hep–th/9609167
BONN–TH–96–11
IASSNS–HEP–96/91

(0,2) Mirror Symmetry

Ralph Blumenhagen[1], Rolf Schimmrigk[2] and Andreas Wißkirchen[3]

[1] *School of Natural Sciences, Institute for Advanced Study,*
Olden Lane, Princeton NJ 08540, USA

[2,3] *Physikalisches Institut der Universität Bonn,*
Nußallee 12, 53115 Bonn, Germany

Abstract

We generalize the previously established (0,2) triality of exactly solvable models, Landau–Ginzburg theories and Calabi–Yau manifolds to a number of different classes of (0,2) compactifications derived from (2,2) vacua. For the resulting models we show that the known (2,2) mirror constructions induce mirror symmetry in the (0,2) context.

9/96

[1] e–mail: blumenha@sns.ias.edu

[2] e–mail: netah@avzw02.physik.uni–bonn.de

[3] e–mail: wisskirc@avzw02.physik.uni–bonn.de

1. Introduction

The discovery of mirror symmetry among Calabi–Yau manifolds seven years ago [1,2] has led to a number of developments which have recently culminated in the circle of ideas related to strong–weak coupling dualities [3–6] and the unification of string vacua [7–9]. The framework of (2,2) compactifications in which mirror symmetry has been formulated is rather restricted however. It has been believed for some time that the class of (2,2) supersymmetric ground states defines only a small part of the moduli space of all consistent string vacua with $N = 1$ spacetime supersymmetry. A natural question therefore is whether mirror symmetry can be extended to (0,2) supersymmetric theories or whether it is indeed an artefact of the restriction to the space of (2,2) vacua, irrelevant for more generic vacua. It is this question which we address in the present paper.

Our discussion will focus primarily on exactly solvable models and Landau–Ginzburg theories. To prepare the ground we first extend the results of [10] concerning the (0,2) triality of string compactification by identifying the Landau–Ginzburg and manifold phases of a subclass of exact (0,2) models constructed in [11]. The (0,2) theories considered there are derived by applying the method of simple currents [12] to the Gepner class of (2,2) tensor models. By choosing the simple currents appropriately the (2,2) world sheet supersymmetry can be broken to (0,2) supersymmetry, necessary and sufficient for $N = 1$ spacetime supersymmetry.

A second longstanding problem in the context of (0,2) compactification of the heterotic string has been the question whether (0,2) $\sigma-$models renormalize to nontrivial conformally invariant fixed points and if so, whether these fixed points are exactly solvable. A related question is whether such (0,2) theories are in fact consistent or whether they are destabilized by instanton corrections [13]. This problem was addressed in [14], where criteria for stability were formulated in the context of the linear $\sigma-$model.

In [10] we solved this problem by establishing (0,2) triality for a simple (0,2) $\sigma-$model whose data can be summarized by its stable bundle structure

$$V_{(1,1,1,1,1;5)} \rightarrow \mathbb{P}_{(1,1,1,1,2,2)}[4\ \ 4] \tag{1.1}$$

over a codimension two Calabi–Yau manifold. It was shown there that the underlying conformal field theory of this $\sigma-$model is described by the exactly solvable theory derived from the Gepner model of five minimal factors at level three, the (2,2) 'quintic' model, in combination with a supersymmetry breaking simple current. This example thus not only gives an independent proof for the existence of conformal fixed points for this type of models but also provides its precise exactly solvable structure. The results of [10] indicate that for this class of (0,2) models the exactly solvable theory directly determines the bundle structure while the structure of the base space is less apparent. In Sections 2–7 we generalize the considerations of [10] to several classes of models. It will become clear that the application of certain types of simple currents amounts to defining maps which result in (0,2) daughter theories for particular types of (2,2) vacua. These maps can be abstracted from the exactly solvable framework and applied to more general Landau–Ginzburg compactifications. Proceeding in this way we construct the first large class of (0,2) Landau–Ginzburg theories.

Armed with (0,2) triality we turn to the discussion of mirror symmetry. In the context of (2,2) Calabi–Yau compactification mirror symmetry usually is formulated as an isomorphism between conformal field theories which, in the large radius limit, leads to an exchange of the Kähler deformation of $\mathrm{H}^{(1,1)}(M)$ and the complex deformations of $\mathrm{H}^{(2,1)}(M)$. Using the Kähler class and the holomorphic 3-form of the Calabi–Yau manifold these cohomology groups are seen to be in 1–1 correspondence with the antigenerations $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM^*)=\mathrm{H}^{(1,1)}(M)$ and generations $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM)=\mathrm{H}^{(2,1)}(M)$. The mirror hypothesis thus states that for every ground state M there exists another vacuum $\widetilde{M}$ such that $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM^*)=\mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},T\widetilde{M})$ and $\mathrm{H}_{\bar\partial}^{(0,1)}(M,TM)=\mathrm{H}_{\bar\partial}^{(0,1)}(M,T\widetilde{M}^*)$.

In (0,2) compactifications the gauge connection is not embedded in the spin connection, hence the gauge bundle is not identified with the tangent bundle of the manifold. We can therefore ask whether for every consistent (0,2) vacuum described by a stable bundle $(V \longrightarrow M)$ over a Calabi–Yau manifold M, with generations in $\mathrm{H}_{\bar\partial}^{(0,1)}(M,V)$ and antigenerations in $\mathrm{H}_{\bar\partial}^{(0,1)}(M,V^*)$, there exists another stable bundle $\widetilde{V} \longrightarrow \widetilde{M}$ such that

$$\begin{aligned} \mathrm{H}_{\bar\partial}^{(0,1)}(M,V^*) &= \mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},\widetilde{V}) \\ \mathrm{H}_{\bar\partial}^{(0,1)}(M,V) &= \mathrm{H}_{\bar\partial}^{(0,1)}(\widetilde{M},\widetilde{V}^*). \end{aligned} \tag{1.2}$$

As in the (2,2) framework the implication here is that the underlying conformal field theories are isomorphic. If this is the case we will call the two bundles $(V \longrightarrow M, \widetilde{V} \longrightarrow \widetilde{M})$ a (0,2) mirror pair.

In the exactly solvable context mirror symmetry can be seen trivially in terms of the order–disorder duality in each of the $N=2$ minimal factors, leading to the usual reversal of the charges. The implementation of this operation does not act on the string vacuum per se and therefore does not lead, in itself, to any insight. Since our construction of (0,2) models derives from (2,2) theories it is natural to ask whether the known mirror construction in the (2,2) context induce mirror pairs for the resulting (0,2) framework. This is indeed the case. Our strategy to obtain mirror pairs of (0,2) models can thus be summarized independently of any particular framework via the diagram

$$\begin{array}{ccc} (0,2) & \longrightarrow & (2,2) \\ \big\downarrow & & \big\downarrow \\ \text{mirror } (0,2) & \longleftarrow & \text{mirror } (2,2) \end{array} \tag{1.3}$$

We review in Section 8 those of the known (2,2) mirror constructions which we generalize in Section 9 to the framework of (0,2) vacua. We will see that (0,2) theories derived from (2,2) mirror pairs induce (0,2) mirror pairs. In the last Section we discuss some open problems and directions for future work. In an appendix we collect the three generation models that result from our constructions.

2. (0,2) Triality

In [11] we proposed a Gepner like construction of string models featuring $(0,2)$ world sheet supersymmetry. This construction was based on the so-called simple current method to obtain heterotic modular invariant partition functions. We obtained a variety of models with different kinds of gauge groups and massless spectra.

(0,2) vacua have been constructed in a number of different contexts but what has been missing was the analog of (2,2) triality: the identification of the underlying exactly solvable models of (0,2) Calabi–Yau manifolds defined by stable bundles with a Landau–Ginzburg phase. In [10] we described such an identification for a (0,2) daughter of the $\left(3^5\right)_{A^5}$ Gepner model based on the simple current[1]

$$J = (0\ 5\ 1)(0\ 0\ 0)^4(1)(0). \tag{2.1}$$

The (0,2) Calabi–Yau theory is defined by the bundle

$$V_{(1,1,1,1,1;5)} \rightarrow \mathbb{P}_{(1,1,1,1,2,2)}[4\ \ 4]\Big|_{(80,0)} \tag{2.2}$$

[1] The notation means $J = \prod_{i=1}^{5}(l_i\, m_i\, s_i)_{N=2}\,(n)_{U(1)_2}(\phi)_{SO(8)}$, where $(l_i\, m_i\, s_i)_{N=2}$ are the quantum numbers of the i^{th} $N=2$ minimal superconformal factor.

constructed from the exact sequence

$$0 \to \ V \to \bigoplus_{a=1}^{5} \mathcal{O}(1) \to \mathcal{O}(5) \to 0, \tag{2.3}$$

over the threefold configuration $\mathbb{P}_{(1,1,1,1,2,2)}[4\ \ 4]$. Here the subscripts denote the number of generations and antigenerations. Not only did the massless spectra agree but also the $\langle 10 \cdot 16 \cdot 16 \rangle$ Yukawa couplings and the spacetime R charges. In [15] the moduli space of this model has been investigated in some detail.

In [10] we confined our analysis to stable bundles defined over Calabi–Yau manifolds which are not only quasismooth but smooth. This left open the problem of establishing (0,2) triality in a more systematic fashion. The first step in the present paper is to remedy this situation. In the process we will construct special classes of $(0,2)$ models with gauge group $SO(10)$, $SU(5)$ and E_3, which will turn out to provide a good testing ground for a number of aspects of $(0,2)$ models, such as mirror symmetry. Before describing a particular class of models we analyze a quasismooth model in some detail.

2.1. A quasismooth $(0,2)$ model

Consider the (0,2) daughter based on the diagonal Gepner model $\left(3 \cdot 8^3\right)_{A^4}$ and the simple current as before

$$J = (0\ 5\ 1)(0\ 0\ 0)^3(1)(0). \tag{2.4}$$

The gauge group is $SO(10) \times SU(2) \times U(1)^3$ with $N_{16} = 113$ generations and $N_{\overline{16}} = 5$ antigenerations. More details of the massless spectrum of this model are summarized in Table 1.

$SO(10)$ Rep.	**1**	**10**	**16**	$\overline{\mathbf{16}}$
Spin 0	561	108	113	5
Spin 1	6	0	0	0

Table 1: *Spectrum of the* $\left(3 \cdot 8^3\right)_{A^4}$ *daughter.*

By carrying out the same analysis of the 113 untwisted states as in [10] one finds that the $(0,2)$ chiral ring is generated by the fields and constraints encoded in the following bundle and threefold data

$$V_{(1,1,1,2,5;10)} \to \mathbb{P}_{(1,1,1,4,4,5)}[8\ \ 8]\Big|_{(113,5)}. \tag{2.5}$$

These bundle data satisfy $c_1(V_{(1,1,1,2,5;10)}) = 0$ as well as the anomaly cancellation condition $c_2(V_{(1,1,1,2,5;10)}) = c_2(T)$ and by following [16] one finds that the linear $\sigma-$model leads to a well behaved Landau–Ginzburg phase for $r \ll 0$. For $r \gg 0$ one still obtains a compact parameter space but the threefold has both an orbifold and a hypersurface singularity at $(0,0,0,0,0,1)$. An interesting question is, whether one can generalize the algorithm for resolving $(0,2)$ orbifold singularities presented in [17] in such a way as to obtain agreement with the spectrum in the Landau–Ginzburg phase. The spectrum of the latter phase can be obtained by the methods in [18,19]. In Table 2 we collect the left and right $U(1)$ charges of the chiral fields and Fermi fields in the linear $\sigma-$model.

Field	$\Phi^{1,2,3}$	$\Phi^{4,5}$	Φ^6	$\lambda^{1,2,3}$	λ^4	λ^5	$\sigma^{1,2}$
q_l	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$-\frac{9}{10}$	$-\frac{4}{5}$	$-\frac{1}{2}$	$-\frac{4}{5}$
q_r	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$

Table 2: *Left and right charges of the fields in the linear $\sigma-$model.*

Furthermore we need the ground state energies and charges of the twisted sectors of the Landau–Ginzburg orbifold. These are contained in Table 3.

l	0	1	2	3	4	5	6	7	8	9	10
q_l	-2	0	2	$-\frac{11}{10}$	$\frac{9}{10}$	$-\frac{7}{5}$	$\frac{3}{5}$	$-\frac{1}{10}$	$-\frac{1}{2}$	$-\frac{6}{5}$	$-\frac{4}{5}$
q_r	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{3}{5}$	$-\frac{3}{5}$	$-\frac{9}{10}$	$-\frac{9}{10}$	$-\frac{3}{5}$	0	$\frac{3}{10}$	$-\frac{3}{10}$
E	0	-1	0	$-\frac{13}{20}$	$-\frac{1}{5}$	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{7}{20}$	0	$-\frac{2}{5}$	0

Table 3: *Ground state energies and charges of the twisted sectors.*
The sectors $l = 11, \ldots, 19$ are determined by CPT invariance.

