\tfrac12|J^\mu_{5a}(0)|p,\tfrac12\rangle, \tag{4.195}$$

one obtaines the modification constant

$$\begin{aligned}\tilde g_{\mathrm{A}} &= \sqrt{\frac{2}{3}}\left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}+\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}+\frac{\kappa_{\mathrm{s}}^2}{2m_{\mathrm{s}}^2}\right)g_{\mathrm{A}0} + \left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}-\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}\right)g_{\mathrm{A}3} \\ &\quad + \frac{1}{\sqrt{3}}\left(\frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2}+\frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2}-\frac{\kappa_{\mathrm{s}}^2}{m_{\mathrm{s}}^2}\right)g_{\mathrm{A}8}.\end{aligned} \tag{4.196a}$$

Assuming isospin symmetry $g^{\mathrm{p}}_{\mathrm{A}0} = g^{\mathrm{n}}_{\mathrm{A}0}$, $g^{\mathrm{p}}_{\mathrm{A}8} = g^{\mathrm{n}}_{\mathrm{A}8}$, and $g^{\mathrm{p}}_{\mathrm{A}3} = -g^{\mathrm{n}}_{\mathrm{A}3} = g_{\mathrm{A}}$ (cf. Eqs. (4.105)), the proton-neutron difference is obtained as

$$\tilde{g}^{\mathrm{p}}_{\mathrm{A}} - \tilde{g}^{\mathrm{n}}_{\mathrm{A}} = 2g_{\mathrm{A}} \left( \frac{\kappa_{\mathrm{u}}^2}{2m_{\mathrm{u}}^2} - \frac{\kappa_{\mathrm{d}}^2}{2m_{\mathrm{d}}^2} \right), \tag{4.196b}$$

where $g_{\mathrm{A}} = 1.26$ is the familiar neutron-beta-decay constant.

### Infinite-momentum limit

To gain something useful from the (as-such needless) finite-momentum GDH sum rule (4.193), one has to assume the legitimacy of the infinite-momentum limit, i.e., function $f_2(\nu, q^2)$ is conjectured to allow for the $p^0 \to \infty$ limit to be dragged into the $\nu$ integral. Then one can write

$$\boxed{-4\pi^2\alpha \left( \frac{\kappa^2}{2M^2} - \tilde{g}_{\mathrm{A}} \right) = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \left( \sigma_{1/2}(\nu) - \sigma_{3/2}(\nu) \right)} \tag{4.197}$$

where the modification constant $\tilde{g}_{\mathrm{A}}$ is given by Eqs. (4.196).

What evidence is there for assuming the legitimacy of the infinite-momentum limit? As pointed out in Sect. 2.2, from the commutator (4.191) nothing can be inferred concerning this problem. Still, the modification, if present, must be expected to survive the infinite-momentum limit. This is because, as claimed by Dicus and Palmer [26], anomalous magnetic moments of quarks also affect current commutators on the light-cone, and as I explained in Sect. 2.3, the light-cone method circumvents the infinite-momentum limit. As a rule of thumb, one may say: "If you have a non-naive commutator of whatever origin, leading to a modification of the finite-momentum GDH sum rule, and if you'd like to know whether this modification will survive the infinite-momentum limit, try and check whether the same kind of commutator emerges within light-cone current-algebra!"[21]

### Discussion

It was Kawarabayashi and Suzuki [10], who first observed that if the current-algebra derivation is applied, anomalous magnetic moments of quarks bring about a modification of the GDH sum rule (Provided, of course, that this modification survives the infinite-momentum limit). Khare [11] gives a more comprehensive treatment, discussing other sum rules, too. Note, however, that the extended electromagnetic current (4.183) involves the piece

$$\frac{1}{3} \left( \frac{\kappa_{\mathrm{u}}}{2m_{\mathrm{u}}} + \frac{\kappa_{\mathrm{d}}}{2m_{\mathrm{d}}} + \frac{\kappa_{\mathrm{s}}}{2m_{\mathrm{s}}} \right) \frac{\partial}{\partial x^\nu} \left( \bar{\psi}(x) \sigma^{\mu\nu} \psi(x) \right), \tag{4.198}$$

i.e., the derivative of the *flavor-singlet* tensor current, which is not included in Refs. 10 and 11. This is a severe deficiency of Refs. 10 and 11, since it restricts their results to the case that the sum of anomalous magnetic moments occurring in Eq. (4.198) vanishes.

[21] Fortunately, there is no need to examine the effect of the *chiral anomaly* on light-cone current commutators, since the investigation presented in Sect. 4.4 clearly shows that the anomaly does not modify the GDH sum rule.

It may be instructive to confront my results with the ones of de Sanctis, Drechsel, and Giannini [101], who effectively perform the equal-times current-algebra derivation of the GDH sum rule, starting from a constituent-quark current that includes relativistic corrections. In Eq. (15) of Ref. 101, a modification of the sum rule is suggested, which has the form (4.197) with

$$\tilde{g}_{\mathrm{A}} = \sum_{i=1}^{3} \frac{\kappa_i^2}{2m_i^2} \, \langle \sigma_{zi} \rangle, \tag{4.199}$$

where $\langle \sigma_{zi} \rangle$ is the spin of the $i$th constituent, projected onto the nucleon's spin. (Note that the upper limit 3 of the sum in Eq. (4.199) is the number of baryon constituents, i.e., $N_{\mathrm{c}}$, *not* the number of flavors considered.) Eq. (4.199) is in qualitative agreement with my Eqs. (4.192) and (4.194), since the operator $\bar{q}\gamma^\mu\gamma_5 q$ tests each constituent of flavor $q$ for its spin.

As I pointed out at the beginning of this section, QCD does not predict a modification of the GDH sum rule due to quark anomalous magnetic moments, since current quarks are pointlike. On the other hand, constituent-quark models apparently predict a modification, as soon as constituent quarks are allowed to have $\kappa_q \neq 0$. What is the solution of this paradox?

In other words: Does Eq. (4.197) really tell us that the GDH experiment will measure the anomalous magnetic moments of constituent quarks? My answer is: definitely no! Rather, the modification constant in Eq. (4.197) should better be written on the other side of the equation, because it gives the "amount of principle incorrectness" of the integral due to neglecting all final states other than three-quark bound states. That is to say, while the QCD current operator may transform a one-nucleon state into any QCD state having the quantum numbers of $\gamma$N, the constituent-quark-model current only transforms the nucleon into another three-quark bound state subsequently decaying into $\pi$N or other hadronic states, i.e., a *resonance*. So, the full non-resonant background is missing from the integral of Eq. (4.197). I call this lack a "principle incorrectness" since I do *not* mean numerical uncertainties coming from such things as narrow-width approximation, simplicity of the nucleon wave function or whatsoever. My persuasion that the missing non-resonant part of the GDH integral equals just the modification constant on the left-hand side of Eq. (4.197) feeds on the following consideration: As mentioned before, anomalous magnetic moments of quarks signal some internal structure. For instance, if you recall the pion-cloud picture of the constituent quark, it is natural that the photon can strike out a pion from the quark. This, however, is a non-resonant process! In the non-relativistic limit, the equality of modification constant on the one hand and non-resonant part of the GDH integral on the other hand, both being generated by anomalous magnetic moments of quarks, was recently proved by Cardarelli, Pasquini, and Simula [104]. Note, however, that if one introduces into Eq. (4.196) the value $g_{\mathrm{A}} = 5/3$, which is standard in nonrelativistic constituent-quark models, then Eq. (12) of Ref. 104 is only half of my result (or the one of Refs. 10, 11). Honestly, I do not know whether this is due to an error or has a deeper meaning.

### Conclusion

I conclude that a modification of the GDH sum rule must not be expected from anomalous magnetic moments of quarks, neither in QCD (because current quarks do not possess anomalous magnetic moments) nor in constituent-quark models (because the apparent modification merely represents a portion missing from the modelled GDH integral). After all, this conclusion restores consistency between constituent-quark models and QCD.

## 4.7 "Localized spontaneous breakdown of electromagnetic gauge symmetry"

In this section, I comment on a proposed modification of the GDH sum rule stated by Ying [16, 105] in 1996. Ying claims that this modification is possible even if both the naive current commutation relation is valid and the infinite momentum limit is legitimate. It is due to "the presence of a localized region inside the nucleon, in which the electromagnetic gauge symmetry is spontaneously broken down" [16]. I mention in advance that I do not agree with this point of view.

### Erratum

To begin with, I want to point out two mistakes in the article under consideration. Firstly, it has to be noted that there are too many invariant amplitudes in Ying's decomposition of the forward virtual Compton amplitude, Ref. 16, Eq. (4). The correct amplitude without requiring gauge invariance reads

$$\begin{aligned} T^{\mu\nu}(p,q) = \bar{u}(p,s)\big[&F_1 g^{\mu\nu} - F_2 q^\mu q^\nu + F_3 p^\mu p^\nu - F_4(p^\mu q^\nu + p^\nu q^\mu) + iF_5\sigma^{\mu\nu} \\ &+ iF_6(p^\mu\sigma^{\nu\alpha}q_\alpha - p^\nu\sigma^{\mu\alpha}q_\alpha) + iF_7(q^\mu\sigma^{\nu\alpha}q_\alpha - q^\nu\sigma^{\mu\alpha}q_\alpha)\big]u(p,s), \end{aligned} \tag{4.200}$$

where I omitted a factor of $1/2M$ on account of differing normalizations of one-nucleon states. (The amplitude (4.200) is dimensionless, while Ying's conventions are such that $T^{\mu\nu}$ has dimension $\mathrm{GeV}^{-1}$.) Ying's eighth tensor $i\varepsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta \not{q}\gamma_5$ is in fact linearly dependent on the others, owing to the relation

$$\begin{aligned} \bar{u}(p,s)\big[&m(\nu^2-q^2)\sigma^{\mu\nu} + \nu(p^\mu\sigma^{\nu\alpha}q_\alpha - p^\nu\sigma^{\mu\alpha}q_\alpha) \\ &- m(q^\mu\sigma^{\nu\alpha}q_\alpha - q^\nu\sigma^{\mu\alpha}q_\alpha) + \varepsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta\not{q}\gamma_5\big]u(p,s) = 0, \end{aligned} \tag{4.201}$$

which is most easily verified by writing $\bar{u}\Gamma u = \mathrm{Tr}(\Gamma u\bar{u})$, where $\Gamma$ is any linear combination of the matrices 1, $\gamma^\mu$, $\sigma^{\mu\nu}$, $\gamma^\mu\gamma_5$, and $\gamma_5$, and by using the relation $u(p,s)\bar{u}(p,s) = \frac{1}{2}(\not{p} + M)(1+\gamma_5\not{s})$, trace theorems, and identity (1.13). Sometimes it is quite tricky to realize the linear dependence of tensors in the $\bar{u}\Gamma u$ notation, and eliminating the spinors in the manner just described often helps. Note that because gauge invariance imposes *three* constraints (see Eqs. (4.203) below), and since, of course, the usual number of *four* invariant amplitudes must remain in the ultimate (gauge invariant) forward virtual Compton amplitude, the total number of $3+4=7$ (not 8) non-gauge-invariant amplitudes must indeed be expected.

As a matter of fact, Ying's amplitude $F_8(\nu, q^2)$ can consistently be dropped throughout the entire paper without affecting the main message.

Secondly, as I demonstrated in Sect. 2.2.1, the naive equal-times commutator of electric current densities reads

$$\big[J^\mu(x), J^\nu(y)\big]_{\mathrm{et}} = 2i\,\varepsilon^{0\mu\nu\lambda}\,\bar{\psi}(x)\,\gamma_\lambda\gamma_5\big(Z^{(\mathrm{q})}\big)^2\psi(x)\,\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{4.202}$$

where $Z^{(\mathrm{q})}$ is the quark charge matrix. The right-hand side of Eq. (4.202) involves an axial-vector current, in contrast to Eq. (17) of Ref. 16, which involves a tensor current. In particular, since $\varepsilon^{00\nu\lambda} = 0$, the charge density $J^0(x)$ naively commutes with *each* component of the current. This is vital since naively, the charge density generates the local gauge transform, and the current is a gauge invariant quantity.

Replacing Ying's erroneous commutator by the correct one, Eq. (4.202), amounts to substituting the axial charge of the nucleon for its tensor charge in the entire paper. Again, this will not affect the message of the paper, since the actual values of these charges are never used.

### Imposing gauge invariance

The gauge invariance condition, $q_\mu T^{\mu\nu} = 0$, imposed on the invariant decomposition (4.200), gives rise to the constraints

$$F_1(\nu, q^2) = q^2 F_2(\nu, q^2) + M\nu F_4(\nu, q^2), \tag{4.203a}$$

$$M\nu F_3(\nu, q^2) = q^2 F_4(\nu, q^2), \tag{4.203b}$$

$$F_5(\nu, q^2) = M\nu F_6(\nu, q^2) + q^2 F_7(\nu, q^2), \tag{4.203c}$$

which are Eqs. (5)–(7) of Ying [16].

Since I am concerned with spin physics, I focus my attention on the antisymmetric part of amplitude (4.200), i.e., the part involving the invariant amplitudes $F_{5,6,7}$. Equating decompositions (2.31) and (4.200), one finds the following relation between the familiar spin-dependent forward Compton amplitudes[22] $A_{1,2}(\nu, q^2)$ and Ying's amplitudes $F_{6,7}(\nu, q^2)$,

$$A_1 = 2M^3 F_6, \tag{4.204a}$$

$$A_2 = 2M^3 F_7. \tag{4.204b}$$

The quantity $F_5$ ought to be regarded as a mere shorthand of the specific combination given by Eq. (4.203c). In view of Eq. (2.36), one has the simple relation $\nu f_2(\nu, q^2) = \alpha F_5(\nu, q^2)$.

### High-energy behavior from naive current commutator?

Ying employs the Bjorken-Johnson-Low (BJL) technique [17, 56] to derive the high-energy behavior[23]

$$F_5(\nu, q^2) \xrightarrow[\nu \to \infty]{} \frac{\text{const}}{\nu} \tag{4.205}$$

from the naive current commutator (4.202). It should be pointed out, however, that using the naive commutator in this context is unjustified, as Johnson and Low already demonstrated in their original paper [56] (see also [66]). In fact, the BJL technique may only be employed to derive a (non-naive) commutator from the high-energy behavior of an amplitude, i.e., just the other way round.

### BJL limit and Regge limit

Ying claims that one can proceed from the BJL limit $q^0 \to \infty$, $\boldsymbol{q}$ fixed, to the desired Regge limit $\nu \to \infty$, $q^2$ fixed, by using an infinite-momentum frame. However, as I demonstrated in Sects. 4.3 and 4.4, the legitimacy of the infinite-momentum technique is a highly fragile assumption. (See Sect. 4.4.4 for an explicit instance of non-coincidence of BJL limit and Regge limit.)

[22] Observe that these are not the functions $A_{1,2}(\nu, q^2)$ of Ying. The relation is $A_1 = 2M^2 A_1^{\text{Ying}}$, $A_2 = 2M^4 A_2^{\text{Ying}}$.

[23] As mentioned above, the constant in this relation is not a tensor charge, as in Eq. (20) of Ref. [16], but an axial charge, or more precisely: a definite linear combination of isovector axial charge and flavor-singlet and flavor-octet isoscalar axial charges.

#### Invalidation of the Burkhardt-Cottingham sum rule

One of the more serious problems with the results of Ref. 16 is that they obviously invalidate the Burkhardt-Cottingham sum rule [52]. This sum rule is a superconvergence relation for the amplitude $A_2(\nu, q^2)$, which requires the high-energy behavior

$$\lim_{\nu\to\infty} \nu A_2(\nu, q^2) = 0, \tag{4.206}$$

while Ying concludes $F_7 \propto \nu$, and hence $\nu A_2 \propto \nu^2$. Although it is indeed not entirely certain whether the Burkhardt-Cottingham integral converges at finite values of $Q^2 = -q^2$ (see Jaffe [106] for an excellent review), such a dramatic behavior vehemently contradicts widely accepted Regge asymptotics of amplitude $A_2$. According to Ioffe, Khoze, and Lipatov [107], the worst-case singularity in complex angular momentum plane that contributes to $A_2$ is a three- or more-pomeron cut, giving rise to the high-energy behavior $A_2 \propto 1/\ln^5 \nu$.