Now, we choose the hypersurfaces W_j and bundle constraints F_a as

$$\begin{gathered} F_1 = \Phi_1^9 \qquad F_2 = \Phi_2^9 \qquad F_3 = \Phi_3^9 \qquad F_4 = \Phi_4^2 \qquad F_5 = \Phi_6 \\ \text{and} \qquad W_1 = \Phi_5^2 \qquad W_2 = \Phi_4 \Phi_5 . \end{gathered} \tag{2.6}$$

The left handed **16**s in the untwisted sector are given by polynomials of degree ten modulo

$$P_{10}(\Phi_0^i) \sim P_{10}(\Phi_0^i) + c^a\, F_a(\Phi_0^i) + s^j\, W_j(\Phi_0^i). \tag{2.7}$$

This leaves exactly 113 generations. Right handed **10**s are given by degree twenty polynomials modulo

$$P_{20}(\Phi_0^i) \sim P_{20}(\Phi_0^i) + c^a\, F_a(\Phi_0^i) + s^j\, W_j(\Phi_0^i), \tag{2.8}$$

yielding $N_{10} = 105$ vectors from the untwisted sector. The $l = 4$ sector contains 3 left handed **10**s, $\lambda^{1,2,3}_{-\frac{1}{5}}|0\rangle$ and 3 left handed $\overline{\mathbf{16}}$s, $\Phi^{1,2,3}_{-\frac{1}{5}}|0\rangle$. The remaining 2 antigenerations appear in the $l = 6$ sector, $\Phi^{4,5}_{-\frac{1}{5}}|0\rangle$. Thus the R–R sectors of the SCFT and LG model are in complete agreement. With a little more effort one can show that also the NS–R sectors, in particular the number of singlets, agree. The numbers of left moving singlets $((q_l, q_r) = (0, -\frac{1}{2}))$ are listed in Table 4.

l	1	3	5	7
Number	$465 + n_g$	48	30	12

Table 4: *Left moving singlets in different sectors. n_g denotes the dimension of the special enhanced gauge group.*

For the choice of the W_j and F_a above we get $n_g = 6$ and thus the desired $N_1 = 561$ massless singlets.

We conclude that the above SCFT based on the $\left(3 \cdot 8^3\right)_{A^4}$ Gepner parent describes a certain point in the LG phase of the linear $\sigma-$model with the data (2.5). Inspection of the untwisted $\overline{\mathbf{16}}$s reveals that the introduction of the simple current in the partition function corresponds to the following modification of the geometric data. First, choose the bundle data of the would-be $(0,2)$ model to be the Calabi–Yau data of the $(2,2)$ model corresponding to the parent Gepner model. In the case at hand the $\left(3 \cdot 8^3\right)_{A^4}$ Gepner model lives on the CY manifold $\mathbb{P}_{(1,1,1,2,5)}[10]$, where the $K = 3$ tensor factor corresponds to the coordinate of weight $k = 2$ in the ambient space and we have introduced a trivial factor. The CY base manifold of the $(0,2)$ model then is obtained by replacing the coordinate corresponding to the $K = 3$ tensor factor by two fields whose weight is twice that of the original coordinate. Finally, the two hypersurfaces have equal weight. This operation which defines a (0,2) theory in terms of the data of certain (2,2) theories can be generalized to the class of Gepner models and, more generally, to all Landau–Ginzburg theories with an appropriate scaling field.

2.2. Odd Theories

The application of the supersymmetry breaking simple current to the exactly solvable model can be encoded directly on the Landau–Ginzburg theory by a generalization of the above operation. We will abbreviate our construction by calling it a 'move'. For the class of Gepner models the prescription is as follows.

Choose a Gepner model which contains at least one factor with odd $K = 2\ell - 1$. Let us assume this factor to be the first one. Let d be the lowest common multiple of the numbers $\{K_1{+}2, K_2{+}2, K_3{+}2, K_4{+}2, K_5{+}2\}$. For models with only four factors set $K_5 = 0$. Then the analysis of the chiral ring reveals that a model obtained by using the following simple currents in the diagonal Gepner parent model

$$J = (0\ K{+}2\ 1)(0\ 0\ 0)^4(1)(0) \tag{2.9}$$

corresponds to a linear $\sigma-$model with the following data

$$\begin{array}{c} V_{\left(\frac{d}{2\ell+1},\,\frac{d}{K_2+2},\,\frac{d}{K_3+2},\,\frac{d}{K_4+2},\,\frac{d}{K_5+2};d\right)} \longrightarrow \\ \mathbb{P}_{\left(\frac{2d}{2\ell+1},\frac{\ell d}{2\ell+1},\frac{d}{K_2+2},\frac{d}{K_3+2},\frac{d}{K_4+2},\frac{d}{K_5+2}\right)}\left[\frac{(\ell+2)d}{2\ell+1},\,\frac{2\ell d}{2\ell+1}\right]. \end{array} \tag{2.10}$$

This set of exactly solvable $(0,2)$ theories can be regarded as the $SO(10)$ analog of the Gepner class of models in the E_6 case. The generalization to Gepner parents with more than five factors or $D-$ and $E-$invariants is straightforward. For instance, the Gepner parent $\left(1^3 \cdot 3^2 \cdot 8\right)_{A^6}$ with one simple current acting on one of the $K = 3$ tensor factors, gives rise to a $(0,2)$ model with input data

$$V_{(3,6,6,10,10,10,15;30)} \rightarrow \mathbb{C}_{(3,6,10,10,10,12,12,15)}[24\ \ 24]. \tag{2.11}$$

We have calculated the massless spectra of all SCFTs in this class. Figure 1 shows $(N_{16} - N_{\overline{16}})$ and $(N_{16} + N_{\overline{16}})$ for all the resulting 455 models which lead to 140 distinct spectra.

Similar to the $(2,2)$ case the class of all Gepner models is not mirror symmetric [20,21]. To get a mirror symmetric plot one had to consider all phase orbifolds including those with discrete torsion [22].

Streamlining the notation, the SCFT analysis has lead us to consider Calabi–Yau hypersurface configurations $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d]$ in weighted projective fourspace with degree $d = \sum_i k_i$, such that at least one of the weights k_i divides the degree as $d/k_i = (2\ell{+}1)$. We will characterize such coordinates and the configurations which contain them, as being 'odd'. For any odd configuration $\mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d]$, for which we can arrange the odd coordinate to be z_1, we define a (0,2) daughter theory by considering the following vector bundle configuration

$$V_{(k_1,k_2,k_3,k_4,k_5;d)} \rightarrow \mathbb{P}_{(2k_1,\ell k_1,k_2,k_3,k_4,k_5)}[(\ell{+}2)k_1\ \ 2\ell k_1], \tag{2.12}$$

where $V_{(k_1,k_2,k_3,k_4,k_5;d)}$ describes a stable bundle over the Calabi–Yau threefold with weights k_i describing the charges of the gauge fermions defined as sections of this bundle. It can be checked that the anomalies of these (0,2) models cancel.

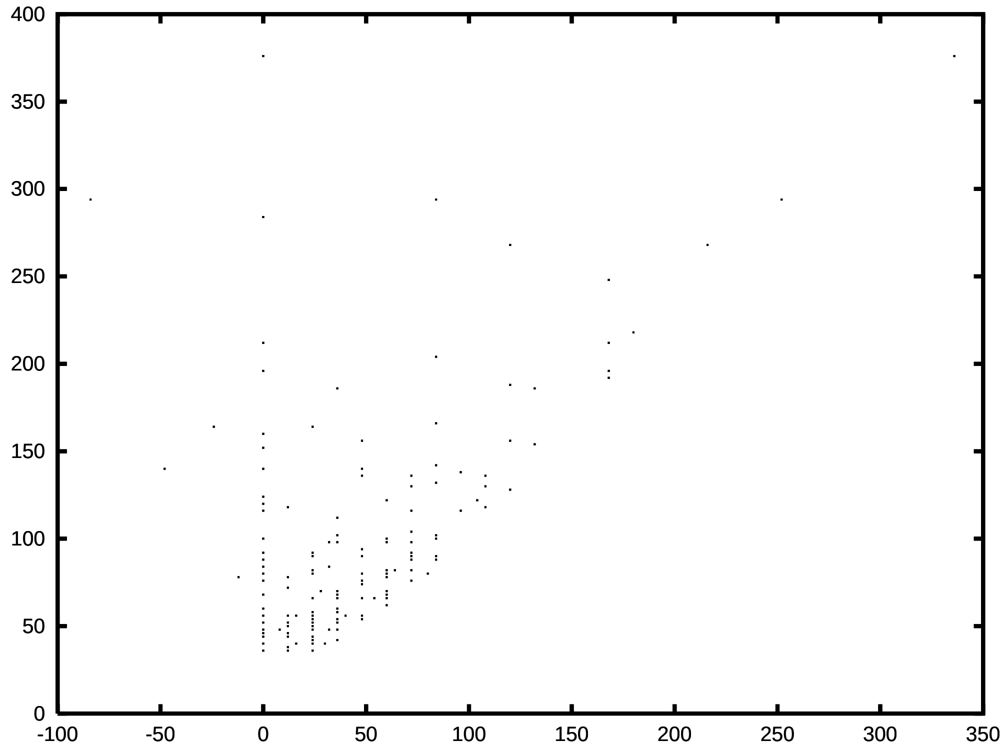

Figure 1: $n_{16} + n_{\overline{16}}$ vs. $n_{16} - n_{\overline{16}}$ for the 140 CFT spectra.

3. Pairing of generations and antigenerations

The identification of the model (2.5) was fairly straightforward. The following example shows that more involved things may happen. Consider the Gepner parent model $(1 \cdot 7 \cdot 25 \cdot 52)_{\mathrm{A^3D}}$, where the index D indicates that for the level $K = 52$ minimal $N = 2$ theory we have chosen the nondiagonal affine $D-$invariant. Applying the move to the $K = 25$ tensor factor should correspond to the following model

$$V_{(1,1,3,9,13;27)} \longrightarrow \mathbb{P}_{(1,2,3,9,13,13)}[15 \quad 26]. \tag{3.1}$$

The spectrum of the superconformal field theory is $(N_{16}, N_{\overline{16}}) = (119, 11)$. The Landau–Ginzburg computation reveals that in the $l = 18$ twisted sector something new appears. The ground state energy is $E = 0$ with charges $(q_l, q_r) = (-\frac{1}{9}, -\frac{11}{18})$. The zero modes are

$$\Phi_0^3,\ \Phi_0^4,\ \overline{\lambda}_0^3,\ \overline{\lambda}_0^4,\ \overline{\sigma}_0^2. \tag{3.2}$$

In the $q_l = 1$ sector we obtain the following sequence

$$0 \longrightarrow \begin{array}{l} \left(\Phi_0^3\right)^2 \left(\overline{\lambda}_0^3\right) \\ \left(\Phi_0^3\right)^4 \left(\overline{\lambda}_0^4\right) \\ \left(\Phi_0^3\right) \left(\Phi_0^4\right) \left(\overline{\lambda}_0^4\right) \\ \left(\Phi_0^3\right)^5 \left(\overline{\sigma}_0^2\right) \\ \left(\Phi_0^3\right)^2 \left(\Phi_0^4\right) \left(\overline{\sigma}_0^2\right) \end{array} \longrightarrow \begin{array}{l} \left(\Phi_0^3\right)^1 0 \\ \left(\Phi_0^3\right)^7 \left(\Phi_0^4\right) \\ \left(\Phi_0^3\right)^4 \left(\Phi_0^4\right)^2 \\ \left(\Phi_0^3\right) \left(\Phi_0^4\right)^3 \end{array} \longrightarrow 0 \tag{3.3}$$

with right moving charges

$$0 \longrightarrow \left(1, -\frac{1}{2}\right) \longrightarrow \left(1, \frac{1}{2}\right) \longrightarrow 0. \tag{3.4}$$

Now, choosing the BRST operator relevant in this sector as

$$\overline{Q}_+ = \lambda_0^3 \left(\Phi_0^3\right)^8 + \lambda_0^4 \left(\Phi_0^4\right)^2 + \sigma_0^2 \left(\Phi_0^3\right)^2 \left(\Phi_0^4\right) \tag{3.5}$$

one gets in this sector exactly one massless field with $(q_l, q_r) = (1, -\frac{1}{2})$. Taking also all the other sectors into account one ends up with the spectrum $(N_{16}, N_{\overline{16}}) = (118, 10)$, which is not expected from the SCFT. However, analogously to the model discussed at the beginning, if we choose the BRST operator to be

$$\overline{Q}_+ = \lambda_0^3 \left(\Phi_0^3\right)^8 + \lambda_0^4 \left(\Phi_0^4\right)^2 + \sigma_0^2 \left(\left(\Phi^2_{-\frac{2}{3}}\right) \left(\Phi^6_{\frac{2}{3}}\right) + \left(\Phi^2_{\frac{2}{3}}\right) \left(\Phi^6_{-\frac{2}{3}}\right)\right) \tag{3.6}$$

the field $\left(\Phi_0^3\right)^7 \left(\Phi_0^4\right)$ is not any longer in the image of $\overline{Q}_+$ and the overall massless spectrum becomes $(N_{16}, N_{\overline{16}}) = (119, 11)$. Thus, this example reveals for the first time an effect for $(0,2)$ models which was long believed to exist. The number of generations and antigenerations is not necessarily constant over the moduli space. This is not an exception, for approximately 13% of all SCFT models have higher numbers (up to 3) of generations than those models obtained by minimal choices of the constraints in the LG models. The number of net generations is constant however.

4. (0,2) Theories from (2,2) Orbifolds

So far we have focused on $(2,2)$ parent models given by Gepner models. We can, however, also establish (0,2) triality for more general models, such as orbifolds. Consider, e. g., the $(2,2)$ orbifold

$$\left(3^5\right)_{A^5} \Big/ \mathbb{Z}_5. \tag{4.1}$$

We will see in Section 7 that this orbifold can alternatively be described, via the fractional transform, as the weighted configuration

$$\mathbb{P}_{(3,4,4,4,5)}[20] = \mathrm{FT}\,(\mathbb{P}_4[5]/\mathbb{Z}_5 : [0\ 4\ 1\ 0\ 0])\,. \tag{4.2}$$

Applying the move to one of the $K = 4$ coordinates leads to

$$V_{(3,4,4,4,5;20)} \longrightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16\ \ 16]\Big|_{(42,2)}. \tag{4.3}$$

We can obtain a superconformal field theory in the moduli space of this model by first implementing the simple current which generates the orbifolds and then break the (2,2) supersymmetry as in the previous models. This leads to the modification of the partition function with

$$\begin{aligned} J_1 &= (0\ 0\ 0)(0\ \ -2\ 0)(0\ 2\ 0)(0\ 0\ 0)^2(0)(0) \\ J_2 &= (0\ 5\ 1)(0\ 0\ 0)^4(1)(0). \end{aligned} \tag{4.4}$$

Indeed, this SCFT has $(N_{16}, N_{\overline{16}}) = (42, 2)$. This model will be reconsidered in the discussion about mirror symmetry.

Another example is provided by the $\left(1 \cdot 5 \cdot 82^2\right)_{A^4}$ parent model describing a point in the moduli space of the threefold configuration $\mathbb{P}_{(1,1,12,28,42)}[84]$. Via fractional transformations one obtains the weighted representation of the orbifold as

$$\mathbb{P}_{(2,3,24,55,84)}[168] = \mathrm{FT}\left(\mathbb{P}_{(1,1,12,28,42)}[84]/\mathbb{Z}_3 : [2\ 0\ 0\ 1\ 0]\right), \tag{4.5}$$

leading to the $(0, 2)$ model

$$V_{(2,3,24,55,84;168)} \longrightarrow \mathbb{P}_{(2,3,48,55,72,84)}[120\ \ 144]\Big|_{(128,32)}. \tag{4.6}$$

The superconformal theory of this model is given by the above Gepner parent modified by the simple currents

$$\begin{aligned} J_1 &= (0\ \ -2\ 0)(0\ 0\ 0)(0\ 56\ 0)(0\ 0\ 0)(0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 7\ 1)(0\ 0\ 0)(0\ 0\ 0)(1)(0). \end{aligned} \tag{4.7}$$

There exist many other examples. Thus, we have found a large class of $(0, 2)$ models featuring the desired CFT/LG/CY triality of description. Further work has to be done to gain a better understanding of the Calabi–Yau phase when the resulting models have hypersurface singularities.

5. SCFTs with gauge group SU(5) and E_3

In [10] we have generalized our $SO(10)$ analysis to models with gauge group $SU(5)$ by considering the (0,2) theory derived from the $\left(3^5\right)_{A^5}$ tensor model via the simple currents

$$\begin{aligned} J_1 &= (0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)^3(1\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)^3(0\ 1)(0). \end{aligned} \tag{5.1}$$

The $(0,2)$ LG theory associated to this exactly solvable model is defined by the following data

$$V_{(0,1,1,1,1,1;5)} \to \mathbb{P}_{(1,1,1,2,2,2,2,5)}[4\ \ 4\ \ 4\ \ 4]\Big|_{(N_{10}=64,N_{\overline{10}}=0)}. \tag{5.2}$$

Here we introduce a new constraint F_1 of weight $k=5$ which resolves the nontransversality of the coordinate of weight $k=5$, implying that the quadratic term $\lambda_1\,\phi_8$ in the superpotential gives no contributions to the cohomology. This constraint furthermore guarantees that the central charge of the internal CFT sector is $(c_l,c_r)=(11,9)$, as needed for a rank 5 gauge bundle. The generalization of the move to these models is obvious, one simply applies it twice on each of the coordinates separately.

Consider the $\left(1\cdot 3^3\cdot 13\right)_{A^5}$ parent model. The simple currents in (5.1) then yield a SCFT with $(N_{10},N_{\overline{10}})=(49,1)$. The same spectrum can be found for the LG model

$$V_{(0,1,3,3,3,5;15)} \longrightarrow \mathbb{P}_{(1,3,5,6,6,6,6,15)}[12\ \ 12\ \ 12\ \ 12]\Big|_{(N_{10}=49,N_{\overline{10}}=1)}. \tag{5.3}$$

Another example is provided by the $\left(1^2\cdot 3\cdot 7\cdot 43\right)_{A^5}$ parent with the simple currents

$$\begin{aligned} J_1 &= (0\ 0\ 0)^2(0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)(1\ 0)(0) \\ J_2 &= (0\ 0\ 0)^2(0\ 0\ 0)(0\ 9\ 1)(0\ 0\ 0)(0\ 1)(0), \end{aligned} \tag{5.4}$$

which corresponds to the LG model

$$V_{(0,1,5,9,15,15;45)} \longrightarrow \mathbb{P}_{(1,10,15,15,18,18,20,45)}[30\ \ 36\ \ 36\ \ 40]\Big|_{(N_{10}=36,N_{\overline{10}}=12)}. \tag{5.5}$$

Finally, the generalization to $E_3=SU(3)\times SU(2)$ is straightforward. The following three simple currents

$$\begin{aligned} J_1 &= (0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)^2(1\ 0\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)(0\ 0\ 0)^2(0\ 1\ 0)(0) \\ J_3 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 5\ 1)(0\ 0\ 0)^2(0\ 0\ 1)(0), \end{aligned} \tag{5.6}$$

applied to the $(3)^5$ Gepner parent yield the spectrum $(N_6, N_{\overline{6}}) = (50,0)$. This corresponds to the LG model

$$V_{(0,0,1,1,1,1,1;5)} \longrightarrow \mathbb{P}_{(1,1,2,2,2,2,2,2,5,5)}[4\ \ 4\ \ 4\ \ 4\ \ 4\ \ 4]\Big|_{(N_6=50,N_{\overline{6}}=0)}, \tag{5.7}$$

in which the move has been applied three times. Since one needs three odd levels in a Gepner model to get such an E_3 model, there do not exist so many of them, in comparison.

6. (0,2) Landau Ginzburg Theories

In this Section we abstract the move map formulated above from its exactly solvable origin and generalize it to the class to $(2,2)$ LG models. A list of all possible weight combinations which allow the choice of a superpotential with an isolated singularity, defining quasismooth projective varieties, was presented in [23]. Given one (2,2) LG theory it is in general possible to define a number of different (0,2) models with different gauge groups, depending on the number of implemented moves. We discuss these possibilities in turn.

6.1. The $SO(10)$ Move

The simplest case generalizes the case in which there is only one supersymmetry breaking simple current. The resulting map can be formulated independent of any exactly solvable considerations in the LG framework as follows:

Move: Suppose one has a $(2,2)$ LG model, denoted as

$$\mathbb{C}_{(k_1,\ldots,k_{\max})}[d] \qquad \text{with } d = \tfrac{2}{\max-3}\textstyle\sum_{i=1}^{\max} k_i$$

for which one of the weights, e. g. k_1, satisfies $\frac{d}{k_1} = (2\ell+1)$ with $\ell = 1,2,\ldots$. Then the choice of the data

$$V_{(k_1,\ldots,k_{\max};d)} \longrightarrow \mathbb{C}_{(2k_1,\ell k_1,k_2,\ldots,k_{\max})}[(\ell+2)k_1\ \ 2\ell k_1]$$

guarantees anomaly cancellation.

Whether such a model really does yield a bona fide $(0,2)$ model with gauge group $SO(10)$ is a more subtle question, since depending on the weights and the constraints effects like nontransversality of the $(0,2)$ LG model or extra massless gauginos in twisted sectors may occur. The latter has been pointed out in [19] where it was considered as the LG analog of the destabilization of the vacuum by world sheet instantons in the CY phase. We will come back to this point below. Note that a move with $\ell = 1$ does not change the model, one simply gets the $(2,2)$ parent. The same can be observed in the SCFT, where the simple current (2.9) in a $K = 1$ tensor factor does not change the Gepner spectrum at all.

Concerning the transversality of the $(0,2)$ LG superpotential, one can make the following general statement. If the $(2,2)$ superpotential can be chosen as

$$W(x_i) = x_1^{2\ell+1} + P(x_2, \ldots, x_{\max}) \tag{6.1}$$

then the choice for the $(0,2)$ constraints

$$F_i = \frac{\partial P}{\partial x_i} \quad \text{for } i \in \{2, \ldots, \max\}, \qquad F_1 = y_1^{\ell},\ W_1 = y_1\, y_2,\ W_2 = y_2^2 \tag{6.2}$$

clearly is transverse. Here the coordinates y_1, y_2 are the new ones of weight $\kappa_1 = 2k_1$ and $\kappa_2 = \ell k_1$, respectively. If the coordinate x_1 also appears in P

$$W(x_i) = x_1^{2\ell+1} + x_1 x_2^{a_2} + Q(x_2, \ldots, x_{\max}) \tag{6.3}$$

then the following choice

$$F_1 = x_2^{a_2},\ F_2 = \frac{\partial Q}{\partial x_2},\ F_i = \frac{\partial Q}{\partial x_i},\ i \in \{3, \ldots, \max\}, \quad W_1 = y_1\, y_2, \quad W_2 = y_1^{\ell} + y_2^2 \tag{6.4}$$

is transverse. It may happen that Q does not depend on x_2. In this case one can either choose another polynomial of the right degree or, if this is not possible, set $F_2 = 0$. One still has (max+1) nontrivial constraints for (max+1) variables and thus the $(0,2)$ LG phase makes perfect sense. Most, but not all, such rather rare examples are inconsistent due to the appearance of extra gauginos, however[2].