#### Discontinuous $q^2$ evolution

Ying's argumentation in favor of a modification of the GDH sum rule proceeds as follows. Define the function

$$\rho(q^2) := -\lim_{\nu\to\infty} \frac{q^2 F_7(\nu, q^2)}{\nu}. \tag{4.207}$$

Since amplitude $F_7(\nu, q^2)$ lacks massless poles, the numerator $q^2 F_7$ vanishes at $q^2 = 0$, and thus $\rho(0) = 0$. On the other hand, the limit

$$\rho_\infty := \lim_{\substack{q^2\to 0\\ q^2\neq 0}} \rho(q^2) \tag{4.208}$$

is claimed to be non-vanishing due to "spontaneous breakdown of electromagnetic gauge symmetry" [16]. Hence the function $q^2 F_7(\nu, q^2)$ is asymptotically linear in $\nu$ for any finite $q^2$. Moreover, as noted above, the sum $F_5 = M\nu F_6 + q^2 F_7$ is pretended to be asymptotically proportional to the inverse of $\nu$. Consequently, amplitude $F_6(\nu, q^2)$ must approach a constant that cancels the high-energy behavior of $F_7$,

$$F_6(\infty, q^2) = -\lim_{\nu\to\infty} \frac{q^2 F_7(\nu, q^2)}{M\nu} = \frac{\rho(q^2)}{M}, \tag{4.209}$$

or

$$A_1(\infty, q^2) = 2M^2 \rho(q^2), \tag{4.210}$$

which enforces a subtraction at infinity in the dispersion relation

$$\frac{1}{4}\big(\bar{A}_1(0, q^2) - A_1(\infty, q^2)\big) = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, G_1(\nu, q^2) =: I_1(q^2). \tag{4.211}$$

Here, $2\pi\, G_1 = \operatorname{Im} A_1$ denotes the familiar spin dependent nucleon structure function (the absorptive part of the forward virtual Compton amplitude), and

$$\bar{A}_1(\nu, q^2) := A_1(\nu, q^2) - \frac{4M^2 q^2\, F_1(q^2) G_{\mathrm{M}}(q^2)}{(2M\nu)^2 - (q^2)^2} \tag{4.212}$$

is the amplitude with its Born pole subtracted (cf. Eq. (2.35c)), obeying the low-energy theorem $\bar{A}_1(0,0) = -\kappa^2$, where $\kappa$ is the anomalous magnetic moment of the nucleon.

Now, if all this was true, then the integral (4.211) would be *discontinuous* as a function of $q^2$,

$$I_1(0) = -\frac{\kappa^2}{4}, \tag{4.213}$$

while

$$I_1(0\pm) := \lim_{\substack{q^2\to 0\\ q^2\neq 0}} I_1(q^2) = -\frac{\kappa^2}{4} - \frac{m^2\rho_\infty}{2}, \tag{4.214}$$

with $\rho_\infty^{\mathrm{p}} \approx -\rho_\infty^{\mathrm{n}} \approx 1\ \mathrm{GeV}^{-2}$ [16]. However, as long as there are no massless particles degenerate with the photon in all quantum numbers, the forward Compton amplitude – and with it the integral of its absorptive part – should be *analytic* in $q^2$ with all cuts and poles well separated from the point $q^2 = 0$. But even if it *was* discontinuous, then the genuine GDH integral, which simply has nothing to do with virtual photons, would certainly not acquire the value $I_1(0\pm)$, but rather $I_1(0)$, Eq. (4.213), which is left unmodified according to Ying. So what?

**Conclusion**

To conclude this section, I stress that the mechanism proposed in Refs. 16 and 105 – albeit possibly relevant with respect to other fields of elementary particle physics – certainly does *not* affect the GDH sum rule, mainly because the author misunderstood the BJL technique, which cannot be utilized to derive the high-energy behavior of an amplitude from a commutator, and secondly because he defined the GDH integral by an inappropriate $q^2$ limiting procedure.

## 4.8 Modified low-energy theorem?

A number of people (including myself) has been fooled by the fact that Low's derivation [36] of the low-energy theorem (LET) (2.9b) *looks like* it would be modified by a non-naive current commutator. In a few instances this delusion even came to publication [10, 108]. However, as already noted by Khare [11], one has to recall that the LET holds for the *physical* Compton amplitude, while the current commutator, in the low-energy limit, has impact on the *T-product* amplitude only. For any specific form of the commutator, the seagull, i.e., the difference of physical and T-product amplitudes, is uniquely determined by the requirement of gauge invariance of the physical amplitude [66]. Taking into account the additional contribution of the seagull to the LET, one soon realizes that it cancels the contribution of the commutator in any reasonable case, albeit in this way the derivation gets unnecessarily awkward. As a matter of fact, the easiest way to proceed is to forget about Low's derivation of the LET (just keeping in mind that it does by no means enforce the current commutator to have its canonical form), focussing instead on the derivation by means of Feynman diagrams given by Gell-Mann and Goldberger [37], or the one based on dispersion relations for helicity amplitudes, as presented 14 years later by Abarbanel and Goldberger [12]. After all, these approaches pleasantly remind one of the fact that LETs follow from symmetries of nature, which are much more fundamental than any kind of anomalous charge-density algebra.

# Chapter 5

# $Q^2$ evolution of the GDH sum rule

In this chapter, I investigate the $Q^2$ evolution of the GDH sum rule, i.e., the transition from the photoabsorption process depicted in Fig. 2.2 (p. 10) to polarized inclusive electron-nucleon scattering in the one-photon-exchange approximation as shown in Fig. 5.1. Experimentally, the charged lepton might as well be a positron or a muon, but this makes no difference here, so I would like to stick to the name "electron" for simplicity. Data on polarized inclusive electroproduction have been acquired at SLAC [19–21, 24] and at CERN [22, 23].

The inclusive electroproduction cross section is differential in the scattering (solid) angle and in the energy of the scattered electron. Owing to the one-photon-exchange approximation, it is possible to express this double differential cross section in terms of a ficticious absorption cross section of a polarized virtual photon on a polarized nucleon. This reduction not only simplifies notations, it also projects out the physically interesting polarization degrees of freedom, namely transverse and longitudinal polarization of the virtual photon. The "virtual photoabsorption" process is depicted in Fig. 5.2.

This chapter is organized as follows. In Sect. 5.1, kinematical variables are defined, and polarized inclusive electroproduction cross sections are presented and expressed in terms of nucleon structure functions $G_{1,2}(\nu, Q^2)$. Moreover, the relation to virtual photoabsorption cross sections is given. In Sect. 5.2, I present different generalizations of the GDH integral to non-zero photon virtuality $Q^2$, focussing particularly on the physical polarization degrees of freedom mentioned above. I argue that at intermediate $Q^2$, large deviations among different generalizations of the GDH integral have to be expected. Sect. 5.3 gives the contribution of

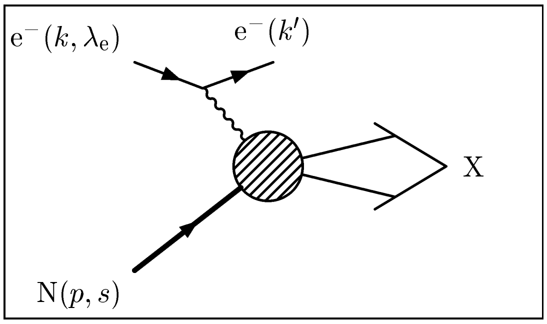



Figure 5.1: Polarized inclusive electron-nucleon scattering in the one-photon-exchange approximation. Cross sections for all final states X are summed over.

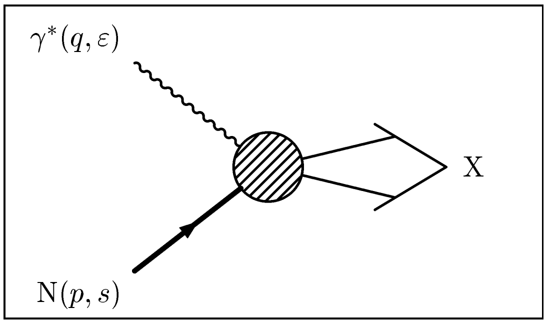



Figure 5.2: "Virtual photoabsorption" on the nucleon. Cross sections for all final states X are summed over. Conventions for the virtual-photon polarization vector $\varepsilon$ are specified in Sect. 5.1.2.

pion electroproduction to these integrals, as well as expansions of this contribution in terms of lowest multipoles. This last section aims at providing some important formulae necessary for performing the saturation of different $Q^2$ evolutions of the GDH integral. The saturation will become feasible as soon as experimental data on polarized pion electroproduction at low $Q^2$ allow for reliably extracting the multipoles. Pertinent experiments are under preparation at Jefferson Lab.

## 5.1 Kinematics and cross sections

The four-momenta of incident and scattered electron are denoted by $k$ and $k'$, respectively. The four-momentum of the virtual photon is $q = k - k'$. Being the difference of two timelike momenta, $q$ is always spacelike, i.e., $Q^2 := -q^2 > 0$. As usual, the electron's mass is neglected in explicit calculations. Lab-frame energies of incident and scattered electron are denoted by $E = p{\cdot}k/M$ and $E' = p{\cdot}k'/M$, respectively. The photon's energy $\nu$ and virtuality $-Q^2$ are related to $E$, $E'$, and the lab-frame scattering angle $\theta_\mathrm{e}$ via

$$\begin{aligned} \nu &= E - E', \\ Q^2 &= 2EE'(1-\cos\theta_\mathrm{e}) = 4EE'\sin^2\frac{\theta_\mathrm{e}}{2}. \end{aligned} \tag{5.1}$$

As indicated in Fig. 5.1, the incoming electron and nucleon are polarized, and all final-state polarizations are summed over. The incident electron's helicity is denoted by $\lambda_\mathrm{e} = \pm\frac{1}{2}$. I discriminate nucleon polarizations with respect to the direction given by the lab-frame three-momentum $\boldsymbol{q}$ of the virtual photon. Adopting the nomenclature of Raskin and Donnelly [109], I distinguish between *longitudinal* and *sideways* nucleon polarization. In the former case, the nucleon's spin $\boldsymbol{s} = \boldsymbol{q}/|\boldsymbol{q}|$, while in the latter case, $\boldsymbol{s}$ is perpendicular to $\boldsymbol{q}$ but coplanar with $\boldsymbol{k}$ and $\boldsymbol{k}'$, i.e., it lies within the scattering plane. Fig. 5.3 shall illustrate this state of affairs.

I now give an explicit coordinate decomposition of the four-vectors $k$, $k'$, $q$, $p$, and $s$. Since I work in the lab frame, the nucleon is at rest,

$$p^\mu = (M,0,0,0), \tag{5.2a}$$

while the photon is taken to be travelling along the positive $\boldsymbol{e}_3$ axis,

$$q^\mu = k^\mu - k'^\mu = (\nu,0,0,\sqrt{\nu^2+Q^2}). \tag{5.2b}$$

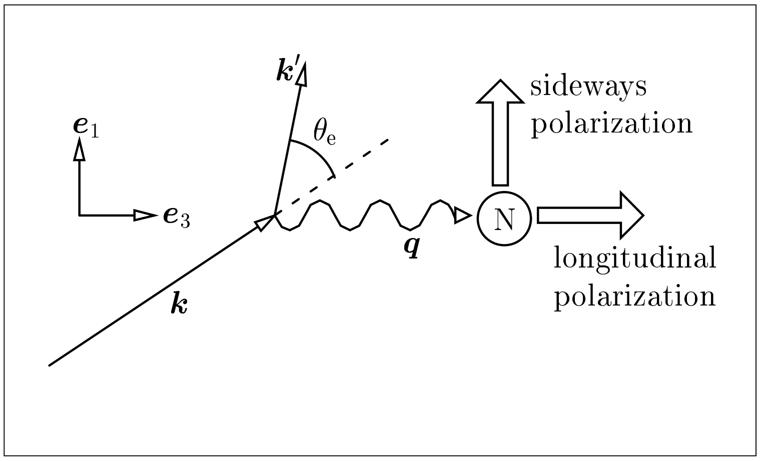



Figure 5.3: Kinematics of polarized inclusive electron-nucleon scattering in the lab frame. The incident electron is taken to have definite helicity, while the spin vector $\boldsymbol{s}$ of the struck nucleon equals $\boldsymbol{e}_1$ or $\boldsymbol{e}_3$ for respective sideways or longitudinal polarization. Final-state hadrons are not shown.

Longitudinal polarization means

$$s^\mu = (0, 0, 0, 1), \tag{5.2c}$$

while sideways polarization is given by

$$s^\mu = (0, 1, 0, 0). \tag{5.2d}$$

The electron momenta are a little more intricate to work out. They read

$$k^\mu = \left( E,\ \frac{EE' \sin\theta_\mathrm{e}}{\sqrt{\nu^2 + Q^2}},\ 0,\ \frac{E(E - E' \cos\theta_\mathrm{e})}{\sqrt{\nu^2 + Q^2}} \right) \tag{5.2e}$$

and

$$k'^\mu = \left( E',\ \frac{EE' \sin\theta_\mathrm{e}}{\sqrt{\nu^2 + Q^2}},\ 0,\ \frac{E'(E \cos\theta_\mathrm{e} - E')}{\sqrt{\nu^2 + Q^2}} \right). \tag{5.2f}$$

The reader may check that $q = k - k'$, $k^2 = k'^2 = 0$ (electron mass neglected!), and $\boldsymbol{k}\cdot\boldsymbol{k}' = |\boldsymbol{k}|\,|\boldsymbol{k}'| \cos\theta_\mathrm{e}$.

### 5.1.1 Polarized inclusive electroproduction

In inclusive electroproduction experiments, energy $E'$ and scattering angles $\theta_\mathrm{e}^\mathrm{lab}$, $\phi_\mathrm{e}^\mathrm{lab}$ of the outgoing electron are detected, i.e., one is interested in the double differential cross section $\mathrm{d}\sigma_{\lambda_\mathrm{e}}/\mathrm{d}E'\mathrm{d}\Omega_\mathrm{e}^\mathrm{lab}$. From now on, I will omit the superscript “lab”, since all quantities are taken to be in the lab frame. To project out polarization observables, cross sections for negative and postive electron helicities are subtracted,

$$\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}}. \tag{5.3}$$

### Leptonic and hadronic tensors

I intend to express these cross sections in terms of polarized nucleon structure functions $G_{1,2}(\nu, Q^2)$. To this end, I introduce the leptonic and hadronic tensors defined by

$$L_{\mu\nu}(\lambda_\mathrm{e}) = \sum_{\lambda'_\mathrm{e}=\pm\frac{1}{2}} \bar{u}(k,\lambda_\mathrm{e})\gamma_\mu u(k',\lambda'_\mathrm{e})\,\bar{u}(k',\lambda'_\mathrm{e})\gamma_\nu u(k,\lambda_\mathrm{e}) \tag{5.4}$$

and

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{4\pi}\sum_\mathrm{X}(2\pi)^4\delta(p+q-p_\mathrm{X})\,\langle p,s|J^\mu(0)|\mathrm{X}\rangle\,\langle\mathrm{X}|J^\nu(0)|p,s\rangle \\ &= \frac{1}{4\pi}\int \mathrm{d}^4x\, e^{iq\cdot x}\big\langle p,s\big|[J^\mu(x),J^\nu(0)]\big|p,s\big\rangle, \end{aligned} \tag{5.5}$$

where the latter equality is obtained by writing

$$(2\pi)^4\delta(q+p-p_\mathrm{X}) = \int \mathrm{d}^4x\, e^{i(q+p-p_\mathrm{X})\cdot x} \tag{5.6}$$

and utilizing translational invariance (1.7), as well as the completeness relation $\sum_\mathrm{X}|\mathrm{X}\rangle\langle\mathrm{X}| = 1$. The reader may recall that $2\pi W_{\mu\nu}$ is the absorptive part of the forward virtual Compton amplitude

$$T^{\mu\nu} = i\int \mathrm{d}^4x\, e^{iq\cdot x}\big\langle p,s\big|\mathrm{T}J^\mu(x)J^\nu(0)\big|p,s\big\rangle. \tag{5.7}$$

The antisymmetric parts of $W^{\mu\nu}$ and $T^{\mu\nu}$ have invariant decompositions

$$W_\mathrm{A}^{\mu\nu} = -\frac{i}{M}\,\varepsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma G_1(\nu,Q^2) - \frac{i}{M^3}\,\varepsilon^{\mu\nu\rho\sigma}q_\rho\big((M\nu)s_\sigma - (q\cdot s)p_\sigma\big)G_2(\nu,Q^2) \tag{5.8}$$

and

$$T_\mathrm{A}^{\mu\nu} = -\frac{i}{M}\,\varepsilon^{\mu\nu\rho\sigma}q_\rho s_\sigma A_1(\nu,Q^2) - \frac{i}{M^3}\,\varepsilon^{\mu\nu\rho\sigma}q_\rho\big((M\nu)s_\sigma - (q\cdot s)p_\sigma\big)A_2(\nu,Q^2), \tag{5.9}$$

respectively, where the forward amplitudes $A_{1,2}(\nu,Q^2)$ and structure functions $G_{1,2}(\nu,Q^2)$ are simply related by

$$\operatorname{Im} A_{1,2}(\nu,Q^2) = 2\pi G_{1,2}(\nu,Q^2). \tag{5.10}$$

Let us now have a closer look at the leptonic tensor (5.4). For an arbitrary spin four-vector $s_\mathrm{e}$ one has

$$u(k,s_\mathrm{e})\,\bar{u}(k,s_\mathrm{e}) = \frac{1+\gamma_5 \not{s}_\mathrm{e}}{2}(\not{k}+m_\mathrm{e}). \tag{5.11}$$