In order to calculate the massless spectrum in the Landau–Ginzburg phase in a systematic manner an algorithm is needed which can be translated into a computer program. The contribution from the individual twisted sectors to the cohomology however depends on the precise form of the constraints W_j and F_a defining the stable bundle

$$V_{(n_1,\ldots,n_{r+1};\ m)} \longrightarrow \mathbb{P}_{(k_1,\ldots,k_{N_i})}\,[d_1 \cdots d_{N_c}] \tag{6.5}$$

[2] There are 26 such examples among 8027 models, 5 of which are consistent

of rank r over a complete intersection space of codimension N_c. More useful in this context is the elliptic genus whose contribution in the $\alpha^{\rm th}$ twisted sector is given by

$$Z^{\alpha}_{LG}(q,y) = {\rm Tr}_{\mathcal{H}_\alpha}\, e^{-\pi i (J_0 - \overline{J}_0)} y^{J_0}\, q^{L_0} \sim \chi_y + O(q), \tag{6.6}$$

its virtue being that it is easily computed. Using the notation of [24] the χ_y genus of a bundle of rank r can be written as

$$\chi^{\alpha}_y = (-1)^{r\alpha} \left. \frac{\prod_a (-1)^{[\alpha\nu_a]} \left(y^{\nu_a} q^{\frac{\beta_a}{2}}\right)^{\{\alpha\nu_a\}} \quad (1-y^{\nu_a} q^{\{\alpha\nu_a\}})(1-y^{-\nu_a} q^{-\beta_a})}{\prod_i (-1)^{[\alpha q_i]} \left(y^{q_i} q^{\frac{\beta_i}{2}}\right)^{\{\alpha q_i\}} \quad (1-y^{q_i} q^{\{\alpha q_i\}})(1-y^{-q_i} q^{-\beta_i})} \right|_* \tag{6.7}$$

where $|_*$ indicates the evaluation of the $q^0 y^n$, $n \in \{0,1,\ldots,r\}$ terms only and

$$\{x\} := x - [x], \quad \beta_a := \{\alpha\nu_a\} - 1, \quad \beta_i := \{\alpha q_i\} - 1. \tag{6.8}$$

Here the charges of the fields are given by $q_i = k_i/m$, $\nu_a = 1 - n_a/m$ and $\nu_{r+1+j} = d_j/m$. Implementing this in a C–code yields for the example discussed in detail in Section 2.1 the result shown in Table 5.

α	χ^α
0	$y^0 + 113y^1 - 113y^3 - y^4$
1	y^4
2	$-3y^2 + 3y^3$
3	$2y^3$
4	0
5	0
6	0
7	$-2y^1$
8	$-3y^1 + 3y^2$
9	$-y^0$
χ	$108y^1 - 108y^3$

Table 5: *The χ_y genus of $\left(3 \cdot 8^3\right)_{A^4}$.*

Using $l = 2\alpha$ one recognizes this as exactly the massless R–R spectrum obtained in Section 2.1. For models which do not lead to a pairing of generations and antigenerations and for which there are no massless gauginos coming from twisted sectors one can read off the number of generations and antigenerations directly from the χ_y genus.

There are circumstances, however, in which extra gauginos do occur. Consider, for instance, the model

$$V_{(5,5,8,9,18;45)} \longrightarrow \mathbb{P}_{(5,5,8,18,18,18)}[36 \ \ 36] \tag{6.9}$$

which contains a state with $(q_l, q_r) = (1, \frac{3}{2})$ in the $\alpha = 17$ sector, yielding gauginos in the spinor representation of $SO(10)$. As pointed out in [19], for restricted choices for the constraints this theory can still make sense, but not as a model with gauge group $SO(10)$. Unfortunately, the χ_y genus is only sensible for the right moving $U(1)$ charge modulo two and consequently can not detect such gauge symmetry enhancement. What is needed is a convenient condition for the consistency of a $(0,2)$ model. One necessary condition is not hard to formulate. To this end, we also have to take explicitly into account the right moving $U(1)$ charge. Generalizing the χ_y genus, the generating function for all states in the LG model before calculating the cohomology is

$$\chi^\alpha = \left.\frac{\prod_a \left(x^{-\rho_a} y^{\nu_a} q^{\frac{\beta_a}{2}}\right)^{\{\alpha\nu_a\}} (1 + x^{-\rho_a} y^{\nu_a} q^{\{\alpha\nu_a\}})(1 + x^{\rho_a} y^{-\nu_a} q^{-\beta_a})}{\prod_i \left(x^{-(1-q_i)} y^{q_i} q^{\frac{\beta_i}{2}}\right)^{\{\alpha q_i\}} (1 - x^{q_i} y^{q_i} q^{\{\alpha q_i\}})(1 - x^{-q_i} y^{-q_i} q^{-\beta_i})}\right|_* \tag{6.10}$$

with the abbreviation

$$\rho_a = 1 - \nu_a. \tag{6.11}$$

For simplicity let us choose a bundle with rank 4 in the following. Since the BRST operator $\overline{Q}_+$ has $(q_l, q_r) = (0, 1)$, one obtains sequences of the form

$$0 \xrightarrow{\overline{Q}_+} \left(q_l, -\frac{3}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, -\frac{1}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, \frac{1}{2}\right) \xrightarrow{\overline{Q}_+} \left(q_l, \frac{3}{2}\right) \xrightarrow{\overline{Q}_+} 0 \tag{6.12}$$

(or longer) with multiplicities

$$0 \to a_{-\frac{3}{2}} \to a_{-\frac{1}{2}} \to a_{\frac{1}{2}} \to a_{\frac{3}{2}} \to 0. \tag{6.13}$$

The BRST operator depends explicitly on the constraints

$$\overline{Q}_+ = \oint dz \sum_a \lambda_a F_a + \sum_j \sigma_j W_j. \tag{6.14}$$

However, if a sequence (6.12) in a sector contains gauginos $(q_l, q_r = \pm\frac{3}{2})$, for $q_l = 0, \pm 1$, such that $a_{q_r} > a_{q_r+1} + a_{q_r-1}$ then these gauginos are inevitable, independently of the detailed form of the constraints. If this inequality is not satisfied then it appears that one has to check the various possible choices in detail.

In Figure 2 we exhibit all $(0, 2)$ models obtained by applying the move on all odd $(2, 2)$ LG models in the table of [23][3]. We have checked all model for 'inevitable' gauginos. We have further analyzed a number of models with 'noninevitable' gauginos and have found no examples where the gauginos could not be avoided by appropriate choices for the constraints.

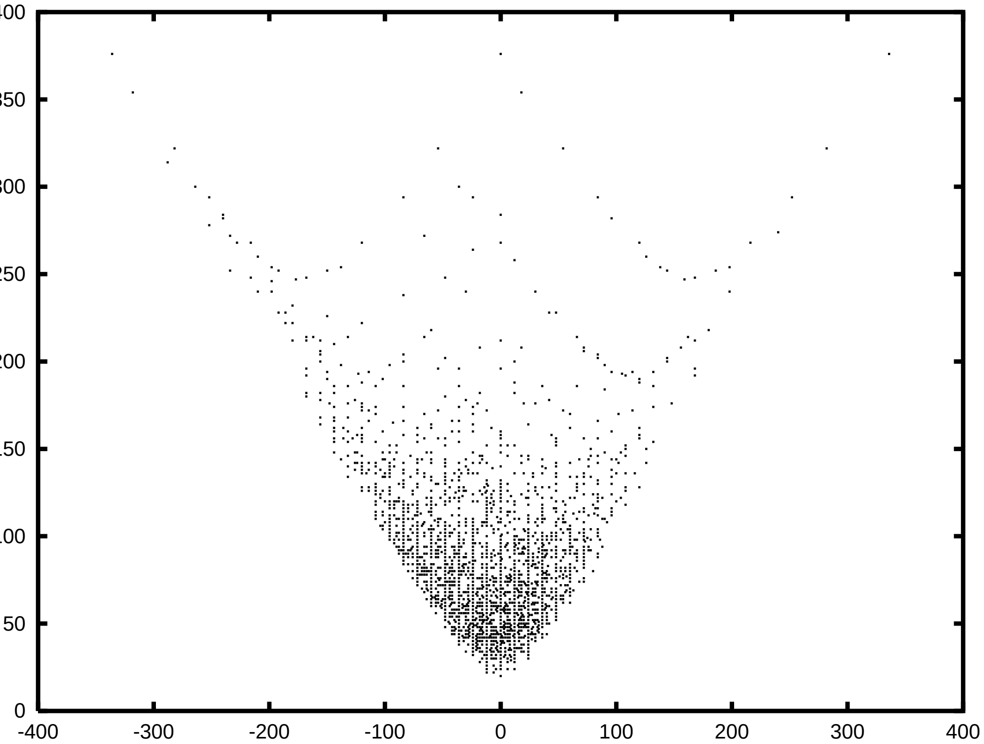

Figure 2: $(n_{16} + n_{\overline{16}})$ vs. $(n_{16} - n_{\overline{16}})$ for all 1757 $SO(10)$ Landau–Ginzburg spectra.

Assuming that all these models are bona fide $(0, 2)$ models with gauge group $SO(10)$ we find a total number of 6328 different models, leading to 1757 different pairs of generations and antigenerations. For 1560 of these 6328 models there exists a mirror partner which results from applying the 'move' to a $(2, 2)$ mirror pair. For further 3009 models there exist a model with flipped $(N_{16}, N_{\overline{16}})$. However, we are not claiming that these models really are mirror pairs for the matching of $(N_{16}, N_{\overline{16}})$ alone can also be a coincidence. Even though only 1759 models are not paired, the plot contains 845 unpaired points, which are

[3] The list of (0,2) models leading to Figure 2 (and also those with smaller gauge groups, cf. Figure 3) can be accessed on the European and US Calabi–Yau web sites [25].

48%. As we will discuss in the last Section such an asymmetry is to be expected for our class of models for a number of reasons.

6.2. $SU(5)$ and E_3

The construction described above can be generalized to smaller gauge groups $SU(5)$ and $E_3 = SU(3) \times SU(2)$ by simply applying the above move iteratively. We obtain 1216 models and 499 distinct spectra with gauge group $SU(5)$ which are shown in Figure 3. For E_3 there are 81 models and 48 distinct spectra. In this case one needs two or three odd coordinates which have to be preserved by the $(2, 2)$ mirror transformation, hence it is clear that we obtain far fewer mirror pairs than for the $SO(10)$ case. Therefore, in the following discussion we will focus on the latter case.

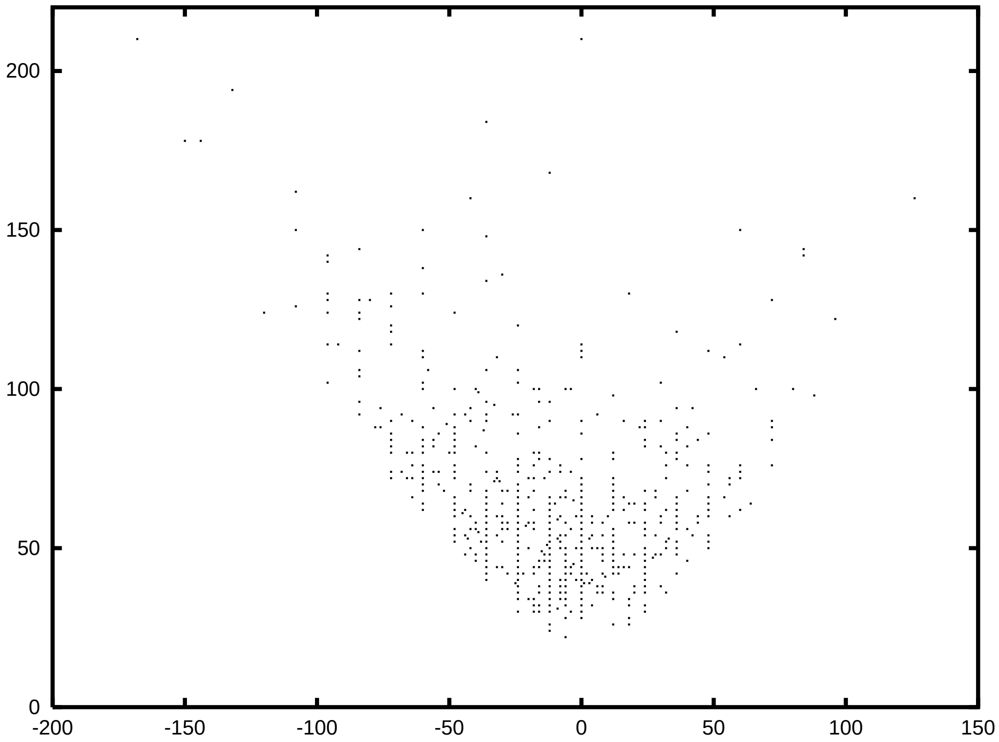

Figure 3: $(n_{10} + n_{\overline{10}})$ vs. $(n_{10} - n_{\overline{10}})$ for the 499 $SU(5)$ spectra.

We finally list the 23 $SO(10)$ and 4 $SU(5)$ three generation models in Table A.1 of the appendix.

7. Codim>1 weighted (0,2) CICYs

The class of Landau–Ginzburg vacua constructed in [1,23] contains potentials with more than five scaling variables and many of these theories have an interpretation as weighted complete intersection Calabi–Yau manifolds described by more than one polynomial. The codimension of these more general weighted CICYs is obtained from the LG theory by the total charge $Q = \sum_i q_i$, where the q_i are the weights of the variables, normalized such that the weight of the potential is unity. This total charge precisely determines the codimension of the Calabi–Yau manifold (if it exists) [20].

A simple class is provided by

$$\mathbb{C}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K] \ni \left\{ \sum_{i=1}^{2} \left(x_i^{K/k} + x_i y_i^2 \right) + \sum_{i=3}^{5} x_i^{K/k_i} = 0 \right\} \tag{7.1}$$

with total charge $Q = 2$. If z_5 is the odd coordinate with $K/k_5 = (2\ell+1)$ the move leads to the (0,2) theory

$$V_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5;2K)} \to \mathbb{C}_{(2k,K-k,2k,K-k,2k_3,2k_4,4k_5,2\ell k_5)}[2(\ell+2)k_5 \quad 4\ell k_5]. \tag{7.2}$$

In manifold speak these (2,2) Landau–Ginzburg theories lead to five-dimensional generalized Calabi–Yau varieties [26,27]

$$\mathbb{P}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K] \tag{7.3}$$

from which one can derive, via the construction of [26], the codimension two Calabi–Yau manifold

$$\begin{matrix} \mathbb{P}_{(1,1)} \\ \mathbb{P}_{(k,k,k_3,k_4,k_5)} \end{matrix} \begin{bmatrix} 2 & 0 \\ k & K \end{bmatrix}, \tag{7.4}$$

defined by polynomials

$$\begin{aligned} p_1 &= y_1^2 x_1 + y_2^2 x_2 \\ p_2 &= x_1^{K/k} + x_2^{K/k} + x_3^{K/k_3} + x_4^{K/k_4} + x_5^{K/k_5}. \end{aligned} \tag{7.5}$$

For $K/k, K/k_i \in \mathbb{N}$ this leads to quasismooth varieties.

8. (2,2) Mirror Symmetry

Our (0,2) mirror construction is based on mirror symmetry in the context of (2,2) theories. In the present Section we therefore briefly review the known mirror constructions before we proceed to generalize them to the (0,2) framework in the following Section.

There have been roughly four different types of constructions of mirror pairs of (2,2) symmetric string vacua. The most precise, if most narrow, framework is provided by considering orbifolds of exactly solvable theories [2]. More general, if less precise, are the Landau–Ginzburg mirror constructions of [28] and [29]. Finally, there exists a toric formulation [30–32] in terms of reflexive polyhedra. The focus in the present paper will be on the first three constructions.

Our first task will be to generalize the orbifolding procedure of the exactly solvable models. As in the (2,2) case we will see that this provides the most precise control over the structure of the mirror. In order to extend the orbifolding analysis of the exactly solvable framework to the more general class of Landau–Ginzburg theories we need some way to extract the weights of the fermions defining the stable bundle for the purported (0,2) mirror Landau–Ginzburg theory. There are two tools available which readily allow us to determine the structure of these gauge bundles. The first is the fractional transform [28] which maps orbifolds with respect to certain discrete group actions into weighted configurations. The second is the transposition construction of Berglund and Hübsch [29] which in particularly simple situations provides the weighted mirror as well.

8.1. Orbifold Mirrors of Gepner Models

Mirror symmetry in the exactly solvable context has been established for the class of Gepner models [33], described by tensor products $\otimes_{i=1}^{r} K_i$ of $N = 2$ superconformal minimal models. The complete set of these models, which has been constructed in [20,21] contains only very few mirror pairs. It was observed in [2], however, that orbifolding individual models with respect to the maximal discrete symmetry group produces the mirror theory.

An alternative way to obtain the exact mirror is by modifying the partition function with appropriate simple currents. The mirror of the quintic model

$$\left(3^5\right)_{A^5} \;\sim\; \mathbb{P}_4[5]^{(1,101)}, \tag{8.1}$$

for instance, can be obtained by taking a $\mathbb{Z}_5^3$ orbifold

$$\left(\mathbb{P}_4[5] \Big/ \mathbb{Z}_5^3 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix}\right)^{(101,1)} . \tag{8.2}$$

It turns out that in the SCFT this orbifold can be obtained by using the following sequence of simple currents

$$\begin{aligned} J_1 &= (0\ \ -2\ 0)(0\ 2\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_2 &= (0\ \ -2\ 0)(0\ \ -2\ 0)(0\ 4\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_3 &= (0\ \ -2\ 0)(0\ \ -2\ 0)(0\ \ -2\ 0)(0\ \ -4\ 0)(0\ 0\ 0)(0) \end{aligned} \tag{8.3}$$

in the $(3^5)_{A^5}$ model. It should be noted that in this case the simple currents can be chosen such that they are local with respect to the GSO projection.

For the tensor product

$$\left(3 \cdot 8^3\right)_{A^4} \quad \sim \quad \mathbb{P}_{(1,1,1,2,5)}[10], \tag{8.4}$$

which we have been discussing repeatedly, the mirror orbifold is obtained by modding with respect to the group $\mathbb{Z}_{10}^2$

$$\left(\mathbb{P}_{(1,1,1,2,5)}[10] \Big/ \mathbb{Z}_{10}^2 : \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.5}$$

The direct generalization of the simple currents in (8.3) to this case leads to a model with spectrum $(99,3)$ instead of $(145,1)$. We have not found a way to implement this $\mathbb{Z}_{10}^2$ orbifold with only two $\mathbb{Z}_{10}$ simple currents which are local with respect to the GSO projection. If one does not insist on locality, then the following simple currents

$$\begin{aligned} J_1 &= (0\ 0\ 0)(0\ 2\ 0)(0\ 0\ 0)(0\ 0\ 0)(0) \\ J_2 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0\ 0\ 0)(0) \\ J_3 &= (0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0) \end{aligned} \tag{8.6}$$

yield the mirror spectrum. In this case one has to use two GSO projections.

An example we will come back to is given by

$$(16 \cdot 7 \cdot 2^2 \cdot 1)_{A^5} \sim \mathbb{P}_{(2,4,9,9,12)}[36]^{(19,43)}. \tag{8.7}$$

The mirror of this model can be obtained by orbifolding with respect to the group action

$$\mathbb{Z}_3: \ [2\ 0\ 0\ 0\ 1], \tag{8.8}$$

which translates into the simple current

$$J_{orb} = (0\ \ -12\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(0\ 2\ 0)(0). \tag{8.9}$$

8.2. Weighted $(2,2)$ *Mirrors via Fractional Transformations*

Mirror symmetry is not restricted to exactly solvable theories, however. Contemporaneously with the orbifolding construction mirror symmetry was discovered in the context of Landau–Ginzburg theories [1] by explicitly constructing a large class of such models. Even though suggestive, the observed mirror symmetry could, in principle, have been coincidental because Hodge number do not identify Calabi–Yau configurations uniquely. In order to put mirror symmetry among Landau–Ginzburg vacua on firmer ground it is necessary to be able to find weighted representations of mirror orbifolds. This is what fractional transformations do [28], thereby identifying mirror pairs among weighted Calabi–Yau configurations.

The simplest starting point again is the set of exactly solvable Gepner theories of products of $N = 2$ minimal theories. It was realized in [34,16] that the Gepner class of models endowed with the diagonal affine invariant in each of the minimal factors describe the renormalization group fixed points of $N = 2$ supersymmetric Landau–Ginzburg theories with superpotentials which are of Fermat type

$$W = \sum_{i=1}^{r} \Phi_i^{a_i+2}, \tag{8.10}$$

where $\Phi_i(z,\theta,\bar{\theta}) = \phi_i(z) + \cdots$ are chiral $N = 2$ superfields. Associated to these superpotentials are affine varieties

$$\mathbb{C}_{(k_1,\ldots,k_r)}[d] \ni \{p(z_i) = \sum_i z_i^{a_i+2} = 0\} \tag{8.11}$$

defined as the zero locus of the polynomials obtained by considering constant lowest components $\phi_i = z_i$ of the superfields.

It was shown in [28] that the mirror symmetry among Landau–Ginzburg theories observed in [1] can be understood by mapping orbifold theories to weighted configurations via the iterated application of the isomorphism given by

$$\begin{array}{l} \mathbb{C}_{\left(\frac{b}{g_{ab}},\frac{a}{g_{ab}}\right)}\left[\frac{ab}{g_{ab}}\right] \ni \left\{z_1^a + z_2^b = 0\right\} \Big/ \mathbb{Z}_b : [\,(b-1) \quad 1\,] \quad \sim \\ \mathbb{C}_{\left(\frac{b^2}{h_{ab}},\frac{(a(b-1)-b)}{h_{ab}}\right)}\left[\frac{ab(b-1)}{h_{ab}}\right] \ni \left\{y_1^{\frac{a(b-1)}{b}} + y_1 y_2^b = 0\right\} \Big/ \mathbb{Z}_{b-1} : [\,1 \quad (b-2)\,] \end{array} \tag{8.12}$$

induced by the fractional transformations[4]

$$\begin{array}{ll} z_1 = y_1^{1-(\frac{1}{b})}, & y_1 = z_1^{\frac{b}{(b-1)}} \\ z_2 = y_1^{\frac{1}{b}} y_2, & y_2 = z_2 z_1^{-\frac{1}{(b-1)}}. \end{array} \tag{8.13}$$

Here g_{ab} is the greatest common divisor of a and b and h_{ab} is the greatest common divisor of b^2 and $(ab-a-b)$. The action of a cyclic group $\mathbb{Z}_b$ of order b denoted by $[m \;\; n]$ indicates that the symmetry acts like $(z_1, z_2) \mapsto (\alpha^m z_1, \alpha^n z_2)$ where α is the b^{th} root of unity.

The isomorphism (8.12) itself generates the mirror only for very few models. The simplest examples of this type are provided by manifolds for which a $\mathbb{Z}_2$ suffices to generate the mirror. In such a situation the fractional transform

$$z_1 = y_1^{1/2}, \qquad z_2 = y_1^{1/2} y_2 \tag{8.14}$$

immediately leads to the weighted representation of the orbifold mirror. In the exactly solvable framework such mirrors are obtained by replacing the diagonal invariant by a $D-$invariant in one of the factors of the tensor model. This is exemplified by the Gepner model

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A^4} \quad \sim \quad \mathbb{P}_{(3,8,33,88,132)}[264] \tag{8.15}$$

with $(h^{(1,1)}, h^{(2,1)}) = (57, 86)$, for which the replacement $86_A \rightarrow 86_D$ leads to the mirror configuration

$$(1 \cdot 6 \cdot 31 \cdot 86)_{A^3 D} \quad \sim \quad \mathbb{P}_{(3,8,66,88,99)}[264]. \tag{8.16}$$

Much more useful is the iterative application of the relation (8.12). For the diagonal model (8.4) the orbifolding with respect to the action $\mathbb{Z}_{10}^2$ leads to the fractional transformations

$$y_1 = z_1^{9/10}, \quad y_2 = z_1^{1/10} z_2^{9/10}, \quad y_3 = z_2^{1/10} z_3, \quad y_4 = z_4, \quad y_5 = z_5 \tag{8.17}$$

[4] This transformation has constant Jacobian and therefore preserves the measure of the path integral, up to an irrelevant factor.

which maps the orbifold into the weighted mirror configuration

$$\mathbb{P}_{(90,80,73,162,405)}[810] \;=\; \mathrm{FT}\left(\mathbb{P}_{(1,1,1,2,5)}[10]\Big/\mathbb{Z}_{10}^2 : \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.18}$$

The fractional transform is not restricted to the Fermat type Landau–Ginzburg theories of exactly solvable minimal models however. As an illustration consider the configuration

$$\mathbb{P}_{(1,1,1,1,3)}[7]^{(2,122)} \ni \{z_1^7 + z_2^7 + z_3^7 + z_4^7 + z_4 z_5^2 = 0\}. \tag{8.19}$$

After orbifolding by $\mathbb{Z}_7^2$ and applying the fractional transform we find the mirror configuration

$$\mathbb{P}_{(31,35,36,42,108)}[252]^{(122,2)} \;=\; \mathrm{FT}\left(\mathbb{P}_{(1,1,1,1,3)}[7]\Big/\mathbb{Z}_7^2 : \begin{bmatrix} 6 & 1 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 \end{bmatrix}\right). \tag{8.20}$$

The relation (8.12) shows that in general we expect some additional modding to be necessary on the fractional transform of the original orbifold in order to find the isomorphic representation. Even though the necessary modding of the FT image becomes part of the projective equivalence of the image in the majority of mirror theories it does happen that the orbifolding has to be performed. An example in case is the mirror orbifold (8.7) whose fractional transform leads to the weighted configuration $\mathbb{P}_{(3,4,9,9,11)}[36]$. This configuration has the cohomology $(h^{(1,1)}, h^{(2,1)}) = (20,32)$. We see that for the resulting weighted space the $\mathbb{Z}_2$ action on the weighted image is not trivial and must by modded out. Doing so results in the mirror spectrum, as it must. Thus we find for the mirror the relation

$$\begin{aligned} &\left(\mathbb{P}_{(2,4,9,9,12)}[36]^{(19,43)}\Big/\mathbb{Z}_3 :\ [2\ 0\ 0\ 0\ 1]\right)^{(43,19)} \sim \\ &\qquad\qquad \left(\mathbb{P}_{(3,4,9,9,11)}[36]^{(20,32)}\Big/\mathbb{Z}_2 :\ [1\ 0\ 0\ 0\ 1]\right)^{(43,19)}. \end{aligned} \tag{8.21}$$

8.3. The Berglund–Hübsch Construction

For manifolds defined by polynomials which are not of Fermat type there exists an alternative construction which sometimes allows to determine the weighted representation of the mirror configuration. This method [29] is based on the 'transposition of polynomials'. In general the transpose of a manifold does not suffice to construct the mirror of a hypersurface and must be accompanied by an additional orbifolding. It does happen for certain configurations, however, that this additional modding is not necessary in which case transposition itself already generates the mirror weights. In such circumstances we are not restricted to the use of fractional transformations in order to generate the weights of the mirror stable bundle but instead can use the transposition prescription or a combination thereof.