In the limit $m_\mathrm{e}\to 0$, individual components of $s_\mathrm{e}$ go to infinity in such a way that $m_\mathrm{e}s_\mathrm{e}$ stays finite. For definite helicity $\lambda_\mathrm{e}$ one has

$$m_\mathrm{e}s_\mathrm{e} \xrightarrow[m_\mathrm{e}\to 0]{} \pm k \quad \text{for} \quad \lambda_\mathrm{e} = \pm\frac{1}{2}, \tag{5.12}$$

so that above projection matrix reads

$$u(k,\pm\tfrac{1}{2})\,\bar{u}(k,\pm\tfrac{1}{2}) = \frac{1\pm\gamma_5}{2}\not{k}, \tag{5.13}$$

and I infer

$$L_{\mu\nu}(\pm\tfrac{1}{2}) = \mathrm{Tr}\left(\gamma_\mu \not{k}'\gamma_\nu\frac{1\pm\gamma_5}{2}\not{k}\right) = 2(k_\mu k'_\nu + k_\nu k'_\mu - \tfrac{1}{2}Q^2 g_{\mu\nu}) \pm 2i\,\varepsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta. \tag{5.14}$$

### Inclusive electroproduction cross sections

Returning now to the differential cross sections (5.3), I apply standard formulae of scattering theory to get [43]

$$\mathrm{d}\sigma_{\lambda_\mathrm{e}} = \frac{1}{4ME}\,\frac{\mathrm{d}^3k'}{(2\pi)^3\,2k'^0} \sum_{\lambda'_\mathrm{e}=\pm\frac{1}{2}} \sum_{\mathrm{X}} \left|T_{\mathrm{eN}\to\mathrm{eX}}(\lambda_\mathrm{e},\lambda'_\mathrm{e})\right|^2 (2\pi)^4\delta(q+p-p_\mathrm{X}), \tag{5.15}$$

where all hadronic final states X are summed over. The invariant scattering amplitude is given by

$$T_{\mathrm{eN}\to\mathrm{eX}} = \bar{u}(k',\lambda'_\mathrm{e})\gamma_\nu u(k,\lambda_\mathrm{e})\,\frac{e^2}{q^2}\,\big\langle\mathrm{X}\big|J^\nu(0)\big|p,s\big\rangle. \tag{5.16}$$

In terms of lab-frame energy and scattering angle, the Lorentz invariant momentum-space element reads

$$\frac{\mathrm{d}^3k'}{(2\pi)^3\,2k'^0} = \frac{E'}{16\pi^3}\,\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}. \tag{5.17}$$

Thus one has

$$\frac{\mathrm{d}\sigma_{\lambda_\mathrm{e}}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} = \frac{\alpha^2}{MQ^4}\,\frac{E'}{E}\,L_{\mu\nu}(\lambda_\mathrm{e})W^{\mu\nu}, \tag{5.18}$$

where definitions (5.4) and (5.5) have been employed. Subtracting electron helicities $\lambda_\mathrm{e} = \pm\frac{1}{2}$, I project out the antisymmetric part of the leptonic tensor (5.14), which in turn vanishes when contracted with the symmetric part of the hadronic tensor, so that

$$\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} = \frac{2\alpha^2}{MQ^4}\,\frac{E'}{E}\,L^\mathrm{A}_{\mu\nu}W^{\mu\nu}_\mathrm{A}, \tag{5.19}$$

where

$$L^\mathrm{A}_{\mu\nu} = \frac{1}{2}\big(L_{\mu\nu}(-\tfrac{1}{2}) - L_{\mu\nu}(+\tfrac{1}{2})\big) = -2i\,\varepsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta, \tag{5.20}$$

and $W^{\mu\nu}_\mathrm{A}$ is given in terms of structure functions $G_{1,2}(\nu,Q^2)$ by Eq. (5.8). The contraction is worked out by means of the relation

$$\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu\rho\sigma}k^\alpha q^\beta = -2k^\rho q^\sigma + 2k^\sigma q^\rho \tag{5.21}$$

and leads to

$$\begin{aligned}\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} = \frac{4\alpha^2}{M^3Q^2}\,\frac{E'}{E}\,\big[&-M\big((k+k')\cdot s\big)G_1(\nu,Q^2)\\ &+2\big(E'(k\cdot s) - E(k'\cdot s)\big)G_2(\nu,Q^2)\big].\end{aligned} \tag{5.22}$$

Inserting the explicit kinematics (5.2), one obtains

$$\boxed{\left[\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}}\right]_\mathrm{long} = \frac{4\alpha^2}{M^3Q^2}\,\frac{E'}{E}\,\frac{E+E'}{\sqrt{\nu^2+Q^2}}\big[M\nu G_1(\nu,Q^2) - Q^2G_2(\nu,Q^2)\big]} \tag{5.23a}$$

and

$$\boxed{\left[\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}}\right]_\mathrm{side} = \frac{8\alpha^2}{M^3Q^2}\,\frac{E'}{E}\,\frac{EE'\sin\theta_\mathrm{e}}{\sqrt{\nu^2+Q^2}}\big[MG_1(\nu,Q^2) + \nu G_2(\nu,Q^2)\big]} \tag{5.23b}$$

### 5.1.2 Virtual photoabsorption

One commonly expresses inclusive electroproduction cross sections in terms of "virtual photoabsorption cross sections" defined by

$$\sigma(\nu, Q^2) = \frac{e^2}{4MK}\,\varepsilon'^*_\mu \varepsilon_\nu \sum_{\mathrm{X}} (2\pi)^4 \delta(p+q-p_{\mathrm{X}})\,\langle p,s|J^\mu(0)|\mathrm{X}\rangle\,\langle \mathrm{X}|J^\nu(0)|p,s\rangle, \tag{5.24}$$

with appropriate choices for the "virtual-photon polarization vectors" $\varepsilon$ and $\varepsilon'$. This definition generalizes, in a minimal way, the photoabsorption cross section (2.14) to non-zero $Q^2$. Since the flux of a virtual particle is not a firmly defined quantity, there is some freedom in choosing the factor $K$ appearing in Eq. (5.24). The convention adopted here,

$$K = \frac{1}{M}\sqrt{(p\cdot q)^2 - p^2 q^2} = \sqrt{\nu^2 + Q^2}, \tag{5.25}$$

follows Ioffe, Khoze, and Lipatov [107]. Different choices of the virtual-photon flux can be found in the literature, all of which reduce to the (well-defined) photoabsorption cross section at photon virtuality $Q^2 = 0$. Using Eq. (5.5), one has

$$\sigma(\nu, Q^2) = \frac{4\pi^2\alpha}{M\sqrt{\nu^2+Q^2}}\,\varepsilon'^*_\mu \varepsilon_\nu W^{\mu\nu}. \tag{5.26}$$

#### Virtual-photon polarization vector

Taking the photon to be propagating in the direction of the positive $\boldsymbol{e}_3$ axis, diverse polarizations are represented as follows. Transverse polarizations are given in the same manner as for real photons, namely

$$\varepsilon^\mu(+1) = \frac{1}{\sqrt{2}}(0,-1,-i,0) \tag{5.27a}$$

for left-circular polarization (helicity +1), and

$$\varepsilon^\mu(-1) = \frac{1}{\sqrt{2}}(0,1,-i,0) \tag{5.27b}$$

for right-circular polarization (helicity −1). Linear polarization is specified by

$$\varepsilon^\mu(\perp) = \frac{i}{\sqrt{2}}\left[\varepsilon^\mu(-1) + \varepsilon^\mu(+1)\right] = (0,0,1,0) \tag{5.27c}$$

(normal to the scattering plane) and

$$\varepsilon^\mu(\|) = \frac{1}{\sqrt{2}}\left[\varepsilon^\mu(-1) - \varepsilon^\mu(+1)\right] = (0,1,0,0) \tag{5.27d}$$

(tangential). Longitudinal[1] polarization is represented by

$$\varepsilon^\mu(0) = \left(\sqrt{\frac{\nu^2}{Q^2}+1},\,0,\,0,\,\frac{\nu}{Q}\right). \tag{5.27e}$$

[1]The reader should not be confused by the different meanings of "longitudinal polarization" for nucleon and photon. A longitudinally polarized nucleon has its *spin* parallel to the momentum, whereas in the case of a longitudinal photon, *electric field* and momentum are parallel, as can be inferred from Eqs. (5.2b) and (5.27e) by considering the electromagnetic field tensor $F^{\mu\nu} = \varepsilon^\mu q^\nu - \varepsilon^\nu q^\mu$. Therefore, if a nucleon and a photon are in helicity eigenstates with both helicities non-zero, then the nucleon is called longitudinal while the photon is called transverse.

Note that the normalization $\varepsilon(0)^2 = 1$ is a matter of convention.

### Virtual-photoabsorption cross sections

I consider two scenarios in more detail. Firstly, I take the nucleon to have longitudinal polarization, Eq. (5.2c), while the photon is assumed to be polarized transversely, $\varepsilon = \varepsilon' = \varepsilon(\pm 1)$. I subtract left-circular from right-circular polarization and define

$$\sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2) = \frac{4\pi^2\alpha}{M\sqrt{\nu^2+Q^2}} \left[\varepsilon^*_\mu(-1)\varepsilon_\nu(-1) - \varepsilon^*_\mu(+1)\varepsilon_\nu(+1)\right] W^{\mu\nu}_{\text{long}} \tag{5.28a}$$

Secondly, the nucleon is taken to have sideways polarization, and the *interference* between longitudinal and transverse (linear) photon polarization is considered,

$$\sigma_{\text{LT}}(\nu, Q^2) = \frac{4\pi^2\alpha}{M\sqrt{\nu^2+Q^2}} \operatorname{Im}\left(\varepsilon^*_\mu(0)\,\varepsilon_\nu(\perp)\,W^{\mu\nu}_{\text{side}}\right). \tag{5.28b}$$

Using decomposition (5.8) together with the explicit kinematics (5.2) yields

$$\boxed{\sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2) = \frac{8\pi^2\alpha}{M^2}\frac{\nu}{\sqrt{\nu^2+Q^2}}\left[G_1(\nu, Q^2) - \frac{Q^2}{M\nu}G_2(\nu, Q^2)\right]} \tag{5.29a}$$

$$\boxed{\sigma_{\text{LT}}(\nu, Q^2) = \frac{4\pi^2\alpha}{M^2}\frac{Q}{\sqrt{\nu^2+Q^2}}\left[G_1(\nu, Q^2) + \frac{\nu}{M}G_2(\nu, Q^2)\right]} \tag{5.29b}$$

### Relation to electroproduction cross sections

By means of Eqs. (5.29), I can cast the inclusive electroproduction cross sections (5.23) into the form

$$\boxed{\left[\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_{\text{e}}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_{\text{e}}}\right]_{\text{long}} = \Gamma\sqrt{1-\varepsilon^2}\left(\sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2)\right)} \tag{5.30a}$$

$$\boxed{\left[\frac{\mathrm{d}\sigma_{-1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_{\text{e}}} - \frac{\mathrm{d}\sigma_{+1/2}}{\mathrm{d}E'\,\mathrm{d}\Omega_{\text{e}}}\right]_{\text{side}} = 2\Gamma\sqrt{2\varepsilon(1-\varepsilon)}\,\sigma_{\text{LT}}(\nu, Q^2)} \tag{5.30b}$$

where

$$\varepsilon = \left[1 + 2\left(1 + \frac{\nu^2}{Q^2}\right)\tan^2\frac{\theta_{\text{e}}}{2}\right]^{-1} \tag{5.31}$$

is the "degree of polarization of the virtual photon", and

$$\Gamma = \frac{\alpha}{2\pi^2}\frac{E'}{E}\frac{\sqrt{\nu^2+Q^2}}{Q^2}\frac{1}{1-\varepsilon} \tag{5.32}$$

is the "virtual-photon flux per electron" (nomenclature of Raskin and Donnelly [109]). The reader may verify the relations

$$\frac{1+\varepsilon}{1-\varepsilon} = \frac{(E+E')^2}{\nu^2+Q^2} \tag{5.33}$$

and

$$\frac{2\varepsilon}{1-\varepsilon} = \frac{4EE' - Q^2}{\nu^2 + Q^2}. \tag{5.34}$$

**The "most physical" linear combinations of structure functions**

As mentioned above, Eqs. (5.30) reduce the differential electroproduction cross sections to "virtual photoabsorption cross sections" depending solely on lab-frame energy $\nu$ and virtuality $-Q^2$ of the photon. The indices 1/2 and 3/2 appearing on the left-hand side of Eq. (5.28a) denote – as in the case of real photoabsorption considered so far – antiparallel and parallel photon-nucleon helicity, respectively. However, the virtual photon can also be polarized longitudinally, i.e., have zero helicity. And since the cross section is the absolute square of the amplitude depicted by Fig. 5.1, which is a sum over all three photon helicities, there is also an interference between longitudinal and transverse polarization, showing up in Eq. (5.28b). The particular combinations of structure functions $G_{1,2}(\nu, Q^2)$ occurring in Eqs. (5.29) are the "most physical" ones owing to their direct relation to the polarization of the exchanged photon. At $Q^2 = 0$, the combination (5.29a) reduces to the familiar combination of real-photoabsorption cross sections

$$\sigma_{1/2}(\nu, 0) - \sigma_{3/2}(\nu, 0) = \frac{8\pi^2\alpha}{M^2} G_1(\nu, 0) = \sigma_{1/2}(\nu) - \sigma_{3/2}(\nu), \tag{5.35}$$

whereas, of course, the LT-interference cross section (5.29b) vanishes for real photons,

$$\sigma_{\mathrm{LT}}(\nu, 0) = 0. \tag{5.36}$$

## 5.2 Generalizations of the GDH integral

This section is devoted to different generalizations of the GDH integral to non-zero photon virtuality. To contact to deep-inelastic-scattering (DIS) sum rules like the Bjorken sum rule [17], I additionally give all integrals in terms of the functions

$$\boxed{g_1(x, Q^2) = \frac{\nu}{M} G_1(\nu, Q^2) \quad \text{and} \quad g_2(x, Q^2) = \frac{\nu^2}{M^2} G_2(\nu, Q^2)} \tag{5.37}$$

where $x = Q^2/2M\nu$ is the Bjorken scaling variable. These structure functions have the simple scaling behavior (modulo logarithms)

$$g_i(x, Q^2) \xrightarrow[Q^2\to\infty]{} g_i(x), \quad i = 1, 2. \tag{5.38}$$

Adopting the notation of Soffer and Teryaev [110, 111] and translating the photon-energy threshold

$$\nu_0 = m_\pi + \frac{m_\pi^2 + Q^2}{2M} \tag{5.39}$$

into a Bjorken-$x$ threshold[2]

$$x_0 = \frac{Q^2}{2M\nu_0} = \frac{Q^2}{2Mm_\pi + m_\pi^2 + Q^2}, \tag{5.40}$$

I define

$$I_1(Q^2) := \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, G_1(\nu, Q^2) = \frac{2M^2}{Q^2} \int_0^{x_0} \mathrm{d}x\, g_1(x, Q^2), \tag{5.41a}$$

$$I_2(Q^2) := \frac{1}{M} \int_{\nu_0}^{\infty} \mathrm{d}\nu\, G_2(\nu, Q^2) = \frac{2M^2}{Q^2} \int_0^{x_0} \mathrm{d}x\, g_2(x, Q^2), \tag{5.41b}$$

and

$$\begin{aligned} I_{1+2}(Q^2) := I_1(Q^2) + I_2(Q^2) &= \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \left( G_1(\nu, Q^2) + \frac{\nu}{M} G_2(\nu, Q^2) \right) \\ &= \frac{2M^2}{Q^2} \int_0^{x_0} \mathrm{d}x \left( g_1(x, Q^2) + g_2(x, Q^2) \right). \end{aligned} \tag{5.41c}$$

Up to a factor of $2M^2/Q^2$, the integrals $I_{1,2}(Q^2)$ are the zeroth moments[3] of structure functions $g_{1,2}(x, Q^2)$. Integral $I_{1+2}(Q^2)$ has a more physical meaning, since it involves the linear combination $G_1 + \frac{\nu}{M} G_2$ of structure functions, which is related to the LT-interference cross section defined above. In fact, on account of Eq. (5.29b) one may write

$$I_{1+2}(Q^2) = \frac{M^2}{4\pi^2\alpha} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \frac{\sqrt{\nu^2 + Q^2}}{Q}\, \sigma_{\mathrm{LT}}(\nu, Q^2). \tag{5.42}$$

Analogously, I define in view of Eq. (5.29a)

$$\begin{aligned} \tilde{I}_{\mathrm{GDH}}(Q^2) &:= \frac{M^2}{8\pi^2\alpha} \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \frac{\sqrt{\nu^2 + Q^2}}{\nu} \left( \sigma_{1/2}(\nu, Q^2) - \sigma_{3/2}(\nu, Q^2) \right) \\ &= \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu} \left( G_1(\nu, Q^2) - \frac{Q^2}{M\nu} G_2(\nu, Q^2) \right) \\ &= \frac{2M^2}{Q^2} \int_0^{x_0} \mathrm{d}x \left( g_1(x, Q^2) - \frac{4M^2 x^2}{Q^2} g_2(x, Q^2) \right). \end{aligned} \tag{5.43a}$$

[2]If $Q^2$ exceeds a few $\mathrm{GeV}^2$, then $x_0$ is very near to unity. Therefore, the deviation from unity is usually ignored in deep-inelastic scattering, and Bjorken-$x$ runs from zero to *one*. This is why the upper bound of the integrals (5.41) may look a little unfamiliar at first sight. Above $x_0$, structure functions $g_{1,2}(x, Q^2)$ vanish except for a delta function at $x = 1$ coming from the elastic contribution to inclusive electroproduction, i.e., a single nucleon as hadronic final state X in Fig. 5.1. The elastic contribution is insignificant at high $Q^2$. For a discussion, see Ji [112].