A simple class of models is described by the n-tadpole polynomials [1]

$$p = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}, \tag{8.22}$$

whose degree matrix can be written as

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}, \tag{8.23}$$

in which the rows indicate the degrees for each of the monomials. The transposed polynomial [29] is defined by the transpose of its degree matrix[5]

$$\begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}. \tag{8.24}$$

Since the dimension of the manifold is to be unchanged under transposition it is apparent that only those polynomials are amenable to transposition for which the number of monomials is equal to the number of variables, excluding the exceptional types discussed in [1].

As mentioned previously, the variety defined by the transposed polynomial is not, in general, the mirror of the original one defined by the polynomial p. Again the modding of an action is the missing ingredient. The orbifolding which produces mirror pairs for pairs of transposed manifolds is determined by splitting off the cyclic group $\mathbb{Z}_d$ whose order is the degree of the polynomial. The symmetry group $D_p = \mathbb{Z}_{a_1 \cdots a_n}$ thus can be decomposed into the quantum phase group [35] $\mathbb{Z}_d$ and the geometric phase group G_p as $D_p = \mathbb{Z}_d \times G_p$

[5] The resulting transformation does not have constant Jacobian.

and $\mathbb{Z}_{d_t} \times G_{p_t}$. The prescription for determining the additional orbifolding is that by going from a manifold to its mirror the role of the quantum and the geometric phase groups are exchanged [36]. Toric descriptions of this construction can be found in [31,32].

We illustrate this with some examples. First consider the manifold

$$\mathbb{P}_{(1,4,16,19,40)}[80]^{(15,127)} \ni \{x_1^{61}x_4 + x_2^{20} + x_3^5 + x_1x_4^4 + x_5^2 = 0\}. \tag{8.25}$$

The matrix of exponents is given by

$$\begin{bmatrix} 61 & 0 & 0 & 1 & 0 \\ 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{8.26}$$

Transposing it leads to the hypersurface $\mathbb{P}_{(10,23,122,150,305)}[610]^{(127,15)}$ with the correct mirror spectrum.

An even simpler example is provided by $\mathbb{P}_{(4,5,13,13,30)}[65]$ for which partial transposition

$$\begin{bmatrix} 13 & 0 & 1 & 0 & 0 \\ 0 & 13 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \oplus \begin{bmatrix} 13 & 1 \\ 0 & 2 \end{bmatrix} \tag{8.27}$$

leads to the mirror configuration $\mathbb{P}_{(5,5,12,13,30)}[65]$.

Finally, we identify a second mirror configuration of the hypersurface (8.19). By taking the transpose of the manifold

$$\mathbb{P}_{(35,43,48,84,126)}[336]^{(122,2)} \ni \{z_1^6z_5 + z_1z_2^7 + z_3^7 + z_4^4 + z_4z_5^2 = 0\} \tag{8.28}$$

one finds a deformation of the 1-tadpole of (8.19) and therefore this configuration provides a second weighted representation of the mirror of this 1-tadpole.

9. (0,2) Mirror Symmetry

In the class of $(2,2)$ vacua a mirror symmetric pair is given by two linear $\sigma-$models or two toric varieties, respectively, which lead to isomorphic superconformal fixed points, differing only by the relative sign of the $U(1)$ charges. In the Calabi–Yau phase this leads to an interchange of complex and Kähler deformations. In the $(0,2)$ setting the first definition still makes sense, except that in the (0,2) Calabi–Yau phase we are lead to more general cohomology groups, such as $H^{(0,1)}_{\bar\partial}(M,V)$ and $H^{(0,1)}_{\bar\partial}(M,V^*)$. It is this pair of groups which is interchanged, as described in the introduction. (0,2) mirror symmetric pairs thus differ both in the threefold and in the vector bundle.

We now apply the (2,2) constructions reviewed in the previous Section in order to construct (0,2) mirror pairs. We will first consider the orbifolding procedure of the exactly solvable models and then determine the weighted (0,2) Landau–Ginzburg mirrors by applying fractional transformations and/or transposition in order to obtain the needed weights of the mirror gauge bundle.

9.1. $(0,2)$ *Orbifold Mirrors*

As in the (2,2) case we can orbifold the exactly solvable tensor model with respect to the cyclic groups $\mathbb{Z}_{K_i+2}$ in each of the minimal factors to generate the exact (2,2) mirror. By applying the supersymmetry breaking simple current (2.9) to this (2,2) mirror we obtain the (0,2) mirror theory. It is clear from this fact that the space of (0,2) theories which we consider is trivially mirror symmetric by virtue of the order–disorder duality of the individual minimal factors [37].

This observation, however, is not particularly useful because this orbifolding is not an operation on the conformal field theory describing the target space. In order to define the modding procedure on the string ground state proper we need to restrict the symmetry actions to leave invariant the holomorphic threeform of the base space. It can be shown that it is always possible to generate the exact mirror of Gepner models with actions of this type and therefore we arrive at the exact (0,2) mirror after implementing the supersymmetry breaking simple current.

As an example we continue our discussion of the model (8.7). The application of the supersymmetry breaking simple current

$$J_{\rm sb} = (0\ 0\ 0)(0\ 9\ 1)(0\ 0\ 0)(0\ 0\ 0)(0\ 0\ 0)(1)(0) \tag{9.1}$$

leads to a (0,2) theory whose LG/CY phase is described by

$$V_{(2,4,9,9,12;\ 36)} \rightarrow \mathbb{P}_{(2,8,9,9,12,16)}[24\ \ 32] \tag{9.2}$$

and whose spectrum can be computed to be $(N_{16}, N_{\overline{16}}) = (34, 10)$. Combining therefore the simple current (8.9) which produces the (2,2) mirror with supersymmetry breaking simple current (9.1) we obtain the (0,2) exactly solvable theory

$$\left(\left(16\cdot 7\cdot 2^2\cdot 1\right)_{A^5} \Big/ J_{\rm orb}\otimes J_{\rm sb}\right)^{(10,34)}. \tag{9.3}$$

Thus we derive the mirror theory by applying the move to the mirror of the (2,2) theory.

9.2. $(0,2)$ *LG Mirrors via Fractional Transformations*

To generalize mirror symmetry to the (0,2) Landau–Ginzburg framework we proceed as follows. Given a (0,2) LG model we first apply the move in order to obtain the (2,2) Calabi–Yau manifold. For this CY space we construct the weighted mirror configuration. We then apply the move in reverse order to get the (0,2) mirror. We thus realize our general strategy described in the introduction in the present context via

$$\begin{array}{ccc} V_{(k_1,k_2,k_3,k_4,k_5;d)} \to \mathbb{P}_{(k_1,k_2,k_3,k_4,2k_5,\ell k_5)}[(\ell+2)k_5 \ \ 2\ell k_5] & \longrightarrow & \mathbb{P}_{(k_1,k_2,k_3,k_4,k_5)}[d] \\ \downarrow & & \downarrow \\ V_{(k_1',k_2',k_3',k_4',k_5';d')} \to \mathbb{P}_{(k_1',k_2',k_3',k_4',2k_5',\ell k_5')}[(\ell+2)k_5' \ \ 2\ell k_5'] & \longleftarrow & \mathbb{P}_{(k_1',k_2',k_3',k_4',k_5')}[d']. \end{array} \tag{9.4}$$

There are a number of subtleties associated with this idea. Consider e.g. the quintic hypersurface. It can be seen via fractional transformations that the mirror of the quintic can be described by

$$\mathbb{P}_{(41,48,51,52,64)}[256] = \mathrm{FT}\left(\mathbb{P}_4[5] \Big/ \mathbb{Z}_5^4 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix}\right) \tag{9.5}$$

. This weighted hypersurface representation of the mirror is not a configuration which is amenable to the application of the move – there are no coordinates such that $d/k_i = (2\ell+1)$. It can be shown, however, that only three of the cyclic groups $\mathbb{Z}_5$ are independent, the fourth one being part of the projective equivalence of the space

$$\mathbb{P}_{(51,64,60,80,65)}[320] = \mathrm{FT}\left(\mathbb{P}_4[5] \Big/ \mathbb{Z}_5^3 : \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \end{bmatrix}\right), \tag{9.6}$$

obtained via fractional transformations from the $\mathbb{Z}_5^3$ orbifold. This space does contain an appropriate coordinate and we can apply the move map which leads us to the mirror candidate

$$V_{(51,64,60,80,65;\ 320)} \to \mathbb{P}_{(51,60,80,65,128,128)}[256 \ \ 256]. \tag{9.7}$$

The first obstacle thus is that the weighted representation of the (2,2) mirror of an odd configuration must itself be odd.

Returning yet again to our example from Section 2 based on $\left(3 \cdot 8^3\right)_{A^4}$ consider

$$V_{(1,1,1,2,5;\ 10)} \to \mathbb{P}_{(1,1,1,4,4,5)}[8 \ \ 8] \tag{9.8}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (113, 5)$. The fractional transform (8.18) of its (2,2) image $\mathbb{P}_{(1,1,1,2,5)}[10]$ leads to the (0,2) mirror configuration

$$V_{(90,80,73,162,405;\ 810)} \rightarrow \mathbb{P}_{(90,80,73,324,324,405)}[648\ \ 648] \tag{9.9}$$

with the reversed (0,2) spectrum.

Next consider

$$V_{(1,1,3,5,5;\ 15)} \rightarrow \mathbb{P}_{(1,1,5,5,6,6)}[12\ \ 12] \tag{9.10}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (80, 8)$. Via a move and the fractional transform of the (2,2) orbifold mirror

$$\mathbb{P}_{(15,13,42,70,70)}[210] = \mathrm{FT}\left(\mathbb{P}_{(1,1,3,5,5)}[10] \Big/ \mathbb{Z}_{15} : [\,14 \quad 1 \quad 0 \quad 0 \quad 0\,]\right) \tag{9.11}$$

we arrive at the (0,2) mirror theory

$$V_{(15,13,42,70,70;\ 210)} \rightarrow \mathbb{P}_{(15,13,70,70,84,84)}[168\ \ 168] \tag{9.12}$$

with the correct mirror spectrum.