[3]The moments $\int \mathrm{d}x\, g_{1,2}(x, Q^2)$ are commonly denoted as $\Gamma_{1,2}(Q^2)$.

and

$$\boxed{\begin{aligned} I_{\mathrm{GDH}}(Q^2) &:= \frac{M^2}{8\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\,\big(\sigma_{1/2}(\nu,Q^2)-\sigma_{3/2}(\nu,Q^2)\big) \\ &= \int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\sqrt{\nu^2+Q^2}}\left(G_1(\nu,Q^2)-\frac{Q^2}{M\nu}G_2(\nu,Q^2)\right) \\ &= \frac{2M^2}{Q^2}\int_0^{x_0}\frac{\mathrm{d}x}{\sqrt{1+\frac{4M^2x^2}{Q^2}}}\left(g_1(x,Q^2)-\frac{4M^2x^2}{Q^2}g_2(x,Q^2)\right)\end{aligned}} \tag{5.43b}$$

where the last definition is most directly related to the (transverse) virtual photoabsorption cross sections.

### GDH, Bjorken, and Burkhardt-Cottingham sum rules

In terms of the integrals (5.41), the GDH sum rule reads (provided it holds, of course)

$$I_1(0) = \int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\,G_1(\nu,0) = \frac{M^2}{8\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu}\,\big(\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\big) = -\frac{\kappa^2}{4}, \tag{5.44}$$

while the Bjorken sum rule [17] (see also Ref. 113) is given by

$$I_1^{\mathrm{p}}(Q^2) - I_1^{\mathrm{n}}(Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2M^2}{Q^2}\int_0^1 \mathrm{d}x\,\big(g_1^{\mathrm{p}}(x)-g_1^{\mathrm{n}}(x)\big) = \frac{2M^2}{Q^2}\,\frac{g_{\mathrm{A}}}{6}, \tag{5.45}$$

where $g_{\mathrm{A}} = 1.26$ is the phenomenological neutron beta-decay constant. Eq. (5.45) is well established experimentally [21].

In contrast to GDH and Bjorken sum rules, the Burkhardt-Cottingham sum rule [52] (see also Jaffe [106]) applies to all values of $Q^2$:

$$I_2(Q^2) = \frac{1}{4}\,F_2(Q^2)\big(F_1(Q^2)+F_2(Q^2)\big) = \frac{1}{4}\,\frac{G_{\mathrm{M}}(Q^2)-G_{\mathrm{E}}(Q^2)}{1+Q^2/4M^2}\,G_{\mathrm{M}}(Q^2). \tag{5.46}$$

Dirac and Pauli form factors $F_{1,2}(Q^2)$ are normalized as $F_1(0) = Z$ and $F_2(0) = \kappa$, respectively. Sachs form factors $G_{\mathrm{E,M}}(Q^2)$ are given by[4]

$$G_{\mathrm{E}}(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2}\,F_2(Q^2), \tag{5.47a}$$

$$G_{\mathrm{M}}(Q^2) = F_1(Q^2) + F_2(Q^2). \tag{5.47b}$$

At $Q^2 = 0$, Eq. (5.46) reduces to $I_2(0) = \frac{1}{4}\kappa(Z+\kappa)$. Hence,

$$I_{1+2}(0) = \frac{Z\kappa}{4}. \tag{5.48}$$

[4] Soffer and Teryaev [110, 111] apparently use a somewhat peculiar definition of Sachs form factors, since they do not include the factor $(1+Q^2/4M^2)^{-1}$ in Eq. (5.46).

### Bjorken sum rule and $J = 1$ fixed pole

At this point, I would like to comment on the question whether a $Q^2 \neq 0$ generalization of the $J = 1$ fixed pole in angular momentum plane, which I discussed in Sect. 4.1 of this thesis, would spoil the Bjorken sum rule. I mention in advance that *it woudn't*, although at first sight, this might be expected.

Consider the forward virtual Compton amplitude $A_1(\nu, Q^2)$ as defined in Eq. (5.9). Its Born contribution is given by Eq. (2.35c). In order to formulate a fixed-$Q^2$ dispersion relation, the Born-*pole* part must be subtracted:

$$\bar{A}_1(\nu, Q^2) := A_1(\nu, Q^2) - \frac{4M^2 Q^2\, F_1(Q^2) G_{\mathrm{M}}(Q^2)}{Q^4 - 4M^2\nu^2}. \tag{5.49}$$

Recall that at $Q^2 = 0$, one has

$$f_2(\nu) = \frac{\alpha}{2M^2}\, A_1(\nu, 0) = \frac{\alpha}{2M^2}\, \bar{A}_1(\nu, 0). \tag{5.50}$$

A $J = 1$ fixed pole would manifest itself in a non-vanishing real part of amplitude $A_1(\nu, Q^2)$ at $\nu \to \infty$, i.e.,

$$A_1(\infty, Q^2) = \bar{A}_1(\infty, Q^2) \neq 0. \tag{5.51}$$

(Except for continuity, I do not imply any assumption on the $Q^2$ dependence of $A_1(\infty, Q^2)$.) In this case, the associated dispersion relation would require a subtraction at infinity. Thus,

$$I_1(Q^2) = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, G_1(\nu, Q^2) = \frac{1}{4}\big(\bar{A}_1(0, Q^2) - A_1(\infty, Q^2)\big), \tag{5.52}$$

since $\operatorname{Im} A_1 = 2\pi G_1$.

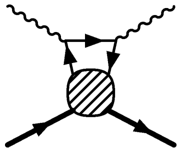
In view of Eq. (5.52), one might be tempted to think that the Bjorken sum rule would receive a modification if $A_1(\infty, Q^2)$ were non-zero. However, while this is true for the GDH sum rule (cf. Eq. (4.5)),

$$I_1(0) = \frac{1}{4}\big(-\kappa^2 - A_1(\infty, Q^2)\big), \tag{5.53}$$

the Bjorken sum rule is left unchanged,

$$I_1^{\mathrm{p}}(Q^2) - I_1^{\mathrm{n}}(Q^2) \xrightarrow[Q^2 \to \infty]{} \frac{2M^2}{Q^2}\, \frac{g_{\mathrm{A}}}{6}, \tag{5.54}$$

due to the following reason. The dispersion-theoretic derivation of the GDH sum rule (see Sect. 2.1) rests upon the Low-Gell-Mann-Goldberger low-energy theorem for the polarized forward Compton amplitude, which presents a statement on the *right-hand side* of the dispersion relation (5.52). The dispersion relation itself enters additionally. In contrast, modern derivation of DIS sum rules (as well as their radiative or higher-twist corrections) by means of operator-product expansion (see, e.g., Cheng and Li [114] or Roberts [115]) yields a direct statement on the integral on the *left-hand side* of the dispersion relation. In terms of the parton model, which represents the lowest order of the operator-product expansion, depicted by the handbag[5] diagram



(5.55)

one may put it this way: The handbag diagram yields the value (5.54) for the proton-neutron difference of the integrals (5.52). It does not tell how this number is distributed among $A_1(0, Q^2)$ and $A_1(\infty, Q^2)$ on the right-hand side of Eq. (5.52).

It should be noted that Bjorken's original derivation [17, 113] of the sum rule is not exactly convenient for the discussion of this point, because it is founded on the validity of the naive current commutator (2.48) *and* on an unsubtracted dispersion relation. As a matter of fact, neither of these assumptions is vital for the validity of the Bjorken sum rule. Moreover, it can be proved that they are highly correlated, in such a way that the modification introduced by a non-naive commutator is exactly cancelled[6] by virtue of its accompanying seagull amplitude. This very simple result of a quite intricate calculation signalizes that the current-algebra derivation of the Bjorken sum rule gets unnecessarily awkward if a non-naive commutator is allowed for.

### 5.2.1 The significance of $G_2(\nu, Q^2)$

Owing to the explicit factor of $Q^2$ preceding structure function $G_2(\nu, Q^2)$ in Eqs. (5.43),[7] all three integrals $I_1(Q^2)$, $I_{\mathrm{GDH}}(Q^2)$, and $\tilde{I}_{\mathrm{GDH}}(Q^2)$ exhibit the same real-photon limit, namely

$$I_1(0) = I_{\mathrm{GDH}}(0) = \tilde{I}_{\mathrm{GDH}}(0) = \int_{\nu_0}^{\infty} \frac{\mathrm{d}\nu}{\nu}\, G_1(\nu, 0). \tag{5.56}$$

Notably, the three integrals also have the same scaling limit, on account of the explicit factor of $1/Q^2$ accompanying function $g_2(x, Q^2)$ in Eqs. (5.43):

$$\lim_{Q^2\to\infty} I_1(Q^2) = \lim_{Q^2\to\infty} I_{\mathrm{GDH}}(Q^2) = \lim_{Q^2\to\infty} \tilde{I}_{\mathrm{GDH}}(Q^2) = \frac{2M^2}{Q^2} \int_0^1 \mathrm{d}x\, g_1(x). \tag{5.57}$$

[5] See footnote 14 on p. 72 of this thesis.

[6] I considered the general non-naive charge-density commutator (4.86), straightforwardly generalized [66, Eq. (4.38)] to all Lorentz components, because the full current-current commutator is needed. Then, in addition to the expression found by Bjorken [17], the non-naive part of the commutator contributes a certain *constant* to the $Q^2 \to \infty$ limit of Compton amplitude $A(0, Q^2)$ occurring in Eq. (5.52). On the other hand, it implies a seagull [66], which is uniquely determined by the requirement of Lorentz invariance and gauge invariance of the amplitude. This seagull is known to dominate the Compton amplitude in the BJL limit (see Sect. 4.2.4). For large $Q^2$, it also dominates in the Regge limit (see discussion on p. 83 of this thesis), so that it necessitates a subtraction at infinity in Eq. (5.52). The subtraction constant $A_1(\infty, Q^2)$ turns out to exactly cancel the correction due to the non-naive commutator, leading back to the naive result (5.54).

[7] Don't be puzzled by the explicit factor of $1/Q^2$ preceding structure function $g_2(x, Q^2)$ in Eqs. (5.43). The small-$g$ structure functions are meaningless at $Q^2 = 0$. Observe that not even Bjorken-$x$ itself is defined for real photons. For a nice re-definition of Bjorken-$x$ that is applicable to both the scaling limit and the real-photon point, see Li and Li [116].

So, the integrals $I_1(Q^2)$, $I_{\rm GDH}(Q^2)$, and $\tilde{I}_{\rm GDH}(Q^2)$ coincide at both ends of the $Q^2$ range under consideration. Nevertheless, I expect them to differ conspicuously at intermediate $Q^2$. In my opinion, this fact is not appropriately dealt with in the literature.

**Beyond leading terms**

From definitions (5.43), the generalized GDH integrals $I_{\rm GDH}(Q^2)$ and $\tilde{I}_{\rm GDH}(Q^2)$ can be expanded about $Q^2=0$ and $Q^2\to\infty$, giving

$$I_{\rm GDH}(Q^2)-\tilde{I}_{\rm GDH}(Q^2)\xrightarrow[Q^2\to 0]{}-\frac{Q^2}{2}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu^3}\,G_1(\nu,0), \tag{5.58a}$$

$$\tilde{I}_{\rm GDH}(Q^2)-I_1(Q^2)\xrightarrow[Q^2\to 0]{}-\frac{Q^2}{M}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu^2}\,G_2(\nu,0), \tag{5.58b}$$

and

$$I_{\rm GDH}(Q^2)\xrightarrow[Q^2\to\infty]{}\frac{2M^2}{Q^2}\left(\int_0^1\mathrm{d}x\,g_1(x,Q^2)-\frac{2M^2}{Q^2}\int_0^1\mathrm{d}x\,x^2\big(g_1(x)+2g_2(x)\big)\right), \tag{5.59a}$$

$$\tilde{I}_{\rm GDH}(Q^2)\xrightarrow[Q^2\to\infty]{}\frac{2M^2}{Q^2}\left(\int_0^1\mathrm{d}x\,g_1(x,Q^2)-\frac{4M^2}{Q^2}\int_0^1\mathrm{d}x\,x^2 g_2(x)\right), \tag{5.59b}$$

respectively. Note that the high-$Q^2$ suppression of the $g_2$ contribution to integrals Eq. (5.59) traces back solely to the scaling property (5.38) and the existence of the second moment $\int\mathrm{d}x\,x^2g_2(x)$. This should not be mistaken with the Burkhardt-Cottingham sum rule, which constrains only the *zeroth* moment of $g_2(x)$ and does not have any effect on the other moments. In Eqs. (5.59), the $Q^2$ dependence of structure function $g_1(x,Q^2)$ has been kept for consistency, since the $1/Q^2$ correction to the zeroth moment of $g_1(x,Q^2)$ is of the same order than the terms involving second moments.

**Historical survey**

The literature provides a number of attempts to connect the limits $Q^2\to 0$ and $Q^2\to\infty$, i.e., Eqs. (5.56) and Eq. (5.57). The former limit is related to the GDH sum rule as reflected by Eq. (5.44), while the latter is governed by the Bjorken [17] and Ellis-Jaffe [18] sum rules and has been measured at SLAC [19–21] and in the EMC [22] and SMC [23] experiments at CERN. Intermediate momentum transfer ($Q^2 = 0.5$ GeV$^2$ and 1.2 GeV$^2$) was recently explored in the SLAC E143 experiment [24] (see Fig. 5.4).

Anselmino, Ioffe, and Leader [117] (see also Ref. 118) proposed a first-guess parametrization of integral $I_1(Q^2)$ inspired by vector-meson dominance, namely a sum of a single and a double pole at the $\rho$-meson mass $Q^2=-m_\rho^2$. The residues of these poles are fitted to the GDH value at $Q^2=0$ and the EMC data at high $Q^2$. Burkert and Ioffe [119–121] refined this model by adding an estimate of the resonance contribution[8] at very low $Q^2$.

[8]Note that there is an error [122] in the resonance contribution depicted in Fig. 1 of Ref. 120. Ioffe [121] used the correct resonance contribution and predicted a sign change of $I_1(Q^2)$ at $Q^2\approx 0.2\ldots0.3$ GeV$^2$, whereas Ref. 120 predicted $Q^2\approx 0.6$ GeV$^2$.

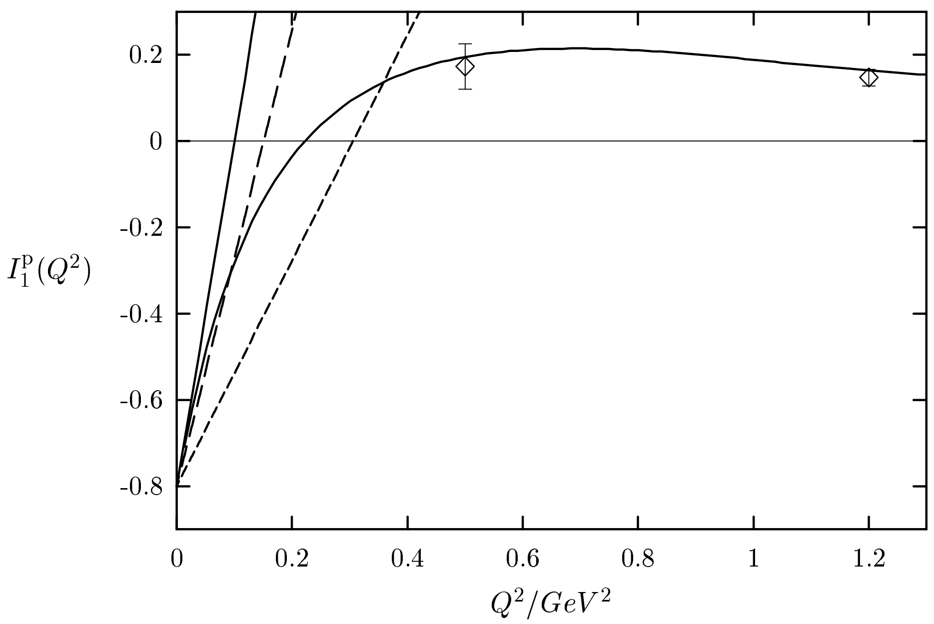



Figure 5.4: The integral $I_1(Q^2)$, Eq. (5.41a), for the proton. Solid lines represent the form given by Soffer and Teryaev [110, 111], as well as its slope at $Q^2 = 0$. Long-dashed and short-dashed lines represent estimates of the slopes of integrals $I^{\mathrm{p}}_{\mathrm{GDH}}(Q^2)$ and $\tilde{I}^{\mathrm{p}}_{\mathrm{GDH}}(Q^2)$, respectively. Data are from SLAC experiment E143 [24]. Details are given in the text.

Within the framework of heavy-baryon chiral perturbation theory, Bernard, Kaiser, and Meißner [123] investigated yet another generalized GDH integral, distinguishing from my definition (5.43b) by the use of a different photon flux.

Li [103] employed a constituent-quark model including relativistic corrections.

Soffer and Teryaev [110, 111] suggested a parametrization of the proton integral $I_1^{\mathrm{p}}(Q^2)$ founded on the requirement that function $I^{\mathrm{p}}_{1+2}(Q^2)$, Eq. (5.41c), should be as smooth and simple as possible. This assumption is motivated[9] from QCD sum rules. Integral $I^{\mathrm{p}}_{1+2}(Q^2)$ is parametrized such that it reproduces the GDH/Burkhardt-Cottingham value (5.48) at $Q^2 = 0$ as well as the asymptotic EMC result. Next, the Burkhardt-Cottingham integral $I_2^{\mathrm{p}}(Q^2)$ is subtracted (employing a reasonable parametrization of the form factors) to gain the integral $I_1^{\mathrm{p}}(Q^2)$. The result is depicted in Fig. 5.4. Note that nothing can be inferred on $I_{\mathrm{GDH}}(Q^2)$ or $\tilde{I}_{\mathrm{GDH}}(Q^2)$ by means of the method of Soffer and Teryaev. Moreover, it should be noted that the main purpose of Refs. 110 and 111 was the *qualitative* statement that the drastic low-$Q^2$ behavior of the generalized GDH integral $I_1(Q^2)$ can be understood in terms of the strong $Q^2$ dependence of the form factors occurring in the Burkhardt-Cottingham sum rule (5.46).

[9] To me, the smoothness assumption of Soffer and Teryaev is quite plausible also with regard to resonance saturation, because the strong $Q^2$ dependence of the GDH integral can be attributed mainly to the $\Delta$ resonance, which is known to die out quickly with increasing $Q^2$ [124]. The $\Delta$ resonance is dominantly an $M_{1+}$ transition, i.e., it "favours" transverse photons. On the other hand, the linear combination of structure functions that occurs in $I_{1+2}(Q^2)$ represents an interference between longitudinal and transverse virtual photons, as reflected by Eq. (5.29b). Thus, the strongly $Q^2$ dependent $\Delta$ resonance will only weakly affect $I_{1+2}(Q^2)$.

### Low $Q^2$

I adopt the parametrization of Soffer and Teryaev [110] for the proton[10] to illustrate some ideas concerning different $Q^2$ evolutions of the GDH integral. Considering the low-$Q^2$ expansion (5.58a) together with identity (5.35), one has that[11]

$$I_{\mathrm{GDH}}(Q^2) - \tilde{I}_{\mathrm{GDH}}(Q^2) \xrightarrow[Q^2\to 0]{} -\frac{M^2Q^2}{16\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu^3}\left(\sigma_{1/2}(\nu)-\sigma_{3/2}(\nu)\right) = -\frac{M^2Q^2}{4\alpha}\,\gamma, \tag{5.60}$$

where $\gamma$, referred to as the spin polarizability [126], is defined as the second coefficient in the Taylor series of forward Compton amplitude $f_2(\nu)$,

$$f_2(\nu) = -\frac{\alpha\kappa^2}{2M^2} + \gamma\,\nu^2 + \mathcal{O}(\nu^4). \tag{5.61}$$

Taking the value

$$\gamma_{\mathrm{p}} = -134\times 10^{-6}\ \mathrm{fm}^4, \tag{5.62}$$

obtained from a recent saturation by pion-production multipoles [7], yields

$$\boxed{I^{\mathrm{p}}_{\mathrm{GDH}}(Q^2) - \tilde{I}^{\mathrm{p}}_{\mathrm{GDH}}(Q^2) \xrightarrow[Q^2\to 0]{} 2.68\,\frac{Q^2}{\mathrm{GeV}^2}} \tag{5.63a}$$

A rough estimate can be found also for the difference $\tilde{I}^{\mathrm{p}}_{\mathrm{GDH}}(Q^2) - I^{\mathrm{p}}_1(Q^2)$ if one recalls that the combination $G_1(\nu,Q^2) + \frac{\nu}{M}G_2(\nu,Q^2)$ is expected to be small. Letting $G_2(\nu,Q^2) \approx -\frac{M}{\nu}G_1(\nu,Q^2)$ as a first guess, Eq. (5.58b) becomes

$$\tilde{I}^{\mathrm{p}}_{\mathrm{GDH}}(Q^2) - I^{\mathrm{p}}_1(Q^2) \underset{Q^2\to 0}{\approx} Q^2\int_{\nu_0}^{\infty}\frac{\mathrm{d}\nu}{\nu^3}\,G^{\mathrm{p}}_1(\nu,0) = \frac{M^2Q^2}{2\alpha}\,\gamma_{\mathrm{p}}. \tag{5.63b}$$

Again, inserting the value (5.62) for the spin polarizability $\gamma_{\mathrm{p}}$ yields

$$\boxed{\tilde{I}^{\mathrm{p}}_{\mathrm{GDH}}(Q^2) - I^{\mathrm{p}}_1(Q^2) \xrightarrow[Q^2\to 0]{} -5.37\,\frac{Q^2}{\mathrm{GeV}^2}} \tag{5.63c}$$

Eqs. (5.63) are to be confronted with the Soffer-Teryaev result

$$\boxed{I^{\mathrm{p}}_1(Q^2) \xrightarrow[Q^2\to 0]{} -0.80 + 7.98\,\frac{Q^2}{\mathrm{GeV}^2}} \tag{5.64}$$

The slope 7.98 $\mathrm{GeV}^{-2}$ is composed of $-0.23$ $\mathrm{GeV}^{-2}$ coming from the smooth parametrization of $I^{\mathrm{p}}_{1+2}(Q^2)$ discussed above, plus 8.21 $\mathrm{GeV}^{-2}$ due to $-I^{\mathrm{p}}_2(Q^2)$, i.e., the proton Burkhardt-Cottingham sum rule. (Throughout, I employ the form-factor parametrization of Mergell,

[10] Soffer and Teryaev also presented a parametrization for the neutron [125], but in my opinion, too much handwaving is done in this case. (In contrast to the proton case, $I^{\mathrm{n}}_{1+2}(0) = 0$ due to Eq. (5.48), so that at least one extremum has to be built in, which somewhat overcharges the method.)

[11] Together with the optical theorem (2.10b) and the Taylor series (5.61), Eq. (5.60) represents the $\nu \to 0$ limit of the subtracted dispersion relation (4.7).

Meißner, and Drechsel [127].) Thus, the sharp rise of integral $I_1^{\mathrm{p}}(Q^2)$ around $Q^2 = 0$ is attributed mainly to the Burkhardt-Cottingham sum rule.

As can be seen from Eqs. (5.63) and (5.64), the integrals $I_1(Q^2)$, $I_{\mathrm{GDH}}(Q^2)$, and $\tilde{I}_{\mathrm{GDH}}(Q^2)$ must indeed be expected to considerably differ at low $Q^2$. This state of affairs is illustrated in Fig. 5.4.

### High $Q^2$

As for the high-$Q^2$ region, structure function $G_2(\nu, Q^2)$ may conspicuously affect $1/Q^2$ higher-twist corrections. To illustrate this, I adopt the notation of Ehrnsperger, Mankiewicz, and Schäfer [128] and wright down zeroth and second moments of $g_{1,2}(x, Q^2)$:

$$\int_0^1 \mathrm{d}x\, g_1(x, Q^2) = \frac{1}{2}\, a_0 + \frac{M^2}{9Q^2}\left(a_2 + 4d_2 + 4f_2\right) + \mathcal{O}(Q^{-4}), \tag{5.65a}$$

$$\int_0^1 \mathrm{d}x\, g_2(x, Q^2) = \mathcal{O}(Q^{-4}), \tag{5.65b}$$

$$\int_0^1 \mathrm{d}x\, x^2\, g_1(x, Q^2) = \frac{1}{2}\, a_2 + \mathcal{O}(Q^{-2}), \tag{5.65c}$$

$$\int_0^1 \mathrm{d}x\, x^2\, g_2(x, Q^2) = -\frac{1}{3}\, a_2 + \frac{1}{3}\, d_2 + \mathcal{O}(Q^{-2}), \tag{5.65d}$$

where $a_{0,2}$, $d_2$, and $f_2$ represent matrix elements of operators of twist two, three, and four, respectively [128]. Eq. (5.65b) states the validity of the Burkhardt-Cottingham sum rule up to twist four. The Bjorken sum rule states that $a_0^{\mathrm{p}} - a_0^{\mathrm{n}} = g_{\mathrm{A}}/3$. Falling back on Eqs. (5.59), I infer

$$\boxed{\begin{aligned} I_1(Q^2) &= \frac{2M^2}{Q^2}\left(\frac{1}{2}\, a_0 + \frac{M^2}{9Q^2}\left(a_2 + 4d_2 + 4f_2\right) + \mathcal{O}(Q^{-4})\right) \\ I_{\mathrm{GDH}}(Q^2) &= \frac{2M^2}{Q^2}\left(\frac{1}{2}\, a_0 + \frac{M^2}{9Q^2}\left(4a_2 - 8d_2 + 4f_2\right) + \mathcal{O}(Q^{-4})\right) \\ \tilde{I}_{\mathrm{GDH}}(Q^2) &= \frac{2M^2}{Q^2}\left(\frac{1}{2}\, a_0 + \frac{M^2}{9Q^2}\left(13a_2 - 8d_2 + 4f_2\right) + \mathcal{O}(Q^{-4})\right) \end{aligned}} \tag{5.66}$$

Observe that $1/Q^2$ corrections may considerably differ among different generalizations of the GDH integral, depending on the relative strength of twist-four matrix elements compared to twist-two and -three matrix elements. At present, estimates on $d_2$ and $f_2$ from polarized DIS data [129, 130] or from QCD sum rules [131] are far too vague to numerically specify the differences between Eqs. (5.66).

### Conclusion

I conclude that one ought to be very conscientious about the question, which of the integrals $I_1(Q^2)$, $I_{\mathrm{GDH}}(Q^2)$, or $\tilde{I}_{\mathrm{GDH}}(Q^2)$ should be considered. Different choices may be appropriate

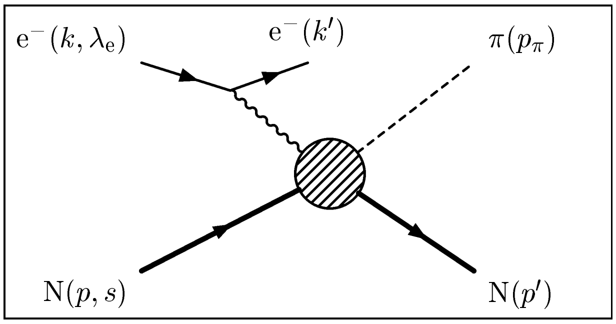



Figure 5.5: Polarized pion electroproduction on the nucleon in the one-photon-exchange approximation.

from different points of view. The crucial questions are: does one want to describe properties of structure function $G_1(\nu, Q^2)$ (corresponding to integral $I_1$), or properties of the nucleon's spin structure as probed with transversely polarized virtual photons (corresponding to $I_{\mathrm{GDH}}$ or $\tilde{I}_{\mathrm{GDH}}$)? And: what is the phenomenological input? Experimental setups with longitudinal target polarization generally access *neither* structure function $G_1(\nu, Q^2)$ *nor* the linear combination $G_1(\nu, Q^2) - \frac{Q^2}{M\nu} G_2(\nu, Q^2)$ (corresponding to asymmetry $A_1$), but rather some other linear combination of $G_1(\nu, Q^2)$ and $G_2(\nu, Q^2)$, whose coefficients involve the beam energy $E$ in addition to $\nu$ and $Q^2$ or $E'$ and $\theta_e$ (see, e.g., Roberts [115]). Additional assumptions about structure function $G_2(\nu, Q^2)$ (or assymetry $A_2$) have to be put in to extract one of the integrals under consideration. There may be some choice that is least affected by the arbitrariness within these assumptions. For instance, the intermediate-$Q^2$ E143 experiment is capable to extract $G_1(\nu, Q^2)$ with much higher accuracy than asymmetry $A_1$, because the former quantity is much less affected by the lack of knowledge on the other structure function or asymmetry [24].

On the other hand, there are estimates of the low-$Q^2$ evolution based on resonance saturation [124], as well as theoretical investigations within the framework of constituent-quark models [103] and chiral perturbation theory [123]. These investigations rather predict the behavior of integrals $I_{\mathrm{GDH}}(Q^2)$ or $\tilde{I}_{\mathrm{GDH}}(Q^2)$ than that of $I_1(Q^2)$, due to the lack of information on longitudinal-photon coupling. (This point has already been stressed by Li and Li [116].)

As I demonstrated in this section, one should at any rate desist from connecting the low-$Q^2$ range of one of these integrals with the high-$Q^2$ range of one of the others.

Further theoretical and experimental investigation of longitudinal photon coupling is badly needed. As far as resonance saturation is concerned, the following section aims at providing a basis for these studies.

## 5.3 The $\pi$N contribution

In view of the saturation of integrals (5.41) and (5.43), I now investigate the $\pi$N contribution to the inclusive electroproduction cross sections (5.23). That is to say, the sum over all hadronic final states X in Fig. 5.1 is approximated by the $\pi$N state as depicted in Fig. 5.5. More precisely, one has $\pi^0$p and $\pi^+$n final states for a proton in the initial state, and $\pi^0$n and $\pi^-$p final states for a neutron. (The discussion of different isospin channels is presented in Sect. 5.3.5.)

The $\pi$N contribution is written in terms of helicity amplitudes, which yield the most

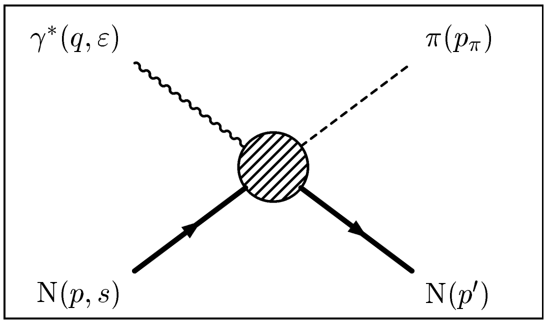



Figure 5.6: "Virtual photoproduction" of pions on the nucleon.

concise and plausible expressions, and in terms of so-called Chew-Goldberger-Low-Nambu (CGLN) amplitudes, being best suited for investigating Born parts.[12] The partial-wave expansion of diverse quantities of interest is computed and their isospin decomposition is presented.

### 5.3.1 Pion electroproduction

I endeavor to comprehensively treat all aspects of pion electroproduction that are needed for the purpose of saturating the integrals (5.41). For any question possibly left open, I recommend the classical review articles of Lyth[13] [133] and of Amaldi, Fubini, and Furlan [134]. The inclusion of polarization degrees of freedom has been discussed in much detail by von Gehlen [51] and by Raskin and Donnelly [109] (see also Drechsel and Tiator [135]).