A second possible obstacle is that in general it can happen that one hypersurface gives rise to more than one (0,2) model and hence to more than one mirror relation. In such a situation it is necessary to be able to use different representations of the mirror by mapping different representations of the mirror orbifold via fractional transformations. As an example consider the manifold $\mathbb{P}_{(1,2,6,18,27)}[54]$ which gives rise to the two (0,2) models

$$\begin{aligned} \ell = 4: &\quad V_{(1,2,6,18,27;\ 54)} \rightarrow \mathbb{P}_{(1,2,12,18,24,27)}[36\ \ 48] \\ \ell = 13: &\quad V_{(1,2,6,18,27;\ 54)} \rightarrow \mathbb{P}_{(1,4,6,18,26,27)}[30\ \ 52], \end{aligned} \tag{9.13}$$

each of which gives rise to a mirror diagram. For $\ell = 4$ we orbifold with respect to the group $\mathbb{Z}_{27} \times \mathbb{Z}_3$ to find the Calabi–Yau mirror $\mathbb{P}_{(1,2,6,18,27)}[54]$. Transforming this (2,2) model into a (0,2) theory leads to

$$\begin{array}{ccc} V_{(1,2,6,18,27;54)} \rightarrow \mathbb{P}_{(1,2,12,18,24,27)}[36\ 48] & \longrightarrow & \mathbb{P}_{(1,2,6,18,27)}[54] \\ \downarrow & & \downarrow \\ V_{(18,51,104,295,468;936)} \rightarrow \mathbb{P}_{(18,51,208,295,416,468)}[624\ 832] & \longleftarrow & \mathbb{P}_{(18,51,104,295,468)}[936]. \end{array} \tag{9.14}$$

For $\ell = 13$ on the other hand we orbifold with respect to $\mathbb{Z}_8 \times \mathbb{Z}_3$ and thereby are led to the (0,2) mirror pair

$$\begin{array}{ccc} V_{(1,2,6,18,27;54)} \rightarrow \mathbb{P}_{(1,4,6,18,26,27)}[30\ 52] & \longrightarrow & \mathbb{P}_{(1,2,6,18,27)}[54] \\ \downarrow & & \downarrow \\ V_{(18,32,141,241,432;864)} \rightarrow \mathbb{P}_{(18,64,141,241,416,432)}[480\ 832] & \longleftarrow & \mathbb{P}_{(18,32,141,241,432)}[864]. \end{array} \tag{9.15}$$

A third possible obstacle is that a weighted representation of the (2,2) mirror must be found such that the resulting (0,2) theory is not inconsistent because of gauginos. Consider the (0,2) image of the move with $\ell = 3$ of the $(2,2)$ mirror pair in (8.20). This leads to

$$V_{(1,1,1,1,3;\ 7)} \rightarrow \mathbb{P}_{(1,1,1,2,3,3)}[5\ \ 6] \tag{9.16}$$

with spectrum $(N_{16}, N_{\overline{16}}) = (86, 2)$ and

$$V_{(31,35,36,42,108;\ 252)} \rightarrow \mathbb{P}_{(31,35,42,72,108,108)}[180\ \ 216]. \tag{9.17}$$

The latter model is not consistent because there inevitably are gauginos in the untwisted sector. In the next Section we will see that the transposition of bundles provides for a (0,2) mirror configuration.

9.3. Transposition of Stable Bundles

We can carry out the transposition of Calabi–Yau manifold almost verbatim. In the context of the examples discussed above we obtain the following. First consider the (0,2) theory

$$V_{(1,4,16,19,40;\ 80)} \rightarrow \mathbb{P}_{(1,4,19,32,32,40)}[64\ \ 64] \tag{9.18}$$

with the move image $\mathbb{P}_{(1,4,16,19,40)}[80]^{(15,127)}$. As shown in (8.25) and (8.26) the transposition rule leads to the hypersurface $\mathbb{P}_{(10,23,122,150,305)}[610]^{(127,15)}$ to which we can again apply a move with $\ell = 2$ to obtain

$$V_{(10,23,122,150,305;\ 610)} \rightarrow \mathbb{P}_{(10,23,150,244,244,305)}[488\ \ 488]. \tag{9.19}$$

An even simpler example is provided by

$$V_{(4,5,13,13,30;\ 65)} \rightarrow \mathbb{P}_{(4,5,13,26,26,30)}[52\ \ 52] \tag{9.20}$$

with spectrum (21,41) and mirror

$$V_{(5,5,12,13,30;\ 65)} \longrightarrow \mathbb{P}_{(5,5,12,26,26,30)}[52\ \ 52] \tag{9.21}$$

with spectrum (41,21), obtained by only partial transposition of

$$\begin{bmatrix} 13 & 0 & 1 & 0 & 0 \\ 0 & 13 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \oplus \begin{bmatrix} 13 & 1 \\ 0 & 2 \end{bmatrix} \tag{9.22}$$

in the fiber.

The last example to discuss in this Section is the $(0,2)$ image of the mirror pairs in (8.28). In contrast to (9.17), in this case the $(0,2)$ mirror of

$$V_{(1,1,1,1,3;\ 7)} \longrightarrow \mathbb{P}_{(1,1,1,2,3,3)}[5\ \ 6] \tag{9.23}$$

with $(N_{16}, N_{\overline{16}}) = (86, 2)$ really is

$$V_{(35,43,48,84,126;\ 336)} \longrightarrow \mathbb{P}_{(35,43,84,96,126,144)}[240\ \ 288] \tag{9.24}$$

with $(N_{16}, N_{\overline{16}}) = (2, 86)$.

In general then the problem of constructing (0,2) Landau–Ginzburg mirror pairs splits into two parts. Given an odd configuration we first need to check whether it is possible to find complete intersection representation of the mirror orbifold. Both the fractional transform and the transposition operation in general need an additional orbifolding on the image theory and and it is a priori not clear whether the necessary orbifolding will be trivial by becoming part of the projective equivalence of the purported weighted mirror configuration. Second, even if one succeeds in finding a hypersurface fractional transform of the orbifold, in order for our construction to work, one still has to make certain that this complete intersection is odd in the way defined above.

It can be checked that for many Landau–Ginzburg theories both of these problems can be solved via the fractional transform, transposition or a combination thereof.

10. Discussion and open problems

By restricting ourselves to $(0,2)$ models which have a LG description and originate from $(2,2)$ models admitting a LG phase we should not expect to get a completely mirror symmetric class of theories. There are a number of reasons for the resulting asymmetry.

Firstly, our focus has been on odd configurations, thereby restricting our considerations to a subclass of Landau–Ginzburg theories. More importantly, however, the class of all $(2,2)$ LG models itself is not mirror symmetric. The missing mirror partners have been shown [32] to be more general hypersurfaces in toric varieties which can formally be written as nontransversal hypersurfaces in weighted projective spaces. Consider, for instance, the manifold $\mathbb{P}_{(5,8,12,15,35)}[75]^{(30,27)}$ contained in the list of [23]. This space has no mirror partner in the class of Landau–Ginzburg theories. It can be described in terms of a reflexive polyhedron as a hypersurface in a toric variety and therefore Batyrev's construction [30] shows that the mirror can be described by its dual polyhedron. It is therefore clear that the weights associated to the dual polyhedron cannot admit a transverse polynomial and must lead to a weighted manifold with hypersurface singularities [32]. It can be shown [29] by combining transposition with fractional transformations that the weighted configuration is given by $\mathbb{P}_{(2,4,5,5,9)}[25]$. All manifolds in this deformation class have a hypersurface singularity at the point $(0,0,0,0,1)$. Applying the move yields the possible $(0,2)$ mirror pair

$$\begin{aligned} V_{(5,8,12,15,35;75)} &\rightarrow \mathbb{P}_{(5,8,12,30,30,35)}[60\ \ 60]\Big|_{(18,30)} \\ V_{(2,4,5,5,9;25)} &\rightarrow \mathbb{P}_{(2,4,5,9,10,10)}[20\ \ 20]. \end{aligned} \tag{10.1}$$

The coordinate of weight $k = 9$ can not appear in the constraints in a transversal way hence this model does not have a $(0,2)$ LG phase and further analysis has to await a more general description of (0,2) models.

Even more strikingly, it can also happen that a nontransversal $(2,2)$ model yields a transversal $(0,2)$ model. For instance, applying the move to the $(2,2)$ toric variety $\mathbb{P}_{(2,2,2,3,5)}[14]$ gives

$$V_{(2,2,2,3,5;\ 14)} \rightarrow \mathbb{P}_{(2,2,3,4,5,6)}[10\ \ 12]\Big|_{(51,3)}. \tag{10.2}$$

Due to the new constraint of weight $d = 10$ the coordinate of weight $k = 5$ can now appear in a transversal way leading to a $(0,2)$ model which has a bona fide LG phase. This example indicates that even in the weighted framework the (0,2) class is much richer than the more restricted (2,2) vacua.

A further phenomenon specific to (0,2) models is that, as we have discussed, a $(2,2)$ LG model can lead to a $(0,2)$ model which is destabilized.

Finally, it can happen that given a $(2,2)$ mirror pair with the same ℓ the resulting $(0,2)$ models are not mirror partners. Consider, for instance, the following two models

$$
\begin{aligned}
&V_{(3,4,4,4,5;20)} \rightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16 \;\; 16]\Big|_{(42,2)} \\
&V_{(13,15,16,16,20;80)} \rightarrow \mathbb{P}_{(13,15,16,20,32,32)}[64 \;\; 64]\Big|_{(8,36)}.
\end{aligned} \tag{10.3}
$$

The $(2,2)$ parents are mirror symmetric, in particular as already discussed they can be realized as orbifolds of the quintic by the discrete subgroups $\mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$, respectively. The puzzle is resolved by realizing that the SCFT based on the $(K=3)^5$ Gepner parent and the following two simple currents

$$
\begin{aligned}
J_1 &= (0\ 5\ 1)(0\ 0\ 0)^4(1)(0) \\
J_2 &= (0\ 2\ 0)(0\ \ {-2}\ 0)(0\ 0\ 0)^3(0)(0)
\end{aligned} \tag{10.4}
$$

yields the $(0,2)$ spectrum $(N_{16}, N_{\overline{16}}) = (36,8)$. This observation naturally leads to the suggestion that the mirror of

$$
V_{(13,15,16,16,20;\ 80)} \rightarrow \mathbb{P}_{(13,15,16,20,32,32)}[64 \;\; 64]\Big|_{(8,36)} \tag{10.5}
$$

is a further $(0,2)$ orbifold of

$$
V_{(3,4,4,4,5;\ 20)} \rightarrow \mathbb{P}_{(3,4,4,5,8,8)}[16 \;\; 16]\Big|_{(42,2)} \tag{10.6}
$$

and vice versa. In order to check this not only on the level of SCFT but on the level of LG models, it would be interesting to develop methods to deal with general $(0,2)$ LG orbifolds.

The same happens with the models

$$
\begin{aligned}
&V_{(4,9,12,15,20;\ 60)} \longrightarrow \mathbb{P}_{(4,9,15,20,24,24)}[48 \;\; 48]\Big|_{(39,3)} \\
&V_{(5,11,12,12,20;\ 60)} \longrightarrow \mathbb{P}_{(5,11,12,20,24,24)}[48 \;\; 48]\Big|_{(13,25)},
\end{aligned} \tag{10.7}
$$

the $(2,2)$ parents of which are orbifolds of the $\left(1 \cdot 3^3 \cdot 15\right)_{A^5}$ Gepner model.

Even though in dealing with $(0,2)$ models one has to be very careful and prepared for surprises, our results show that the status of $(0,2)$ models is on a par with (2,2) compactifications. As in the latter class we have established mirror symmetry and shown that it can be understood by extending known $(2,2)$ mirror constructions to the $(0,2)$ case. Thus mirror symmetry does in fact generalize to (0,2) theories.

Acknowledgements

It is a pleasure to thank M. Kreuzer and M. Lynker for discussions. R. S. thanks the physics department at Indiana University, South Bend, for hospitality during the course of part of this work. This work is supported by U. S. DOE grant No. DE–FG05–85ER–40219 and NSF PHY–9513835.