#### Kinematics

Exclusive processes like pion electroproduction are conveniently described in the center-of-mass frame of the hadronic final state, in our case $\pi\mathrm{N}'$, where one has

$$\boldsymbol{q} + \boldsymbol{p} = \boldsymbol{p}_\pi + \boldsymbol{p}' = 0. \tag{5.67}$$

Note that this is *not* the center-of-mass frame of the actual reaction, $\mathrm{eN} \to \mathrm{e}'\pi\mathrm{N}'$ (where, instead, $\boldsymbol{k} + \boldsymbol{p}$ would be vanishing). The one-photon-exchange approximation enables me to employ the notion of "virtual photoproduction" of pions, analogously to the reduction of inclusive electroproduction to "virtual photoabsorption" presented in Sect. 5.1.2 and expressed by Eqs. (5.28) and (5.30). Pictorially, this corresponds to talking about Fig. 5.5 in terms of Fig. 5.6.

The full kinematics of the process is presented in Fig. 5.7. I adopt the usual distinction between *scattering plane*, defined by the electron momenta, and *reaction plane*, defined by the momenta of the outgoing hadrons. The angle enclosed by scattering plane and reaction plane is called the pion azimuthal scattering angle $\phi_\pi$. In the center-of-mass frame, it clearly

[12] The Born contribution to pion electroproduction amplitudes have been given by several authors. I recommend von Gehlen [51]. The article of Dennery [132] should be corrected for a sign error: The $\pm$ sign in Eq. (13) must be removed.

[13] Although Lyth's report is very detailed and instructive, it unfortunately containes a number of misprints. Explicit formulae should therefore not be adopted without verification.

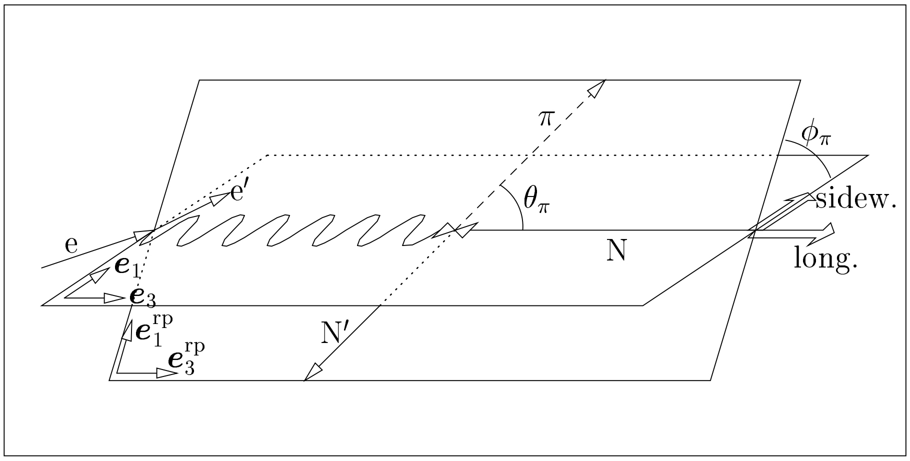



Figure 5.7: Kinematics of pion electroproduction in the center-of-mass frame. The electron momenta are scaled down in magnitude in order not to overshoot the dimensions of the plot. Apart from the dreibein $\boldsymbol{e}_k$ related to the scattering plane, the dreibein $\boldsymbol{e}_k^{\rm rp}$ related to the reaction plane is shown. (The respective second vectors are such that both dreibeine are orthonormal and right-handed.) Sideways and longitudinal polarizations of the incident nucleon are indicated.

has the same value as in the lab. The angle between photon and pion momenta is the frame-dependent zenith scattering angle $\theta_\pi$, which is taken to be measured in the center-of-mass frame. The right-handed coordinate dreibein $\boldsymbol{e}_k$ is defined such that the photon propagates in the direction of $\boldsymbol{e}_3$, the scattering plane is perpendicular to $\boldsymbol{e}_2$, and the electron momenta have positive first components. For later use, an auxiliary right-handed coordinate dreibein $\boldsymbol{e}_k^{\rm rp}$ is defined such that $\boldsymbol{e}_3^{\rm rp} = \boldsymbol{e}_3$, the *reaction* plane is perpendicular to $\boldsymbol{e}_2^{\rm rp}$, and the first component of the *pion* momentum is positive. These coordinate systems are related by

$$\begin{aligned} \boldsymbol{e}_1^{\rm rp} &= \boldsymbol{e}_1 \cos\phi_\pi + \boldsymbol{e}_2 \sin\phi_\pi, \\ \boldsymbol{e}_2^{\rm rp} &= \boldsymbol{e}_2 \cos\phi_\pi - \boldsymbol{e}_1 \sin\phi_\pi. \end{aligned} \tag{5.68}$$

### Cross sections

In pion electroproduction one measures triple differential cross sections $\mathrm{d}\sigma_{\lambda_{\rm e}}/\mathrm{d}E'\mathrm{d}\Omega_{\rm e}\mathrm{d}\Omega_\pi$. Here I am interested in the double differential cross section

$$\frac{\mathrm{d}\sigma_{\lambda_{\rm e}}^{(\pi\mathrm{N})}}{\mathrm{d}E'\,\mathrm{d}\Omega_{\rm e}} = \int \mathrm{d}\Omega_\pi \, \frac{\mathrm{d}\sigma_{\lambda_{\rm e}}^{(\pi\mathrm{N})}}{\mathrm{d}E'\mathrm{d}\Omega_{\rm e}\mathrm{d}\Omega_\pi}, \tag{5.69}$$

where electron scattering angles are measured in the lab frame and pion scattering angles are measured in the center-of-mass frame. Just like in the case of inclusive electroproduction, the difference of electron helicities $\lambda_{\rm e} = \pm\frac{1}{2}$ is considered. The analoga of Eqs. (5.28)–(5.30) read

$$\sigma_{1/2}^{(\pi\mathrm{N})}(\nu,Q^2) - \sigma_{3/2}^{(\pi\mathrm{N})}(\nu,Q^2) = \frac{4\pi^2\alpha}{M\sqrt{\nu^2+Q^2}} \left[\varepsilon_\mu^*(-1)\varepsilon_\nu(-1) - \varepsilon_\mu^*(+1)\varepsilon_\nu(+1)\right] W_{(\pi\mathrm{N})\mathrm{long}}^{\mu\nu}, \tag{5.70a}$$

$$\sigma_{\rm LT}^{(\pi\mathrm{N})}(\nu,Q^2) = \frac{4\pi^2\alpha}{M\sqrt{\nu^2+Q^2}} \,\mathrm{Im}\left(\varepsilon_\mu^*(0)\,\varepsilon_\nu(\perp)\,W_{(\pi\mathrm{N})\mathrm{side}}^{\mu\nu}\right), \tag{5.70b}$$

$$\sigma_{1/2}^{(\pi\mathrm{N})}(\nu,Q^2) - \sigma_{3/2}^{(\pi\mathrm{N})}(\nu,Q^2) = \frac{8\pi^2\alpha}{M^2}\,\frac{\nu}{\sqrt{\nu^2+Q^2}}\left[G_1^{(\pi\mathrm{N})}(\nu,Q^2) - \frac{Q^2}{M\nu}G_2^{(\pi\mathrm{N})}(\nu,Q^2)\right], \tag{5.71a}$$

$$\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})}(\nu,Q^2) = \frac{4\pi^2\alpha}{M^2}\,\frac{Q}{\sqrt{\nu^2+Q^2}}\left[G_1^{(\pi\mathrm{N})}(\nu,Q^2) + \frac{\nu}{M}G_2^{(\pi\mathrm{N})}(\nu,Q^2)\right], \tag{5.71b}$$

$$\left[\frac{\mathrm{d}\sigma_{-1/2}^{(\pi\mathrm{N})}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}^{(\pi\mathrm{N})}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}}\right]_{\mathrm{long}} = \Gamma\sqrt{1-\varepsilon^2}\left(\sigma_{1/2}^{(\pi\mathrm{N})}(\nu,Q^2) - \sigma_{3/2}^{(\pi\mathrm{N})}(\nu,Q^2)\right), \tag{5.72a}$$

$$\left[\frac{\mathrm{d}\sigma_{-1/2}^{(\pi\mathrm{N})}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}} - \frac{\mathrm{d}\sigma_{+1/2}^{(\pi\mathrm{N})}}{\mathrm{d}E'\,\mathrm{d}\Omega_\mathrm{e}}\right]_{\mathrm{side}} = 2\Gamma\sqrt{2\varepsilon(1-\varepsilon)}\,\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})}(\nu,Q^2). \tag{5.72b}$$

Eqs. (5.71) define the $\pi$N contribution to structure functions $G_{1,2}(\nu,Q^2)$. The $\pi$N contribution $W^{\mu\nu}_{(\pi\mathrm{N})}$ to the hadronic tensor can be read off from Eq. (5.5),

$$W^{\mu\nu}_{(\pi\mathrm{N})} = \frac{1}{4\pi}\sum_{\lambda'=\pm\frac{1}{2}}\int\frac{\mathrm{d}^3p_\pi}{(2\pi)^3 2p_\pi^0}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2p'^0}\,(2\pi)^4\delta(p+q-p'-p_\pi)$$
$$\times\,\langle p,s|J^\mu(0)|p_\pi;p',\lambda'\rangle\,\langle p_\pi;p',\lambda'|J^\nu(0)|p,s\rangle, \tag{5.73}$$

where, as before, the incident nucleon's spin $s$ is to be taken longitudinal (negative-helicity eigenstate) in Eq. (5.70a) and sideways in Eq. (5.70b). Of course, the helicity $\lambda'$ of the outgoing nucleon is summed over. The integrals run over *all* pion and nucleon momenta, while the four-dimensional delta function attends to energy and momentum conservation. The integration over $\boldsymbol{p}'$ is trivial,

$$W^{\mu\nu}_{(\pi\mathrm{N})} = \frac{1}{64\pi^3}\sum_{\lambda'=\pm\frac{1}{2}}\int\frac{\mathrm{d}^3p_\pi}{p_\pi^0\,p'^0}\,\delta(p^0+q^0-p'^0-p_\pi^0)$$
$$\times\,\langle p,s|J^\mu(0)|p_\pi;p',\lambda'\rangle\,\langle p_\pi;p',\lambda'|J^\nu(0)|p,s\rangle, \tag{5.74}$$

whereas the $\boldsymbol{p}_\pi$ integration requires some caution. I first write

$$\mathrm{d}^3p_\pi = \mathrm{d}\Omega_\pi\,\mathrm{d}|\boldsymbol{p}_\pi|\,\boldsymbol{p}_\pi^2 \tag{5.75}$$

in the center-of-mass frame. Now observe that $p'^0 = \sqrt{M^2+\boldsymbol{p}_\pi^2}$ and $p_\pi^0 = \sqrt{m_\pi^2+\boldsymbol{p}_\pi^2}$ are functions of $|\boldsymbol{p}_\pi|$. Hence the argument of the delta function is quite intricate if written in terms of $|\boldsymbol{p}_\pi|$. The trick is to substitute $|\boldsymbol{p}_\pi|$ for $W' := p'^0 + p_\pi^0$. Then the argument of the delta function is simply $W - W'$, where the center-of-mass energy $W$ of the hadronic final state obeys

$$W^2 = \left(p^0+q^0\right)^2 = (p+q)^2 = M^2 + 2M\nu - Q^2. \tag{5.76}$$

One has $p'^0\,\mathrm{d}p'^0 = p_\pi^0\,\mathrm{d}p_\pi{}^0 = |\boldsymbol{p}_\pi|\,\mathrm{d}|\boldsymbol{p}_\pi|$, which implies $p_\pi^0\,p'^0\mathrm{d}W' = W'\,|\boldsymbol{p}_\pi|\,\mathrm{d}|\boldsymbol{p}_\pi|$, eventually leading to

$$W^{\mu\nu}_{(\pi\mathrm{N})} = \frac{|\boldsymbol{p}_\pi|}{64\pi^3 W}\sum_{\lambda'=\pm\frac{1}{2}}\int\mathrm{d}\Omega_\pi\,\langle p,s|J^\mu(0)|p_\pi;p',\lambda'\rangle\,\langle p_\pi;p',\lambda'|J^\nu(0)|p,s\rangle. \tag{5.77}$$

The magnitude of the pion momentum can be expressed in terms of the total energy $W$,

$$|\boldsymbol{p}_\pi| = \frac{\sqrt{(W^2 - M^2 - m_\pi^2)^2 - 4M^2 m_\pi^2}}{2W}. \tag{5.78}$$

For later use, I also give the virtual photon's momentum,

$$|\boldsymbol{q}| = \frac{\sqrt{(W^2 - M^2 + Q^2)^2 + 4M^2 Q^2}}{2W} = \frac{M}{W}\sqrt{\nu^2 + Q^2}. \tag{5.79}$$

### 5.3.2 Helicity amplitudes

The $\pi$N contribution (5.77) to the hadronic tensor can now be inserted into the pion production cross sections (5.70). Strikingly simple expressions are obtained if one introduces helicity amplitudes [136] defined by

$$\boxed{f_{\lambda',\lambda\lambda_\gamma}(W, Q^2, \theta_\pi) = \frac{ie}{8\pi W}\left[\varepsilon_\mu(\lambda_\gamma)\left\langle p_\pi; p', \lambda'\middle| J^\mu(0)\middle| p, \lambda\right\rangle\right]_{\mathrm{rp}}} \tag{5.80}$$

where the subscript "rp" means that the phases of the states are fixed with respect to the reaction plane. Since an angle-$\phi_\pi$ rotation about $\boldsymbol{q}$ transforms from scattering plane to reaction plane, the relation between photon polarization vectors incorporated in Eqs. (5.70) and those of Eq. (5.80) is simply

$$\varepsilon(\lambda_\gamma) = e^{i\lambda_\gamma\phi_\pi}\varepsilon_{\mathrm{rp}}(\lambda_\gamma), \tag{5.81}$$

and the incident-nucleon's state transforms as

$$|p, \lambda\rangle = e^{-i\lambda\phi_\pi}|p, \lambda\rangle_{\mathrm{rp}}. \tag{5.82}$$

This will be relevant to the case of sideways polarization.[14]

From parity conservation, one has

$$f_{-\lambda',-\lambda-\lambda_\gamma} = -(-1)^{\lambda_\gamma-\lambda+\lambda'} f_{\lambda',\lambda\lambda_\gamma}, \tag{5.83}$$

so that six of the twelve amplitudes $f_{\lambda',\lambda\lambda_\gamma}$ are mutually independent. One commonly replaces the indices $\pm\frac{1}{2}$ and $\pm 1$ by simple $\pm$'s and chooses the four transverse amplitudes $f_{\lambda',\lambda+}$ and the two longitudinal amplitudes $f_{\lambda',+0}$ to be the independent ones occurring explicitly.

For longitudinal nucleon polarization, Eq. (5.70a), one lets $\lambda = -\frac{1}{2}$, sums over $\lambda'$, and subtracts the result for $\lambda_\gamma = +1$ from the result for $\lambda_\gamma = -1$, arriving at

$$\boxed{\sigma_{1/2}^{(\pi\mathrm{N})} - \sigma_{3/2}^{(\pi\mathrm{N})} = \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\int \mathrm{d}\Omega_\pi \left(|f_{+,++}|^2 + |f_{-,++}|^2 - |f_{+,-+}|^2 - |f_{-,-+}|^2\right).} \tag{5.84}$$

The case of sideways nucleon polarization, Eq. (5.70b), requires some more caution with respect to phases. Let $\boldsymbol{S}$ denote the spin operator. Then, according to Jacob and Wick

[14] Fortunately, the transformation of the $\pi$N state need not be inspected since only the phase-independent projection operator $|p_\pi; p', \lambda'\rangle\langle p_\pi; p', \lambda'|$ appears in the hadronic tensor (5.77).

[33], the phase relation between different-helicity[15] nucleon states is such that the operators $S^1 \pm iS^2$ switch between states without giving rise to an additional phase,

$$\begin{aligned} (S^1 \pm iS^2)\, |p, \mp\tfrac{1}{2}\rangle &= 0, \\ (S^1 \pm iS^2)\, |p, \pm\tfrac{1}{2}\rangle &= |p, \mp\tfrac{1}{2}\rangle. \end{aligned} \tag{5.85}$$

The sideways-polarized state is an eigenstate of $S^1$ corresponding to eigenvalue $+\frac{1}{2}$, which is easily constructed from relations (5.85) to be

$$\begin{aligned} |p, \mathrm{side}\rangle &= \frac{1}{\sqrt{2}} \left( |p, +\tfrac{1}{2}\rangle + |p, -\tfrac{1}{2}\rangle \right) \\ &= \frac{1}{\sqrt{2}} \left( e^{-i\phi_\pi/2}\, |p, +\tfrac{1}{2}\rangle_{\mathrm{rp}} + e^{+i\phi_\pi/2}\, |p, -\tfrac{1}{2}\rangle_{\mathrm{rp}} \right). \end{aligned} \tag{5.86}$$

Inserting this into Eq. (5.70b) using Eq. (5.77), one obtains

$$\begin{aligned} \sigma_{\mathrm{LT}}^{(\pi\mathrm{N})} &= \frac{1}{2\sqrt{2}} \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \sum_{\lambda', \lambda_1, \lambda_2 = \pm\frac{1}{2}} \sum_{\lambda_\gamma = \pm 1} \int \mathrm{d}\Omega_\pi\, \mathrm{Im}\big( i f^*_{\lambda', \lambda_1 0} f_{\lambda', \lambda_2 \lambda_\gamma} e^{i(\lambda_1 - \lambda_2 + \lambda_\gamma)\phi_\pi} \big) \\ &= \frac{1}{\sqrt{2}} \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \int \mathrm{d}\Omega_\pi \Big[ \mathrm{Re}\big( f^*_{-,+0} f_{+,++} - f^*_{+,+0} f_{-,++} \big) \\ &\qquad - \mathrm{Im}\big( f^*_{-,+0} (f_{-,++} + f_{+,-+}) + f^*_{+,+0} (f_{+,++} - f_{-,-+}) \big) \sin\phi_\pi \\ &\qquad + \mathrm{Re}\big( f^*_{-,+0} f_{-,-+} + f^*_{+,+0} f_{+,-+} \big) \cos(2\phi_\pi) \Big], \end{aligned} \tag{5.87}$$

where $\phi_\pi$ dependent terms have been kept to render the possibility of expressing the triple differential pion electroproduction cross section in terms of helicity amplitudes. Owing to the fact that both $\sin\phi_\pi$ and $\cos(2\phi_\pi)$ integrate to zero, Eq. (5.87) simplifies to

$$\boxed{\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})} = \frac{1}{\sqrt{2}} \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \int \mathrm{d}\Omega_\pi\, \mathrm{Re}\big( f^*_{-,+0} f_{+,++} - f^*_{+,+0} f_{-,++} \big).} \tag{5.88}$$

Eqs. (5.84) and (5.88) may be inserted into Eqs. (5.71) to obtain the $\pi$N contribution to structure functions $G_{1,2}(\nu, Q^2)$ in terms of helicity amplitudes.

### 5.3.3 Chew-Goldberger-Low-Nambu amplitudes

In view of the calculation of the Born contribution to cross sections $\sigma_{1/2}^{(\pi\mathrm{N})} - \sigma_{3/2}^{(\pi\mathrm{N})}$ and $\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})}$, a different set of pion-electroproduction amplitudes suggests itself, namely the amplitudes $\mathcal{F}_{1\ldots6}(W, Q^2, \theta_\pi)$ defined by [132]

$$\begin{aligned} &\frac{-ie}{8\pi W} \langle p_\pi; p', \lambda' \big| \boldsymbol{J}(0) \big| p, \lambda \rangle = \\ &\quad \chi^\dagger(p', \lambda') \big[ \boldsymbol{\sigma}^\perp \mathcal{F}_1 - i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}_\pi)\, (\boldsymbol{\sigma}\times\hat{\boldsymbol{q}})\, \mathcal{F}_2 + \hat{\boldsymbol{p}}_\pi^\perp\, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\, \mathcal{F}_3 + \hat{\boldsymbol{p}}_\pi^\perp\, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}_\pi)\, \mathcal{F}_4 \\ &\qquad + \hat{\boldsymbol{q}}\, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\, \mathcal{F}_5 + \hat{\boldsymbol{q}}\, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}_\pi)\, \mathcal{F}_6 \big]\, \chi(p, \lambda), \end{aligned} \tag{5.89}$$

[15] Again, the reader may recall that nucleon polarization in *positive* $x^3$ direction corresponds to *negative* helicity in the center-of-mass frame. Moreover, the boost transforming from the lab to the center of mass leaves Eqs. (5.85) unaltered.

where the circumflex accent denotes a unit vector, e.g. $\hat{\boldsymbol{q}} = \boldsymbol{q}/|\boldsymbol{q}|$, and the superscript $\perp$ stands for projection onto the plane perpendicular to the photon momentum, e.g. $\boldsymbol{\sigma}^{\perp} = \boldsymbol{\sigma} - (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\,\hat{\boldsymbol{q}}$. Current conservation $q\cdot J = 0$ then fixes the charge-density matrix element

$$\frac{-ie}{8\pi W}\,\langle p_\pi; p', \lambda' | J^0(0) | p, \lambda\rangle = \frac{|\boldsymbol{q}|}{q^0}\,\chi^\dagger(p', \lambda')\,\left[(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{q}})\,\mathcal{F}_5 + (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}_\pi)\,\mathcal{F}_6\right]\chi(p, \lambda). \tag{5.90}$$

This indicates the purely longitudinal nature[16] of amplitudes $\mathcal{F}_{5,6}$.

The six electroproduction amplitudes defined by Eq. (5.89) are commonly called CGLN amplitudes, although Chew et al. [137] merely considered photoproduction, where only four amplitudes $\mathcal{F}_{1\ldots4}$ occur. Dennery [132] was actually the first to treat electroproduction, where also longitudinal degrees of freedom come in.

Introducing explicit photon-polarization vectors $\varepsilon(\lambda_\gamma)$ and Pauli spinors $\chi^\dagger(p', \lambda')$, $\chi(p, \lambda)$ into definitions (5.80) and (5.89), one obtains the following connection between helicity amplitudes and CGLN amplitudes [136]:

$$f_{+,++} = f_{-,--} = \sqrt{2}\,\sin\frac{\theta_\pi}{2}\,\left(\mathcal{F}_1 + \mathcal{F}_2 + \cos^2\frac{\theta_\pi}{2}\,(\mathcal{F}_3 + \mathcal{F}_4)\right), \tag{5.91a}$$

$$f_{-,++} = -f_{+,--} = -\sqrt{2}\,\cos\frac{\theta_\pi}{2}\,\left(\mathcal{F}_1 - \mathcal{F}_2 - \sin^2\frac{\theta_\pi}{2}\,(\mathcal{F}_3 - \mathcal{F}_4)\right), \tag{5.91b}$$

$$f_{+,-+} = -f_{-,+-} = \sqrt{2}\,\cos\frac{\theta_\pi}{2}\,\sin^2\frac{\theta_\pi}{2}\,(\mathcal{F}_3 - \mathcal{F}_4), \tag{5.91c}$$

$$f_{-,-+} = f_{+,+-} = -\sqrt{2}\,\sin\frac{\theta_\pi}{2}\,\cos^2\frac{\theta_\pi}{2}\,(\mathcal{F}_3 + \mathcal{F}_4), \tag{5.91d}$$

$$f_{+,+0} = -f_{-,-0} = \frac{Q}{q^0}\,\cos\frac{\theta_\pi}{2}\,(\mathcal{F}_5 + \mathcal{F}_6), \tag{5.91e}$$

$$f_{-,+0} = f_{+,-0} = \frac{Q}{q^0}\,\sin\frac{\theta_\pi}{2}\,(\mathcal{F}_5 - \mathcal{F}_6). \tag{5.91f}$$

Employing these relations, the pion-production cross sections (5.84) and (5.88) can be written

$$\boxed{\begin{aligned}\sigma^{(\pi\mathrm{N})}_{1/2} - \sigma^{(\pi\mathrm{N})}_{3/2} = 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\int \mathrm{d}\Omega_\pi\,\Big[&|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 - 2\,\mathrm{Re}(\mathcal{F}_1^*\mathcal{F}_2)\cos\theta_\pi \\ &+ \mathrm{Re}(\mathcal{F}_1^*\mathcal{F}_4 + \mathcal{F}_2^*\mathcal{F}_3)\sin^2\theta_\pi\Big]\end{aligned}} \tag{5.92a}$$

and

$$\boxed{\begin{aligned}\sigma^{(\pi\mathrm{N})}_{\mathrm{LT}} = \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\,\frac{Q}{q^0}\int \mathrm{d}\Omega_\pi\,\Big[&\mathrm{Re}(\mathcal{F}_5^*\mathcal{F}_1 - \mathcal{F}_6^*\mathcal{F}_2) - \mathrm{Re}(\mathcal{F}_5^*\mathcal{F}_2 - \mathcal{F}_6^*\mathcal{F}_1)\cos\theta_\pi \\ &+ \tfrac{1}{2}\,\mathrm{Re}(\mathcal{F}_5^*\mathcal{F}_4 - \mathcal{F}_6^*\mathcal{F}_3)\sin^2\theta_\pi\Big]\end{aligned}} \tag{5.92b}$$

respectively. Eqs. (5.92) may be inserted into Eqs. (5.71) to obtain the πN contribution to structure functions $G_{1,2}(\nu, Q^2)$ in terms of CGLN amplitudes.

[16] Some authors omit the projection indicated by superscript $\perp$ in Eq. (5.89), which has the effect of contaminating amplitudes $\mathcal{F}_5$ and $\mathcal{F}_6$ with transverse pieces $-\mathcal{F}_1 - \cos\theta_\pi\mathcal{F}_3$ and $-\cos\theta_\pi\mathcal{F}_4$, respectively. Most of these authors subsequently define primed amplitudes $\mathcal{F}'_{5,6}$, which have the transverse portions subtracted and which coincide with my unprimed ones. Definition (5.89) is adopted from Drechsel and Tiator [135].

### 5.3.4 Partial-wave expansion

Partial-wave expansion of amplitudes corresponds to transforming the pion scattering-angle dependence into the dependence on the discrete total angular-momentum quantum number. It is most concisely written for helicity amplitudes [136]:

$$f_{\pm,++} = -\sqrt{2} \sum_{J\geq\frac{1}{2}} (J+\tfrac{1}{2}) \left(A_{(J-\frac{1}{2})+} \pm A_{(J+\frac{1}{2})-}\right) d^J_{\frac{1}{2}\mp\frac{1}{2}}(\theta_\pi), \tag{5.93a}$$

$$f_{\pm,-+} = \frac{1}{\sqrt{2}} \sum_{J\geq\frac{3}{2}} (J+\tfrac{1}{2}) \sqrt{(J-\tfrac{1}{2})(J+\tfrac{3}{2})} \left(B_{(J-\frac{1}{2})+} \pm B_{(J+\frac{1}{2})-}\right) d^J_{\frac{3}{2}\mp\frac{1}{2}}(\theta_\pi), \tag{5.93b}$$

$$f_{\pm,++} = \sum_{J\geq\frac{1}{2}} (J+\tfrac{1}{2}) \left(C_{(J-\frac{1}{2})+} \mp C_{(J+\frac{1}{2})-}\right) d^J_{-\frac{1}{2}\mp\frac{1}{2}}(\theta_\pi). \tag{5.93c}$$

The half-integer number $J$ corresponds to the total angular momentum of the $\pi$N state. The *helicity multipoles* occurring in Eq. (5.93) are related to conventional electric, magnetic, and scalar (or longitudinal) multipoles via

$$A_{l+}(W,Q^2) = \frac{l+2}{2}\, E_{l+} + \frac{l}{2}\, M_{l+}, \tag{5.94a}$$

$$A_{l-}(W,Q^2) = -\frac{l-1}{2}\, E_{l-} + \frac{l+1}{2}\, M_{l-}, \tag{5.94b}$$

$$B_{l+}(W,Q^2) = E_{l+} - M_{l+}, \tag{5.94c}$$

$$B_{l-}(W,Q^2) = E_{l-} + M_{l-}, \tag{5.94d}$$

$$C_{l+}(W,Q^2) = \frac{Q}{|\boldsymbol{q}|}\,(l+1)\, S_{l+} = \frac{Q}{q^0}\,(l+1)\, L_{l+}, \tag{5.94e}$$

$$C_{l-}(W,Q^2) = -\frac{Q}{|\boldsymbol{q}|}\, l\, S_{l-} = -\frac{Q}{q^0}\, l\, L_{l-}, \tag{5.94f}$$

where $l$ represents the *orbital* angular momentum of the $\pi$N system. In the real-photon limit, multipoles $A_{l\pm}$ and $B_{l\pm}$ match the notation of Walker [138].

As far as the rotation matrix elements $d^J_{MM'}$ are concerned, I adopt the conventions of Brink and Satchler [139]. In particular, the orthogonality condition reads (indices $M, M'$ *not summed*)

$$\int \mathrm{d}\Omega\, d^J_{MM'}(\theta)\, d^{J'}_{MM'}(\theta) = \frac{4\pi}{2J+1}\, \delta_{JJ'}. \tag{5.95}$$

Inserting expansions (5.93) into cross sections (5.84) and (5.88), the integration over pion scattering angles is easily performed, yielding

$$\boxed{\begin{aligned} \sigma^{(\pi\mathrm{N})}_{1/2} - \sigma^{(\pi\mathrm{N})}_{3/2} = 8\pi\, \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \sum_{J\geq\frac{1}{2}} \Big[ & (J+\tfrac{1}{2})\left(|A_{(J-\frac{1}{2})+}|^2 + |A_{(J+\frac{1}{2})-}|^2\right) \\ & - \frac{1}{4}(J-\tfrac{1}{2})(J+\tfrac{1}{2})(J+\tfrac{3}{2})\left(|B_{(J-\frac{1}{2})+}|^2 + |B_{(J+\frac{1}{2})-}|^2\right)\Big] \end{aligned}} \tag{5.96a}$$

and

$$\boxed{\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})} = 4\pi \, \frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \sum_{J\geq\frac{1}{2}} (J+\tfrac{1}{2}) \, \mathrm{Re}\big(C^*_{(J-\frac{1}{2})+} A_{(J-\frac{1}{2})+} + C^*_{(J+\frac{1}{2})-} A_{(J+\frac{1}{2})-}\big)} \tag{5.96b}$$

Eqs. (5.96) may be inserted into Eqs. (5.71) to obtain the πN contribution to structure functions $G_{1,2}(\nu, Q^2)$ in terms of multipoles.

### 5.3.5 Isospin decomposition

There are four isospin channels of the pion-electroproduction process, viz.

$$\begin{aligned} \gamma^*\mathrm{p} &\rightarrow \begin{cases} \pi^0\mathrm{p} \\ \pi^+\mathrm{n} \end{cases} \\ \gamma^*\mathrm{n} &\rightarrow \begin{cases} \pi^0\mathrm{n} \\ \pi^-\mathrm{p} \end{cases} \end{aligned} \tag{5.97}$$

Assuming isospin symmetry, it is convenient to define amplitudes for the absorption of isoscalar and isovector photons (see, e.g., Lyth [133]). For instance, the CGLN amplitudes $\mathcal{F}_{1\ldots6}(W, Q^2, \theta_\pi)$ defined in Sect. 5.3.3 are decomposed according to

$$\begin{aligned} \mathcal{F}_i^{(\pi^0\mathrm{p})} &= \mathcal{F}_i^{(0)} + \mathcal{F}_i^{(+)}, \\ \mathcal{F}_i^{(\pi^+\mathrm{n})} &= \sqrt{2}\big(\mathcal{F}_i^{(0)} + \mathcal{F}_i^{(-)}\big), \\ \mathcal{F}_i^{(\pi^0\mathrm{n})} &= -\mathcal{F}_i^{(0)} + \mathcal{F}_i^{(+)}, \\ \mathcal{F}_i^{(\pi^-\mathrm{p})} &= \sqrt{2}\big(\mathcal{F}_i^{(0)} - \mathcal{F}_i^{(-)}\big), \end{aligned} \tag{5.98}$$

where superscripts (±) refer to isovector photons, and (0) corresponds to an isoscalar photon. Analogous decompositions apply to helicity amplitudes and partial-wave multipoles. Observe that isospin symmetry reduces the number of independent channels from four to three.