Appendix

Model			$N_{16/10}$	$N_{\overline{16}/\overline{10}}$
$V_{(1,9,14,18,21;63)}$	$\longrightarrow$	$\mathbb{P}_{(1,14,18,18,21,27)}[45 \;\; 54]$	35	32
$V_{(1,7,11,14,16;49)}$	$\longrightarrow$	$\mathbb{P}_{(1,11,14,14,16,21)}[35 \;\; 42]$	35	32
$V_{(1,21,30,38,45;135)}$	$\longrightarrow$	$\mathbb{P}_{(2,21,30,38,45,67)}[69 \;\; 134]$	31	28
$V_{(3,4,21,35,42;105)}$	$\longrightarrow$	$\mathbb{P}_{(4,6,21,35,42,51)}[57 \;\; 102]$	31	28
$V_{(1,6,14,21,21;63)}$	$\longrightarrow$	$\mathbb{P}_{(2,6,14,21,21,31)}[33 \;\; 62]$	31	28
$V_{(2,3,11,18,23;57)}$	$\longrightarrow$	$\mathbb{P}_{(2,6,11,18,23,27)}[33 \;\; 54]$	30	27
$V_{(3,6,16,21,23;69)}$	$\longrightarrow$	$\mathbb{P}_{(6,6,16,21,23,33)}[39 \;\; 66]$	27	24
$V_{(3,5,8,8,21;45)}$	$\longrightarrow$	$\mathbb{P}_{(5,6,8,8,21,21)}[27 \;\; 42]$	26	23
$V_{(3,4,13,27,34;81)}$	$\longrightarrow$	$\mathbb{P}_{(4,6,13,27,34,39)}[45 \;\; 78]$	25	22
$V_{(3,4,8,9,21;45)}$	$\longrightarrow$	$\mathbb{P}_{(4,6,8,9,21,21)}[27 \;\; 42]$	25	22
$V_{(4,5,7,25,34;75)}$	$\longrightarrow$	$\mathbb{P}_{(4,7,10,25,34,35)}[45 \;\; 70]$	24	21
$V_{(6,15,21,28,35;105)}$	$\longrightarrow$	$\mathbb{P}_{(6,21,28,30,35,45)}[75 \;\; 90]$	21	18
$V_{(3,11,14,26,27;81)}$	$\longrightarrow$	$\mathbb{P}_{(6,11,14,26,27,39)}[45 \;\; 78]$	17	14
$V_{(1,9,10,30,49;99)}$	$\longrightarrow$	$\mathbb{P}_{(2,9,10,30,49,49)}[51 \;\; 98]$	54	57
$V_{(3,4,9,20,27;63)}$	$\longrightarrow$	$\mathbb{P}_{(3,4,18,20,27,27)}[45 \;\; 54]$	33	36
$V_{(3,4,14,21,39;81)}$	$\longrightarrow$	$\mathbb{P}_{(4,6,14,21,39,39)}[45 \;\; 78]$	31	34
$V_{(1,18,32,39,45;135)}$	$\longrightarrow$	$\mathbb{P}_{(2,18,32,39,45,67)}[69 \;\; 134]$	28	31
$V_{(3,12,21,34,35;105)}$	$\longrightarrow$	$\mathbb{P}_{(6,12,21,34,35,51)}[57 \;\; 102]$	27	30
$V_{(3,15,18,26,31;93)}$	$\longrightarrow$	$\mathbb{P}_{(6,15,18,26,31,45)}[51 \;\; 90]$	21	24
$V_{(3,9,19,24,26;81)}$	$\longrightarrow$	$\mathbb{P}_{(6,9,19,24,26,39)}[45 \;\; 78]$	21	24
$V_{(3,8,21,30,31;93)}$	$\longrightarrow$	$\mathbb{P}_{(6,8,21,30,31,45)}[51 \;\; 90]$	18	21
$V_{(3,4,13,18,19;57)}$	$\longrightarrow$	$\mathbb{P}_{(4,6,13,18,19,27)}[33 \;\; 54]$	17	20
$V_{(4,5,5,7,14;35)}$	$\longrightarrow$	$\mathbb{P}_{(4,5,7,10,14,15)}[25 \;\; 30]$	16	19
$V_{(0,2,3,3,6,7;21)}$	$\longrightarrow$	$\mathbb{P}_{(2,6,6,6,7,9,9,21)}[15 \;\; 15 \;\; 18 \;\; 18]$	21	18
$V_{(0,1,1,3,4,6;15)}$	$\longrightarrow$	$\mathbb{P}_{(2,2,3,4,6,7,7,15)}[9 \;\; 9 \;\; 14 \;\; 14]$	17	14
$V_{(0,1,1,2,2,5;11)}$	$\longrightarrow$	$\mathbb{P}_{(2,2,2,2,5,5,5,11)}[7 \;\; 7 \;\; 10 \;\; 10]$	31	34
$V_{(0,1,9,14,18,21;63)}$	$\longrightarrow$	$\mathbb{P}_{(2,14,18,18,21,27,31,63)}[33 \;\; 45 \;\; 54 \;\; 62]$	21	24

Table A.1: *Three generation models with gauge group $SO(10)$ and $SU(5)$.*

References

[1] P. Candelas, M. Lynker and R. Schimmrigk, *Calabi–Yau manifolds in weighted* $\mathbb{P}_4$, Nucl. Phys. **B341** (1990) 383

[2] B. R. Greene and R. Plesser, *Duality in Calabi–Yau moduli space*, Nucl. Phys. **B338** (1990) 15

[3] C. Hull and P. Townsend, *Unity of superstring dualities*, Nucl. Phys. **B438** (1995) 109, hep–th/9410167;
E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B443** (1995) 85, hep–th/9503124

[4] J. Schwarz, *The power of M theory*, Phys. Lett. **B367** (1996) 97, hep–th/9510086;
P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. **B460** (1996) 506, hep–th/9510209; *Eleven-dimensional supergravity on a manifold with boundary*, hep–th/9603142

[5] S. Kachru and C. Vafa, *Exact results for* $N = 2$ *compactifications of heterotic strings*, Nucl. Phys. **B450** (1995) 69, hep–th/9505105

[6] C. Vafa, *Evidence for F–theory*, Nucl. Phys. **B469** (1996) 403, hep–th/9602022;
D. R. Morrison and C. Vafa, *Compactifications of F–theory on Calabi–Yau threefolds I,II*, hep–th/9602114, hep–th/9603161

[7] P. Candelas, A. M. Dale, C. A. Lütken and R. Schimmrigk, *Complete intersection Calabi–Yau manifolds*, Nucl. Phys. **B298** (1988) 493

[8] P. Candelas, P. S. Green and T. Hübsch, *Finite distances between distinct Calabi–Yau vacua: (other worlds are just around the corner)*, Phys. Rev. Lett. **62** (1989) 1956; *Rolling among Calabi–Yau vacua*, Nucl. Phys. **B330** (1990) 49

[9] A. Strominger, *Massless black holes and conifolds in string theory*, Nucl. Phys. **B451** (1995) 96, hep–th/9504090;
B. R. Greene, D. R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, Nucl. Phys. **B451** (1995) 109, hep–th/9504145

[10] R. Blumenhagen, R. Schimmrigk and A. Wißkirchen, *The* $(0, 2)$ *exactly solvable structure of chiral rings, Landau–Ginzburg theories and Calabi–Yau manifolds*, Nucl. Phys. **B461** (1996) 460, hep–th/9510055

[11] R. Blumenhagen and A. Wißkirchen, *Exactly solvable* $(0, 2)$ *supersymmetric string vacua with GUT gauge groups*, Nucl. Phys. **B454** (1995) 561, hep–th/9506104

[12] A. N. Schellekens and S. Yankielowicz, *New modular invariants for* $N = 2$ *tensor products and four-dimensional strings*, Nucl. Phys. **B330** (1990) 103

[13] M. Dine, N. Seiberg, X. Wen and E. Witten, *Nonperturbative effects on the string world sheet I,II*, Nucl. Phys. **B278** (1986) 769; *ibid.* **B289** (1987) 319;
J. Distler, *Resurrecting* $(2, 0)$ *compactifications*, Phys. Lett. **B188** (1987) 431;

J. Distler and B. R. Greene, *Aspects of* $(2,0)$ *string compactifications*, Nucl. Phys. **B304** (1988) 1

[14] J. Distler and S. Kachru, *Singlet couplings and* $(0,2)$ *models*, Nucl. Phys. **B430** (1994) 13, hep–th/9406090;
E. Silverstein and E. Witten, *Criteria for conformal invariance of* $(0,2)$ *models*, Nucl. Phys. **B444** (1995) 161, hep–th/9503212

[15] R. Blumenhagen and A. Wißkirchen, *Exploring the moduli space of* $(0,2)$ *strings*, Nucl. Phys. **B475** (1996) 225, hep-th/9604140

[16] E. Witten, *Phases of* $N=2$ *theories in two dimensions*, Nucl. Phys. **B403** (1993) 159, hep–th/9301042

[17] J. Distler, B. R. Greene and D. R. Morrison, *Resolving singularities in* $(0,2)$ *models*, hep–th/9605222

[18] S. Kachru and E. Witten, *Computing the complete massless spectrum of a Landau–Ginzburg orbifold*, Nucl. Phys. **B407** (1993) 637, hep–th/9307038

[19] J. Distler and S. Kachru, $(0,2)$ *Landau-Ginzburg theory*, Nucl. Phys. **B413** (1994) 213, hep–th/9309110

[20] M. Lynker and R. Schimmrigk, *On the spectrum of* $(2,2)$ *compactifications of the heterotic string on conformal field theories*, Phys. Lett. **B215** (1988) 681; *A–D–E quantum Calabi–Yau manifolds*, Nucl. Phys. **B339** (1990) 121

[21] J. Fuchs, A. Klemm, C. Scheich and M. Schmidt, *Gepner models with arbitrary invariants and the associated Calabi–Yau spaces*, Phys. Lett. **B232** (1989) 317; *Spectra and symmetries of Gepner models compared to Calabi–Yau compactifications*, Ann. Phys. **204** (1990) 1

[22] M. Kreuzer and H. Skarke, *ADE string vacua with discrete torsion*, Phys. Lett. **B318** (1993) 305, hep–th/9307145

[23] A. Klemm and R. Schimmrigk, *Landau–Ginzburg string vacua*, Nucl. Phys. **B411** (1994) 559, hep–th/9204060;
M. Kreuzer and H. Skarke, *No mirror symmetry in Landau–Ginzburg spectra!*, Nucl. Phys. **B388** (1992) 113, hep–th/9205004

[24] T. Kawai and K. Mohri, *Geometry of* $(0,2)$ *Landau–Ginzburg orbifolds*, Nucl. Phys. **B425** (1994) 191, hep–th/9402148

[25] http://thew02.physik.uni–bonn.de/˜netah/cy.html;
http://www.math.okstate.edu/˜katz/CY

[26] R. Schimmrigk, *Critical superstring vacua from noncritical manifolds: a novel framework for string compactification and mirror symmetry*, Phys. Rev. Lett. **70** (1993) 3688, hep–th/9210062; *Mirror symmetry and string vacua from a class of Fano varieties*, Int. J. Mod. Phys. **A11** (1996) 3049, hep–th/9405086

[27] P. Candelas, E. Derrick and L. Parkes, *Generalized Calabi–Yau manifolds and the*

mirror of a rigid manifold, Nucl. Phys. **B407** (1993) 115, hep–th/9304045

[28] M. Lynker and R. Schimmrigk, *Landau–Ginzburg theories as orbifolds*, Phys. Lett. **B249** (1990) 237; *Conifold transitions and aspects of duality*, hep–th/9511058

[29] P. Berglund and T. Hübsch, *A generalized construction of mirror manifolds*, Nucl. Phys. **B393** (1993) 377, hep–th/9201014

[30] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3** (1994) 493, alg–geom/9310003

[31] P. Berglund and S. Katz, *Mirror symmetry constructions: a review*, to appear in *Essays on mirror manifolds II*, hep–th/9406008

[32] P. Candelas, X. de la Ossa and S. Katz, *Mirror symmetry for Calabi–Yau hypersurfaces in weighted* $\mathbb{P}_4$ *and extensions of Landau–Ginzburg theory*, Nucl. Phys. **B450** (1995) 267, hep–th/9412117

[33] D. Gepner, *Space–time supersymmetry in compactified string theory and superconformal models*, Nucl. Phys. **B296** (1988) 757

[34] D. Gepner, *Exactly solvable string compactification on manifolds of* $SU(n)$ *holonomy*, Phys. Lett. **B199** (1987) 380;
E. Martinec, *Algebraic geometry and effective Lagrangians*, Phys. Lett. **B217** (1989) 431;
C. Vafa and N. Warner, *Catastrophes and the classification of conformal theories*, Phys. Lett. **B218** (1989) 51;
B. R. Greene, C. Vafa and N. Warner, *Calabi–Yau manifolds and renormalization group flows*, Nucl. Phys. **B324** (1989) 371;
E. Witten, *On the Landau–Ginzburg description of* $N = 2$ *minimal models*, Int. J. Mod. Phys. **A9** (1994) 4783, hep–th/9304026

[35] C. Vafa, *Quantum symmetries of string vacua*, Mod. Phys. Lett. **A4** (1989) 1615

[36] P. Berglund and M. Henningson, *Landau-Ginzburg orbifolds, mirror symmetry and the elliptic genus*, Nucl. Phys. **B433** (1995) 311, hep–th/9401029;
M. Kreuzer, *The mirror map for invertible LG models*, Phys. Lett. **B328** (1994) 312, hep–th/9402114;
B. R. Greene, R. Plesser and S. S. Roan, *New constructions of mirror manifolds: Probing moduli space far from Fermat points*, in *Essays on mirror manifolds*, International Press, 1992

[37] D. Gepner and Z. Qiu, *Modular invariant partition functions for parafermionic field theories*, Nucl. Phys. **B285** (1987) 423