Since I am concerned with virtual photoabsorption cross sections, amplitudes are squared and subsequently added in pairs. For example, the cross-section difference (5.92a) can be cast into the compact form

$$\sigma_{1/2}^{(\pi\mathrm{N})} - \sigma_{3/2}^{(\pi\mathrm{N})} = 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \int \mathrm{d}\Omega_\pi \sum_{i,j=1}^{4} K_{ij} \mathcal{F}_i^* \mathcal{F}_j, \tag{5.99}$$

where I have introduced the real symmetric matrix

$$K = \begin{pmatrix} 1 & -\cos\theta_\pi & 0 & \frac{1}{2}\sin^2\theta_\pi \\ -\cos\theta_\pi & 1 & \frac{1}{2}\sin^2\theta_\pi & 0 \\ 0 & \frac{1}{2}\sin^2\theta_\pi & 0 & 0 \\ \frac{1}{2}\sin^2\theta_\pi & 0 & 0 & 0 \end{pmatrix}. \tag{5.100}$$

For pion electroproduction on the proton, one has final states $\pi^0$p and $\pi^+$n. Accordingly,

$$\Big[\sigma_{1/2}^{(\pi\mathrm{N})} - \sigma_{3/2}^{(\pi\mathrm{N})}\Big]_\mathrm{p} = 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|} \int \mathrm{d}\Omega_\pi \sum_{i,j=1}^{4} K_{ij} \Big(\mathcal{F}_i^{(\pi^0\mathrm{p})*} \mathcal{F}_j^{(\pi^0\mathrm{p})} + \mathcal{F}_i^{(\pi^+\mathrm{n})*} \mathcal{F}_j^{(\pi^+\mathrm{n})}\Big]$$

$$= 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\int \mathrm{d}\Omega_\pi \sum_{i,j=1}^{4} K_{ij}\Big[3\mathcal{F}_i^{(0)*}\mathcal{F}_j^{(0)} + 2\mathcal{F}_i^{(-)*}\mathcal{F}_j^{(-)} + \mathcal{F}_i^{(+)*}\mathcal{F}_j^{(+)}$$
$$+ 2\,\mathrm{Re}\Big(\mathcal{F}_i^{(0)*}\big(2\mathcal{F}_j^{(-)} + \mathcal{F}_j^{(+)}\big)\Big)\Big], \qquad (5.101a)$$

while for the neutron one has

$$\Big[\sigma_{1/2}^{(\pi\mathrm{N})} - \sigma_{3/2}^{(\pi\mathrm{N})}\Big]_{\mathrm{n}} = 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\int \mathrm{d}\Omega_\pi \sum_{i,j=1}^{4} K_{ij}\Big[\mathcal{F}_i^{(\pi^0\mathrm{n})*}\mathcal{F}_j^{(\pi^0\mathrm{n})} + \mathcal{F}_i^{(\pi^-\mathrm{p})*}\mathcal{F}_j^{(\pi^-\mathrm{p})}\Big]$$
$$= 2\,\frac{|\boldsymbol{p}_\pi|}{|\boldsymbol{q}|}\int \mathrm{d}\Omega_\pi \sum_{i,j=1}^{4} K_{ij}\Big[3\mathcal{F}_i^{(0)*}\mathcal{F}_j^{(0)} + 2\mathcal{F}_i^{(-)*}\mathcal{F}_j^{(-)} + \mathcal{F}_i^{(+)*}\mathcal{F}_j^{(+)}$$
$$- 2\,\mathrm{Re}\Big(\mathcal{F}_i^{(0)*}\big(2\mathcal{F}_j^{(-)} + \mathcal{F}_j^{(+)}\big)\Big)\Big]. \qquad (5.101b)$$

Analogous formulae apply to the decompositions of $\sigma_{\mathrm{LT}}^{(\pi\mathrm{N})}$ and to the expressions of virtual-photoabsorption cross sections in terms of helicity amplitudes or multipoles.

### 5.3.6 Outlook

As noted in the introduction to this chapter, the present section aims at providing formulae for the resonance saturation of generalized GDH integrals. In order to reliably perform the saturation, a few additional investigations are desirable. The high-energy behavior of structure functions $G_{1,2}(\nu, Q^2)$ should be analyzed by means of Reggeization. Moreover, the influence of final states other than $\pi$N ought to be estimated. In particular, the role of double-pion production should be clarified.

# Chapter 6

# Summary and conclusion

This thesis presents an attempt to comprehensively treat the theoretical aspects of the GDH sum rule, in such a way that all proposed modifications can rigorously be checked. The ingredients of the current-algebra derivation of the sum rule, in particular the infinite-momentum limit, turn out to be most important.

### Derivations of the GDH sum rule

In Ch. 2, I presented all derivations of the GDH sum rule that can be found in the literature [1–3, 25, 26]. I commented on the impact of the lowest-order consideration of electromagnetic coupling (see pp. 9, 10, and 12). As far as equal-times current algebra is concerned, I presented a derivation that exhibits the infinite-momentum limit as its last step – in contrast to Refs. 3 and 25, where this limit is employed in between. The *finite-momentum GDH sum rule* (2.75) presents the form that the sum rule takes before the infinite-momentum limit is taken. At finite momenta, the GDH integration path is a parabola in $(\nu, q^2)$ plane as depicted in Fig. 2.5, p. 25. Since the legitimacy of the infinite-momentum limit cannot be questioned within the scope of current algebra, some perturbative model has to be adopted as regards this problem. The finite-momentum GDH sum rule is well suited for this investigation. It re-appears at several other places within this thesis.

### The GDH sum rule within perturbative models

In Ch. 3, I presented some important tests of the GDH sum rule, concerning its validity within three renormalizable field theoretical models: a pion-nucleon model with pseudoscalar $\pi$N coupling and minimal electromagnetic interaction [28], quantum electrodynamics [29, 30], and the Weinberg-Salam model of electro-weak interactions of leptons [29]. In all three instances, both sides of the GDH sum rule are considered to lowest non-trivial order in the respective coupling constant $g$ and $e$. Since the fermion's anomalous magnetic moment is of the order $g^2$ in the pion-nucleon model and $e^2$ in QED and the Weinberg-Salam model, the left-hand side of the sum rule vanishes to lowest order in all these models. Then, I generalized the electromagnetic coupling of the pion-nucleon model by introducing an anomalous magnetic moment $\kappa_0$ to the nucleon on the vertex level, which gives rise to a divergent GDH integral, because due to the derivative coupling of the anomalous magnetic moment, the polarized photoabsorption cross section approaches a non-zero constant at large photon energy.

In QED and in the Weinberg-Salam model, there is no hadronic interaction. Hence, the lowest non-trivial order is $\alpha^2$. (In the pion-nucleon model, it is $\alpha g^2$.) On the other hand, the GDH sum rule of the physical nucleon is explicitly considered to order $\alpha$ only. That is to say, electromagnetism is treated to lowest order, while hadronic interactions are, of course, non-perturbative. The reader may note that this does not reduce the instructive power of testing the GDH sum rule by means of electro-weak processes. Rather, the following conclusion ought to be kept in mind. In all models considered, the GDH integral either adopts the predicted value (viz., zero), or both sides of the sum rule diverge due to anomalous magnetic moments. No instance of a "subtraction at infinity", which would spoil the GDH sum rule (see Sect. 4.1), is detected. Moreover, the confirmation of the sum rule at next-to-leading order ($\alpha^2$) supports the idea that in principle, the forward Compton amplitude could be considered to all orders in $\alpha$. This is relevant to the discussion of a possible $J = 1$ fixed pole in angular-momentum plane.

To order $\alpha g^4$ within the pion-nucleon model, or order $\alpha^3$ within QED and the Weinberg-Salam model, the left-hand side of the sum rule would be non-vanishing. Thus, a study of the respective next orders within these models would in principle be interesting, but quite painstaking, too.

### Possible sources of modifications

Ch. 4 has been devoted to a thorough discussion of all possible sources of modifications of the GDH sum rule. Particular attention has been paid to claimed modifications within the current-algebra approach.

But first, in Sect. 4.1, I discussed the effect of a fixed Regge singularity [12] at $J = 1$ in the complex angular-momentum plane, emphasizing the crucial role of spin. Such a singularity would give rise to a "subtraction at infinity", where the subtraction constant is essentially the residue of the fixed pole. I showed that the subtraction constant is directly related to the high-energy behaviour of a certain Compton-scattering cross section, see Eq. (4.13). However, I stressed that neither of the perturbative models considered in Ch. 3 yields a fixed pole. Assuming that the forward Compton amplitude can be defined to all orders in the electromagnetic coupling and that each order obeys a dispersion relation possibly subtracted at infinity, I argued that a very peculiar dependence among fundamental parameters of the standard model would emerge if not *all* subtraction constants were simultaneously vanishing (see Eq. (4.19)).

In Sect. 4.2, I thoroughly discussed the possibility of a correction to the GDH sum rule due to an anomalous commutator of electric charge densities. After an introduction into general features of Schwinger terms and anomalous commutators, I presented the vital tools for their computation. I particularly focussed on the investigation presented by Chang, Liang, and Workman [13–15], which, at the second glance, turns out to be intimately related to a theory containing spin-1 mesons as gauge fields of chiral transformations. Finally, I presented the most general non-naive charge-density commutator relevant to the GDH sum rule. I calculated the modification of the *finite-momentum* GDH sum rule that is brought about by the non-naive charge-density commutator, emphasizing the fact that the legitimacy of the infinite-momentum limit remains to be checked in each concrete case under consideration. This legitimacy corresponds to the possibility of interchanging the infinite-momentum limit $p^0 \to \infty$ with the photon-energy integration.

In Sect. 4.3, I illustrated the possibility of a finite modification due to the infinite-momentum limit used in the current-algebra derivation of the GDH sum rule. I stressed that the only instance of such a correction is found in the Weinberg-Salam model, where it *compensates* a modification formerly brought about by an anomalous charge-density algebra. This has to be regarded as a very strong evidence for the validity of the unmodified GDH sum rule.

Sect. 4.4 comprises most of the original work of this thesis. I presented the calculation of the anomalous equal-times commutator of electric charge-densities within the Weinberg-Salam model. The anomaly is due to the fermion-triangle loop incorporated in the $Z^0$-exchange graphs displayed in Fig. 4.3(d), p. 71. The one-electron matrix element of the charge-density commutator is given in Eq. (4.136). It translates into the electric-dipole-moment commutator presented in Eq. (4.145). Thereupon, the modified finite-momentum GDH sum rule (4.146) arises.

On the other hand, the model considered allows to investigate the infinite-momentum limit by calculating the forward virtual Compton amplitude $f_2(\nu, q^2)$ for all values of photon energy $\nu$ and virtuality $q^2$. That is to say, I inspected the GDH integral at *finite* electron energies $p^0$, where the integration runs along the parabolae shown in Fig. 4.4 (p. 79), and took the $p^0 \to \infty$ limit subsequently. Then, I examined whether this limit deviates from the *genuine* GDH integral, which is taken along the straight line at $q^2 = 0$. This is indeed the case, due to the $Z^0$-exchange graphs. The contribution from these graphs to amplitude $f_2(\nu, q^2)$ is given by Eq. (4.160). I explicitly showed that the finite-momentum GDH integral picks up a certain constant from the part of the integration path that lies above the two-electron threshold $q^2 = 4m^2$. On the other hand, below this threshold, the contribution of $Z^0$ exchange to the GDH integrand vanishes identically (cf. Fig. 4.5, p. 81). This illustrates that indeed, the $p^0 \to \infty$ limit cannot be interchanged with the $\nu$ integration, i.e., the finite-momentum GDH integral is related in a non-naive way to the genuine GDH integral with integrand $\sigma_{1/2}(\nu) - \sigma_{3/2}(\nu)$, as expressed by Eq. (4.166). Inserting then the modified finite-momentum sum rule (4.146), one recovers the unmodified version (3.27).

This calculation has revealed the very important fact that the naive infinite-momentum limit can*not* be applied if anomalous charge-density commutators occur. Rather, it has to be expected that the infinite-momentum limit *compensates* any modification obtained from anomalous charge-density commutators.

In Sect. 4.5, I returned to hadron physics and investigated the exchange of axial-vector mesons in the $t$ channel. The purpose of this section has been twofold. Firstly, I reminded to the fact that a possible $J = 1$ fixed pole in complex angular-momentum plane cannot be attributed to $t$-channel exchange of spin-1 particles. Secondly, I pointed out that axial-vector exchange is analogous to $Z^0$ exchange considered in the Salam-Weinberg model. Therefore, the same compensation mechanism is found. That is to say, one obtaines an anomalous-commutator correction at finite nucleon momenta, which is compensated by the infinite-momentum limit. On the other hand, I proved in Sect. 4.5 that the result of Chang, Liang, and Workman [13–15] obtained from anomalous charge-density commutators can also be understood by means of axial-vector exchange. This again shows that the modification of the GDH sum rule that is claimed in Refs. 13–15 actually appears only at finite momenta and does not survive the infinite-momentum limit. In particular, Sect. 4.5 proves that the central result of Sect. 4.4 does not crucially depend on the fact that higher orders of $\alpha$ have to be considered in the Weinberg-Salam model.

In Sect. 4.6, I investigated the role of possible anomalous magnetic moments $\kappa_q$ of quarks. I calculated the modified charge-density algebra (4.191) that is brought about by non-vanishing $\kappa_q$. Applying the current-algebra derivation, this gives rise to a definite modification of the

GDH sum rule, expressed by Eq. (4.197) with Eqs. (4.196). I argued that in this instance, the modification must be expected to survive the infinite-momentum limit.

I emphasized that, for fundamental reasons, anomalous magnetic moments of *current* quarks vanish, while *constituent* quarks may have $\kappa_q \neq 0$. In view of the GDH sum rule, this leads to an apparent inconsistency between constituent-quark models and QCD. I argued that this paradox can be remedied by recalling that the non-resonant background is missing from the GDH integral in constituent-quark models. Thus, the putative modification constant is just this missing portion of the integral, and an actual modification of the sum rule must not be expected.

In Sect. 4.7, I examined a modification of the GDH sum rule claimed in the literature [16] due to "the presence of a localized region inside the nucleon, in which the electromagnetic gauge symmetry is spontaneously broken down" [16]. I argued that the proposed mechanism does actually *not* affect the GDH sum rule.

In Sect. 4.8, I pointed out that non-naive current algebras have *no* effect on the low-energy theorem of Low, Gell-Mann, and Goldberger, although this may be expected upon considering Low's derivation [36] of the theorem. In fact, the putative modification of the low-energy theorem is remedied by the contribution from the seagull amplitude that is induced by the non-naive commutator. I want to stress that caution is advisable as soon as statements on *amplitudes* are derived from current commutators. Very often, a commutator correction is cancelled by the corresponding seagull contribution. Generally, this signalizes that there is a less awkward approach. (The Bjorken sum rule [17] presents another example of this rule of thumb.)

### The state of the art

The results of Ch. 4 may be summarized by stating that none of the hitherto proposed modifications of the GDH sum rule survives a close inspection. This is the state of the art after three decades of hard thinking! Thus, I conclude that a modification appears highly unlikely.

### $Q^2$ evolution of the GDH sum rule

In Ch. 5, I investigated the $Q^2$ evolution of the GDH sum rule, i.e., the transition from polarized photoabsorption to polarized inclusive electroproduction in the one-photon-exchange approximation. After introducing the pertinent kinematics and cross sections, I discussed different generalizations of the GDH integral, which coincide both at the real-photon point $Q^2 = 0$ and in the scaling limit $Q^2 \to \infty$. In particular, I considered the standard integral $I_1(Q^2)$ defined in Eq. (5.41a) in terms of structure function $G_1(\nu, Q^2)$, as well as the more physically motivated integrals $I_{\text{GDH}}(Q^2)$ and $\tilde{I}_{\text{GDH}}(Q^2)$ defined in Eqs. (5.43), which involve the linear combination $G_1(\nu, Q^2) - \frac{Q^2}{M\nu} G_2(\nu, Q^2)$. As demonstrated in Sect. 5.1.2, this combination is directly related to the absorption of transversely polarized virtual photons, while structure function $G_1(\nu, Q^2)$ also incorporates the interference between longitudinal and transverse photon polarizations.

At $Q^2 = 0$, the so-defined integrals coincide due to explicit factors of $Q^2$. At $Q^2 \to \infty$, they coincide due to the scaling behavior of $G_2(\nu, Q^2)$. I emphasized that at intermediate $Q^2$, substantial deviation must be expected. Therefore, in Sect. 5.2.1, I studied the limits $Q^2 \to 0$ and $Q^2 \to \infty$ beyond leading terms. Using phenomenologically plausible assumptions, I showed that indeed the *slopes* at $Q^2 = 0$, i.e., $I_1'(0)$, $I_{\rm GDH}'(0)$ and $\tilde{I}_{\rm GDH}'(0)$, considerably differ in the case of the proton (presently, data are to poor to consider the neutron case), see Eqs. (5.63), (5.64) and Fig. 5.4.

As for the high-$Q^2$ region, I considered higher-twist corrections to generalized GDH integrals $I_1(Q^2)$, $I_{\rm GDH}(Q^2)$, and $\tilde{I}_{\rm GDH}(Q^2)$. In Eqs. (5.66), these corrections are expressed in terms of coefficients $a_2$, $d_2$, and $f_2$ [128], which represent nucleon matrix elements of operators of twist two, three, and four, respectively. Thus, depending on the relative strength of these coefficients, conspicuous deviations among the generalized GDH integrals have to be expected also in the high-$Q^2$ domain.

I conclude that due to the vital role of structure function $G_2(\nu, Q^2)$, one should pay more attention to the differences between the generalized GDH integrals $I_1(Q^2)$, $I_{\rm GDH}(Q^2)$, and $\tilde{I}_{\rm GDH}(Q^2)$. Further theoretical and experimental investigation of longitudinal photon coupling is desirable.

In Sect. 5.3, the contribution from single-pion production to polarized inclusive electroproduction has been investigated. Expressions in terms of different sets of amplitudes and multipoles have been derived. The isospin decomposition has been reviewed. Sect. 5.3 gives a collection of formulae as a basis for investigating the saturation of diverse generalizations of the GDH integral.

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